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Estimation of domains of attraction: A global optimization approach

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ABSTRACT

In this paper a methodology for the estimation of domains of attraction of stable equilibriums based on maximal Lyapunov functions is proposed. The basic idea consists in finding the best level set of a Lyapunov function which is fully contained in the region of negative definiteness of its time derivative. An optimization problem is formulated, which includes a tangency requirement between the level sets and constraints on the sign of the numerator and denominator of the Lyapunov function. Such constraints help in avoiding a large number of potential dummy solutions of the nonlinear optimization model. Moreover, since global optimality is also required for proper estimation, a deterministic global optimization solver of the branch and bound type is adopted. The methodology is applied to several examples to illustrate different aspects of the approach.

1. Introduction

The Domain of Attraction (DOA) of an asymptotically stable equilibrium point of a dynamic system is the portion of the states space where trajectories that converge to such equilibrium originate. Some knowledge of its size and shape is usually required for the proper planning of the operation of the nonlinear system [1]. However, for the general nonlinear case, the DOA is a complicated set that does not admit analytical representation.

Many techniques have been proposed to address the estimation of DOAs. Broadly, such techniques can be classified in simulation based, Lyapunov based and non-Lyapunov methodologies. The first family approaches the problem by characterizing the trajectories that lie on the boundary of the DOA. Typically, such boundaries are the stable manifolds of saddle nodes and limit cycles. The second family is constructed on elements of the Lyapunov stability theory. Non-Lyapunov methods do not employ explicitly Lyapunov functions. A comprehensive review of the existing methodologies is beyond the scope of this paper. The interested reader is referred to [2] for a clear classification of the techniques and a review of classic work on the topic. In this contribution a Lyapunov based methodology, which addresses the estimation of the DOA as a level set of a Lyapunov function, is adopted.

Among the Lyapunov based techniques, those whose rationale is to approximate the DOA by a level set of a Lyapunov function of the equilibrium point have been particularly studied. Specifically, many approaches have been proposed to identify the best level set of a given Lyapunov function by solving an optimization model. This problem is very challenging for the general case, since the resulting mathematical formulation corresponds to a nonlinear semi-infinite optimization model.

In the seminal contribution of Vannelli and Vidyasagar [3] the authors introduced the so-called Maximal Lyapunov functions which are rational type functions instead of polynomials. In such functions the denominator has a "blow up" effect near the boundaries of the DOA, with the result that their level sets closely represent the region of stability in that part of the states space. In that paper the authors proposed a recursive algorithm to simultaneously construct the maximal function and compute the estimation of the domain.

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Chesi and co-workers used Linear Matrix Inequalities (LMI) optimization [4] to address the estimation of DOAs of dynamic systems based on level sets of Lyapunov functions. In [5] a methodology was proposed that approximates the DOA as the union of an infinite number of level sets of Lyapunov functions instead of only one, as done in existing methods. A parameter dependent Lyapunov function was introduced and the corresponding levels sets computed by solving convex LMI optimization problems. In [6] a strategy for estimating DOAs of non-polynomial systems was proposed. The idea is to convert the non-polynomial optimization problem into a polynomial one, which can be addressed via convex LMI techniques. Non-polynomial terms are approximated by truncated Taylor series and the worst-case reminders are taken into account by parameterizing them into a convex polytope.

Formulations that make use of results on deterministic global optimization based on the theory of moments were recently proposed [7,8]. The technique allows the identification of the best possible level set of a rational Lyapunov function that constitutes an estimation of the DOA for nonlinear dynamic systems of polynomial type.

In this contribution, an extension of the formulation presented in [8] is proposed. An optimization model is formulated, which includes additional constraints to avoid possible dummy solutions, and makes use of global optimization software to address nonlinear systems of the general type. The technique is firstly illustrated by a two states system that presents a very rich nonlinear behavior and is then applied to several dynamic systems including a three states model. Some preliminary results of this technique can be found in [9].

The remaining of the paper is organized as follow. Section 2 introduces basic definitions and theorems used along the paper. In Section 3 the DOA estimation problem is presented. Section 4 describes the proposed approach. In Section 5 the proposed approach is applied to some illustrative examples. A conclusions section concludes the paper.

2. Background definitions and theorems

In this section, the basic definitions and theorems required to support the proposed contribution are introduced [1,10].

Consider the following autonomous nonlinear dynamic system:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = f(\mathbf{x}), \quad \mathbf{x} \in \Re^n,\ \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1}$$

being $\mathbf{x} = \mathbf{x}^*$, an asymptotically stable equilibrium point of (1).

Definition 2.1 (*Equilibrium Point*)**.** A point $\mathbf{x}^* \in \Re^n$ is called an equilibrium point of system (1) if $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$.

Remark 2.1. In what follows, we assume without loss of generality, that the equilibrium point under study coincides with the origin of the states space of $\Re^n$, ($\mathbf{x}^* = \mathbf{0}$).

Definition 2.2 (*Asymptotic Stability*)**.** Let $\mathbf{x}(t, \mathbf{x}_0)$ denote the trajectory initiated at state $\mathbf{x}_0$ in time t_0. Equilibrium $\mathbf{x}^* = \mathbf{0}$ of system (1) is asymptotically stable if there exists a $\eta > 0$ such that:

$$\lim_{t\to\infty} \mathbf{x}(t, \mathbf{x}_0) = \mathbf{0}, \quad \text{whenever } \|\mathbf{x}_0\| < \eta.$$

Lyapunov stability theory provides a tool to assess stability of equilibrium points by means of the so-called Lyapunov functions.

Theorem 2.1 (*Asymptotic Stability in the Lyapunov Sense*)**.** *If there exists a Lyapunov function $V(\mathbf{x})$ for equilibrium point $\mathbf{x} = \mathbf{0}$ of system (1), then $\mathbf{x} = \mathbf{x}^* = \mathbf{0}$ is asymptotically stable.*

Definition 2.3 (*Lyapunov Function*)**.** Let $V(\mathbf{x})$ be a continuously differentiable real-valued function defined on a domain $R(\mathbf{0}) \subseteq \Re^n$ containing equilibrium $\mathbf{x} = \mathbf{0}$. Function $V(\mathbf{x})$ is a Lyapunov function of equilibrium $\mathbf{x} = \mathbf{0}$ of system (1) if the following conditions hold:

- $V(\mathbf{x})$ is positive definite on $R(\mathbf{0})$.
- The time derivative of $V(\mathbf{x})$, $\frac{\mathrm{d}V(\mathbf{x})}{\mathrm{d}t} = [\nabla V(\mathbf{x})]^{\mathrm{T}}\, \mathbf{f}(\mathbf{x})$, is negative definite on $R(\mathbf{0})$.

