

The Jamming Constant of Uniform Random Graphs

Paola Bermolen^a, Matthieu Jonckheere^b, Pascal Moyal^{c,*}

^a*Universidad de La Republica, Av. De Julio 1968, Montevideo, Uruguay*

^b*CONICET, Mathematics Department, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1482 Pabellon I, Ciudad Universitaria Buenos Aires, Argentina*

^c*Laboratoire de Mathematiques Appliquees LMAC, Universite de Technologie de Compiègne, Rue du Dr Schweitzer CS 60319, 60203 Compiègne, France*

Abstract

By constructing jointly a random graph and an associated exploration process, we define the dynamics of a “parking process” on a class of uniform random graphs as a measure-valued Markov process, representing the empirical degree distribution of non-explored nodes. We then establish a functional law of large numbers for this process as the number of vertices grows to infinity, allowing us to assess the jamming constant of the considered random graphs, i.e. the size of the maximal independent set discovered by the exploration algorithm. This technique, which can be applied to any uniform random graph with a given degree distribution, can be seen as a generalization in the space of measures, of the differential equation method introduced by Wormald.

Keywords: Random graph; Configuration model; Parking process; Measure-valued Markov process; Hydrodynamic limit.

1. Introduction

Consider a finite graph G for which \mathcal{V} is the set of nodes or sites. The parking process in continuous time on G may be described as follows. At time 0, all sites are vacant. They all have independent exponential clocks. When the clock of a given vacant site rings and all of its neighbors are vacant, this site turns occupied. Otherwise, nothing happens. Once occupied, a site remains so for ever. The process goes on until all sites are either occupied or have at least one of their neighbors occupied. The final state of the process is often referred to as the jamming limit of G , and the final proportion of occupied sites, its jamming constant.

Our motivation to study the parking process on random graphs is twofold. On the one hand, these dynamics are the simplest procedure to discover maximal independent sets and have been extensively studied for some specific graphs. Explicit results have been obtained for regular graphs [24], exploiting their very specific structure (see also [10] for graphs with large girths). In the Erdős Rényi case, a similar differential method can be employed thanks to the great amount of independence and symmetry of the collection of edges, to get an explicit jamming constant (see Theorem 2.2 and the references in [16]). Hence, to look at “uniform” random graphs having a given asymptotic degree distribution, but much less structure and symmetry, is a natural continuation of this research avenue.

*Corresponding author

Email addresses: paola@fing.edu.uy (Paola Bermolen), matthieu.jonckheere@gmail.com (Matthieu Jonckheere), pascal.moyal@utc.fr (Pascal Moyal)

On the other hand, parking processes have received a great amount of attention in the case of spatial structures. It has been considered on discrete structures like \mathbb{Z}^d [18, 8] and on point processes [17, 8, 2, 22]. In physics and biological sciences, where it is usually referred to as random sequential absorption, it models phenomena of deposition of colloidal particles or proteins on surfaces (see [7]). In communication sciences and in wireless networks in particular, it allows to represent the number of connections for CSMA-like algorithms in a given time-slot, for a given spatial configuration of terminals (see [14] for a classical reference on the definition of the protocol). The general idea of CSMA is to schedule transmissions in such a way that nodes that interfere each other would not transmit simultaneously, see for instance [2] for a stochastic geometry-based model in which CSMA is approximated by a Matérn-like process. Unfortunately, spatial models are in general very difficult to study theoretically and to the best of our knowledge, there is no efficient way to compute the jamming constant in most cases. Studying the jamming constant of uniformly chosen random graphs with a given asymptotic degree distribution (we make this notion more precise in the sequel) allows to make a first step in this direction, by studying a “first order” model which grasps only the bonds between points but no further correlations. The techniques and analysis presented here are at the core of the performance evaluation analysis of wireless systems, as developed in a companion applied paper [3].

In this paper we focus on the parking process for a class of random graphs having given deterministic asymptotic degree distributions, and derive a computable characterization of the jamming constant, as the number of vertices grows to infinity. To describe the evolution of the parking process in a Markovian fashion, without keeping track of a too large set of information, our strategy is the following: we start from the degree distribution of the graph, and then construct simultaneously the random graph and the associated parking process. In both cases, the underlying (multi-)graph is constructed similarly to the configuration model, see [25, 21, 11] and the references therein. A similar approach was considered in [6], to construct a random social network together with a SIR process which propagates on it.

Following these ideas, we first define these dynamics for a graph having a fixed number of nodes n and study the time-evolution of the empirical measure of the degrees of the vacant sites, which defines a measure-valued Markov process. Then, under the assumption that the initial empirical measure of degrees converges to a measure having mild moment assumptions, we take n to infinity and prove a functional law of large numbers on the evolution of the empirical measures of degrees. We show in particular that given our assumptions on the initial random degree distribution, the limit is unique, and defined as the solution of a non-linear infinite-dimensional system of differential equations. This can be seen as a generalization in the space of point measures, of the differential equation method introduced by Wormald [24], providing simple and insightful characteristics of the random graphs under consideration, as solutions of finite-dimensional differential equations. To the best of our knowledge, this is the first such general result, embracing in particular, several particular cases investigated in the literature.

In the case of the Poisson distribution, we are able to explicitly compute the measure-valued flow of unexplored nodes, which turns out to be an inhomogeneous Poisson measure. We then retrieve the jamming constant of the Erdős-Rényi (ER) graph. (Note however that our construction does not lead to a proper ER graph, only to a random multi-graph having the same degree distribution.) We also retrieve a constant calculated by Rényi [9, 20] for a spatial model on \mathbb{Z} , showing that both models share the same jamming limit.

The proof of the functional law of large numbers is based on successive approximations for the generator of the infinite-dimensional Markov process, relying in particular on quantifying the probability to obtain self-loops and multi-edges. Note that this difficulty is inherent to configuration model constructions, which have the disadvantage of constructing a multi-graph rather than a simple one, though elegant arguments have shown that with a probability independent of the size of the graph, a simple graph is obtained [11] (see also the monograph [21]). The uniqueness of the deterministic limiting measure-valued flow is not immediate and has to be proved using an adequate norm on the

spaces of solutions.

The rest of the paper is organized as follows. In Section 2, we describe the simultaneous construction of the parking process and the random graph. In Section 3 we calculate the generator of the induced measure-valued Markov process, and the corresponding semi-martingale decomposition is introduced. In Section 4, we state our main result and its consequences. In particular we show how the latter leads to closed-forms, or at least to computable characterizations of the jamming constant in various cases. Section 5 is devoted to the proofs of our main results.

Notation. Let us introduce the main notation used throughout the paper.

- We denote by \mathbb{R} the set of real numbers, and \mathbb{R}_+ (respectively, \mathbb{R}^*) the subset of non-negative (resp., non-null) real numbers. Let also \mathbb{N} be the set of non-negative integers and \mathbb{N}^* , the subset of positive integers. For any $x, y \in \mathbb{R}$, let $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and $x^+ = x \vee 0$. Let also for any $a, b \in \mathbb{N}$, $\llbracket a, b \rrbracket = \{a, a+1, \dots, b\}$. We denote by $o(\cdot)$, a function: $\mathbb{N} \rightarrow \mathbb{R}$ such that $n^\beta o(n) \xrightarrow{n \rightarrow \infty} 0$ for any real number $\beta < 1$.
- Let \mathcal{B}_b be the set of Borel bounded functions: $\mathbb{R} \rightarrow \mathbb{R}$. For all $\phi \in \mathcal{B}_b$, denote

$$\|\phi\| = \sup_{x \in \mathbb{R}} |\phi(x)|.$$

Denote for any Borel set A , $\mathbb{1}_A$ the indicator function of A . Denote by $\mathbf{1}$, the real function constantly equal to 1 and for any $k \in \mathbb{N}$, χ^k the function $x \mapsto x^k$. For all $\phi \in \mathcal{B}_b$, we also denote by $\Delta\phi$ the *discrete gradient* of ϕ , i.e.

$$\Delta\phi(i) = \phi(i) - \phi(i-1), \forall i \in \mathbb{N}^*.$$

- Let $\mathcal{M}_F(\mathbb{N})$ be the set of finite measures on \mathbb{N} . We write $\mu(i) := \mu(\{i\})$ for any $\mu \in \mathcal{M}_F(\mathbb{N})$ and any $i \in \mathbb{N}$. The null measure is denoted $\mathbf{0}$. For all $\mu \in \mathcal{M}_F(\mathbb{N})$ and all $\phi : \mathbb{R} \rightarrow \mathbb{R}$, $\langle \mu, \phi \rangle$ denotes the integral of ϕ with respect to μ :

$$\langle \mu, \phi \rangle = \int \phi(x) \mu(dx) = \sum_{i \in \mathbb{N}} \phi(i) \mu(i).$$

In this way, for any such μ and any $A \subset \mathbb{N}$, $\langle \mu, \mathbb{1}_A \rangle = \mu(A)$ is the measure of A , $\langle \mu, \mathbf{1} \rangle = \mu(\mathbb{N})$ is the total mass of μ , and for any $k \in \mathbb{N}^*$, $\langle \mu, \chi^k \rangle$ is the k -th moment of μ . For any $\mu \in \mathcal{M}_F(\mathbb{N})$, denote F_μ the cumulative function associated to μ and F_μ^{-1} , its generalized inverse. For any counting measure $\mu \in \mathcal{M}_F(\mathbb{N})$, we will be led to order and index the atoms of μ as follows:

- we denote for any $\ell \in \{1, \dots, \mu(0)\}$, $v_\ell(\mu)$ the ℓ th atom of degree 0 ranked in arbitrary order;
- by induction, for any $i \in \mathbb{N}$ and any $\ell \in \llbracket 1, \mu(i+1) \rrbracket$, $v_{\sum_{j=1}^i \mu(j) + \ell}(\mu)$, the ℓ th atom of degree $i+1$, in arbitrary order, in a way that

$$\mu = \sum_{j=1}^{\langle \mu, \mathbf{1} \rangle} \delta_{v_j(\mu)}. \quad (1)$$

- For any Polish space E , $\mathcal{D}([0, T], E)$ is the Polish space of rcll functions from $[0, T]$ to E , and $\mathcal{C}([0, T], E)$ is the space of continuous functions from $[0, T]$ to E . Both $\mathcal{D}([0, T], E)$ and $\mathcal{C}([0, T], E)$ are equipped with the Skorokhod J_1 -topology.
- Unless explicitly mentioned, throughout all the random variables (r.v.'s, for short) are defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$. On the latter, let us write “ \Rightarrow ” for weak convergence of r.v.'s, and “ $\xrightarrow{(\mathcal{P})}$ ” for convergence in probability. Finally, let us denote $(\langle M \rangle_t)_{t \geq 0}$ the quadratic variation of the $\mathcal{D}([0, T], E)$ -valued martingale $(M_t)_{t \geq 0}$.

2. Construction of the graph and Markov representation

In this section we present our construction of a random graph of prescribed degree distribution, simultaneously with the parking process on the latter graph. The basic objects of our construction are:

- (i) a probability measure ν on \mathbb{N} having support $\llbracket 0, n-1 \rrbracket$ (where $n \geq 1$), which will be referred to as the *degree distribution*;
- (ii) a n -independent sample $\mathbf{d} := (d(1), \dots, d(n))$ of the distribution ν , termed *degree vector*.¹ The *empirical degree distribution* is the following random point measure,

$$\mu_0 = \sum_{i=1}^n \delta_{d(i)};$$

- (iii) the set of *nodes* \mathcal{V} , whose elements are denoted $u_0(1), \dots, u_0(n)$ (the use of this notation will become clear in a few lines). We set a one-to-one relation between the nodes and the atoms of μ_0 as follows: to the node $u_0(i)$ is associated the element $d(\gamma(i))$, where γ is a permutation of $\llbracket 1, n \rrbracket$ arranging d in increasing order. We then say that $d(\gamma(i))$ is the *degree* of $u_0(i)$ and we write $d_{u_0(i)} := d(\gamma(i))$.

At time 0, all the nodes of \mathcal{V} are disconnected: the *associated graph* of our construction at time 0, denoted G_0 , thus consists in the set of nodes \mathcal{V} , without any edge. At this point, the nodes are all said *unexplored* (we say that they are “U-nodes”). We consider that each node has as many **unmatched half-edges** as its degree - we then say that the node is the *ego* of its half-edges.

Let us define for all $t \geq 0$, \mathcal{U}_t , \mathcal{A}_t and \mathcal{B}_t the sets of *unexplored*, *active* and *blocked* nodes, respectively. At $t = 0$, we thus fix $\mathcal{U}_0 = \mathcal{V}$ (hence the notation above), and set $\mathcal{A}_0 = \mathcal{B}_0 = \emptyset$. Let us also define for all $t \geq 0$,

$$\mathcal{H}_t = \left\{ \text{unmatched half-edges at time } t \right\}.$$

Let for any t and any $j \in \mathcal{U}_t$, $d_j(\mathcal{U}_t)$ denote the number of unmatched half-edges of j at time t . Define also the following element of $\mathcal{M}_F(\mathbb{N})$,

$$\mu_t = \sum_{j \in \mathcal{U}_t} \delta_{d_j(\mathcal{U}_t)},$$

termed *empirical degree distribution at t* . Notice that the cardinality U_t of \mathcal{U}_t , and the cardinality H_t of \mathcal{H}_t can respectively be retrieved from μ_t by

$$U_t = \langle \mu_t, \mathbf{1} \rangle \quad \text{and} \quad H_t = \langle \mu_t, \chi \rangle.$$

All these time-dependent quantities will be updated, by induction on the event times, as will be described hereafter.

Fix $\lambda > 0$, and let ξ_0 be a random variable of exponential distribution of rate λn . As long as this exponential clock does not ring, the system remains unchanged: we set $\mu_t = \mu_0$ and likewise, $\mathcal{U}_t, \mathcal{A}_t, \mathcal{B}_t$ and \mathcal{H}_t equal their initial value, for any $t \in [0, \xi_0)$. The dynamics of the system is then determined by induction, as follows: assume that a clock rings at time t . Then, several state changes occur instantaneously at t , but following a given sequence. To represent these changes, all characteristics of the system just before the event occurs are indexed by t^- and then, to distinguish between the different steps of the state actualization, by t^{-+} , t^{-++} and finally t .

¹We assume independence of the degrees for simplicity, however it should be noted that the results hereafter hold in larger generality: for assumption (28) to hold true, we only need the convergence of the Cesàro means of the vector \mathbf{d} as n goes large.

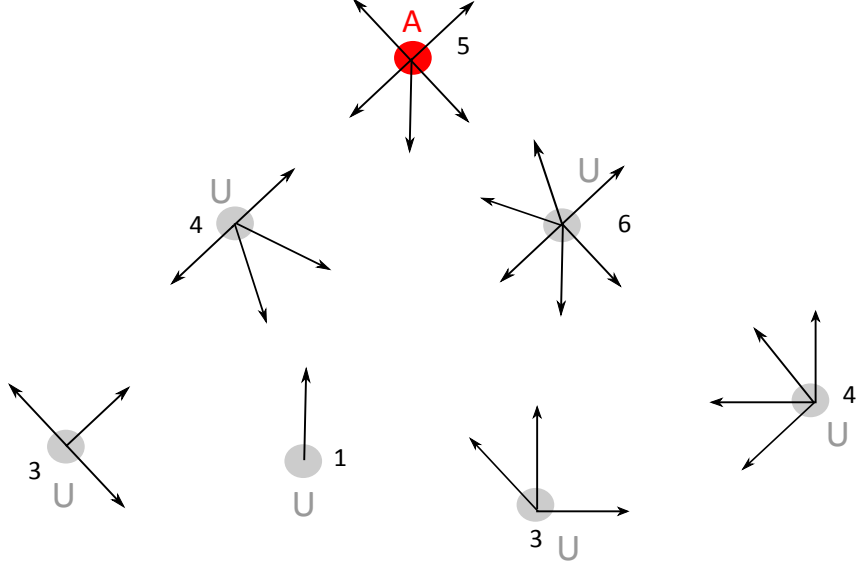


Figure 1: Step 1 - the new active node is selected.

