On the timelike Liouville three-point function

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Abstract

In a recent paper, D. Harlow, J. Maltz, and E. Witten showed that a particular proposal for the timelike Liouville three-point function, originally due to Al. Zamolodchikov and to I. Kostov and V. Petkova, can actually be computed by the original Liouville path integral evaluated on a new integration cycle. Here, we discuss a Coulomb gas computation of the timelike three-point function and show that an analytic extension of the Selberg type integral formulas involved reproduces the same expression, including the adequate normalization. A notable difference with the spacelike calculation is pointed out.

Liouville field theory finds its application in various areas of theoretical physics, including string theory [1], three-dimensional general relativity [2], string theory in Anti-de Sitter space [3], and supersymmetric gauge theory [4]. The timelike version of Liouville theory, on the other hand, has interesting applications as well. For instance, timelike Liouville was considered in holographic quantum cosmology [5], in the study of tachyon condensation [6] and in other time-dependent scenarios of string theory. In a recent paper [7] timelike Liouville theory was reexamined from a more general point of view. After a thoughtful analysis of the analytic continuation of Liouville field theory, the question was raised as to what extent the peculiar case of timelike Liouville theory can actually be regarded as a conformal field theory. A crucial ingredient in the discussion is the three-point correlation function, which is computed in [7] within the path integral approach. The computation of this observable is a non-trivial problem since, contrary to the naive expectation, the structure constants of the timelike Liouville theory are not the analytic extension of the structure constants of spacelike Liouville theory. In this note, with the aim of contributing to the discussion, we perform a computation of the timelike three-point function, alternative to that in [7], and show that it reproduces the same result, including the adequate normalization. We discuss the difference with respect to the spacelike three-point function computation.

Let us begin by briefly reviewing Liouville theory. The action of Liouville theory formulated on a closed manifold $\mathcal{C}$ is

$$S_{\mathrm{L}}[\varphi, \mu] = \frac{1}{4\pi} \int_{\mathcal{C}} d^2x \sqrt{g} \left(\sigma g^{ab} \partial_a \varphi \partial_b \varphi + Q_\sigma R \varphi + 4\pi \mu e^{2b\varphi} \right) \tag{1}$$

where b and μ are two real parameters, and $Q_\sigma = b + \sigma b^{-1}$ with $\sigma = \pm 1$. The value $\sigma = +1$ corresponds to the standard Liouville theory while $\sigma = -1$ corresponds to the theory with the *wrong sign* kinetic term. Using the string theory terminology, we refer to these theories as the spacelike and the timelike models, respectively. Action (1), at least in its spacelike version $\sigma = +1$ that we understand better, defines a non-compact conformal field theory. Primary operators of the theory are given by the exponential fields $V_\alpha(z) = e^{2\alpha\varphi(z)}$, which create states with conformal dimension $\Delta_\alpha = \sigma \ \alpha(Q_\sigma - \alpha)$. From this, it is possible to check that the self-interaction term in (1) represents a marginal deformation. The central charge of the theory is given by $c = 1 + 6\sigma Q_\sigma^2$. Notice also that the timelike model $\sigma = -1$ can be alternatively obtained from the standard case $\sigma = +1$ by going to imaginary values of the parameter $b \to ib$ and the

Liouville field $\varphi \to -i\varphi$. This is, indeed, a convenient way of thinking about the timelike model; however, here we prefer to keep σ in the formulae below and consider $b \in \mathbb{R}$.

The n-point correlation functions of local operators on a curve $\mathcal{C}$ are defined by

$$\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \rangle_{\mathcal{C}} = \int_{\varphi(\mathcal{C})} \mathcal{D}\varphi \; e^{-S_{\mathrm{L}}[\varphi,\mu]} \prod_{i=1}^{n} e^{2\alpha_i \varphi(z_i)}. \tag{2}$$