Definition 2.4 (*Positive and Negative Definite Functions*)**.** A continuously differentiable real-valued function $\varphi(\mathbf{x})$ defined on a domain $R(\mathbf{0}) \subseteq \Re^n$ containing point $\mathbf{x} = \mathbf{0}$ is called positive definite if the following conditions hold:

- $\varphi(\mathbf{0}) = 0$.
- $\varphi(\mathbf{x}) > 0\ \forall \mathbf{x} \in \{R(\mathbf{0}) \backslash \mathbf{0}\}$.

Function $\varphi(\mathbf{x})$ is negative definite if $-\varphi(\mathbf{x})$ is positive definite.

Remark 2.2. In the following, the symbol $\succ 0 (\prec 0)$ is used to denote positive (negative) definiteness of functions.

Definition 2.5 (*Domain of Attraction*)**.** The DOA of the equilibrium point $\mathbf{x} = \mathbf{0}$ is given by:

$$\mathrm{DOA}(\mathbf{0}) = \{\mathbf{x}_0 \in \Re^n : \lim_{t\to\infty} \mathbf{x}(t, \mathbf{x}_0) \to \mathbf{0}\}. \tag{2}$$

Lyapunov stability theory provides the basis of a family of techniques for estimation of regions of asymptotic stability whose rationale is to approximate the DOA($\mathbf{0}$) by a level set of a Lyapunov function of the equilibrium point.

Theorem 2.2 (*Estimation of the Domain of Attraction*)**.** *Let $V(\mathbf{x})$ be a Lyapunov function for equilibrium $\mathbf{x} = \mathbf{0}$ of system (1). Consider that $\mathrm{d}V(\mathbf{x})/\mathrm{d}t$ is negative definite in the region:*

$$S(\mathbf{0}) = \{\mathbf{x} : V(\mathbf{x}) = c,\ c > 0\} \tag{3}$$

Then, every trajectory initiated within region $S(\mathbf{0})$ tends to $\mathbf{x} = \mathbf{0}$ as time tends to infinity.

3. Estimation of DOAs

Consider that a Lyapunov function $V(\mathbf{x})$ is given, meaning that:

$$V(\mathbf{x}) \succ 0 \quad \text{in } R(\mathbf{0}) \tag{4a}$$

$$\frac{\mathrm{d}V(\mathbf{x})}{\mathrm{d}t} \prec 0 \quad \text{in } R(\mathbf{0}). \tag{4b}$$

According to (3) the larger the level set value c, the better the estimation of DOA($\mathbf{0}$). The calculation of the maximum level set of the Lyapunov function which is still an estimation of DOA($\mathbf{0}$) can be obtained by solving a problem whose pseudo-optimization formulation is as follows:

$$\max_{c,\mathbf{x}}\ c$$

$$\text{s.t. } \{\mathbf{x} \text{ belong to level set } V(\mathbf{x}) - c = 0\} \tag{5a}$$

$$\{\mathbf{x} \text{ is any point belonging to region } S(\mathbf{0}) \text{ contained in } R(\mathbf{0})\}. \tag{5b}$$

The idea behind problem (5) is to find the maximum level set of $V(\mathbf{x})$ (5a) which is fully contained in the region of negative definiteness of $\mathrm{d}V(\mathbf{x})/\mathrm{d}t$ (5b). This can be obtained by solving the following minimization problem to global optimality [8]:

$$\begin{aligned} &\min_{c,\mathbf{x}}\ c \\ \text{s.t.}\quad & V(\mathbf{x}) - c = 0 \\ & \frac{\mathrm{d}V(\mathbf{x})}{\mathrm{d}t} = 0 \\ & c > 0. \end{aligned} \tag{6}$$

The rationale behind problem (6) is to find the minimum level set of function $V(\mathbf{x})$ which is contained within level set $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$. The desired solution is a single point in the states space, which corresponds to a contact of level sets $V(\mathbf{x}) = c$ and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$. It should be noted that model (6) is nonlinear and therefore it may have many local solutions. In order to converge to the right estimation, global optimality has to be ensured.

To graphically illustrate this approach consider the Vanderpol oscillator (Eq. (7)) [3] and a given Lyapunov function for such system (8).

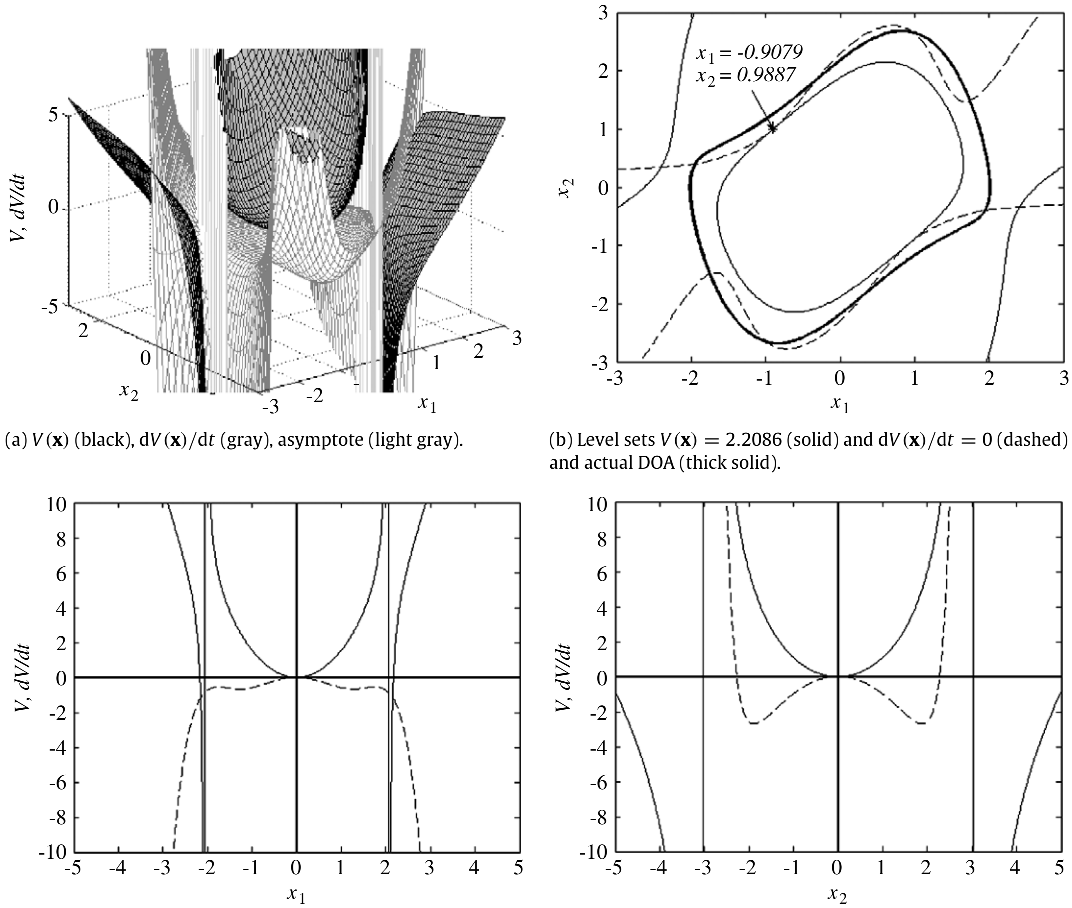
$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= -x_2 \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_1 - x_2(1 - x_1^2) \end{aligned} \tag{7}$$