Step 1. A node a becomes active: we draw uniformly an index i in $\llbracket 1, U_{t-} \rrbracket$. The corresponding node $a := u_{t-}(i)$ of \mathcal{U}_{t-} becomes active and we set

$$\mathcal{U}_{t-+} = \mathcal{U}_{t-} \setminus \{a\}, \quad \mathcal{A}_{t-+} = \mathcal{A}_{t-} \cup \{a\} \quad \text{and} \quad \mathcal{B}_{t-+} = \mathcal{B}_{t-}.$$

Let $K(\mu_{t-}) = d_a$, the number of unmatched half-edges of a at t^- . As a is no longer a U-node, we update the measure μ_{t-} as follows,

$$\mu_{t-+} = \mu_{t-} - \delta_{K(\mu_{t-})}. \quad (2)$$

Example 2.1. Let the measure at t^- be

$$\mu_{t-} = \delta_1 + 2\delta_3 + 2\delta_4 + \delta_5 + \delta_6,$$

so that the associated graph has $n = \langle \mu_{t-}, \mathbf{1} \rangle = 7$ nodes and $\langle \mu_{t-}, \chi \rangle = 26$ half-edges (see Fig. 1). A clock rings at t . The new A-node a has degree $K(\mu_{t-}) = 5$ and the measure is updated to

$$\mu_{t-+} = \delta_1 + 2\delta_3 + 2\delta_4 + \delta_6.$$

Step 2. The neighbors of a become of class B: the neighbors of the new A-node are blocked, we say that they become B-nodes. The identity of these new B-nodes is determined by matching the $K(\mu_{t-})$ elements of \mathcal{H}_{t-} of ego a , with half-edges of \mathcal{H}_{t-} , as follows:

- a first half-edge of \mathcal{H}_{t-} of ego a is matched with another one, drawn uniformly among the $\langle \mu_{t-}, \chi \rangle - 1$ possible ones;
- on and on, as long as all half-edges of \mathcal{H}_{t-} of ego a have not been matched, we take one of those, and match it with another half-edge of \mathcal{H}_{t-} which has not yet been matched, drawn uniformly in the latter set.

Notice that at each step, we may match couples of half-edges emanating from a together - hence creating self-loops around a . At the end of this procedure, we let $\tilde{K}(\mu_{t-})$ be the number of

edges linking a to other nodes. Clearly, $\tilde{K}(\mu_{t-})$ cannot exceed the number of half-edges of ego a , nor the number of half-edges of \mathcal{H}_{t-} of egos different from a , in other words

$$\tilde{K}(\mu_{t-}) \leq K(\mu_{t-}) \wedge (\langle \mu_{t-}, \chi \rangle - K(\mu_{t-})). \quad (3)$$

We have thus fixed the identity of the q new B-nodes (where $q \leq \tilde{K}(\mu_{t-})$), which are the egos $u_{t-}(i_1), u_{t-}(i_2), \dots, u_{t-}(i_q)$ different from a , of the $\tilde{K}(\mu_{t-})$ half-edges matched with the $\tilde{K}(\mu_{t-})$ half-edges of ego a . We then set

$$\begin{cases} \mathcal{U}_{t-++} &= \mathcal{U}_{t-+} \setminus \{u_{t-}(i_1), u_{t-}(i_2), \dots, u_{t-}(i_q)\}; \\ \mathcal{A}_{t-++} &= \mathcal{A}_{t-+}; \\ \mathcal{B}_{t-++} &= \mathcal{B}_{t-+} \cup \{u_{t-}(i_1), u_{t-}(i_2), \dots, u_{t-}(i_q)\}. \end{cases}$$

For all $j \in \llbracket 1, q \rrbracket$, let $N_j(\mu_{t-})$ be the number of edges shared by $u_{t-}(i_j)$ with a . Let us define the two following point measures,

$$Y(\mu_{t-}) = \sum_{j=1}^q N_j(\mu_{t-}) \delta_{d_{u_{t-}(i_j)}}; \quad (4)$$

$$\tilde{Y}(\mu_{t-}) = \sum_{j=1}^q \delta_{d_{u_{t-}(i_j)}}. \quad (5)$$

In other words, for any i , $\tilde{Y}(\mu_{t-})(i)$ (resp., $Y(\mu_{t-})(i)$) is the number of neighbors of a (resp., of edges shared by a with its neighbors) having i unmatched half-edges at t^- . Thus, $\langle Y(\mu_{t-}), \mathbf{1} \rangle = \tilde{K}(\mu_{t-})$ is the number of half-edges of ego a and $\langle \tilde{Y}(\mu_{t-}), \mathbf{1} \rangle = q$ is the number of neighbors of a . As the new B-nodes are no longer unexplored, their degree must be erased from the measure μ_{t-+} , which is updated as follows,

$$\mu_{t-++} = \mu_{t-+} - \tilde{Y}(\mu_{t-}). \quad (6)$$

There remain $\langle \mu_{t-}, \chi \rangle - K(\mu_{t-}) - \tilde{K}(\mu_{t-})$ unmatched half-edges at this point.

Example 2.2 (Example 2.1 continued). *The uniform selection of the neighbors of the new A-node a results in a loop around it, so we have $\tilde{K}(\mu_{t-}) = 3$, $q = 2$,*

$$Y(\mu_{t-}) = \delta_4 + 2\delta_6 \quad \text{and} \quad \tilde{Y}(\mu_{t-}) = \delta_4 + \delta_6,$$

as there exists a double-edge between a and its neighbor of degree 6 (see Fig. 2). We then have

$$\mu_{t-++} = \delta_1 + 2\delta_3 + \delta_4.$$

Observe that $\tilde{K}(\mu_{t-}) = 3$ and the number of unmatched half-edges at this point is $\langle \mu_{t-}, \chi \rangle - K(\mu_{t-}) - \tilde{K}(\mu_{t-}) = 26 - 5 - 3 = 18$. Between parenthesis is indicated the number of still unmatched half-edges of the B-nodes.

Step 3. Updating of the number of unmatched half-edges: the available half-edges at this point, *i.e.* the elements of \mathcal{H}_{t-++} , either emanate from B-nodes and do not point to a , or emanate from nodes of \mathcal{U}_{t-++} . Let us denote

$$\mathcal{H}_{t-++}^B := \left\{ \text{half-edges of } \mathcal{H}_{t-++} \text{ having ego in } \mathcal{B}_{t-++} \right\} \subset \mathcal{H}_{t-++},$$

and observe that

$$|\mathcal{H}_{t-++}^B| = \langle \tilde{Y}(\mu_{t-}), \chi \rangle - \tilde{K}(\mu_{t-}) = \langle \tilde{Y}(\mu_{t-}), \chi \rangle - \langle Y(\mu_{t-}), \mathbf{1} \rangle. \quad (7)$$

We now fully attach the new B-nodes to the associated graph, *i.e.* we match all the half-edges of \mathcal{H}_{t-++}^B , either with other elements of \mathcal{H}_{t-++}^B , or with elements of \mathcal{H}_{t-++} having ego in \mathcal{U}_{t-++} . This is done according to the following procedure:

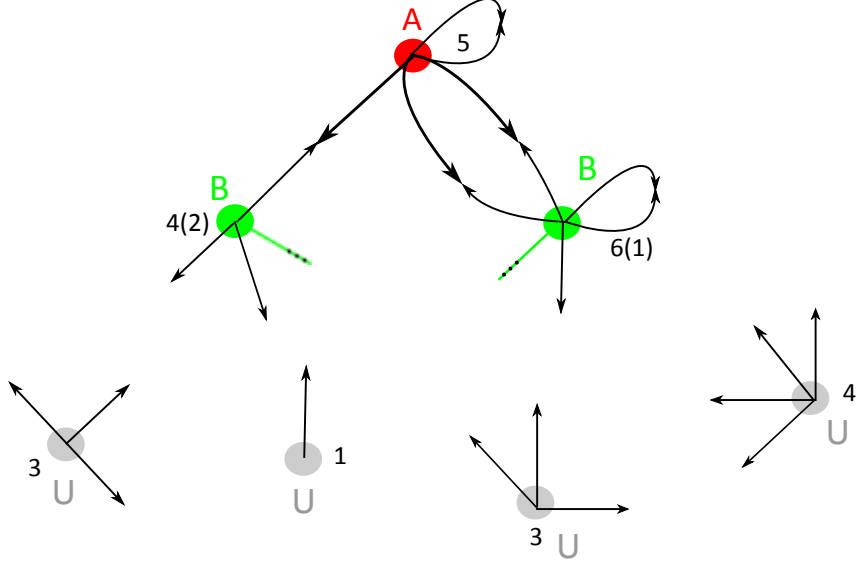


Figure 2: Step 2 - the blocked nodes are attached to the active node.

- draw an integer uniformly at random in $\llbracket 1, q \rrbracket$, say ℓ . We match the remaining $d_{u_t^-(i_\ell)} - N_\ell(\mu_{t-})$ open half-edges of $u_t^-(i_\ell)$ exactly as those of a : we take these open half-edges one by one; each time, we draw uniformly at random a match for the latter in all available half-edges (emanating from a node of \mathcal{U}_{t-++} , another node of \mathcal{B}_{t-++} or from $u_t^-(i_\ell)$ itself), until all half-edges of $u_t^-(i_\ell)$ are matched;
- then, draw at random another integer m in $\llbracket 1, q \rrbracket \setminus \{\ell\}$, and match all available half-edges of $u_t^-(i_m)$ in the same manner, and so on... until all half-edges of \mathcal{H}_{t-++}^B have been matched, to form edges of the associated graph.

At the end of this operation, we have possibly created edges between the new B-nodes and the remaining U-nodes. Let us denote $X(\mu_{t-})$ the number of such edges, in other words

$$X(\mu_{t-}) = \text{Card} \left\{ \text{half-edges of } \mathcal{H}_{t-++}^B \text{ matched with half-edges of } \mathcal{H}_{t-++} \setminus \mathcal{H}_{t-++}^B \right\}. \quad (8)$$

To update μ_{t-++} , we have to subtract these $X(\mu_{t-})$ half-edges to the number of available half-edges of the remaining U-nodes. To formalize this operation, it is convenient to index the remaining U-nodes in the following way: for all $i \in \llbracket 1, n-1 \rrbracket$ and all $\ell \in \llbracket 1, \mu_{t-++}(i) \rrbracket$, we let $u_{t-++}(i, \ell)$ be the ℓ -th node of \mathcal{U}_{t-++} having i unmatched half-edges at $t-++$ (if any), ranked in an arbitrary order. Then, we define

$$W(\mu_{t-})(i, \ell) = \text{Card} \left\{ \text{edges shared by } u_{t-++}(i, \ell) \text{ with nodes of } \mathcal{B}_{t-++} \right\}, \quad (9)$$

and observe that

$$\sum_{i=1}^{n-1} \sum_{\ell=1}^{\mu_{t-++}(i)} W(\mu_{t-})(i, \ell) = X(\mu_{t-}).$$

The quantity $W(\mu_{t-})(i, \ell)$ has to be subtracted from the number of open half-edges of each unexplored node $u_{t-++}(i, \ell)$, hence we finally write

$$\mu_t = \mu_{t-++} - \sum_{i=1}^{n-1} \sum_{\ell=1}^{\mu_{t-++}(i)} \left(\delta_i - \delta_{i-W(\mu_{t-})(i, \ell)} \right). \quad (10)$$

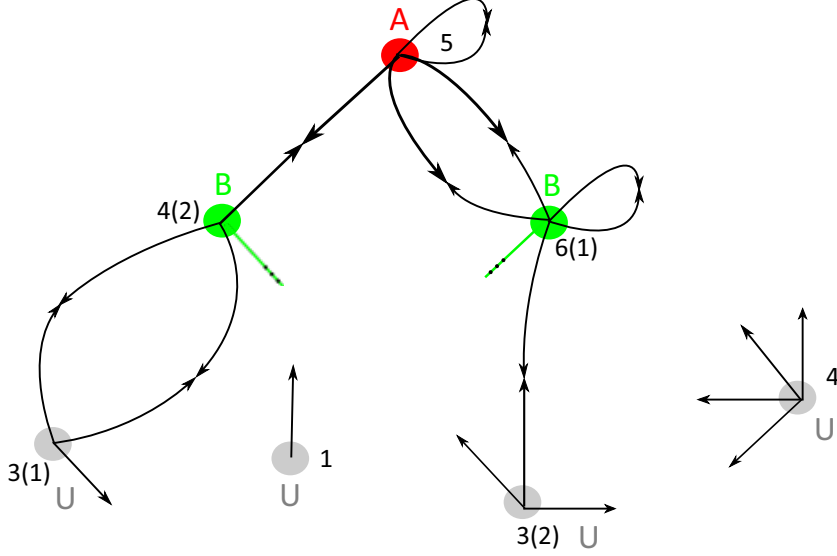


Figure 3: Step 3 - the new blocked nodes are connected between each other, and with remaining unexplored nodes

Example 2.3 (Example 2.1 concluded). In Fig. 3 we see that $|\mathcal{H}_{t-++}^B| = \langle \tilde{Y}(\mu_{t+}), \chi \rangle - \tilde{K}(\mu_{t-}) = 10 - 3 = 7$ and $X(\mu_{t-}) = 3$, since there are 4 half-edges emanating from B-nodes that are matched together. The three remaining half-edges are matched with the remaining U-nodes as follows: if among the U-nodes of degree 3, the one on the left has label 1 and that on the right has label 2, then

$$W(\mu_{t-})(3, 1) = 2 \text{ and } W(\mu_{t-})(3, 2) = 1.$$

The updated measure is then

$$\mu_t = 2\delta_1 + \delta_2 + \delta_4.$$

There remain $\langle \mu_t, \mathbf{1} \rangle = 4$ unexplored nodes and $\langle \mu_t, \chi \rangle = 8$ unmatched half-edges.

After Steps 1-3, we end up with a measure μ_t where $1 + q$ atoms have been erased with respect to μ_{t-} , and whose first moment (*i.e.* the number of open half-edges) has the same parity as that of μ_{t-} . The associated graph G_t equals G_{t-} , plus all edges that have been drawn between a and its neighbors, between its neighbors with one another, and between its neighbors and the remaining U-nodes. Moreover, only the remaining U-nodes still have open half-edges at t and the measure μ_t provides the repartition of the latter half-edges among \mathcal{U}_t . At this point, we re-index all elements of \mathcal{U}_t in the order of increasing number of open half-edges, as was done above:

$$\mathcal{U}_t = \{u_t(1), \dots, u_t(U_t)\}.$$

We now draw a new exponential r.v. ξ_t of parameter λU_t , independently of everything else. As above, the system remains constant until time $t + \xi_t$, at which we re-iterate Step 1-3, and so on.

The procedure ends at the stopping time

$$T_0 = \inf\{t \geq 0; U_t = 0\},$$

which clearly is a.s. finite. At that instant, if $\sum_{i=1}^n d(i)$ was odd there remain a single unmatched half-edge, which we remove. At that time T_0 , we thus end up with $\mu_{T_0} = \mathbf{0}$, $\mathcal{U}_{T_0} = \mathcal{H}_{T_0} = \emptyset$ and $|\mathcal{A}_{T_0} \cup \mathcal{B}_{T_0}| = n$. The final associated graph G_{T_0} is a multi-graph of degree vector \mathbf{d} (up to the deletion of a single half-edge in the case mentioned above). The set \mathcal{A}_{T_0} is the *jammed limit* of the latter graph (and of the parking process) and $|\mathcal{A}_{T_0}|/n$, its *jammed constant*.