Here we are interested in the three-point correlation functions on the sphere. The three-point function defines the structure constants of the theory, hereafter denoted as $C_b(\alpha_1, \alpha_2, \alpha_3)$. Namely

$$\left\langle \prod_{i=1}^{3} V_{\alpha_i}(z_i) \right\rangle_{\mathbb{S}^2} = \prod_{i<j}^{3} |z_i - z_j|^{\Delta_{ij}} \; C_b(\alpha_1, \alpha_2, \alpha_3), \tag{3}$$

with $\Delta_{ij} = \Delta_{\alpha_1} + \Delta_{\alpha_2} + \Delta_{\alpha_3} - 2\Delta_{\alpha_i} - 2\Delta_{\alpha_j}$. It follows that

$$C_b(\alpha_1, \alpha_2, \alpha_3) = \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle_{\mathbb{S}^2} = \int_{\varphi(\mathbb{CP}^1)} \mathcal{D}\varphi \; e^{-S_{\mathrm{L}}[\varphi,\mu]} \; e^{2\alpha_1\varphi(0)} e^{2\alpha_2\varphi(1)} e^{2\alpha_3\varphi(\infty)} \tag{4}$$

where projective invariance is invoked and used to fix the three insertions on the Riemann sphere, namely $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$. On the Riemann sphere the field configurations are conditioned to obey the asymptotic $\varphi(z) \sim -2Q_\sigma \log|z|$ for $|z| >> 1$. In (4), a factor $|z_3|^{-2\Delta_3}$ when taking the limit $z_3 \to \infty$ was omitted for short.

The three-point function (3) of spacelike Liouville theory has been calculated by H. Dorn and H. Otto in Ref. [8] and independently by the brothers Zamolodchikov in Ref. [9]; this is why the explicit formula for $C_b(\alpha_1, \alpha_2, \alpha_3)$ is usually referred to as the DOZZ formula. Further details of the calculation of the structure constants were given in Refs. [10, 11].

The solution of the timelike model has been early investigated by A. Strominger and T. Takayanagi in [6] within the context of the closed string theory tachyon dynamics. Afterwards, V. Schomerus in a beautiful paper [12] proposed the first satisfactory answer for the timelike three-point function, namely for the structure constants of the theory with $c < 1$ (which corresponds to non-real values of b in the standard Liouville.) The proposal of [12] was obtained by taking the limit from the expressions valid for $c \geq 25$ carefully. The limit is actually delicate as the theory with $c \leq 1$ happens not to depend smoothly on c. In fact, the timelike structure constants proposed in [12] are not analytic functions of the momenta α_i, and the spacelike and timelike theories cannot be related simply by Wick rotation analytically continuing in the parameters α_i and b.

The analytic extension of Liouville three-point function to non-real values of b was also investigated in Refs. [13, 14, 15, 16, 17]. In particular, in [13] Al. Zamolodchikov proposed a timelike version of the DOZZ formula which is not the naive analytic continuation of its spacelike analogue. A similar proposal was independently given by I. Kostov and V. Petkova [14]. In a recent a paper by D. Harlow, J. Maltz, and E. Witten [7], it is showed that the timelike Liouville three-point function proposed in [13] can be obtained by the original Liouville path integral evaluated on a new integration cycle.

In this note, with the aim of contributing to the discussion of the timelike model and in particular to the discussion of the timelike three-point function, we will rederive this quantity using the Coulomb gas approach. The Coulomb gas calculation of the $c < 1$ three-point function was also discussed in [14] (see the discussion around Eq. (3.9) therein.) Here, we will show that a natural analytic extension of the Selberg type integral formulas involved reproduces the expression of [13] with the same normalization as in [7]. We will perform a detailed comparison between the spacelike and timelike calculations and point out a notable difference. The difference comes from a divergent factor that arises in the integration over the zero-mode. While in the spacelike case the DOZZ formula is obtained by considering the full correlation function, in the timelike case the analogous formula proposed in [13] is reproduced by the residues associate to resonant correlators, meaning that a simple pole has to be extracted.