$$V(\mathbf{x}) = \frac{0.593x_1^2 - 0.364x_1x_2 + 0.437x_2^2 - 0.1253x_1^4 + 0.2885x_1^3x_2 - 0.0537x_1^2x_2^2 + 0.0581x_1x_2^3 - 0.0196x_2^4}{1 - 0.0001x_1 + 0.0001x_2 - 0.2685x_1^2 + 0.3217x_1x_2 - 0.1163x_2^2}. \tag{8}$$

In Fig. 1(a) it is shown the Lyapunov function and its time derivative (in black and gray respectively). The asymptote to the region of non-definition of the Lyapunov function is shown in light gray. Lateral views of $V(\mathbf{x})$ (solid line) and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t$ (dashed line) are shown in Fig. 1(c) and (d) to better visualize the functions. In Fig. 1(b) the level sets of interest are depicted together with the actual DOA($\mathbf{0}$) (thick solid line). The closed curve $V(\mathbf{x}) = 2.2086$ (solid line) contained in the level set $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$ (dashed line) constitutes the desired solution of problem (6) for the system under study. It should be noted that in this particular system there are two points in the states space for the same $V(\mathbf{x}) = c$, due to symmetry. Since both solutions are equivalent regarding the resulting estimation, only one of them is reported.

The methodology described requires the availability of a Lyapunov function for the system under study. Hachicho (2007) [8], proposed the use of maximal Lyapunov functions in problem (6). Maximal Lyapunov functions have been presented in [3]. A maximal Lyapunov "candidate" is a rational function with the following structure:

$$V(\mathbf{x}) = \frac{N(\mathbf{x})}{D(\mathbf{x})} = \frac{\sum_{i=2}^{\infty} R_i(\mathbf{x})}{1 + \sum_{i=1}^{n-2} Q_i(\mathbf{x})} \tag{9}$$



(a) $V(\mathbf{x})$ (black), $\mathrm{d}V(\mathbf{x})/\mathrm{d}t$ (gray), asymptote (light gray).

(b) Level sets $V(\mathbf{x}) = 2.2086$ (solid) and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$ (dashed) and actual DOA (thick solid).

(c) Lateral view of $V(\mathbf{x})$ (solid) and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t$ (dashed) at $x_2 = 0$.

(d) Lateral view of $V(\mathbf{x})$ (solid) and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t$ (dashed) at $x_1 = 0$.

Fig. 1. Lyapunov function and time derivative for system (7)–(8).

where $N(\mathbf{x})$ and $D(\mathbf{x})$ are polynomials defined through $R_i(x)$ and $Q_i(x)$, which are homogeneous functions of degree i for $n \geqslant 2$. Note that $V(\mathbf{x})$ is defined such that it "blows up" when $D(\mathbf{x})$ approaches zero, unlike most Lyapunov candidates which are usually defined everywhere in the states space. By representing the system of nonlinear equations (1) in terms of a series of homogeneous functions of degree i, $F_i(x)$:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}) = \sum_{i=1}^{\infty} F_i(\mathbf{x}) \tag{10}$$

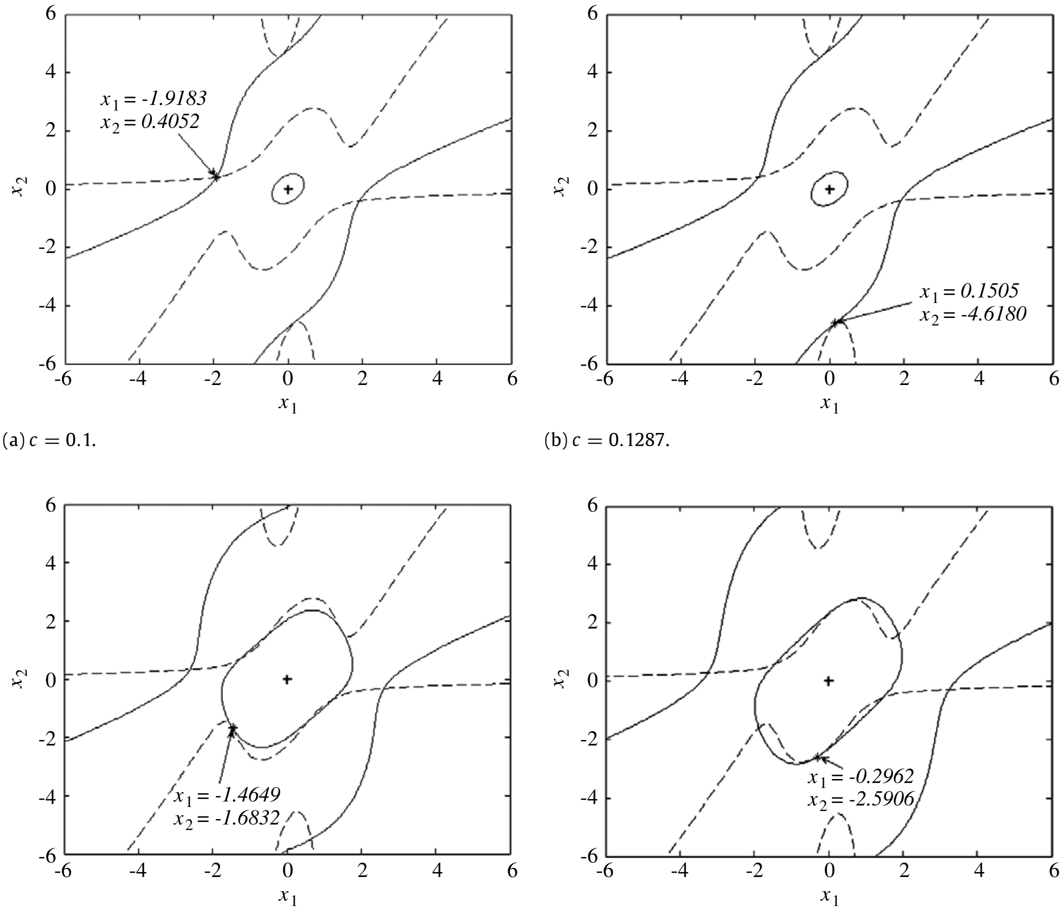
a Lyapunov function can be obtained with the procedure proposed in [3]. The estimations of DOAs by means of maximal functions are usually better than those calculated with other types of Lyapunov functions as demonstrated in [3]. Therefore, maximal Lyapunov functions are adopted in this contribution.