3. Generator and semi-martingale decomposition

It follows from the construction of Section 2, that the process $(\mu_t)_{t \geq 0}$ is Markov in $\mathcal{M}_F(\mathbb{N})$. Its infinitesimal generator \mathcal{Q} is defined for all $F : \mathcal{M}_F(\mathbb{N}) \rightarrow \mathbb{R}$ in the domain of \mathcal{Q} , by

$$\mathcal{Q}F(\mu) = \lim_{h \rightarrow 0} \frac{1}{h} \left(\mathbf{E}[F(\mu_h) \mid \mu_0 = \mu] - F(\mu) \right), \mu \in \mathcal{M}_F(\mathbb{N}). \quad (11)$$

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded function, and denote for all $\mu \in \mathcal{M}_F(\mathbb{N})$, $\Pi_\phi(\mu) = \langle \mu, \phi \rangle$. We show that Π_ϕ belongs to the domain of \mathcal{Q} and deduce $\mathcal{Q}\Pi_\phi$ from (2,6,10), following Steps 1-3.

Step 1. The probability that a node of degree k is drawn, equals the proportion of atoms at level k among all atoms of the measure μ_0 . In other words, for any μ and k ,

$$\mathbb{P}_K(k \mid \mu) := \mathbf{P}[K(\mu_0) = k \mid \mu_0 = \mu] = \frac{\mu(k)}{\langle \mu, \mathbf{1} \rangle}. \quad (12)$$

Denote for any μ , k , and any integer $\tilde{k} \leq k \wedge (\langle \mu, \chi \rangle - k)$,

$$\mathbb{P}_{\tilde{K}}(\tilde{k} \mid \mu, k) := \mathbf{P}[\tilde{K}(\mu_0) = \tilde{k} \mid \mu_0 = \mu, K(\mu) = k] \quad (13)$$

and $\mathbb{E}_{\tilde{K}}[\cdot \mid \mu, k]$ the corresponding conditional expectation. We won't need a precise expression of the latter distribution, but will show instead that \tilde{K} coincides with K with overwhelming probability as the size n of the graph goes large. Or which is equivalent, when the number of nodes goes to infinity the probability of choosing two half-edges of the new A-node to match together is arbitrarily small.

To see this, consider a system in which all half-edges emanating from the new A-node a are duplicated, and where we match the $K(\mu_0)$ half-edges of a uniformly, either with half-edges emanating from the other nodes, or with their duplicata. Hence, given μ and k , in this auxiliary system the number $\tilde{K}(\mu_0)$ of half-edges of ego a matched with half-edges of other egos (and not with their duplicata) follows an hypergeometric distribution with parameters k , $\langle \mu, \chi \rangle$ and $\langle \mu, \chi \rangle - k$. Moreover the probability of matching as many half-edges of ego a with half-edges of other egos as possible (*i.e.* $K(\mu_0) \wedge (\langle \mu_0, \chi \rangle - K(\mu_0))$ half-edges) is clearly larger in the actual system than in the auxiliary one, thus

$$\begin{aligned} \mathbf{P}[\text{the new A-node at 0 shares its } K(\mu_0) \text{ edges with other nodes} \mid \mu_0 = \mu, K(\mu) = k] \\ = \mathbb{P}_{\tilde{K}}(k \mid \mu, k) \\ \geq \mathbf{P}[\tilde{K}(\mu_0) = k \mid \mu_0 = \mu, K(\mu) = k] \\ = \frac{\binom{\langle \mu, \chi \rangle - k}{k}}{\binom{\langle \mu, \chi \rangle}{k}} \mathbb{1}_{\{k \leq \langle \mu, \chi \rangle - k\}}, \end{aligned} \quad (14)$$

which will be proven to tend to 1 as n goes large.

Step 2. Conditionally to $\{\mu_0 = \mu\} \cap \{K(\mu_0) = k\} \cap \{\tilde{K}(\mu_0) = \tilde{k}\}$, $Y(\mu_0)$ follows a multivariate hypergeometrical distribution on $\mathcal{M}_F(\mathbb{N})$, of parameters $(\tilde{k}, \langle \mu, \chi \rangle - k, n - 1, P)$ (see Appendix A), where P is given by

$$P(i) = i(\mu(i) - \delta_k(i)), i \in \llbracket 0, n - 1 \rrbracket.$$

In other words, we have for all $y \in \mathcal{M}_F(\llbracket 0, n - 1 \rrbracket)$ such that $\langle y, \mathbf{1} \rangle = \tilde{k}$ and $y(i) \leq P(i)$ for all i ,

$$\begin{aligned}\mathbb{P}_Y(y \mid \mu, k, \tilde{k}) &:= \mathbf{P} \left[Y(\mu_0) = y \mid \mu_0 = \mu, K(\mu_0) = k, \tilde{K}(\mu_0) = \tilde{k} \right] \\ &= \frac{\prod_{i \in \llbracket 0, n-1 \rrbracket} \binom{i(\mu(i) - \delta_k(i))}{y(i)}}{\binom{\langle \mu, \chi \rangle - k}{\tilde{k}}},\end{aligned}\quad (15)$$

and we denote $\mathbb{E}_Y \left[\cdot \mid \mu, k, \tilde{k} \right]$ the corresponding conditional expectation. For all $\tilde{y} \in \mathcal{M}_F(\mathbb{N})$, we denote

$$\mathbb{P}_{\tilde{Y}}(\tilde{y} \mid \mu, k, \tilde{k}, y) := \mathbf{P} \left[\tilde{Y}(\mu_0) = \tilde{y} \mid \mu_0 = \mu, K(\mu) = k, \tilde{K}(\mu) = \tilde{k}, Y(\mu) = y \right], \quad (16)$$

and define $\mathbb{E}_{\tilde{Y}} \left[\cdot \mid \mu, k, \tilde{k}, y \right]$ accordingly. Here again, the precise form of this distribution is not needed explicitly, since $\tilde{Y}(\cdot)$ and $Y(\cdot)$ tend to coincide for large graphs, *i.e.* $\mathbb{P}_{\tilde{Y}}(y \mid \mu, k, \tilde{k}, y)$ (the probability that all edges between the new active node and its neighbors are simple) tends to 1 for large n . This will be shown using the following lower-bound: recalling (1), we have that

$$\begin{aligned}\mathbb{P}_{\tilde{Y}}(y \mid \mu, k, \tilde{k}, y) &= \sum_{\substack{\beta \in (\{1, \dots, \langle \mu, \mathbf{1} \rangle - 1\})^{\tilde{k}}; \\ \beta(1) \neq \beta(2) \neq \dots \neq \beta(k)}} \frac{\prod_{i=1}^{\tilde{k}} d_{v_{\beta(i)}}(\mu)}{\mathcal{A}_{\langle \mu, \chi \rangle - k}^{\tilde{k}}} \\ &= \frac{\prod_{i \in \mathbb{N}^*} i^{y(i)}}{\mathcal{A}_{\langle \mu, \chi \rangle - k}^{\tilde{k}}} \\ &\geq 1 - \binom{\tilde{k}}{2} \sum_{\ell=1}^{\langle \mu, \mathbf{1} \rangle - 1} \frac{d_{v_\ell}(\mu - \delta_k)(d_{v_\ell}(\mu - \delta_k) - 1)}{(\langle \mu, \chi \rangle - k)(\langle \mu, \chi \rangle - k - 1)} \\ &\geq 1 - \binom{\tilde{k}}{2} \frac{\langle \mu, \chi^2 \rangle}{(\langle \mu, \chi \rangle - k)(\langle \mu, \chi \rangle - (k+1))},\end{aligned}\quad (17)$$

since the number of (ordered) configurations entailing a multiple edge between a and a given U-vertex u , is less than the number of pairs of half edges starting from u , times the number of couples of half-edges matched with the \tilde{k} half-edges of a .

Step 3. Let us first address the r.v. $X(\cdot)$ defined by (8). Denote for all y, \tilde{y} and all integers $x \leq \langle \tilde{y}, \chi \rangle - \langle y, \mathbf{1} \rangle$,

$$\mathbb{P}_X(x \mid \mu, k, \tilde{k}, y, \tilde{y}) := \mathbf{P} \left[X(\mu_0) = x \mid \mu_0 = \mu, K(\mu) = k, \tilde{K}(\mu) = \tilde{k}, Y(\mu) = y, \tilde{Y}(\mu) = \tilde{y} \right]. \quad (18)$$

Again, the exact distribution is not needed, as we will see that $X(\mu_0)$ tends to coincide with the number of open half-edges emanating from the new B-nodes (in other words, the latter nodes tend not to share any edge with one another - which would have created triangles including a). Indeed, noticing that the total number of available half-edges after the connexion between the new A node and its neighbors equals $\langle \mu_0, \chi \rangle - K(\mu_0) - \tilde{K}(\mu_0)$ and using (7), we can show exactly as for (14) that

$$\begin{aligned}\mathbb{P}_X(\langle \tilde{y}, \chi \rangle - \tilde{k} \mid \mu, k, \tilde{k}, y, \tilde{y}) &= \mathbf{P} \left[\text{the new B-nodes do not share any edge with one another} \right. \\ &\quad \left. \mid \mu_0 = \mu, K(\mu) = k, \tilde{K}(\mu) = \tilde{k}, Y(\mu) = y, \tilde{Y}(\mu) = \tilde{y} \right] \\ &\geq \frac{\binom{\langle \mu, \chi \rangle - k - \langle \tilde{y}, \chi \rangle}{\langle \tilde{y}, \chi \rangle - \tilde{k}}}{\binom{\langle \mu, \chi \rangle - k - \tilde{k}}{\langle \tilde{y}, \chi \rangle - \tilde{k}}} \mathbb{1}_{\{\langle \tilde{y}, \chi \rangle - \tilde{k} \leq \langle \mu, \chi \rangle - k - \langle \tilde{y}, \chi \rangle\}},\end{aligned}\quad (19)$$

which is the probability that an hypergeometric random variable with parameters $\langle \tilde{y}, \chi \rangle - \tilde{k}$, $\langle \mu, \chi \rangle - k - \langle \tilde{y}, \chi \rangle$ and $\langle \mu, \chi \rangle - k - \tilde{k}$ takes the value $\langle \tilde{y}, \chi \rangle - \tilde{k}$. As above, the latter quantity will be shown to tend to one for large values of n .

Now, let $\mu, k, \tilde{k}, y, \tilde{y}$ and x as above, and define the set

$$\mathcal{W}(\mu, k, \tilde{k}, y, \tilde{y}, x) := \left\{ w \in \mathbb{N}^{\llbracket 1, n-1 \rrbracket^2}; \sum_{i=1}^{n-1} \sum_{\ell=1}^{n-1} w_{i\ell} = x \text{ and for all } i \in \llbracket 1, n-1 \rrbracket, \right. \\ \left. w(i, \ell) \leq i \text{ and } w(i, \ell) = 0 \text{ for all } \ell > \mu(i) - \delta_k(i) - \tilde{y}_i. \right\}.$$

Clearly, the double-indexed sequence $W(\mu)$ defined by (9) is drawn according to an hypergeometric choice among the “bunches” of half-edges represented by the remaining U-nodes. So we obtain readily that for all $w \in \mathcal{W}(\mu, k, \tilde{k}, y, \tilde{y}, x)$,

$$\begin{aligned} \mathbb{P}_W(w \mid \mu, k, \tilde{k}, y, \tilde{y}, x) \\ = \mathbf{P} \left[W(\mu_0) = w \mid \mu_0 = \mu, K(\mu_0) = k, \tilde{K}(\mu_0) = \tilde{k}, Y(\mu_0) = y, \tilde{Y}(\mu_0) = \tilde{y}, X(\mu_0) = x \right] \\ = \frac{\prod_{i \in \llbracket 1, n-1 \rrbracket} \prod_{\ell=1}^{\mu(i) - \delta_k(i) - \tilde{y}(i)} \binom{i}{w_{i\ell}}}{\binom{\langle \mu, \chi \rangle - k - \langle \tilde{y}, \chi \rangle}{x}}. \end{aligned} \quad (20)$$

We also need to introduce the following probability,

$$\begin{aligned} \mathbb{Q}_W(\mu, k, \tilde{k}, y, \tilde{y}, x) &= \sum_{w; w(i, \ell) \leq 1 \forall i, \ell} \mathbb{P}_W(w \mid \mu, k, x, y) \\ &= \mathbf{P} \left[\text{no new B-vertex at 0 has a multiple edge towards a remaining U-node} \right. \\ &\quad \left. \mu_0^n = \mu, K(\mu_0) = k, \tilde{K}(\mu_0) = \tilde{k}, Y(\mu_0) = y, \tilde{Y}(\mu_0) = \tilde{y}, X(\mu_0^n) = x \right]. \end{aligned} \quad (21)$$

Just as (17), we obtain that

$$\begin{aligned} \mathbb{Q}_W(\mu, k, \tilde{k}, y, \tilde{y}, x) &\geq 1 - \binom{x}{2} \sum_{\ell=1}^{\langle \mu, \mathbf{1} \rangle - 1 - k} \frac{d_{v_\ell}(\mu - \delta_k - \sum_i \tilde{y}(i) \delta_i) (d_{v_\ell}(\mu - \delta_k - \sum_i \tilde{y}(i) \delta_i) - 1)}{(\langle \mu, \chi \rangle - k - \langle y, \chi \rangle) (\langle \mu, \chi \rangle - k - \langle y, \chi \rangle - 1)} \\ &\geq 1 - \binom{x}{2} \frac{\langle \mu, \chi^2 \rangle}{(\langle \mu, \chi \rangle - k - \langle y, \chi \rangle) (\langle \mu, \chi \rangle - k - \langle y, \chi \rangle - 1)}. \end{aligned} \quad (22)$$

We finally introduce, for all $i \in \llbracket 1, n-1 \rrbracket$,

$$Z(\mu_0)(i) = \sum_{\ell=1}^{n-1} W(i, \ell)(\mu_0), \quad (23)$$

the number of half-edges of U-nodes having initially i unmatched half-edges, that have been matched with half-edges emanating from the new B-nodes. It then follows from (20) that, given $\mu, k, \tilde{k}, y, \tilde{y}$ and x , the integer measure $Z(\mu_0)$ follows a multivariate hypergeometric distribution of parameters $(x, \langle \mu, \chi \rangle - k - \langle \tilde{y}, \chi \rangle, n-1, P')$, where

$$P'(i) = i(\mu(i) - \delta_k(i) - \tilde{y}(i)), \quad i \in \llbracket 1, n-1 \rrbracket.$$

In other words, for all $z \in \mathcal{M}_F(\mathbb{N})$,

$$\begin{aligned} \mathbb{P}_W \left(Z(\mu) = z \mid \mu, k, \tilde{k}, y, \tilde{y}, x \right) \\ = \mathbf{P} \left[Z(\mu_0) = z \mid \mu_0 = \mu, K(\mu_0) = k, \tilde{K}(\mu_0) = \tilde{k}, Y(\mu_0) = y, \tilde{Y}(\mu_0) = \tilde{y}, X(\mu_0) = x \right] \\ = \frac{\prod_{i \in \llbracket 1, n-1 \rrbracket} \binom{i(\mu(i) - \delta_k(i) - \tilde{y}(i))}{z(i)}}{\binom{\langle \mu, \chi \rangle - k - \langle \tilde{y}, \chi \rangle}{x}}. \end{aligned} \quad (24)$$