The Coulomb gas calculation of Liouville correlation functions was early discussed in Ref. [18]. This consists in expanding the interaction term of the Liouville action and performing the Wick contraction of the operators using the free field theory. After integrating the zero mode φ_0 of the Liouville field, we can write (4) as follows

$$C_b^{(\sigma)}(\alpha_1, \alpha_2, \alpha_3) = \Gamma(-s_\sigma)\mu^{s_\sigma} b^{-1} \int_{\varphi_{(\mathbb{CP}^1)}} \mathcal{D}\widetilde{\varphi}\ e^{-S_{\mathrm{L}}[\widetilde{\varphi},\mu=0]}\ e^{2\alpha_1\widetilde{\varphi}(0)} e^{2\alpha_2\widetilde{\varphi}(1)} e^{2\alpha_3\widetilde{\varphi}(\infty)} \textstyle\prod_{r=1}^{s_\sigma} e^{2b\widetilde{\varphi}(w_r)} \quad (5)$$

where $\widetilde{\varphi} = \varphi - \varphi_0$ are the fluctuation of the field, and $s_\pm = b^{-1}(Q_\pm - \alpha_1 - \alpha_2 - \alpha_3)$. The superindices and subindices (σ) indicate whether a given expression corresponds to the spacelike $(+)$ or to the timelike $(-)$ case.

Since the right hand side of (5) is actually a correlator of a free theory, we can easily write it down using the Wick contractions of the exponential fields. The free field propagator in this case is $\langle \varphi(z_i)\varphi(z_j)\rangle = -\sigma \log|z_i - z_j|$, so that

$$C_b^{(\sigma)}(\alpha_1, \alpha_2, \alpha_3) = \Gamma(-s_\sigma)\mu^{s_\sigma} b^{-1} \textstyle\prod_{l=1}^{s_\sigma} \displaystyle\int_{\mathbb{C}} d^2 w_l \textstyle\prod_{r=1}^{s_\sigma} |w_r|^{-4\sigma b\alpha_1} |1 - w_r|^{-4\sigma b\alpha_2} \prod_{t'<t}^{s_\sigma} |w_{t'} - w_t|^{-4\sigma b^2}.$$

This multiple integral of the Selberg type can be solved explicitly for generic $s_\pm \in \mathbb{Z}_{>0}$. This was done by Dotsenko and Fateev in the context of the Minimal Models in Ref. [19]. The result adapted to Liouville theory reads

$$C_b^{(\sigma)}(\alpha_1,\alpha_2,\alpha_3) = (-1)^{s_\sigma}\Gamma(-s_\sigma)\Gamma(1+s_\sigma)b^{-1}(\pi\mu b^4\gamma(\sigma b^2))^{s_\sigma}\frac{\prod_{r=1}^{s_\sigma}\gamma(-\sigma r b^2)}{\prod_{i=1}^{3}\prod_{t=1}^{s_\sigma}\gamma(2\sigma b\alpha_i+\sigma(t-1)b^2)} \tag{6}$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. The overall factor $\Gamma(-s_\pm)$ in the expressions above comes from the integration of the zero-mode over the non-compact target space, and it is interpreted as in [20] (see the discussion around Eq. (2.10) therein.) Notice that the factor $\Gamma(-s_\pm)$ would in principle introduce a divergence provided $s_\pm \in \mathbb{Z}_{\geq 0}$. However, as we will see below, in the spacelike case $\sigma = +1$ such divergent factor nicely cancels out with a contribution coming from the products in (6). The timelike model, in contrast, exhibits a special feature and the divergence has to be extracted.

Using the property $\gamma(x) = \gamma^{-1}(1-x)$, we can write the products appearing on the right hand side of (6) as follows

$$\prod_{l=1}^{s_\sigma}\gamma(-\sigma l b^2)\prod_{i=1}^{3}\left(\frac{\prod_{r=1}^{\beta_i^\sigma}\gamma(rb^2)}{\prod_{t=1}^{\beta_i^\sigma+s_\sigma}\gamma(tb^2)}\right)^{\sigma}, \tag{7}$$

with $\beta_i^\pm = 2\alpha_i b^{-1} - 1 + (1\mp 1)b^{-2}/2$. Here we assumed $2\alpha_i b^{-1} \in \mathbb{Z}_{>0}$ and $b^{-2} \in \mathbb{Z}_{>0}$ to work out the products; however, we will eventually extend the expression to generic values of α_i and b. To do this, we consider the following expression

$$\prod_{r=1}^{n}\gamma(rb^2) = \frac{\Upsilon_b(nb+b)}{\Upsilon_b(b)}b^{n(b^2(n+1)-1)}, \tag{8}$$

where $\Upsilon_b(x)$ is the special function introduced in [9], which admits to be written in terms of the Barnes' double Γ-functions $\Gamma_2(x|y)$ [21] as follows

$$\Upsilon_b(x) \equiv \Gamma_2^{-1}(x|b,b^{-1})\Gamma_2^{-1}(b+b^{-1}-x|b,b^{-1}),$$

with the definition

$$\log\Gamma_2(x|y_1,y_2) = \lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon}\sum_{k_1,k_2\in\mathbb{Z}_{\geq 0}}(x+k_1y_1+k_2y_2)^{-\varepsilon};$$

see also (13) below.