4. Proposed approach

Formulation (6) is a nonlinear optimization problem which may have many solutions that are not proper estimations of the DOA for the system under study. Consider for example the different solutions of problem (6) for system (7)–(8) shown in Fig. 2. In Fig. 2 (a)–(d) different level sets of the Lyapunov function (Eq. (8)) are shown in solid line. In all cases the level set of the time derivative at value zero is shown as dashed lines. Also note that as in the previous case, there are two solutions for the same $V(\mathbf{x}) = c$. Only one of them is reported in the following description.

The solution depicted in Fig. 2(a) is a solution of problem (6) with a value of c lower than that of the desired estimation of Fig. 1(b). This solution corresponds to a transversal intersection between the level set $V(\mathbf{x}) = 0.1$ and the level set $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$. It can be noticed that the intersection between the level sets takes place far from the origin and therefore a very small portion of the level set of $V(\mathbf{x})$ is in fact an estimation of the actual DOA (the small circle around the origin).

Unlike the solution of Fig. 2(a), that of Fig. 2(b) verifies the desired tangential intersection between the level sets. However, the estimation of the size of the DOA is also poor, because the contact between level sets takes place far from the origin.



(a) $c = 0.1$.

(b) $c = 0.1287$.

(c) $c = 2.8$.

(d) $c = 4.7652$.

Fig. 2. Dummy solutions of system (7)–(8). Level sets $V(\mathbf{x}) = c$ (solid) and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$ (dashed).

It can be observed in both previous solutions that although the constraints of problem (6) hold, both $V(\mathbf{x})$ and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t$ verify a transition of sign in an intermediate portion of the states space (see asymptote in Fig. 1(a)). Specifically $V(\mathbf{x})$ is positive definite in some domain around the origin, becomes negative in an intermediate region and recovers positive sign far from the origin (solid lines in Fig. 1(c) and (d)). Similarly $\mathrm{d}V(\mathbf{x})/\mathrm{d}t$ remains negative definite close the origin but becomes positive as it moves away in the states space (dashed lines in Fig. 1(d)). While the small circle is an estimation of the DOA since it is fully contained in the negative definiteness region of $\mathrm{d}V(\mathbf{x})/\mathrm{d}t$, the fact that the actual solution occurs beyond the transition of sign of the functions is clearly undesirable.

Fig. 2(c) and (d) are also solutions of problem (6) which present tangential intersections between the level sets $V(\mathbf{x}) = c$ and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$. However, none of them are actual estimations of the DOA since the level sets of $V(\mathbf{x})$ (solid lines) are not fully contained in the region of negative definiteness of $\mathrm{d}V(\mathbf{x})/\mathrm{d}t$ (dashed line). Since all solutions of Fig. 2 are valid solutions of problem (6) but none is the desired estimation of the actual DOA, they are considered "dummy solutions" of system (7)–(8).

From the above qualitative description it can be concluded that for a certain level set $V(\mathbf{x}) = c$ to be an optimal estimation of the DOA, it should verify the following conditions:

Condition 1. The level sets $V(\mathbf{x}) = c$ and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$ intersect tangentially.

Condition 2. The solution belongs to the portion of the states space before a transition of sign of $V(\mathbf{x})$ and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t$ takes place as $|\mathbf{x}| \to \infty$.

Condition 3. Level set $V(\mathbf{x}) = c$ is the global minimum.

Condition 1 avoids dummy solutions such as the one in Fig. 2(a). Condition 2 ensures that solutions of type Fig. 1(b) are preferred over a solution of type Fig. 2(b). Finally, the global minimum Condition 3 forces the solution of Fig. 1(b) over the tangential solutions of Fig. 2(c) and (d).

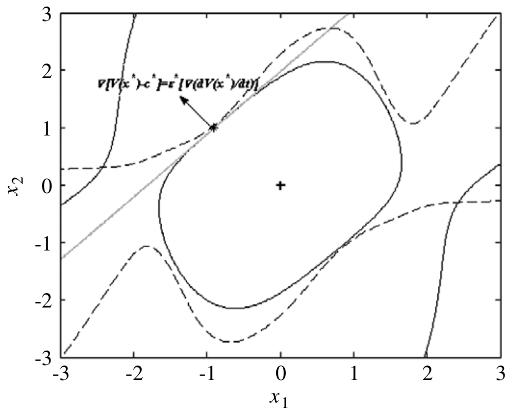


Fig. 3. Condition 1. Tangential contact between level sets $V(\mathbf{x}) = c$ (solid) and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$ (dashed).

While problem (6) explicitly implies global optimality (Condition 3), neither Condition 1 nor Condition 2 are considered. Therefore, in order to avoid the convergence to any of the dummy solutions previously discussed a new formulation that excludes these undesirable situations and ensures the identification of the right solution to the DOA estimation problem, is proposed in Eq. (11):

$$\min_{c,\mathbf{x},\varepsilon}\ c$$

$$\text{s.t. } V(\mathbf{x}) = \frac{N(\mathbf{x})}{D(\mathbf{x})} \tag{11a}$$

$$V(\mathbf{x}) - c = 0 \tag{11b}$$

$$\frac{\mathrm{d}V(\mathbf{x})}{\mathrm{d}t} = 0 \tag{11c}$$

$$\frac{\partial[V(\mathbf{x}) - c]}{\partial x_i} = \varepsilon \frac{\partial\left[\frac{\mathrm{d}V(\mathbf{x})}{\mathrm{d}t}\right]}{\mathrm{d}x_i}, \quad i = 1, \ldots, n \tag{11d}$$