Therefore, for all $h > 0$, from (2,6,10) and (12) we obtain that for all $\mu \in \mathcal{M}_F(\mathbb{N})$,

$$\begin{aligned} \mathcal{Q}\Pi_\phi(\mu) &= \lim_{h \rightarrow 0} \frac{1}{h} \mathbf{E} \left[(\langle \mu_h, \phi \rangle - \langle \mu, \phi \rangle) \mathbb{1}_{\{\xi_0 < h\}} \mid \mu_0 = \mu \right] \\ &= \lambda \langle \mu, \mathbf{1} \rangle \left[- \sum_{k \in \mathbb{N}} \frac{\mu(k)}{\langle \mu, \mathbf{1} \rangle} \right. \\ &\quad \times \left\{ \phi(k) + \sum_{\tilde{k} \in \llbracket 0, k \rrbracket} \mathbb{P}_{\tilde{K}}(\tilde{k} \mid \mu, k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = \tilde{k}}} \mathbb{P}_Y(y \mid \mu, k, \tilde{k}) \sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} \mid \mu, k, \tilde{k}, y) \right. \\ &\quad \times \sum_{i=1}^{n-1} \left(\tilde{y}(i) \phi(i) \right. \\ &\quad \left. \left. + \sum_{x \in \mathbb{N}^*} \mathbb{P}_X(x \mid \mu, k, \tilde{k}, y, \tilde{y}) \sum_{w \in \mathbb{N}[\llbracket 1, n-1 \rrbracket]^2} \mathbb{P}_W(w \mid \mu, k, \tilde{k}, y, \tilde{y}, x) \sum_{\ell=1}^{n-1} (\phi(i) - \phi(i - w(i, \ell))) \right) \right\} \Bigg] \\ &= \lambda \left[- \langle \mu, \phi \rangle - \sum_{k \in \mathbb{N}} \mu(k) \right. \\ &\quad \times \left\{ \sum_{\tilde{k} \in \llbracket 0, k \rrbracket} \mathbb{P}_{\tilde{K}}(\tilde{k} \mid \mu, k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = \tilde{k}}} \mathbb{P}_Y(y \mid \mu, k, \tilde{k}) \sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} \mid \mu, k, \tilde{k}, y) \right. \\ &\quad \times \sum_{i=1}^{n-1} \left(\tilde{y}(i) \phi(i) \right. \\ &\quad \left. \left. + \sum_{x \in \mathbb{N}^*} \mathbb{P}_X(x \mid \mu, k, \tilde{k}, y, \tilde{y}) \sum_{w \in \mathbb{N}[\llbracket 1, n-1 \rrbracket]^2} \mathbb{P}_W(w \mid \mu, k, \tilde{k}, y, \tilde{y}, x) \sum_{\ell=1}^{n-1} (\phi(i) - \phi(i - w(i, \ell))) \right) \right\} \Bigg], \end{aligned} \quad (25)$$

where $\mathbb{P}_{\tilde{K}}(\tilde{k} \mid \mu, k)$, $\mathbb{P}_Y(y \mid \mu, k, \tilde{k})$, $\mathbb{P}_{\tilde{Y}}(\tilde{y} \mid \mu, k, \tilde{k}, y)$, $\mathbb{P}_X(x \mid \mu, k, \tilde{k}, y, \tilde{y})$ and $\mathbb{P}_W(w \mid \mu, k, \tilde{k}, y, \tilde{y}, x)$ are respectively defined by (13), (15), (16), (18) and (20).

Semi-martingale decomposition

Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration of $(\mu_t)_{t \geq 0}$. From (25) it is easy to check using Lemma 3.5.1 and Corollary 3.5.2 of [5] that $(\mu_t)_{t \geq 0}$ is a Feller-Dynkin process of the space $\mathcal{D}([0, \infty), \mathcal{M}_F(\mathbb{N}))$. It then follows from standard stochastic calculus that for any bounded $\phi : \mathbb{R} \rightarrow \mathbb{R}$, the following is a \mathcal{F} -martingale:

$$t \mapsto M(\phi)_t = \langle \mu_t, \phi \rangle - \langle \mu_0, \phi \rangle - \int_0^t \mathcal{Q}\Pi_\phi(\mu_s) ds, \quad (26)$$

of quadratic variation process given by

$$\begin{aligned} t \mapsto \langle\langle M(\phi) \rangle\rangle_t &= \int_0^t \left(\mathcal{Q}(\Pi_\phi)^2(\mu_s) - 2\Pi_\phi(\mu_s)\mathcal{Q}\Pi_\phi(\mu_s) \right) ds \\ &= \lambda \int_0^t \left[\sum_{k \in \mathbb{N}} \mu_s(k) \left\{ \phi(k) + \sum_{\tilde{k} \in \llbracket 0, k \rrbracket} \mathbb{P}_{\tilde{K}}(\tilde{k} | \mu_s, k) \sum_{y \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_Y(y | \mu_s, k) \right. \right. \\ &\quad \times \sum_{i=1}^{n-1} \left(\sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} | \mu_s, k, y) \tilde{y}(i) \phi(i) \right. \\ &\quad \left. \left. + \sum_{x \in \mathbb{N}^*} \mathbb{P}_X(x | \mu_s, k, y) \sum_{w \in \mathbb{N}^{\llbracket 1, n-1 \rrbracket^2}} \mathbb{P}_W(w | \mu_s, k, x, y) \sum_{\ell=1}^{n-1} (\phi(i) - \phi(i - w_{i, \ell})) \right) \right\}^2 \right] ds. \quad (27) \end{aligned}$$

4. Main results and consequences

4.1. Hydrodynamic limit

We are interested in the behavior of the measure-valued process $(\mu_t)_{t \geq 0}$ as the size of the graph grows to infinity. We consider a sequence of models, where the size of the n -th graph equals n , and add a superscript n to all parameters and processes relative to the n -th system. Then, we scale the n -th process of empirical degree distributions as follows:

$$\bar{\mu}_t^n = \frac{1}{n} \mu_t^n, \quad t \geq 0$$

and we denote the normalized versions of the derived processes accordingly, *i.e.* for all $t \geq 0$,

$$\bar{U}_t^n = \frac{U_t^n}{n}; \quad \bar{H}_t^n = \frac{H_t^n}{n}; \quad \bar{M}^n(\phi)_t = \frac{1}{n} M^n(\phi)_t, \quad \phi \in \mathcal{B}_b.$$

Our main result is the following.

Theorem 4.1. *Assume that for all $\phi \in \mathcal{B}_b \cup \{\chi^6\}$,*

$$\langle \bar{\mu}_0^n, \phi \rangle \xrightarrow[n \rightarrow \infty]{(\mathcal{P})} \langle \zeta, \phi \rangle, \quad (28)$$

where ζ is a deterministic element of $\mathcal{M}_F(\mathbb{N})$ such that

$$0 < \langle \zeta, \chi \rangle \quad \text{and} \quad \langle \zeta, \chi^6 \rangle < \infty. \quad (29)$$

Then, for all T and all $\phi \in \mathcal{B}_b$ we have

$$\sup_{t \in [0, T]} |\langle \bar{\mu}_t^n, \phi \rangle - \langle \bar{\mu}_t, \phi \rangle| \xrightarrow[n \rightarrow \infty]{(\mathcal{P})} 0,$$

where $\bar{\mu}$ is the unique element of $\mathcal{C}(\mathbb{R}+, \mathcal{M}_F(\mathbb{N}))$ satisfying the following infinite dimensional differential system: for all $t \geq 0$ and all bounded ϕ ,

$$\begin{aligned} \langle \bar{\mu}_0, \phi \rangle &= \langle \zeta, \phi \rangle; \\ \frac{d}{dt} \langle \bar{\mu}_t, \phi \rangle &= \langle \Psi(\bar{\mu})_t, \phi \rangle \\ &:= \begin{cases} -\lambda \left[\langle \bar{\mu}_t, (\mathbf{1} + \chi) \phi \rangle + \langle \bar{\mu}_t, \chi \Delta \phi \rangle \left(\frac{\langle \bar{\mu}_t, \chi^2 \rangle}{\langle \bar{\mu}_t, \chi \rangle} - 1 \right) \right] & \text{if } \langle \bar{\mu}_t, \chi \rangle > 0; \\ -\lambda \bar{\mu}_t(0) \phi(0) & \text{if } \langle \bar{\mu}_t, \chi \rangle = 0. \end{cases} \end{aligned} \quad (30)$$

Remark 4.2. A consequence of the proof of this theorem is the existence of a solution to (30). Its uniqueness is proved separately.

Remark 4.3. By our very assumptions, $\langle \mu_0^n, \mathbf{1} \rangle = n$ and thus $\langle \bar{\mu}_0^n, \mathbf{1} \rangle = 1$ for all n . It thus follows from (28) and (29) that $\bar{\mu}_0$ is a probability measure. In particular we have

$$\langle \bar{\mu}_0, \mathbf{1} \rangle \vee \langle \bar{\mu}_0, \chi^6 \rangle < \infty,$$

a fact that will be used at several points of the proofs.

4.2. Main characteristics, and the jamming constant

It follows from Theorem 4.1 that the sequence $\{\bar{U}^n\}$ tends uniformly over compact time sets to the deterministic functions \bar{u} given for all t by $\bar{u}_t = \langle \bar{\mu}_t, \mathbf{1} \rangle$ and which, from (30), satisfies

$$\dot{\bar{u}}_t = -\lambda (\bar{u}_t + \bar{h}_t), \quad t \geq 0. \quad (31)$$

On another hand, as a simple consequence of Theorem 4.1, we have the weak convergence

$$\bar{\mu}^n \Rightarrow \bar{\mu} \text{ in } \mathcal{D}(\mathbb{R}+, \mathcal{M}_F(\mathbb{N})).$$

It then follows from the continuity of the mapping

$$\begin{cases} \mathcal{D}(\mathbb{R}+, \mathcal{M}_F(\mathbb{N})) & \longrightarrow & \mathcal{D}(\mathbb{R}+, \mathbb{R}) \\ \mu & \longmapsto & \langle \mu, \chi \rangle \end{cases}$$

and from the Continuous Mapping Theorem, that $\bar{H}^n \Rightarrow \bar{h}$ in $\mathcal{D}([0, T], \mathbb{R})$, where $\bar{h}_t = \langle \bar{\mu}_t, \chi \rangle$ for all $t \geq 0$. Again from (30), we can easily check that

$$\dot{\bar{h}}_t = -2\lambda \langle \bar{\mu}_t, \chi^2 \rangle, \quad t \geq 0.$$

Jamming constant. Denote for all $n \in \mathbb{N}^*$ and all $t \geq 0$,

$$J_t^n := |\mathcal{A}_t^n| \quad (32)$$

the number of active nodes at t . The *jamming constant* \bar{J}^n of the associated graph, is the proportion of active nodes at the ending time T_0^n of the exploration process. In other words, it is given by

$$\bar{J}^n = \frac{J_{T_0^n}^n}{n}. \quad (33)$$

The following result can be deduced from Theorem 4.1. Its proof is provided in section 5.5.

Corollary 4.4 (Jamming constant of random graphs). *Under the assumption of Theorem 4.1, we have that*

$$\bar{J}^n \xrightarrow[n \rightarrow \infty]{} c_\zeta \text{ in } L^1,$$

where

$$c_\zeta = \lambda \int_0^\infty \bar{u}_t dt = \lambda \int_0^\infty \langle \bar{\mu}_t, \mathbf{1} \rangle dt \quad (34)$$

and $(\bar{\mu}_t)_{t \geq 0}$ is the only solution to (30).

Remark 4.5. *As expected, it readily follows from (34) and (31) that $c_\zeta = \int_0^\infty \bar{u}_{t/\lambda} dt$ does not depend on λ - we hence fix $\lambda = 1$ in section 4.4 without loss of generality.*

4.3. Connexion with the configuration model

As far as the structure of the associated random graph is concerned, our construction mimics the so-called *configuration model* (or uniform model). More precisely, denote $\text{CM}(n, \mathbf{d}^n)$ the random (multi-)graph obtained by the uniform mapping of half-edges in a graph of n nodes having degree vector \mathbf{d}^n , as described in [21, 25]. We have the following.

Proposition 4.6. *The associated random multi-graph constructed jointly with the exploration process in Section 2 equals $\text{CM}(n, \mathbf{d}^n)$ in distribution.*

Proof. The result follows from the so-called *independence property* of the configuration model (see e.g. [25]): choosing whatever rule for matching the half-edges as long as a given half-edge is matched uniformly among all the unmatched half-edges at each step, provides a realization of $\text{CM}(n, \mathbf{d}^n)$. This is exactly what is done here: at each step, first all half-edges of the new active node a are matched with other half-edges that are chosen uniformly among available ones (including other half-edges of a), and then all the remaining open half-edges of the neighbors b_1 , and then b_2, b_3, \dots and finally b_q (q being the number of neighbors of a) are matched according to the same rule. Therefore, at the end of the algorithm the associated multi-graph we have drawn is nothing but a realization of $\text{CM}(n, \mathbf{d}^n)$. \square

We can now link our construction with the more usual one, consisting in fixing the graph beforehand, and then building an independent set on the latter. The *parking process* on a uniform graph, mentioned in the introduction, can be formalized as follows: we first fix \tilde{G} , a realization of $\text{CM}(n, \mathbf{d}^n)$. We then construct sequentially the independent set on \tilde{G} according to the following procedure,

- (i) At time 0, set all nodes of \tilde{G} as unexplored, and initiate an exponential clock of intensity λn ;
- (ii) Each time an exponential clock rings, select a new active node a uniformly at random among all unexplored nodes;
- (iii) The neighbors of a in \tilde{G} all become of class B;
- (iv) We set another clock of intensity λU , where U is the cardinality of the set of unexplored nodes at that instant, and go to step (ii).

The algorithm is terminated as soon as the set of unexplored nodes of \tilde{G} is empty, and the jamming constant of the graph is obtained as the proportion of active nodes at that time.

Let for all t , $\tilde{\mathcal{U}}_t^n$ the set of unexplored nodes at t and for any $j \in \tilde{\mathcal{U}}_t^n$, $d_j(\tilde{\mathcal{U}}_t^n)$ the number of neighbors of j in $\tilde{\mathcal{U}}_t^n$ at t . Let also the following random point measure,

$$\tilde{\mu}_t^n = \sum_{j \in \tilde{\mathcal{U}}_t^n} \delta_{d_j(\tilde{\mathcal{U}}_t^n)}. \quad (35)$$

Observe that only the order of exploration of the nodes differs in the present construction with respect to ours, this order being itself drawn according to a uniform choice on the set of unexplored nodes at each time in both cases. Therefore, as a simple consequence of the invariance in distribution of any permutation of the coordinates of \mathbf{d}^n , and of Proposition 4.6,

Corollary 4.7. *The sequence of measure-valued processes $\{\tilde{\mu}^n\}$ defined by (35) coincides in distribution with $\{\mu^n\}$, and so do the jamming constants of the two models.*

It is significant that, although the two processes μ^n and $\tilde{\mu}^n$ have the same distribution, the first one is Markov but the second one is not, since the knowledge of the multi-graph \tilde{G} is needed.

4.4. Jamming constants of particular graphs

Characterizing the jamming constant of parking problems has a long history in mathematics, see e.g. [17]. One of the most studied problem in the field is the so-called random sequential absorption on discrete structures. We show hereafter how our result can be adapted to regular graphs, *i.e.* graphs with fixed degree. We then focus on the case of the Poisson distribution, and relate our approach to the Erdős-Rényi graph.

Regular graph of degree 2. When for any n , the root degree distributions ν^n (and therefore, ζ) is deterministic and equal to δ_2 , we can solve exactly the three-dimensional limiting differential system:

$$\begin{cases} \frac{d}{dt}\bar{\mu}_t(2) &= -\bar{\mu}_t(2)(1 + 2L(\bar{\mu}_t)); \\ \frac{d}{dt}\bar{\mu}_t(1) &= -\bar{\mu}_t(1)(1 + L(\bar{\mu}_t)) + 2(L(\bar{\mu}_t) - 1)\bar{\mu}_t(2); \\ \frac{d}{dt}\bar{\mu}_t(0) &= -\bar{\mu}_t(0) + (L(\bar{\mu}_t) - 1)\bar{\mu}_t(1), \end{cases} \quad (36)$$

with

$$L(\bar{\mu}_t) = \frac{\langle \bar{\mu}_t, \chi^2 \rangle}{\langle \bar{\mu}_t, \chi \rangle} = \frac{4\bar{\mu}_t(2) + \bar{\mu}_t(1)}{2\bar{\mu}_t(2) + \bar{\mu}_t(1)}, \quad t \geq 0.$$

After tedious but simple calculus, one obtains

$$\begin{cases} \bar{\mu}_t(2) &= e^{-3t-2+2e^{-t}}; \\ \bar{\mu}_t(1) &= 2(e^t - 1)\bar{\mu}_t(2); \\ \bar{\mu}_t(0) &= (e^t - 1)^2\bar{\mu}_t(2). \end{cases}$$

Therefore the jamming constant for δ_2 equals

$$c_{\delta_2} = \int_0^\infty \sum_{i=0}^2 \bar{\mu}_t(i) dt = \int_0^\infty e^{-t-2+2e^{-t}} dt = \frac{1 - e^{-2}}{2},$$

which coincides with the famous Rényi parking constant of \mathbb{Z} (see [9] Section 5.3.1). Informally, this can be explained by the fact that in the resulting configuration graph with distribution δ_2 , only cycles of different sizes can appear while the number of small cycles (say smaller than a given constant) present in the resulting random graph is going to be very small compared to n with overwhelming probability. We do not go further into these calculations, which are beyond the scope of this paper.