Notice that while Eq. (8) only makes sense for $n \in \mathbb{Z}_{>0}$, the expression on the right hand side is defined on a continuous range. This is a crucial step in extending the Coulomb gas integral expression to more general values of α_i and b. This extension is clearly not unique; for instance, one could add a phase $e^{2\pi i n}$ to the right hand side of (8). In turn, the analytic continuation of the integral formulas has to be regarded modulo certain type of contributions.

Function $\Upsilon_b(x)$ presents simple zeros at $x = mb + nb^{-1}$ if $m, n \in \mathbb{Z}_{\leq 0}$ or $m, n \in \mathbb{Z}_{>0}$. Using the properties of $\Gamma_2(x|y)$ it is possible to show that the $\Upsilon_b(x)$ function satisfies the shift relations

$$\Upsilon_b(x+b) = \gamma(bx) b^{1-2bx} \Upsilon_b(x), \qquad \Upsilon_b(x+b^{-1}) = \gamma(x/b) b^{-1+2x/b} \Upsilon_b(x) \tag{9}$$

as well as the inversion relations

$$\Upsilon_b(x) = \Upsilon_{b^{-1}}(x), \qquad \Upsilon_b(x) = \Upsilon_b(b + b^{-1} - x). \tag{10}$$

Expression (8) follows from iterating (9).

Now, let us use the results above to compute the three-point function. Let us first analyze the spacelike three-point function; namely, consider first the case $\sigma = +1$. In this case, we can use (8) to write (7) as follows

$$\prod_{l=1}^{s_+} \gamma(-lb^2) \prod_{i=1}^{3} \frac{\prod_{r=1}^{2b^{-1}\alpha_i - 1} \gamma(rb^2)}{\prod_{t=1}^{2b^{-1}\alpha_i + s_+ - 1} \gamma(tb^2)} = \frac{\Upsilon_b(Q_+) b^{-2s(b^2+1)}}{\Upsilon_b(\sum_{k=1}^{3} \alpha_k - Q_+)} \prod_{i=1}^{3} \frac{\Upsilon_b(2\alpha_i)}{\Upsilon_b(\sum_{j=1}^{3} \alpha_j - 2\alpha_i)}, \tag{11}$$

which allows us to analytically extend the expression to values $2\alpha_i/b \notin \mathbb{Z}$.

Then, using (8) and the functional properties (10) we find the final expression

$$C_b^{(+)}(\alpha_1, \alpha_2, \alpha_3) = \left(\pi\mu\gamma(b^2) b^{2-2b^2}\right)^{s_+} \frac{\Upsilon_b'(0)}{\Upsilon_b(\sum_{k=1}^{3} \alpha_k - Q_+)} \prod_{i=1}^{3} \frac{\Upsilon_b(2\alpha_i)}{\Upsilon_b(\sum_{j=1}^{3} \alpha_j - 2\alpha_i)} \tag{12}$$

with $Q_+ = b + b^{-1}$, $s_+ = 1 + b^{-2} - b^{-1} \sum_{i=1}^{3} \alpha_i$, and $\Upsilon_b'(x) = \frac{\partial}{\partial x} \Upsilon_b(x)$. To derive (12) we used the following integral expression for the $\Upsilon_b(x)$ function,

$$\log \Upsilon_b(x) = \int_{\mathbb{R}_{>0}} \frac{d\tau}{\tau} \left(\left(\frac{b}{2} + \frac{1}{2b} - x \right)^2 e^{-\tau} - \frac{\sinh^2((\frac{b}{2} + \frac{1}{2b} - x)\frac{\tau}{2})}{\sinh(\frac{b\tau}{2}) \sinh(\frac{\tau}{2b})} \right) \tag{13}$$

for $0 < \mathrm{Re}(x) < (b + b^{-1})/2$. In the boundary of this range we find that the $\Upsilon_b(x)$ function behaves like $\Upsilon_b(Q_+) = \Upsilon_b(0) \sim \Upsilon_b'(0)/\Gamma(0)$. The $\Gamma(0)$ appearing here cancels the one coming from $(-1)^{s_+}\Gamma(-s_+)$.