$$N(\mathbf{x}) > 0 \tag{11e}$$

$$D(\mathbf{x}) > 0 \tag{11f}$$

$$\mathbf{x} \neq 0 \tag{11g}$$

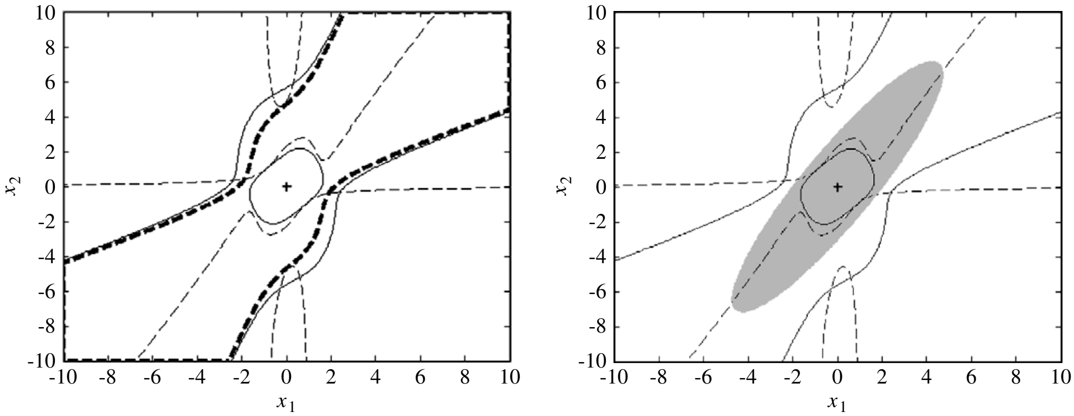
$$c > 0. \tag{11h}$$

Condition 1 is explicitly imposed in the formulation as a geometric constraint by requiring that the solution point on both level sets belong to the same hyper-plane in the states space. This constraint is mathematically expressed as the linear dependence of the gradients of the level sets at the solution (11d), where ε is an auxiliary variable and n the number of states of the dynamic system. Using vector notation, constraint (11d) can be expressed as: $\nabla\,[V(\mathbf{x}) - c] = \varepsilon\nabla\left[\frac{\mathrm{d}V(\mathbf{x})}{\mathrm{d}t}\right]$. This tangency condition is illustrated in Fig. 3 for the system under study. At the solution the normal vectors to the tangent plane are: $\nabla[V(\mathbf{x}) - c] = [-1.9838, 1.8135]^{\mathrm{T}}$ and $\nabla[\frac{\mathrm{d}V(x)}{\mathrm{d}t}] = [-0.8080, 0.7386]^{\mathrm{T}}$ and parameter ε is 2.4550.

Condition 2 on the other hand does not posses obvious mathematical expressions to be included within the optimization problem. In this contribution it is proposed to include a strict positive constraint on both the numerator and the denominator of function (9) to account for such condition (Eqs. (11e) and (11f)). In Fig. 4(a) it is shown the constraint $N(\mathbf{x}) > 0$ (region delimited by the thick dashed line) and in Fig. 4(b) the corresponding to $D(\mathbf{x}) > 0$ (shaded region).

Fig. 5 shows the intersection between the regions defined by constraints $N(\mathbf{x}) > 0$ and $D(\mathbf{x}) > 0$ (region delimited by the thick solid line). Such a region contains the desired solution, Fig. 5(a), but excludes the dummy solution shown in Fig. 2(b) as can be appreciated from Fig. 5(b).

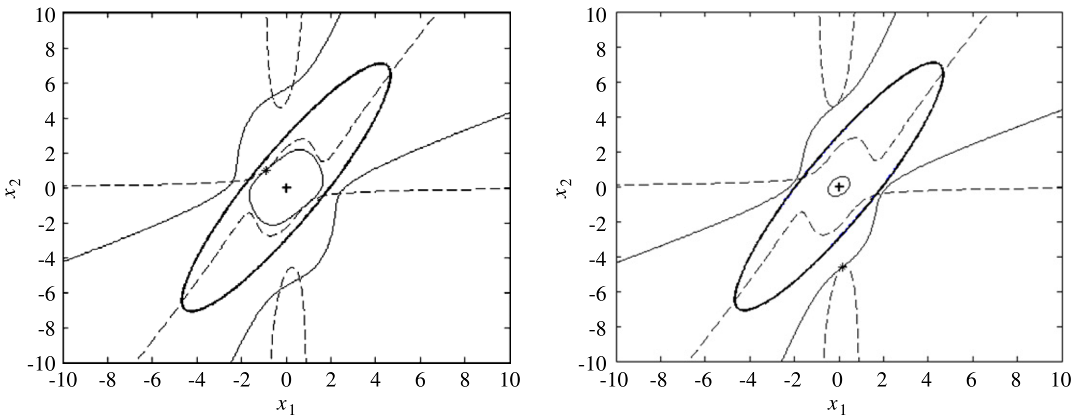
Finally, global optimality Condition 3 is ensured by solving the NLP problem (11) with state of the art global optimization software. In particular the GAMS platform [11] with the global optimization solver BARON [12] is adopted. BARON implements a deterministic global optimization algorithm of the branch and bound type, which guarantees to provide the global optima under fairly general assumptions. For a complete presentation of the theory behind the BARON solver see [13]. Constraints (11g) and (11h) are included to avoid getting trapped in the origin of the states space which is a meaningless solution.



(a) $N(\mathbf{x}) > 0$ (thick dashed).

(b) $D(\mathbf{x}) > 0$ (shaded).

Fig. 4. Condition 2. Strict positive sign on numerator and denominator of the Lyapunov function.



(a) Desired solution included.

(b) Dummy solution excluded.

Fig. 5. Intersection of regions $N(\mathbf{x}) > 0$ and $D(\mathbf{x}) > 0$ (thick solid line).

Problem (11) can be reformulated as in Eq. (12) by defining $\frac{\mathrm{d}V(\mathbf{x})}{\mathrm{d}t} = \frac{\hat{N}(\mathbf{x})}{\hat{D}(\mathbf{x})}$ and carrying out appropriate algebraic manipulations.

$$\min_{c,\mathbf{x},\varepsilon} c$$

$$\text{s.t. } N(\mathbf{x}) - cD(\mathbf{x}) = 0 \tag{12a}$$

$$\hat{N}(\mathbf{x}) = 0 \tag{12b}$$

$$\nabla\left[N(\mathbf{x}) - cD(\mathbf{x})\right] = \varepsilon\nabla\left[\hat{N}(\mathbf{x})\right] \tag{12c}$$

$$N(\mathbf{x}) > 0 \tag{12d}$$

$$D(\mathbf{x}) > 0 \tag{12e}$$

$$\mathbf{x} \neq 0 \tag{12f}$$

$$c > 0. \tag{12g}$$

5. Examples

In this section the approach described is applied to the five dynamic systems presented in Table 1. For each system, a rational Lyapunov function was calculated following the procedure described in [3]. The functions obtained are reported in Table 2. The solution of problem (12) is reported for each example in Table 1. In all cases (Figs. 6–10) the solid line represents the level set $V(\mathbf{x}) = c$ and the dashed line the level set $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$. In order to illustrate constraints (12d) and (12e) the thick dashed line delimits region $N(\mathbf{x}) > 0$ and the shaded region corresponds to $D(\mathbf{x}) > 0$.