Regular graphs for $d \geq 3$. More generally, jamming constants of regular graphs for $d \geq 3$ have been shown to be asymptotically equivalent to $\frac{1-(d-1)^{-\frac{d-2}{d-1}}}{2}$, for a random graph $\text{CM}(n, \mathbf{d}^n)$ of degree distribution $\nu^n = \delta_d$ (i.e. $\mathbf{d}^n = (d, d, \dots, d)$), see [24].

Let us quickly reformulate in our terminology the approach in [24], and compare it with ours. The algorithm of *random pairing* as termed in [24] is similar to the one that is presented here, the main difference in the construction of $\text{CM}(n, \mathbf{d}^n)$ being that at any time, all unexplored nodes have exactly d available half-edges. This is obtained as follows: each time a new active node is selected (uniformly at random among all unexplored nodes), its d half-edges are matched uniformly at random with d other ones, exactly as we do so. At this point, according to the method of deferred decisions (see [15]), and unlike our construction, the connectivity of the neighbors of the new active node with the rest of the graph is not completed yet. At the following instant, a new active node is selected among all unexplored nodes, and the same procedure is reiterated on and on, until there is no more unexplored node. It is important to observe that, at the end of this algorithm, only the edges between active and blocked nodes have been build in the associated graph, and that the blocked nodes may have unmatched half-edges (precisely as many as d minus the number of edges they share with their active neighbor). Then, the associated graph is completed by creating edges between blocked nodes in arbitrary order, following uniform choices of half-edges (which does not change the jamming constant of the graph). Under such a dynamics, it appears clearly that the process $((J_t^n, U_t^n))$ is Markov (recall the notation 32). In particular, using the fact that the possible neighbors of the new active node must have d available half-edges, it is easy to observe that whenever a node becomes active in $[t, t+h)$ we have that

$$\mathbf{E}[U_{t+h}^n - U_t^n \mid (J_t^n, U_t^n)] = -1 - d \frac{U_t^n}{n - 2J_t^n}, \quad (37)$$

which leads to a simple one-dimensional asymptotic ODE for $t \mapsto U_t^n$, that is solved explicitly.

However, for a general degree distribution the relation (37) no longer holds, and the process $((J_t^n, U_t^n))$ is no longer Markov. In fact, one can observe that the measure-valued process (μ_t^n) itself is not Markov, when constructing the graph as is done in [24], since one needs to know which nodes are active and which are blocked in the associated graph, to complete the connectivity of the new active node at each instant - which is why we need to complete the neighboring between blocked nodes at each step.

The price to pay for working in such generality, is that, due to a more intricate dynamics we do not obtain a closed-form formula for the function $t \mapsto \bar{u}_t$ in the particular case of regular graphs. However, though an exact computation of (\bar{u}_t) becomes more and more involved when the degree d grows, we can easily retrieve the asymptotic value of [24] by solving numerically the system corresponding to (36), as is shown for $d = 3$ and 4 in Table 1.

The Poisson distribution. In the case where the asymptotic initial empirical degree distribution is Poisson, we obtain a closed-form expression for the function $t \mapsto \bar{\mu}_t$.

Proposition 4.8. *If ζ is a Poisson distribution with parameter p (we denote $\zeta = \mathcal{P}(p)$), then*

$$\bar{\mu}_t(i) = v_t \frac{(pv_t)^i}{i!} \exp(-pv_t), \quad t \geq 0, \quad i \in \mathbb{N},$$

where v is the solution of the differential equation

$$\dot{v} = -v(1 + pv).$$

Moreover the jamming constant reads

$$c_{\mathcal{P}(p)} = \frac{1}{p} \log(1 + p).$$

Proof. Let us define the following Poisson measures for all t ,

$$\kappa_t(i) = \frac{(pv_t)^i}{i!} \exp(-pv_t), \quad i \in \mathbb{N}.$$

For all $t \geq 0$, using the definition of v_t and $\bar{\mu}_t$ we obtain

$$\begin{aligned} \frac{d}{dt} \langle \bar{\mu}_t, \phi \rangle &= \dot{v}_t \exp(-pv_t) \left[\sum_{i \geq 0} (i+1) \frac{p^i}{i!} v_t^i \phi(i) - p \sum_{i \geq 0} \frac{p^i}{i!} v_t^{i+1} \phi(i) \right], \\ &= -\lambda v_t (1 + pv_t) \left[\langle \kappa_t, (1 + \chi) \phi \rangle - \sum_{i \geq 0} (i+1) \kappa_t(i+1) \phi(i) \right], \\ &= -(1 + pv_t) \left[\langle \bar{\mu}_t, (1 + \chi) \phi \rangle - \sum_{j \geq 1} j \bar{\mu}_t(j) \phi(j-1) \right], \\ &= - \left[\langle \bar{\mu}_t, (1 + \chi) \phi \rangle + pv_t \langle \bar{\mu}_t, \chi \Delta \phi \rangle + pv_t \langle \bar{\mu}_t, \phi \rangle - \sum_{j \geq 1} j \bar{\mu}_t(j) \phi(j-1) \right], \\ &= - \left[\langle \bar{\mu}_t, (1 + \chi) \phi \rangle + \left(\frac{\langle \bar{\mu}_t, \chi^2 \rangle}{\langle \bar{\mu}_t, \chi \rangle} - 1 \right) \langle \bar{\mu}_t, \chi \Delta \phi \rangle \right], \end{aligned}$$

where we use in the last identity that

$$pv_t = \frac{\langle \bar{\mu}_t, \chi^2 \rangle}{\langle \bar{\mu}_t, \chi \rangle} - 1$$

and

$$pv_t \langle \bar{\mu}_t, \phi \rangle = \sum_{i \geq 0} (i+1) \frac{p^{i+1}}{(i+1)!} v_t^{i+1} \exp(-pv_t) \phi(i) = \sum_{j \geq 1} j \frac{p^j}{j!} v_t^j \exp(-pv_t) \phi(j-1) = \sum_{j \geq 1} j \bar{\mu}_t(j) \phi(j-1).$$

The jamming constant then simply follows by integration of $t \mapsto v_t$.

□

The asymptotic empirical degree measure of unexplored nodes thus turns out to be an inhomogeneous Poisson measure. A typical example where Proposition 4.8 applies is that of a binomial degree distribution $\nu^n = \text{Bin}(n-1, p^n)$ for all n (as is the case for Erdős-Rényi graphs), where $np^n \xrightarrow{n \rightarrow \infty} p$.

It is then interesting to observe that we retrieve the jamming constant of the Erdős-Rényi graph of parameter p , which was given in [16]. Notice however that we do *not* construct a proper Erdős-Rényi graph and in fact, no uniform construction based on a prescribed degree distribution can do so, since the independence assumption for the existence of the various edges cannot be fulfilled.

Table 1 gathers in the middle column, our results for several jamming constants (exact for $\zeta = \mathcal{P}(1)$ and δ_2 , and numerically computed for exponential, δ_3 and δ_4).

Degree distribution	JC of Random Graphs	JC of specific deterministic Graphs
Exponential (1/2)	0.7599203270	
Poisson (1)	$\log(2) = 0.6931472$	
δ_2	$\frac{1-e^{-2}}{2} = 0.4323323583$	$\frac{1-e^{-2}}{2} (\mathbb{Z})$
δ_3	3/8	0.37913944 (Honeycomb)
δ_4	1/3	0.3641323 (\mathbb{Z}^2)

Table 1: Jamming constants for different degree distributions and their counterparts on deterministic graphs (simulation values for deterministic graphs are taken from [23]).

5. Proof of Theorem 4.1

From (29), there exists two real numbers $\alpha > 0$ and $M > 1$ such that

$$\langle \zeta, \chi^6 \rangle < M; \quad (38)$$

$$\langle \zeta, \chi \rangle > \alpha, \quad (39)$$

where ζ is defined by (28). Let us define the following subsets of $\mathcal{M}_F(\mathbb{N})$:

$$\mathcal{M}_{\alpha, M} = \left\{ \mu \in \mathcal{M}_F(\mathbb{N}); \langle \mu, \mathbf{1} \rangle \vee \langle \mu, \chi^6 \rangle < M \text{ and } \langle \mu, \chi \rangle > \alpha \right\} \quad (40)$$

and for all $n \in \mathbb{N}^*$,

$$n\mathcal{M}_{\alpha, M} = \left\{ \mu \in \mathcal{M}_F(\mathbb{N}); \frac{1}{n}\mu \in \mathcal{M}_{\alpha, M} \right\}.$$

The strategy of the proof is as follows. After showing that the generator can be approximated for large n by the (non-linear) application Ψ (at least on the subset $n\mathcal{M}_{\alpha, M}$), we show uniqueness of the possible limit - which is solution of the deterministic system of equations (30) - and finally the convergence in probability towards this limit.

5.1. Generator approximations

Recall the definition of the map $\Psi : \mathcal{C}(\mathbb{R}^+, \mathcal{M}_F(\mathbb{N})) \rightarrow \mathcal{C}(\mathbb{R}^+, \mathcal{M}_F(\mathbb{N}))$ in (30). We show that the finite variation part $\frac{1}{n}\mathcal{Q}^n \Pi_\phi(\mu^n)$ of $\{\bar{\mu}^n\}$ can be approximated by $\Psi(\bar{\mu}^n)$ as long as $\bar{\mu}^n$ takes values in $\mathcal{M}_{\alpha, M}$. To show this, we first write for all n, t and ϕ ,

$$\Psi(\bar{\mu}^n)_t =: -\lambda \left\{ \langle \bar{\mu}_t^n, \phi \rangle + \langle \bar{A}^n(\mu_t^n), \phi \rangle + \langle \bar{B}^n(\mu_t^n), \phi \rangle \right\}, \quad (41)$$

where for all μ such that $\langle \mu, \chi \rangle > 0$,

$$\langle \bar{A}^n(\mu), \phi \rangle = \frac{1}{n} \langle \mu, \chi \phi \rangle; \quad (42)$$

$$\langle \bar{B}^n(\mu), \phi \rangle = \frac{1}{n} \langle \mu, \chi \Delta \phi \rangle \left(\frac{\langle \mu, \chi^2 \rangle}{\langle \mu, \chi \rangle} - 1 \right). \quad (43)$$

Then, we decompose the rescaled generator into two terms $\bar{\mathcal{A}}^n$ and $\bar{\mathcal{B}}^n$ that will be shown to get close to \bar{A}^n and \bar{B}^n as n goes large, when applied to a measure $\mu \in n\mathcal{M}_{\alpha, M}$. For doing so, the combinatorial approximation arguments for large n will turn out to be valid only if the first moment of the measure remains ‘of order n ’ after the transformation corresponding to the deletion of the new

A-vertex and the new B-vertices. We thus have to tease apart the cases where the degrees of the latter vertices are too big, which correspond to a third term $\bar{\mathcal{C}}^n$ that will be proven to vanish for large n . More precisely, we write for all n , $\mu \in \mathcal{M}_F(\mathbb{N})$ and all $\phi \in \mathcal{B}_b$,

$$\frac{1}{n} \mathcal{Q}^n \Pi_\phi(\mu) =: -\lambda \left\{ \frac{1}{n} \langle \mu, \phi \rangle + \langle \bar{\mathcal{A}}^n(\mu), \phi \rangle + \langle \bar{\mathcal{B}}^n(\mu), \phi \rangle + \langle \bar{\mathcal{C}}^n(\mu), \phi \rangle \right\}, \quad (44)$$

where

$$\begin{aligned} \langle \bar{\mathcal{A}}^n(\mu), \phi \rangle &= \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) \sum_{\tilde{k} \in \llbracket 0, k \rrbracket} \mathbb{P}_{\tilde{K}}(\tilde{k} | \mu, k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k}} \mathbb{P}_Y(y | \mu, k, \tilde{k}) \\ &\quad \times \sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} | \mu, k, \tilde{k}, y) \langle \tilde{y}, \phi \rangle; \end{aligned} \quad (45)$$

$$\begin{aligned} \langle \bar{\mathcal{B}}^n(\mu), \phi \rangle &= \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\tilde{k} \in \llbracket 0, k \rrbracket} \mathbb{P}_{\tilde{K}}(\tilde{k} | \mu, k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y | \mu, k, \tilde{k}) \sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} | \mu, k, \tilde{k}, y) \\ &\quad \times \sum_{i=1}^{n-1} \sum_{x \in \mathbb{N}^*} \mathbb{P}_X(x | \mu, k, \tilde{k}, y, \tilde{y}) \sum_{w \in \mathbb{N}[\mathbf{1}, n-1]^2} \mathbb{P}_W(w | \mu, k, \tilde{k}, y, \tilde{y}, x) \sum_{\ell=1}^{n-1} (\phi(i) - \phi(i - w(i, \ell))); \end{aligned} \quad (46)$$

$$\begin{aligned} \langle \bar{\mathcal{C}}^n(\mu), \phi \rangle &= \frac{1}{n} \left(\sum_{\substack{k \in \mathbb{N}^*; \\ k > n\alpha/2}} \mu(k) \sum_{\tilde{k} \in \llbracket 0, k \rrbracket} \mathbb{P}_{\tilde{K}}(\tilde{k} | \mu, k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = \tilde{k}}} \mathbb{P}_Y(y | \mu, k, \tilde{k}) \right. \\ &\quad \times \sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} | \mu, k, \tilde{k}, y) \langle \tilde{y}, \phi \rangle \Big) \\ &\quad + \frac{1}{n} \left(\sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\tilde{k} \in \llbracket 0, k \rrbracket} \mathbb{P}_{\tilde{K}}(\tilde{k} | \mu, k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = \tilde{k}; \\ \langle y, \chi \rangle > n\alpha/2}} \mathbb{P}_Y(y | \mu, k, \tilde{k}) \sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} | \mu, k, \tilde{k}, y) \right. \\ &\quad \times \sum_{i=1}^{n-1} \sum_{x \in \mathbb{N}^*} \mathbb{P}_X(x | \mu, k, \tilde{k}, y, \tilde{y}) \sum_{w \in \mathbb{N}[\mathbf{1}, n-1]^2} \mathbb{P}_W(w | \mu, k, \tilde{k}, y, \tilde{y}, x) \sum_{\ell=1}^{n-1} (\phi(i) - \phi(i - w(i, \ell))) \Big) \\ &\quad + \frac{1}{n} \left(\sum_{\substack{k \in \mathbb{N}^*; \\ k > \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\tilde{k} \in \llbracket 0, k \rrbracket} \mathbb{P}_{\tilde{K}}(\tilde{k} | \mu, k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = \tilde{k}}} \mathbb{P}_Y(y | \mu, k, \tilde{k}) \sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} | \mu, k, \tilde{k}, y) \right. \\ &\quad \times \sum_{i=1}^{n-1} \sum_{x \in \mathbb{N}^*} \mathbb{P}_X(x | \mu, k, \tilde{k}, y, \tilde{y}) \sum_{w \in \mathbb{N}[\mathbf{1}, n-1]^2} \mathbb{P}_W(w | \mu, k, \tilde{k}, y, \tilde{y}, x) \sum_{\ell=1}^{n-1} (\phi(i) - \phi(i - w(i, \ell))) \Big) \\ &=: \langle \bar{\mathcal{C}}_1^n(\mu), \phi \rangle + \langle \bar{\mathcal{C}}_2^n(\mu), \phi \rangle + \langle \bar{\mathcal{C}}_3^n(\mu), \phi \rangle. \end{aligned} \quad (47)$$

We first have the following.