Expression (12) is the spacelike three-point function [8, 9]. That is, the Coulomb gas approach based on the free field calculation exactly reproduces the DOZZ formula, provided one analytically extends the products standing in the integral formula (6) using (8).

Now, let us proceed in the same way for the timelike case $\sigma = -1$. In this case, we can write (7) as follows

$$\prod_{r=1}^{s_-} \gamma(rb^2) \prod_{i=1}^{3} \frac{\prod_{t=1}^{2\alpha_i b^{-1}+b^{-2}+s_- -1} \gamma(tb^2)}{\prod_{r=1}^{2\alpha_i b^{-1}+b^{-2}-1} \gamma(rb^2)} = \frac{\Upsilon_b(-\sum_{k=1}^{3} \alpha_k + Q_- + b)}{\Upsilon_b(b) b^{2s(1-b^2)}} \prod_{i=1}^{3} \frac{\Upsilon_b(2\alpha_i - \sum_{j=1}^{3} \alpha_j + b)}{\Upsilon_b(b - 2\alpha_i)}. \tag{14}$$

From this we notice that, unlike the spacelike case, in which the contribution $\Upsilon_b(0) \sim \Upsilon_b'(0)/\Gamma(0)$ cancels the divergence coming from the factor $\Gamma(-s_+) \sim (-1)^{s_+}\Gamma(0)/\Gamma(1+s_+)$, in the timelike case there is no $\Upsilon_b(0)$ factor coming from (14); instead, a finite contribution $\Upsilon_b^{-1}(b)$ stands and the overall $\Gamma(-s_-)$ factor then is not cancelled; in addition, a factor $(-1)^{s_-}$ survives. The Coulomb gas calculation of the timelike case does yield a finite result if, instead, we calculate the residues of the resonant correlators. This amounts to extract the simple poles at $s_- \in \mathbb{Z}_{\geq 0}$ in the momenta $\mathbb{C}$ plane. This is achieved by excluding the divergent overall factor $\Gamma(0)$; recall $\Gamma(-s_-) \sim (-1)^{s_-}\Gamma(0)/s_-!$. The factor $1/s_-! = \Gamma^{-1}(s_- + 1)$ can alternatively be thought of as a multiplicity factor coming from the permutation of the screening operators $\mu e^{2b\varphi(w)}$ in the Feigin-Fuchs type realization [22, 19] (see for instance Eq. (3.15) in [9]; see also [18]). Then, using the properties of the $\Upsilon_b(x)$ functions, we find

$$C_b^{(-)}(\alpha_1,\alpha_2,\alpha_3) = \left(-\pi\mu\gamma(-b^2)b^{2+2b^2}\right)^{s_-} \frac{\Upsilon_b(-\sum_{k=1}^{3}\alpha_k + Q_- + b)}{b\Upsilon_b(b)} \prod_{i=1}^{3} \frac{\Upsilon_b(2\alpha_i - \sum_{j=1}^{3}\alpha_j + b)}{\Upsilon_b(b-2\alpha_i)} \tag{15}$$

with $Q_- = b - b^{-1}$, $s_- = 1 - b^{-2} - b^{-1}\sum_{i=1}^{3}\alpha_i$. The equality in (15) has to be understood after having computed the residue of the expression.

Up to a phase $e^{-i\pi s_-}$, (15) turns out to reproduce exactly the timelike three-point function recently discussed in [7] and originally proposed in [13] (notice that to translate our notation into that used in [7] one has to do $\alpha_i \to -\widehat{\alpha}_i$, $b \to \widehat{b}$, and $Q_- \to -\widehat{Q}$.) About the phase, we already mentioned that a prescription to analytically extend the products as in (8) is not sensitive to phases like $e^{2\pi i n}$. This is even more drastic in the case of multiple products like those in (14), so that expression (15) has to be regarded up to such a phase ambiguity. In this aspect, our result agrees with that of [15], which does not exhibit the phase $e^{-i\pi s_-}$ either. It is worthwhile emphasizing that, up to the phase, the expression we obtained for $C_b^{(-)}(\alpha_1,\alpha_2,\alpha_3)$