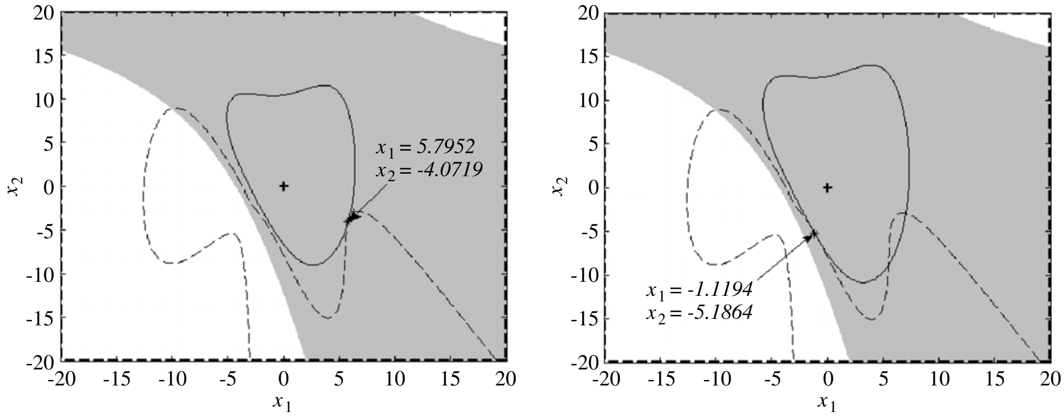
Table 1
Dynamic systems and solutions.

Example	System	Solution
E1 Genesio et al. (1985) [2]	$\frac{dx_1}{dt} = -x_1 + x_2$ $\frac{dx_2}{dt} = 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3$	$x_1 = 5.7952$ $x_2 = -4.0719$ $c = 28.7343$ $\varepsilon = 0.3780$
E2 Bequette (1998) [14]	$\frac{dx_1}{dt} = 7.6411 - x_1 - 3.493 \times 10^7 \exp\left(-\frac{5960.2415}{x_2+368.0628}\right)(x_1 + 2.3589)$ $\frac{dx_2}{dt} = -91.0816 - 1.3x_2 + 4.1637 \times 10^8 \exp\left(-\frac{5960.2415}{x_2+368.0628}\right)(x_1 + 2.3589)$	$x_1 = 4.7561$ $x_2 = -33.8799$ $c = 639.5575$ $\varepsilon = 0.4745$
E3 Wang and Ruan (2004) [15]	$\frac{dx_1}{dt} = -0.84x_1 - 1.44x_2 - 0.3x_1x_2$ $\frac{dx_2}{dt} = 0.54x_1 + 0.34x_2 + 0.3x_1x_2$	$x_1 = -3.2213$ $x_2 = 3.4096$ $c = 12.9345$ $\varepsilon = 0$
E4 Chesi (2009) [6]	$\frac{dx_1}{dt} = -x_1 + x_2 + 0.5(\exp(x_1) - 1)$ $\frac{dx_2}{dt} = -x_1 - x_2 + x_1x_2 + x_1\cos(x_1)$	$x_1 = 1.3010$ $x_2 = -0.6175$ $c = 1.2251$ $\varepsilon = 0.3675$
E5 Hachicho and Tibken (2002) [7]	$\frac{dx_1}{dt} = -x_1 + x_2 + x_3^2$ $\frac{dx_2}{dt} = -x_2 + x_1x_2$ $\frac{dx_3}{dt} = -x_3$	$x_1 = -1.0661$ $x_2 = 1.7120$ $x_3 = -1.3328$ $c = 1.3209$ $\varepsilon = 0$

Table 2
Lyapunov functions for the systems of Table 1.

Example	Terms of function (9)
E1	$R_2(\mathbf{x}) = 0.5184x_1^2 + 0.3684x_1x_2 + 0.3421x_2^2$ $R_3(\mathbf{x}) = -0.0144x_1^3 - 0.1646x_1^2x_2 - 0.0143x_1^2x_2^2 - 0.0005x_2^3$ $R_4(\mathbf{x}) = 0.0303x_1^4 + 0.0043x_1^3x_2 + 0.0037x_1^2x_2^2 - 0.0012x_1x_2^3$ $Q_1(\mathbf{x}) = 0.2269x_1 + 0.0529x_2$ $Q_2(\mathbf{x}) = -0.0029x_1^2 - 0.0144x_1x_2 - 0.0026x_2^2$
E2	$R_2(\mathbf{x}) = 325.4977x_1^2 + 71.4479x_1x_2 + 4.2514x_2^2$ $R_3(\mathbf{x}) = 69.4087x_1^3 + 22.9472x_1^2x_2 + 2.1447x_1x_2^2 + 0.0404x_2^3$ $R_4(\mathbf{x}) = 1.7325x_1^4 + 0.7648x_1^3x_2 + 0.0591x_1^2x_2^2 + 0.0008x_1x_2^3 + 0.0005x_2^4$ $Q_1(\mathbf{x}) = 0.7341x_1 + 0.0809x_2$ $Q_2(\mathbf{x}) = 0.0877x_1^2 + 0.0195x_1x_2 + 0.0007x_2^2$
E3	$R_2(\mathbf{x}) = 1.8276x_1^2 + 3.8341x_1x_2 + 6.6488x_2^2$ $R_3(\mathbf{x}) = -0.0309x_1^3 + 0.3806x_1^2x_2 + 0.4899x_1^2x_2^2 - 0.1068x_2^3$ $R_4(\mathbf{x}) = -0.0195x_1^4 - 0.0527x_1^3x_2 - 0.1971_1^2x_2^2 - 0.1271x_1x_2^3 - 0.1349x_2^4$ $Q_1(\mathbf{x}) = 0.1856x_1 + 0.7640x_2$ $Q_2(\mathbf{x}) = -0.0394x_1^2 - 0.0289x_1x_2 - 0.0899x_2^2$
E4	$R_2(\mathbf{x}) = x_1^2 + 1.3333x_1x_2 + 1.1667x_2^2$ $R_3(\mathbf{x}) = -0.2272x_1^3 - 0.1396x_1^2x_2 + 0.3785x_1^2x_2^2 - 0.1798x_2^3$ $R_4(\mathbf{x}) = 0.0136x_1^4 - 0.2864x_1^3x_2 + 0.1918x_1^2x_2^2 - 0.0530x_1x_2^3 + 0.0172x_2^4$ $Q_1(\mathbf{x}) = -0.5605x_1 - 0.7255x_2^2$ $Q_2(\mathbf{x}) = 0.3254x_1^2 + 0.0910x_1x_2 + 0.1015x_2^2$
E5	$R_2(\mathbf{x}) = 0.5x_1^2 + 0.5x_2^2 + 0.5x_3^2$ $R_3(\mathbf{x}) = -0.0739x_1^3 + 0.2594x_1x_2^2 - 0.0739x_1x_3^2$ $R_4(\mathbf{x}) = -0.0301x_1^4 + 0.0573x_1^2x_2^2 - 0.0301x_1^2x_3^2 + 0.2501x_1x_2x_3^2 - 0.03x_2^4 - 0.03x_2^2x_3^2$ $Q_1(\mathbf{x}) = -0.1478x_1$ $Q_2(\mathbf{x}) = -0.0602x_1^2 - 0.06x_2^2$