Lemma 5.1. *For all sufficiently large n , all $\mu \in n\mathcal{M}_{\alpha,M}$ and all bounded ϕ ,*

$$\langle \bar{\mathcal{C}}^n(\mu), \phi \rangle = o(n).$$

Proof. Fix ϕ , $n > \frac{4}{3\alpha}$ and $\mu \in n\mathcal{M}_{\alpha,M}$ throughout the proof. First, observe that

$$\langle \bar{\mathcal{C}}_1^n(\mu), \phi \rangle \leq \frac{\|\phi\|}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k > n\alpha/2}} \mu(k)k \leq \frac{\|\phi\|}{n} \frac{2}{n\alpha} \sum_{\substack{k \in \mathbb{N}^*; \\ k > n\alpha/2}} \mu(k)k^2 \leq \frac{2\|\phi\|}{n^2\alpha} \langle \mu, \chi^2 \rangle \leq \frac{2\|\phi\|}{n\alpha} M. \quad (48)$$

Second, as $\langle \mu, \chi \rangle > n\alpha$ we have that

$$k \leq \frac{\langle \mu, \chi \rangle}{4} \Rightarrow \langle \mu, \chi \rangle - k \geq \frac{3n\alpha}{4}. \quad (49)$$

Hence, in view of the distribution of $Y(\cdot)$ in (15) and from (A.3), we obtain that for all k and \tilde{k} ,

$$\mathbb{E}_Y \left[\langle Y(\mu), \chi \rangle^2 \mid \mu, k, \tilde{k} \right] \leq \frac{k(k-1) \langle \mu, \chi \rangle^2}{(\langle \mu, \chi \rangle - k)(\langle \mu, \chi \rangle - (k+1))} \leq k(k-1) \frac{16M^2}{3\alpha(3\alpha - 4/n)}. \quad (50)$$

Therefore,

$$\begin{aligned} \langle \bar{\mathcal{C}}_2^n(\mu), \phi \rangle &\leq \frac{2}{n} \|\phi\| \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\tilde{k} \in \llbracket 0, k \rrbracket} \mathbb{P}_{\tilde{K}}(\tilde{k} \mid \mu, k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = \tilde{k}; \\ \langle y, \chi \rangle > n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k) \langle y, \chi \rangle \\ &\leq \frac{2}{n} \|\phi\| \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \frac{2}{n\alpha} \mathbb{E}_Y \left[\langle Y(\mu), \chi \rangle^2 \mid \mu, k, \tilde{k} \right] \\ &\leq \frac{64 \|\phi\| M^2}{3n\alpha^2(3\alpha - 4/n)} \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k)k(k-1) \leq \frac{64 \|\phi\| M^3}{3n\alpha^2(3\alpha - 4/n)}, \end{aligned} \quad (51)$$

where we reason as for (48) in the second upper-bound.

Last, as for all k and y , $\langle \mu, \chi \rangle - k - \langle y, \chi \rangle \geq 0$, we have that

$$k > \langle \mu, \chi \rangle / 4 \Rightarrow \langle y, \chi \rangle \leq \frac{3\langle \mu, \chi \rangle}{4} < 3k. \quad (52)$$

Therefore,

$$\begin{aligned} \langle \bar{\mathcal{C}}_3^n(\mu), \phi \rangle &\leq 2 \|\phi\| \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k > \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\tilde{k} \in \llbracket 0, k \rrbracket} \mathbb{P}_{\tilde{K}}(\tilde{k} \mid \mu, k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = \tilde{k}}} \mathbb{P}_Y(y \mid \mu, k, \tilde{k}) \langle y, \chi \rangle \\ &\leq 2 \|\phi\| \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k > \langle \mu, \chi \rangle / 4}} \mu(k)3k \leq 2 \|\phi\| \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k > n\alpha/4}} \mu(k)3k \leq \frac{24 \|\phi\| M}{n\alpha}, \end{aligned} \quad (53)$$

applying again the same argument as for (48). The proof is complete by gathering (48), (51) and (53).

□

We now turn to the terms $\bar{\mathcal{A}}^n$ and $\bar{\mathcal{B}}^n$.

Lemma 5.2. For all sufficient large n , all $\mu \in n\mathcal{M}_{\alpha,M}$ and all bounded ϕ ,

$$|\langle \mathcal{A}^n(\mu), \phi \rangle - \langle \bar{A}^n(\mu), \phi \rangle| = o(n).$$

Proof. Fix ϕ , $n > \frac{2}{\alpha}$ and $\mu \in n\mathcal{M}_{\alpha,M}$ throughout the proof. First, for all k such that $k \leq n\alpha/2$ we have that

$$\langle \mu, \chi \rangle - k \geq \frac{n\alpha}{2},$$

which will be used in several steps of this proof. In particular, from (17) we clearly have that

$$\mathbb{P}_{\tilde{Y}}(y | \mu, k, k, y) \geq 1 - \binom{k}{2} 4 \frac{M}{n\alpha \left(\alpha - \frac{2}{n}\right)}. \quad (54)$$

Recall (14), and let us rewrite

$$\begin{aligned} \langle \mathcal{A}^n(\mu), \phi \rangle &= \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) \mathbb{P}_{\tilde{K}}(k | \mu, k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k}} \mathbb{P}_Y(y | \mu, k, k) \sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} | \mu, k, k, y) \langle \tilde{y}, \phi \rangle \\ &\quad + \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) \sum_{\tilde{k} \in \llbracket 0, k-1 \rrbracket} \mathbb{P}_{\tilde{K}}(\tilde{k} | \mu, k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = \tilde{k}}} \mathbb{P}_Y(y | \mu, k, \tilde{k}) \sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} | \mu, k, \tilde{k}, y) \langle \tilde{y}, \phi \rangle \\ &= \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k}} \mathbb{P}_Y(y | \mu, k, k) \sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} | \mu, k, k, y) \langle \tilde{y}, \phi \rangle \\ &\quad + \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) (\mathbb{P}_{\tilde{K}}(k | \mu, k) - 1) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k}} \mathbb{P}_Y(y | \mu, k, k) \sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} | \mu, k, k, y) \langle \tilde{y}, \phi \rangle \\ &\quad + \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) \sum_{\tilde{k} \in \llbracket 0, k-1 \rrbracket} \mathbb{P}_{\tilde{K}}(\tilde{k} | \mu, k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = \tilde{k}}} \mathbb{P}_Y(y | \mu, k, \tilde{k}) \sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} | \mu, k, \tilde{k}, y) \langle \tilde{y}, \phi \rangle \\ &=: \langle \mathcal{A}_1^n(\mu), \phi \rangle + \langle \mathcal{A}_2^n(\mu), \phi \rangle + \langle \mathcal{A}_3^n(\mu), \phi \rangle. \quad (55) \end{aligned}$$

The last two terms vanish as a consequence of the fact that $\tilde{K}(\mu)$ and $K(\mu)$ tend to coincide as the size of the graph goes large. To see this, from (14) we have that for all $k \leq n\alpha/2$,

$$\begin{aligned} \mathbb{P}_{\tilde{K}}(k | \mu, k) &\geq \frac{\binom{\langle \mu, \chi \rangle - k}{k}}{\binom{\langle \mu, \chi \rangle}{k}} \mathbb{1}_{\{k \leq \langle \mu, \chi \rangle - k\}} \\ &= \frac{\binom{\langle \mu, \chi \rangle - k}{k}}{\binom{\langle \mu, \chi \rangle}{k}} \\ &= \frac{(\langle \mu, \chi \rangle - k) \dots (\langle \mu, \chi \rangle - 2k + 1)}{(\langle \mu, \chi \rangle) \dots (\langle \mu, \chi \rangle - k + 1)} \\ &\geq \left(1 - \frac{k}{\langle \mu, \chi \rangle - k + 1}\right)^k \\ &\geq 1 - \frac{k^2}{\langle \mu, \chi \rangle - k + 1}. \quad (56) \end{aligned}$$

Therefore, as $\langle \tilde{y}, \mathbf{1} \rangle \leq k$ for any k, \tilde{k} and \tilde{y} we have that

$$|\langle \mathcal{A}_2^n(\mu), \phi \rangle + \langle \mathcal{A}_3^n(\mu), \phi \rangle| \leq \frac{2}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) k (1 - \mathbb{P}_{\tilde{K}}(k | \mu, k)) \leq \frac{2}{n} \frac{\langle \mu, \chi^3 \rangle}{\langle \mu, \chi \rangle - k + 1} \leq \frac{4M}{n\alpha}. \quad (57)$$

Let us now address the term $\bar{\mathcal{A}}_1^n$. We have

$$\begin{aligned}
\langle \bar{\mathcal{A}}_1^n(\mu), \phi \rangle &= \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k}} \mathbb{P}_Y(y \mid \mu, k) \mathbb{E}_{\tilde{Y}} \left[\langle \tilde{Y}(\mu), \phi \rangle \mathbb{1}_{\{\tilde{Y}(\mu) \neq y\}} \mid \mu, k, y \right] \\
&\quad + \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) \sum_{y \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_Y(y \mid \mu, k) \mathbb{P}_{\tilde{Y}}(y \mid \mu, k, k, y) \langle y, \phi \rangle \\
&= \frac{1}{n} \left\{ \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k}} \mathbb{P}_Y(y \mid \mu, k) \right. \\
&\quad \times \left(\mathbb{E}_{\tilde{Y}} \left[\langle \tilde{Y}(\mu), \phi \rangle \mathbb{1}_{\{\tilde{Y}(\mu) \neq y\}} \mid \mu, k, y \right] + \langle y, \phi \rangle \left(\mathbb{P}_{\tilde{Y}}(y \mid \mu, k, k, y) - 1 \right) \right) \Big\} \\
&\quad + \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) \mathbb{E}_Y[\langle Y(\mu), \phi \rangle \mid \mu, k] \\
&=: \langle \bar{\mathcal{A}}_{11}^n(\mu), \phi \rangle + \langle \bar{\mathcal{A}}_{12}^n(\mu), \phi \rangle. \quad (58)
\end{aligned}$$

First, as $\langle y, \mathbf{1} \rangle \leq k$, from (54) we have that

$$\begin{aligned}
|\langle \bar{\mathcal{A}}_{11}^n(\mu), \phi \rangle| &\leq \frac{2 \|\phi\|}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) \sum_{y \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_Y(y \mid \mu, k) \left(1 - \mathbb{P}_{\tilde{Y}}(y \mid \mu, k, k, y) \right) k \\
&\leq \frac{2 \|\phi\|}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) \binom{k}{2} 4 \frac{M}{n\alpha \left(\alpha - \frac{2}{n} \right)} k \leq \frac{4 \|\phi\| M^2}{n\alpha \left(\alpha - \frac{2}{n} \right)}. \quad (59)
\end{aligned}$$

Second, using again the hypergeometric distribution of $Y(\cdot)$ and (A.1) we have that

$$\begin{aligned}
|\langle \bar{\mathcal{A}}_{12}^n(\mu), \phi \rangle - \langle \bar{A}^n(\mu), \phi \rangle| &= \frac{1}{n} \left| \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) \sum_{i=1}^{n-1} \phi(i) \mathbb{E}_Y[Y(\mu)(i) \mid \mu, k] - \langle \mu, \phi \chi \rangle \right| \\
&= \frac{1}{n} \left| \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) \sum_{i=1}^{n-1} \phi(i) k \frac{i\mu(i) - i\delta_k(i)}{\langle \mu, \chi \rangle - k} - \langle \mu, \phi \chi \rangle \right| \\
&\leq \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \mu(k) \frac{\|\phi\| k}{\langle \mu, \chi \rangle - k} + \frac{\langle \mu, \phi \chi \rangle}{n} \left| \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \frac{\mu(k) k}{\langle \mu, \chi \rangle - k} - 1 \right| \\
&\leq \frac{2M \|\phi\|}{n\alpha} + \frac{\langle \mu, \phi \chi \rangle}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq n\alpha/2}} \frac{\mu(k) k^2}{\langle \mu, \chi \rangle (\langle \mu, \chi \rangle - k)} + \frac{\langle \mu, \phi \chi \rangle}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k > n\alpha/2}} \frac{\mu(k) k}{\langle \mu, \chi \rangle} \\
&\leq \frac{2M \|\phi\|}{n\alpha} + \frac{2M^2 \|\phi\|}{n\alpha^2} + \frac{M \|\phi\|}{\alpha} \frac{2M}{n\alpha},
\end{aligned}$$

where the upper-bound of the third term of the last inequality follows from the same argument as for (48). This, together with (59) in (58), and then with (57) in (55), concludes the proof. \square

Lemma 5.3. *For all n , all $\mu \in n\mathcal{M}_{\alpha,M}$ and all bounded ϕ ,*

$$|\langle \bar{\mathcal{B}}^n(\mu), \phi \rangle - \langle \bar{B}^n(\mu), \phi \rangle| = o(n).$$

Proof. Fix $\phi \in \mathcal{B}_b$, $n > \frac{4}{\alpha}$ and $\mu \in n\mathcal{M}_{\alpha,M}$. We split $\bar{\mathcal{B}}^n$ as follows:

$$\langle \bar{\mathcal{B}}^n(\mu), \phi \rangle = \sum_{i=1}^4 \langle \bar{\mathcal{B}}_i^n(\mu), \phi \rangle, \quad (60)$$

where

$$\begin{aligned} \langle \bar{\mathcal{B}}_1^n(\mu), \phi \rangle = \frac{1}{n} & \left(\sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \sum_{x \in \mathbb{N}^*} \mathbb{P}_X(x \mid \mu, k, k, y, y) \right. \\ & \left. \times \sum_{w \in \mathbb{N}^{\llbracket 1, n-1 \rrbracket^2}} \mathbb{P}_W(w \mid \mu, k, k, y, y, x) \sum_{i=1}^{n-1} Z(\mu)(i) \Delta \phi(i) \right); \quad (61) \end{aligned}$$

$$\begin{aligned} \langle \bar{\mathcal{B}}_2^n(\mu), \phi \rangle = \frac{1}{n} & \left(\sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \sum_{x \in \mathbb{N}^*} \mathbb{P}_X(x \mid \mu, k, k, y, y) \right. \\ & \times \sum_{\substack{w \in \mathbb{N}^{\llbracket 1, n-1 \rrbracket^2}; \\ \exists (i_0, \ell_0) \in \llbracket 1, n-1 \rrbracket^2; w(i, \ell_0) \geq 2}} \mathbb{P}_W(w \mid \mu, k, k, y, y, x) \sum_{i=1}^{n-1} \left(\sum_{\ell=1}^{n-1} (\phi(i) - \phi(i - w(i, \ell))) - Z(\mu)(i) \Delta \phi(i) \right) \Bigg); \quad (62) \end{aligned}$$