reproduces the normalization of Ref. [7], which differs from that in [13]; cf. Ref. [14]. More precisely, in (15) we find exactly the same factor $(-\mu\pi\gamma(-b^2)b^{2+2b^2})^{s_-}$, as in [7], and find no additional b-dependent factors, in contrast with [13]. The only aspect about the normalization we may find puzzling is the divergent $\Gamma(0)$ factor in which the spacelike and the timelike differ. Despite one can easily keep track of such divergent factor through the calculation, we do not find a simple way of explaining why it appears in the timelike computation while it cancels out in the spacelike computation. To try to understand this, we can check whether the same happens in the partition function: The number of integrated screening operators in that case would be $s_\sigma - 3 = -2 + b^2$, instead of s_σ. This is because to compute the genus-zero zero-point function we have to consider the correlator with three local operators $e^{2b\varphi(z)}$ inserted at fixed points, say $z_1 = 0$, $z_2 = 1$, and $z_3 = \infty$. This stabilizes the sphere compensating the volume of the conformal Killing group. Then, assuming $s_\sigma \in \mathbb{Z}_{>3}$, the Liouville partition function on the sphere topology reads

$$Z_b^{(\sigma)} = \frac{\mu^{s_\sigma}}{b}\Gamma(-s_\sigma)\Gamma(s_\sigma - 2)(-\pi\gamma(\sigma b^2)b^4)^{s_\sigma - 3}\prod_{r=1}^{s_\sigma - 3}\frac{\gamma(-\sigma r b^2)\gamma(\sigma(s_\sigma + r - 1)b^2 - 1)}{\gamma^2(\sigma(1+r)b^2)}.$$

Noticing that $1 - \sigma r b^2 = \sigma b^2(s_\sigma - 1 - r)$ we can rearrange the products of Γ-functions and eventually find for the timelike case

$$Z_b^{(-)} = \frac{(1+b^2)\left(\pi\mu\gamma(-b^2)\right)^{Q_-/b}}{\gamma(-b^2)\gamma(-b^{-2})\pi^3 Q_-}, \tag{16}$$

recall $Q_- = b - b^{-1}$. A remarkable feature is that this expression is not invariant under Liouville self-duality $b \to b^{-1}$ [26]. It should not be a surprise that (16) agrees with the expression obtained by replacing in the standard Liouville partition function as $Q_+ \to Q_-$ and $b^2 \to -b^2$. This is because, unlike what happens with the three-point function, both the zero- and the two-point function admit a natural analytic continuation to negative values of b^2.

It is interesting to compare the three-point functions (12) and (15). It was early noticed in [13] and [14] for the case of Minimal Gravity that the timelike Liouville structure constants are not the naive analytic continuation of their spacelike analogues. In fact, as emphasized in [13], contrary to one's expectation, the timelike structure constants turn out to be, roughly speaking, the *inverse* of spacelike structure constants, in the sense that the product of both timelike and spacelike quantities yields a remarkably simple factorized expression. This peculiar relation between timelike and spacelike structure constants is particularly expressed by the fact that the dependences on the $\Upsilon_b(x)$ functions in (12) and in (15) are, *mutatis mutandis*, inverse of the other. This intriguing feature is nicely explained in the Coulomb gas calculation as it directly follows from the property $\gamma(x) = \gamma^{-1}(1-x)$ of the Γ-functions. Exactly the same happens in string theory on $\mathrm{AdS}_3 \times \mathbb{S}^3 \times \mathbb{T}^4$, where the three-point functions of chiral states take a remarkably simple expression due to surprising cancellations of $\Upsilon_b(x)$ functions that take place between the $\mathbb{H}_+^3$ and the $\mathbb{S}^3$ pieces [23, 24]; in [25] such result was also reproduced in the Coulomb gas approach. Here, this approach led us to reproduce the formula of [7] for the timelike Liouville three-point function. The fact that the analytic extension of the integral formulas standing in the Coulomb gas calculation yields the correct answer both for the spacelike and for the timelike Liouville structure constants is notable because, as already mentioned, the latter are not simply obtained by analytic continuation from the former.

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