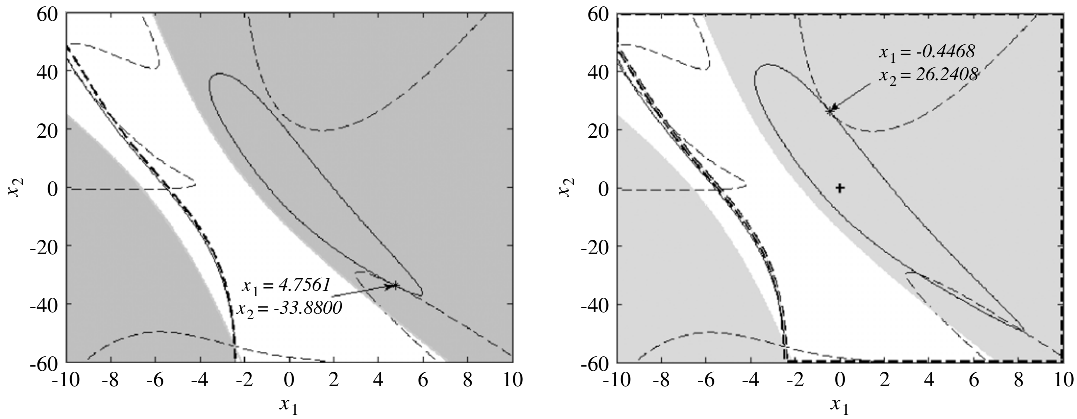
Fig. 6(a) shows the estimation of DOA($\mathbf{0}$) for example E1. As expected the solution verifies tangency between the level sets $V(\mathbf{x}) = c$ and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$ and is included in the intersection between regions $N(\mathbf{x}) > 0$ and $D(\mathbf{x}) > 0$. For this system $N(\mathbf{x})$ is greater than 0 for the whole optimization domain while the constraint on $D(\mathbf{x})$ prunes a large part of it. Note that a



(a) Solution, $c = 28.7343$.

(b) Dummy solution $c = 43.2379$.

Fig. 6. Example E1. Level sets $V(\mathbf{x}) = c$ (solid) and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$ (dashed).



(a) Solution, $c = 639.5575$.

(b) Dummy solution $c = 840.1966$.

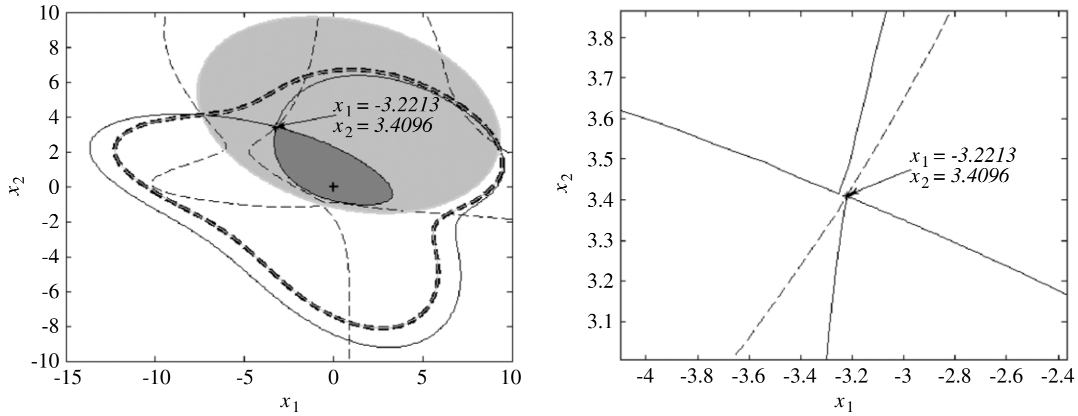
Fig. 7. Example E2. Level sets $V(\mathbf{x}) = c$ (solid) and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$ (dashed).

close "dummy solution" exist in this problem (Fig. 6(b)). This solution is effectively excluded by ensuring global optimality of problem (12).

Fig. 7(a) shows the level sets and regions of interest for system E2 and the actual solution of this problem. In this particular example it can be seen that region $D(\mathbf{x}) > 0$ is split in two zones in the analyzed portion of the states space. Constraint $N(\mathbf{x}) > 0$ prunes the left bottom part and reduces the feasible region of the model to the right top part of $D(\mathbf{x}) > 0$ where the solution belongs. However, since a close "dummy solution" exists (Fig. 7(b)) the requirement of global optimality is a must to avoid miscalculation of the estimation of the DOA($\mathbf{0}$).

Level sets and regions are presented for system E3 in Fig. 8(a). A detail on the solution (Fig. 8(b)) shows that the inner part of the level set $V(\mathbf{x}) = c$ is the closed region that estimates the domain (dark gray) while the outer part of the set does not touch $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$. As expected the solution belongs to the intersection of regions defined by constraints $N(\mathbf{x}) > 0$ (thick dashed line) and $D(\mathbf{x}) > 0$ (shaded region). It should be noted that auxiliary variable ε has value 0, which means that the gradient of $V(\mathbf{x}) - c = 0$ is zero at the solution.

System E4 involves the cosine function (Table 1). Since the BARON solver does not support trigonometric functions, a procedure in two steps was adopted for this example. First, the cosine function was approximated by its Taylor series expansion, and model (12) solved with BARON. The solution obtained ($x_1 = 1.2668$, $x_2 = -0.6229$, $c = 1.1762$, $\varepsilon = 0.3528$) is global for the approximate problem and is expected to be close to the solution of the original problem. Then, in order to calculate the real actual solution, problem (12) was solved for the original system E4 with the local nonlinear solver CONOPT [12] (which supports trigonometric functions) from the starting point corresponding to the solution of the approximate problem. The final solution is reported in Table 1 and shown in Fig. 9. The level sets show the expected behavior: tangential contact at the solution, which belongs to the intersection of regions defined by constraints $N(\mathbf{x}) > 0$ and $D(\mathbf{x}) > 0$ (Fig. 9).



(a) Solution, $c = 12.9345$. (b) Detail on solution of Fig. 8(a).

Fig. 8. Example E3. Level sets $V(\mathbf{x}) = c$ (solid) and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$ (dashed).

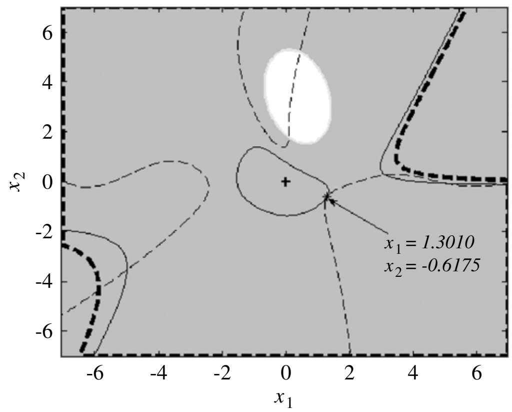


Fig. 9. Example E4. Level sets $V(\mathbf{x}) = c$ (solid) and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$ (dashed).