$$\begin{aligned}
\langle \bar{\mathcal{B}}_3^n(\mu), \phi \rangle &= \frac{1}{n} \left(\sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) (\mathbb{P}_{\tilde{Y}}(y \mid \mu, k, k, y) - 1) \right. \\
&\quad \times \sum_{x \in \mathbb{N}^*} \mathbb{P}_X(x \mid \mu, k, k, y, y) \sum_{w \in \mathbb{N}[\![1, n-1]\!]^2} \mathbb{P}_W(w \mid \mu, k, k, y, y, x) \sum_{i=1}^{n-1} \sum_{\ell=1}^{n-1} (\phi(i) - \phi(i - w(i, \ell))) \Big) \\
&+ \frac{1}{n} \left(\sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \sum_{\substack{\tilde{y} \in \mathcal{M}_F(\mathbb{N}); \\ \tilde{y} \neq y}} \mathbb{P}_{\tilde{Y}}(\tilde{y} \mid \mu, k, k, y) \right. \\
&\quad \times \sum_{x \in \mathbb{N}^*} \mathbb{P}_X(x \mid \mu, k, k, y, \tilde{y}) \sum_{w \in \mathbb{N}[\![1, n-1]\!]^2} \mathbb{P}_W(w \mid \mu, k, k, y, \tilde{y}, x) \sum_{i=1}^{n-1} \sum_{\ell=1}^{n-1} (\phi(i) - \phi(i - w(i, \ell))) \Big); \quad (63)
\end{aligned}$$

$$\begin{aligned}
\langle \bar{\mathcal{B}}_4^n(\mu), \phi \rangle &= \frac{1}{n} \left(\sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) (\mathbb{P}_{\tilde{K}}(k \mid \mu, k) - 1) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} \mid \mu, k, \tilde{k}, y) \right. \\
&\quad \times \sum_{x \in \mathbb{N}^*} \mathbb{P}_X(x \mid \mu, k, k, y, \tilde{y}) \sum_{w \in \mathbb{N}[\![1, n-1]\!]^2} \mathbb{P}_W(w \mid \mu, k, k, y, \tilde{y}, x) \sum_{i=1}^{n-1} \sum_{\ell=1}^{n-1} (\phi(i) - \phi(i - w(i, \ell))) \Big) \\
&+ \frac{1}{n} \left(\sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\tilde{k} \in \llbracket 0, k-1 \rrbracket} \mathbb{P}_{\tilde{K}}(\tilde{k} \mid \mu, k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = \tilde{k}; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, \tilde{k}) \sum_{\tilde{y} \in \mathcal{M}_F(\mathbb{N})} \mathbb{P}_{\tilde{Y}}(\tilde{y} \mid \mu, k, \tilde{k}, y) \right. \\
&\quad \times \sum_{x \in \mathbb{N}^*} \mathbb{P}_X(x \mid \mu, k, \tilde{k}, y, \tilde{y}) \sum_{w \in \mathbb{N}[\![1, n-1]\!]^2} \mathbb{P}_W(w \mid \mu, k, \tilde{k}, y, \tilde{y}, x) \sum_{i=1}^{n-1} \sum_{\ell=1}^{n-1} (\phi(i) - \phi(i - w(i, \ell))) \Big). \quad (64)
\end{aligned}$$

In the latter partition, the last three terms $\bar{\mathcal{B}}_4^n(\cdot)$, $\bar{\mathcal{B}}_3^n(\cdot)$ and $\bar{\mathcal{B}}_2^n(\cdot)$, which gather respectively the cases of existence of self-loops around the new A-node, of multiple edges between the new A-node and the new B-nodes, and of multi-edges between the new B-nodes and the remaining U-nodes, will be shown to be small w.r.t. n . Only the term $\bar{\mathcal{B}}_1^n(\cdot)$ is non degenerate, and as we will prove, is at a distance to $\bar{B}^n(\cdot)$ which is also small w.r.t. n .

To prove this, let us first recall the definition of $Z(\mu)$ in (23) and its distribution in (24): for any i , there are at most $Z(\mu)(i)$ indexes $\ell \in \llbracket 1, n-1 \rrbracket$ such that $W(\mu)(i, \ell)$ differs from zero. Therefore

we have that

$$\begin{aligned} \sum_{i=1}^{n-1} \sum_{\ell=1}^{n-1} (\phi(i) - \phi(i - w(i, \ell))) &\leq 2 \|\phi\| \sum_{i=1}^{n-1} Z(\mu)(i) \\ &\leq 2 \|\phi\| X(\mu) \leq 2 \|\phi\| (\langle Y(\mu), \chi \rangle - K(\mu)). \end{aligned} \quad (65)$$

On another hand, notice that for all k ,

$$\begin{aligned} &\sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k) (\langle y, \chi \rangle - k) \\ &= \mathbb{E}_Y [\langle Y(\mu), \chi \rangle - k \mid \mu, k] - \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle > n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k) (\langle y, \chi \rangle - k) \\ &= \sum_{i=1}^{n-1} ik \frac{i(\mu(i) - \delta_k(i))}{\langle \mu, \chi \rangle - k} - k - \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle > n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k) (\langle y, \chi \rangle - k) \\ &= k \left(\frac{\langle \mu, \chi^2 \rangle}{\langle \mu, \chi \rangle - k} - 1 \right) - \frac{k^3}{\langle \mu, \chi \rangle - k} - \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle > n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k) (\langle y, \chi \rangle - k) \\ &= k \left(\frac{\langle \mu, \chi^2 \rangle}{\langle \mu, \chi \rangle - k} - 1 \right) + k^3 o_3(n) + k(k-1) o_4(n), \end{aligned} \quad (66)$$

where the equivalent $o_4(n)$ follows from (50), just as in (51). Clearly, (66) implies in particular that

$$\sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k) (\langle y, \chi \rangle - k) \leq k \frac{4M}{3\alpha}. \quad (67)$$

We are now in position to address the vanishing terms in the decomposition (60), from bottom to top. First, as for (57), the term $\bar{\mathcal{B}}_4^n(\cdot)$ vanishes since $\tilde{K}(\mu)$ and $K(\mu)$ tend to coincide. More precisely, we have with (65) that

$$\begin{aligned} |\langle \bar{\mathcal{B}}_4^n(\mu), \phi \rangle| &\leq \frac{4 \|\phi\|}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) (1 - \mathbb{P}_{\tilde{K}}(k \mid \mu, k)) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) (\langle y, \chi \rangle - k) \\ &\leq \frac{4 \|\phi\|}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \frac{k^2}{\langle \mu, \chi \rangle - k + 1} \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) (\langle y, \chi \rangle - k) \\ &\leq \frac{4 \|\phi\|}{n} \frac{2}{2n\alpha + 1} \frac{4M}{3\alpha} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) k^3 \leq 4 \|\phi\| \frac{2}{2n\alpha + 1} \frac{4M^2}{3\alpha}, \end{aligned} \quad (68)$$

where we use (56) in the second upper-bound and (67) in the last one.

The term $\bar{\mathcal{B}}_3^n(\cdot)$ can be treated analogously. Applying again (65) and then (54), we have that

$$\begin{aligned}
|\langle \bar{\mathcal{B}}_3^n(\mu), \phi \rangle| &\leq \frac{4 \|\phi\|}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) (1 - \mathbb{P}_{\tilde{Y}}(y \mid \mu, k, k, y)) (\langle y, \chi \rangle - k) \\
&\leq \frac{4 \|\phi\|}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \binom{k}{2} 4 \frac{M}{n\alpha(\alpha - \frac{2}{n})} \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) (\langle y, \chi \rangle - k) \\
&\leq \frac{8 \|\phi\|}{n} \frac{M}{n\alpha(\alpha - \frac{2}{n})} \frac{4M}{3\alpha} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) k^2 \leq \frac{32 \|\phi\|}{3n} \frac{M^3}{\alpha^2(\alpha - \frac{2}{n})}, \tag{69}
\end{aligned}$$

where we use again (67) in the third upper-bound.

We now examine the term $\bar{\mathcal{B}}_2^n(\cdot)$ defined by (62). For all $k \leq \langle \mu, \chi \rangle / 4$ and y such that $\langle y, \chi \rangle \leq \frac{n\alpha}{2}$, we have that

$$\langle \mu, \chi \rangle - (k + \langle y, \chi \rangle) \geq \frac{n\alpha}{4}. \tag{70}$$

Thus, likewise (54), from (22) we have that

$$\mathbb{Q}_W(\mu, k, k, y, y, x) \geq 1 - \binom{x}{2} \frac{M}{\frac{n\alpha}{4}(\frac{\alpha}{4} - \frac{1}{n})} =: 1 - x^2 o_1(n).$$

Therefore, from the second inequality in (65) we have that

$$\begin{aligned}
|\langle \bar{\mathcal{B}}_2^n(\mu), \phi \rangle| &\leq \frac{2 \|\phi\|}{n} \left(\sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \right. \\
&\quad \times \sum_{x \in \mathbb{N}^*} \mathbb{P}_X(x \mid \mu, k, k, y, y) (1 - \mathbb{Q}_W(\mu, k, k, y, y, x)) 2x \Big) \\
&\leq \frac{4 \|\phi\|}{n} o_1(n) \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \mathbb{E}_Y \left[\langle Y(\mu), \chi \rangle^3 \mid \mu, k, k \right].
\end{aligned}$$

But (A.2) implies that

$$\begin{aligned}
\mathbb{E}_Y \left[\langle Y(\mu), \chi \rangle^3 \mid \mu, k, k \right] &\leq \frac{3k^5 \langle \mu, \mathbf{1} \rangle^2 \langle \mu, \chi^6 \rangle}{(\langle \mu, \chi \rangle - k)^3} + \frac{3k^4 \langle \mu, \mathbf{1} \rangle \langle \mu, \chi^5 \rangle}{(\langle \mu, \chi \rangle - k)^2} + \frac{3k^3 \langle \mu, \chi^4 \rangle}{\langle \mu, \chi \rangle - k} \\
&\leq \frac{3k^5 M^3 4^3}{\alpha^3} + \frac{3k^4 M^2 4^2}{\alpha^2} + \frac{12k^3 M}{\alpha},
\end{aligned}$$

so injecting this in (71) we obtain that

$$|\langle \bar{\mathcal{B}}_2^n(\mu), \phi \rangle| \leq 4 \|\phi\| C o_1(n), \tag{71}$$

where the constant C is given by

$$C = \frac{3M^4 4^3}{\alpha^3} + \frac{3M^3 4^2}{\alpha^2} + \frac{12M^2}{\alpha}.$$

Let us finally focus on the first term (61). Recalling the distribution of $Z(\cdot)$ in (24) and using (A.1), the latter can be rewritten as follows,

$$\begin{aligned}
& \langle \bar{\mathcal{B}}_1^n(\mu), \phi \rangle \\
&= \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \mathbb{E}_X[X(\mu) \mid \mu, k, k, y, y] \sum_{i=1}^{n-1} \Delta\phi(i) \frac{i(\mu(i) - \delta_k(i) - y(i))}{\langle \mu, \chi \rangle - \langle y, \chi \rangle - k} \\
&= \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) (\langle Y(\mu), \chi \rangle - k) \sum_{i=1}^{n-1} \Delta\phi(i) \frac{i(\mu(i) - \delta_k(i) - y(i))}{\langle \mu, \chi \rangle - \langle y, \chi \rangle - k} \\
&+ \frac{1}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \sum_{i=1}^{n-1} \Delta\phi(i) \frac{i(\mu(i) - \delta_k(i) - y(i))}{\langle \mu, \chi \rangle - \langle y, \chi \rangle - k} \\
&\times \left(\mathbb{E}_X[X(\mu) \mathbb{1}_{\{X(\mu) \neq \langle y, \chi \rangle - k\}} \mid \mu, k, k, y, y] + (\langle y, \chi \rangle - k) \left(\mathbb{P}_X(\langle y, \chi \rangle - k \mid \mu, k, k, y, y) - 1 \right) \right) \\
&=: \langle \bar{\mathcal{B}}_{11}^n(\mu), \phi \rangle + \langle \bar{\mathcal{B}}_{12}^n(\mu), \phi \rangle. \quad (72)
\end{aligned}$$

Let us prove that $\langle \bar{\mathcal{B}}_{12}^n(\mu), \phi \rangle$ also vanishes. Likewise (56), we have that for all k and y ,

$$\mathbb{P}_X(\langle y, \chi \rangle - k \mid \mu, k, k, y, y) \geq 1 - \frac{(\langle y, \chi \rangle - k)^2}{\langle \mu, \chi \rangle - \langle y, \chi \rangle - k + 1}.$$

This implies using (70) that

$$\begin{aligned}
\langle \bar{\mathcal{B}}_{12}^n(\mu), \phi \rangle &\leq \frac{4 \|\phi\|}{n} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \frac{(\langle y, \chi \rangle - k)^2}{\langle \mu, \chi \rangle - \langle y, \chi \rangle - k + 1} (\langle y, \chi \rangle - k) \\
&\leq \frac{16 \|\phi\|}{n(n\alpha + 4)} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \mathbb{E}_Y[\langle Y(\mu), \chi \rangle^3 \mid \mu, k, k], \quad (73)
\end{aligned}$$

which is again a $o(n)$, just as (71). It only remains to prove that $|\langle \bar{\mathcal{B}}_{11}^n(\mu), \phi \rangle - \langle \bar{B}^n(\mu), \phi \rangle|$ also is a

$o(n)$. We have

$$\begin{aligned}
& |\langle \bar{\mathcal{B}}_{11}^n(\mu), \phi \rangle - \langle \bar{B}^n(\mu), \phi \rangle| \\
& \leq \frac{1}{n} \left| \langle \mu, \chi \Delta \phi \rangle \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \left(\frac{\langle y, \chi \rangle - k}{\langle \mu, \chi \rangle - \langle y, \chi \rangle - k} \right) - \left(\frac{\langle \mu, \chi^2 \rangle}{\langle \mu, \chi \rangle} - 1 \right) \right| \\
& + \frac{1}{n} \left| \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) (\langle y, \chi \rangle - k) \sum_{i=1}^{n-1} \Delta \phi(i) \frac{i(\delta_k(i) + y(i))}{\langle \mu, \chi \rangle - \langle y, \chi \rangle - k} \right| \\
& \leq \frac{1}{n} \left| \langle \mu, \chi \Delta \phi \rangle \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \frac{\langle y, \chi \rangle - k}{\langle \mu, \chi \rangle - \langle y, \chi \rangle - k} - \frac{\langle y, \chi \rangle - k}{\langle \mu, \chi \rangle} \right| \\
& + \frac{1}{n} \left| \langle \mu, \chi \Delta \phi \rangle \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \frac{\langle y, \chi \rangle - k}{\langle \mu, \chi \rangle} - \left(\frac{\langle \mu, \chi^2 \rangle}{\langle \mu, \chi \rangle} - 1 \right) \right| \\
& + \frac{2 \|\phi\|}{n} \left| \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) (\langle y, \chi \rangle - k) \frac{k + \langle y, \chi \rangle}{\langle \mu, \chi \rangle - \langle y, \chi \rangle - k} \right| \\
& \quad \quad \quad =: \langle \bar{\mathcal{B}}_{111}^n(\mu), \phi \rangle + \langle \bar{\mathcal{B}}_{112}^n(\mu), \phi \rangle + \langle \bar{\mathcal{B}}_{113}^n(\mu), \phi \rangle. \quad (74)
\end{aligned}$$