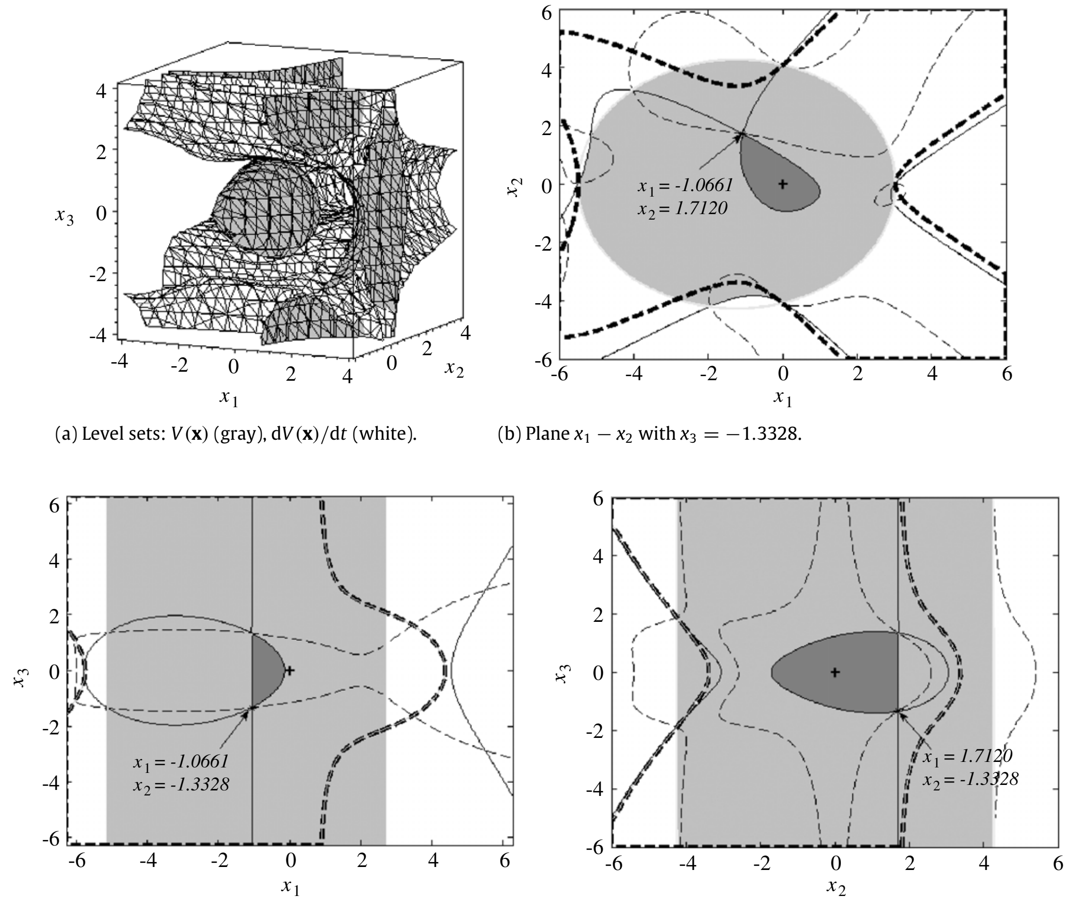
In Fig. 10 the solution is graphically shown for the three states example E5. In Fig. 10(a) the level sets $V(\mathbf{x}) = c$ and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$ are shown in gray and white respectively. The solid ball around the origin constitutes the estimation of the DOA($\mathbf{0}$). In Fig. 10(b), (c) and (d) three different views of the level sets are plotted together with the corresponding constraints $N(\mathbf{x}) > 0$ and $D(\mathbf{x}) > 0$. Each view is parameterized at the value of the solution in the remaining state in order to appreciate that the solution obtained effectively verifies tangency and belongs to the feasible region determined by $N(\mathbf{x}) > 0$ and $D(\mathbf{x}) > 0$. Note that in each view, the region corresponding to the estimation is dark-shaded to facilitate its visualization.

6. Conclusions

A new formulation based on a constrained global optimization approach was proposed to estimate DOAs of stable equilibriums of systems of dynamic equations. Major contributions regarding previous approaches are:

1. A tangency constraint was included in order to skip transversal intersections of level sets $V(\mathbf{x}) = c$ and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t = 0$. In this way a large subset of possible "dummy solutions" of nonlinear problem (11) is effectively avoided.
2. Constraints on functions $N(\mathbf{x})$ and $D(\mathbf{x})$ were included in formulation (11) sensibly reducing the feasible space and therefore avoiding getting trapped in solutions beyond regions where changes of definiteness of $V(\mathbf{x})$ and $\mathrm{d}V(\mathbf{x})/\mathrm{d}t$ take place.
3. Finally, in order to avoid solutions that verify both previous requirements but produce level sets not fully contained in the region of negative definiteness of $\mathrm{d}V(\mathbf{x})/\mathrm{d}t$, problem (11) is solved to global optimality with the use of a global optimization software. While global optimality is also guaranteed in previous approaches [8] the adoption of GAMS/BARON allows the treatment of nonlinear systems of a more general type rather than limited to polynomial functions as generally required.

The importance of pruning the different types of dummy solutions with the use of Conditions 1–3 was demonstrated through several examples.



(a) Level sets: $V(\mathbf{x})$ (gray), $dV(\mathbf{x})/dt$ (white).

(b) Plane $x_1 - x_2$ with $x_3 = -1.3328$.

(c) Plane $x_1 - x_3$ with $x_2 = 1.7120$.

(d) Plane $x_2 - x_3$ with $x_1 = -1.0661$.

Fig. 10. Example E5. Level sets $V(\mathbf{x}) = c$ (solid) and $dV(\mathbf{x})/dt = 0$ (dashed).

The estimation of the DOA of stable equilibriums is an important problem in itself. However, it may be considered as a sub-problem of a more challenging one: the actual design of the nonlinear system. Besides ensuring a large DOA, the design problem necessarily requires the asymptotic stability of the resulting equilibrium point at open or closed loop modes. The "design for stability" is by itself a very challenging problem usually studied in terms of eigenvalue optimization [16]. Moreover, while important from a dynamic point of view, stability and large DOAs are not the only interest for real systems operations. Many systems and processes usually require the optimization of economic objectives while ensuring proper dynamic behavior either open loop or under control. Future work on this topic should address such multi-objective optimization problems with application to larger systems in the states and parameter spaces.

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