We first have that

$$\begin{aligned}
\langle \bar{\mathcal{B}}_{111}^n(\mu), \phi \rangle & \leq 2M \|\phi\| \left| \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \frac{\langle y, \chi \rangle^2}{\langle \mu, \chi \rangle (\langle \mu, \chi \rangle - \langle y, \chi \rangle - k)} \right| \\
& + 2M \|\phi\| \left| \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \frac{k^2}{\langle \mu, \chi \rangle (\langle \mu, \chi \rangle - \langle y, \chi \rangle - k)} \right| \\
& = o(n), \quad (75)
\end{aligned}$$

since

$$\left| \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \frac{k^2}{\langle \mu, \chi \rangle (\langle \mu, \chi \rangle - \langle y, \chi \rangle - k)} \right| \leq \frac{4}{(n\alpha)^2} \langle \mu, \chi^2 \rangle \leq \frac{4}{n\alpha^2} M$$

and

$$\begin{aligned} & \left| \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \frac{\langle y, \chi \rangle^2}{\langle \mu, \chi \rangle (\langle \mu, \chi \rangle - \langle y, \chi \rangle - k)} \right| \\ & \leq \frac{4}{(n\alpha)^2} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \mathbb{E}(\langle Y(\mu), \chi \rangle^2) \leq \frac{4}{(n\alpha)^2} \frac{16M^2}{3\alpha(3\alpha - 4/n)} \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) k^2 \leq \frac{4}{n\alpha^2} \frac{16M^3}{3\alpha(3\alpha - 4/n)}, \end{aligned}$$

where we also used (50). We now focus on the second term of (74). From (66) we have that

$$\begin{aligned} & \langle \bar{\mathcal{B}}_{112}^n(\mu), \phi \rangle \\ & \leq 2M \|\phi\| \left| \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \frac{\mu(k)}{\langle \mu, \chi \rangle} \left(k \left(\frac{\langle \mu, \chi^2 \rangle}{\langle \mu, \chi \rangle - k} - 1 \right) + k^3 o_3(n) + k(k-1) o_4(n) \right) - \left(\frac{\langle \mu, \chi^2 \rangle}{\langle \mu, \chi \rangle} - 1 \right) \right| \\ & \leq 2M \|\phi\| \left(\sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \frac{k\mu(k)}{\langle \mu, \chi \rangle} \left| \frac{\langle \mu, \chi^2 \rangle}{\langle \mu, \chi \rangle - k} - \frac{\langle \mu, \chi^2 \rangle}{\langle \mu, \chi \rangle} \right| + \frac{1}{\alpha} M o_3(n) + \frac{1}{\alpha} M o_4(n) \right. \\ & \quad \left. + \sum_{\substack{k \in \mathbb{N}^*; \\ k > \langle \mu, \chi \rangle / 4}} \frac{k\mu(k)}{\langle \mu, \chi \rangle} \left| \frac{\langle \mu, \chi^2 \rangle}{\langle \mu, \chi \rangle} - 1 \right| \right) \\ & \leq 2M \|\phi\| \left(\sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \frac{k\mu(k)}{\langle \mu, \chi \rangle} \frac{k \langle \mu, \chi^2 \rangle}{\langle \mu, \chi \rangle (\langle \mu, \chi \rangle - k)} + \frac{1}{\alpha} M (o_3(n) + o_4(n)) + \frac{M}{(n\alpha^2)} \frac{4}{(n\alpha)^2} M \right) \\ & \leq 2M \|\phi\| \left(\frac{4M^2}{3n\alpha^3} + \frac{1}{\alpha} M (o_3(n) + o_4(n)) + \frac{4M^2}{n\alpha^3} \right), \quad (76) \end{aligned}$$

where we have used the same argument as for (48) for the last term. Finally,

$$\begin{aligned} \langle \bar{\mathcal{B}}_{113}^n(\mu), \phi \rangle & \leq \frac{2 \|\phi\|}{n} \left| \sum_{\substack{k \in \mathbb{N}^*; \\ k \leq \langle \mu, \chi \rangle / 4}} \mu(k) \sum_{\substack{y \in \mathcal{M}_F(\mathbb{N}); \\ \langle y, \mathbf{1} \rangle = k; \\ \langle y, \chi \rangle \leq n\alpha/2}} \mathbb{P}_Y(y \mid \mu, k, k) \frac{\langle y, \chi \rangle^2 - k^2}{\langle \mu, \chi \rangle - \langle y, \chi \rangle - k} \right| \\ & \leq \frac{2 \|\phi\|}{n} \left(\frac{4}{n\alpha} \frac{16M^2}{3\alpha(3\alpha - 4/n)} + \frac{4M}{\alpha} \right), \end{aligned}$$

which also follows from (50). This, together with (75) and (76) in (74), concludes the proof. \square

5.2. Tightness

Proposition 5.4. *The sequence of measure-valued processes $\left\{(\bar{\mu}_t^n)_{t \geq 0}\right\}_{n \in \mathbb{N}^*}$ is tight in $D([0, T], \mathcal{M}_F(\mathbb{N}))$.*

Proof. First fix $t \geq 0$. It is clear that the family of random measure $\{\bar{\mu}_t^n\}_{n \in \mathbb{N}^*}$ is tight. Indeed we have almost surely for any finite subset A of \mathbb{N} , $\bar{\mu}_t^n(A) \leq \mu_0^n(A)$ for all n . Hence, the family of random variables $\{\bar{\mu}_t^n(A)\}_{n \in \mathbb{N}^*}$ is tight for all such A , which implies in turn that the family of random measures $\{\bar{\mu}_t^n\}_{n \in \mathbb{N}^*}$ is tight (see Lemma 14.15 in [13]).

Therefore, from Roelly's criterion [19], it suffices to show that $\left\{(\langle \bar{\mu}_t^n, \phi \rangle_t)_{t \geq 0}\right\}_{n \in \mathbb{N}^*}$ is tight in $D([0, T], \mathbb{R})$ for all $\phi \in \mathcal{B}_b$. For this, we exploit the semi-martingale decomposition (26) as in Joffe and Métivier (Corollary 2.3.3 in [12]) and apply Rebolledo-Aldous's criterion ([1]) for the finite variation part $\frac{1}{n} \int_0^\cdot \mathcal{Q}\Pi_\phi(\mu_s^n) ds$ and the quadratic variation process $\langle \bar{M}^n(\phi) \rangle$. In detail, we aim at showing that for all $\varepsilon > 0$ and $\eta > 0$, there exists $\delta > 0$ and n_0 such that

$$\sup_{n \geq n_0} \mathbf{P} \left[\left| \frac{1}{n} \int_{\sigma_n}^{\tau_n} \mathcal{Q}\Pi_\phi(\mu_s^n) ds \right| \geq \eta \right] \leq \varepsilon; \quad (77)$$

$$\sup_{n \geq n_0} \mathbf{P} \left[|\langle \bar{M}^n(\phi) \rangle_{\tau_n} - \langle \bar{M}^n(\phi) \rangle_{\sigma_n}| \geq \eta \right] \leq \varepsilon, \quad (78)$$

for any two sequences $\{\tau_n\}_{n \in \mathbb{N}^*}$ and $\{\sigma_n\}_{n \in \mathbb{N}^*}$ of stopping times such that $\tau_n < \sigma_n < \tau_n + \delta$ for all $n \in \mathbb{N}^*$. First, it readily follows from (44) and (26) that for all such $n \in \mathbb{N}^*$,

$$\begin{aligned} & \mathbf{P} \left[\left| \frac{1}{n} \int_{\sigma_n}^{\tau_n} \mathcal{Q}\Pi_\phi(\mu_s^n) ds \right| \geq \eta \right] \\ & \leq \frac{\mathbf{E} \left[\frac{1}{n} \int_{\sigma_n}^{\tau_n} |\mathcal{Q}\Pi_\phi(\mu_s^n)| ds \right]}{\eta} \\ & \leq \frac{\lambda}{\eta} \mathbf{E} \left[\int_{\sigma_n}^{\tau_n} \langle \bar{\mu}_s^n, \phi \rangle ds + \int_{\sigma_n}^{\tau_n} \langle \bar{\mathcal{A}}^n(\mu_s^n), \phi \rangle ds \right. \\ & \quad \left. + \int_{\sigma_n}^{\tau_n} \langle \bar{\mathcal{B}}^n(\mu_s^n), \phi \rangle ds + \int_{\sigma_n}^{\tau_n} \langle \bar{\mathcal{C}}^n(\mu_s^n), \phi \rangle ds \right] \\ & \leq \frac{\lambda \|\phi\|}{\eta} \mathbf{E} \left[\int_{\sigma_n}^{\tau_n} \sum_k \bar{\mu}_s^n(k) \left\{ 1 + \mathbb{E}_Y[\langle Y(\mu_s^n), \mathbf{1} \rangle \mid \mu_s^n, k] + \mathbb{E}_Y[\langle Y(\mu_s^n), \chi \rangle \mid \mu_s^n, k] \right\} ds \right] \\ & \leq \frac{\lambda \|\phi\|}{\eta} \mathbf{E} \left[(\tau_n - \sigma_n) \left(\langle \bar{\mu}_0^n, \mathbf{1} \rangle + \langle \bar{\mu}_0^n, \chi \rangle + \langle \bar{\mu}_0^n, \chi \rangle \langle \bar{\mu}_0^n, \chi^2 \rangle \right) \right] \\ & \leq \frac{\lambda \|\phi\| (2M + M^2)}{\eta} \mathbf{E} [\tau_n - \sigma_n], \quad (79) \end{aligned}$$

using successively Markov's inequality and the hard bounds in the proofs of Lemmas 5.2 and 5.3. All

the same, using (27) we obtain that for some constant C ,

$$\begin{aligned}
& \mathbf{P} \left[\left| \langle \bar{M}^n(\phi) \rangle_{\tau_n} - \langle \bar{M}^n(\phi) \rangle_{\sigma_n} \right| \geq \eta \right] \\
& \leq \frac{C\lambda \|\phi\|^2}{n\eta} \mathbb{E} \left[\int_{\sigma_n}^{\tau_n} \sum_k \bar{\mu}_s^n(k) \left\{ 1 + (\mathbb{E}_Y [\langle Y(\mu_s^n), \mathbf{1} \rangle \mid \mu_s^n, k])^2 \right. \right. \\
& \quad \left. \left. + (\mathbb{E}_Y [\langle Y(\mu_s^n), \chi \rangle \mid \mu_s^n, k])^2 \right\} ds \right] \\
& \leq \frac{C\lambda \|\phi\|^2}{n\eta} \mathbf{E} \left[(\tau_n - \sigma_n) \left(\langle \bar{\mu}_0^n, \mathbf{1} \rangle + \langle \bar{\mu}_0^n, \chi^2 \rangle + \langle \bar{\mu}_0^n, \chi^2 \rangle (\langle \bar{\mu}_0^n, \chi^2 \rangle)^2 \right) \right] \\
& \leq \frac{C\lambda \|\phi\|^2 (2M + M^3)}{n\eta} \mathbf{E} [\tau_n - \sigma_n]. \quad (80)
\end{aligned}$$

Clearly, from (79) and (80) we can choose δ small enough so that (77) and (78) hold for $n_0 = 1$, which concludes the proof. \square

5.3. Uniqueness

Let $(\bar{\mu}_t)_{t \geq 0}$ and $(\bar{\nu}_t)_{t \geq 0}$, two solutions of (30) in $\mathcal{C}(\mathbb{R}_+, \mathcal{M}_F(\mathbb{N}))$, with the same initial condition ζ , and let us denote $\gamma_t = \bar{\mu}_t - \bar{\nu}_t$ for all t . Denote also for all β such that $0 < \beta < \alpha$,

$$\begin{aligned}
t_\beta^\mu &= \sup \left\{ t \geq 0; \langle \bar{\mu}_t, \chi \rangle > \beta \right\}; \\
t_\beta^\nu &= \sup \left\{ t \geq 0; \langle \bar{\nu}_t, \chi \rangle > \beta \right\}.
\end{aligned}$$

Recall, that in view of (38) we have that

$$\langle \bar{\mu}_t, \chi^5 \rangle \vee \langle \bar{\nu}_t, \chi^5 \rangle \leq M, \quad t \geq 0.$$

Thus, denoting again for all $\mu \in \mathcal{M}_F(\mathbb{N})$, $L(\mu) = \frac{\langle \mu, \chi^2 \rangle}{\langle \mu, \chi \rangle}$ we have that

$$L(\bar{\mu}_t) \vee L(\bar{\nu}_t) \leq \frac{M}{\beta}, \quad t \in [0, t_\beta^\mu \wedge t_\beta^\nu].$$

Let for all $t \geq 0$,

$$\Gamma_t = \sum_{i \geq 0} i^8 \gamma_t(i)^2.$$

Fix β such that $0 < \beta < \alpha$ and $t \in [0, t_\beta^\mu \wedge t_\beta^\nu]$. From (30), we have that

$$\begin{aligned}
\frac{d}{dt}\Gamma_t &= \sum_{i \geq 0} i^8 \frac{d}{dt} \gamma_t(i)^2 \\
&= 2 \sum_{i \geq 0} i^8 \gamma_t(i) \frac{d}{dt} \gamma_t(i) \\
&= 2\lambda \sum_{i \geq 0} \left\{ (i+1)(L(\bar{\mu}_t) - 1) \gamma_t(i) \gamma_t(i+1) - i^8 (1 + iL(\bar{\mu}_t)) (\gamma_t(i))^2 \right\} \\
&\quad - 2\lambda \sum_{i \geq 0} \left\{ (L(\bar{\nu}_t) - L(\bar{\mu}_t)) i^8 \gamma_t(i) (i\bar{\nu}_t(i) - (i+1)\bar{\nu}_t(i+1)) \right\} \\
&= A_t + B_t.
\end{aligned} \tag{81}$$

We first deal with the term A_t . Using that

$$(i+1)\gamma_t(i+1)\gamma_t(i) = \frac{1}{2}(i+1)\gamma_t(i+1)^2 + \frac{1}{2}(i+1)\gamma_t(i)^2 - \frac{1}{2}(\sqrt{i+1}\gamma_t(i+1) - \sqrt{i+1}\gamma_t(i))^2,$$

we get

$$\begin{aligned}
\sum_{i \geq 0} i^8 (i+1)\gamma_t(i+1)\gamma_t(i) &\leq \frac{1}{2} \sum_{i \geq 0} i^8 (i+1)\gamma_t(i+1)^2 + \frac{1}{2} \sum_{i \geq 0} i^8 (i+1)\gamma_t(i)^2 \\
&\leq \frac{1}{2} \sum_{i \geq 0} (i+1)^9 \gamma_t(i+1)^2 + \frac{1}{2} \sum_{i \geq 0} i^9 \gamma_t(i)^2 + \frac{1}{2} \sum_{i \geq 0} i^8 \gamma_t(i)^2 \\
&\leq \sum_{i \geq 0} i^9 \gamma_t(i)^2 + \frac{1}{2} \sum_{i \geq 0} i^8 \gamma_t(i)^2.
\end{aligned}$$

Hence, as $L(\bar{\mu}_t) \geq 1$,

$$\begin{aligned}
A_t &= 2\lambda \sum_i \left\{ i^8 (L(\bar{\mu}_t) - 1) (i+1)\gamma_t(i+1)\gamma_t(i) + i^8 (1 + iL(\bar{\mu}_t)) (\gamma_t(i))^2 \right\} \\
&\leq 2\lambda(L(\bar{\mu}_t) - 1) \left(\sum_{i \geq 0} i^9 \gamma_t(i)^2 + \frac{1}{2} \sum_{i \geq 0} i^8 \gamma_t(i)^2 \right) - 2\lambda L(\bar{\mu}_t) \sum_{i \geq 0} i^9 \gamma_t(i)^2 \\
&\leq \lambda L(\bar{\mu}_t) \sum_i i^8 \gamma_t(i)^2 \\
&\leq \frac{\lambda M}{\beta} \Gamma_t.
\end{aligned} \tag{82}$$

Let us now deal with B_t . First observe that

$$\begin{aligned}
&|L(\bar{\mu}_t) - L(\bar{\nu}_t)| \\
&= \left| \frac{\sum_i i^2 \bar{\mu}_t(i) \sum_i i \bar{\nu}_t(i) - \sum_i i \bar{\mu}_t(i) \sum_i i^2 \bar{\nu}_t(i)}{\sum_i i \bar{\mu}_t(i) \sum_i i \bar{\nu}_t(i)} \right| \\
&= \left| \frac{\sum_i i^2 \gamma_t(i) \sum_i i \bar{\nu}_t(i) + \sum_i i^2 \bar{\nu}_t(i) \sum_i i \bar{\nu}_t(i) - \sum_i i \gamma_t(i) \sum_i i^2 \bar{\nu}_t(i) - \sum_i i^2 \bar{\nu}_t(i) \sum_i i \bar{\nu}_t(i)}{\sum_i i \bar{\mu}_t(i) \sum_i i \bar{\nu}_t(i)} \right| \\
&\leq \frac{\sum_i i^2 |\gamma_t(i)| \sum_i i \bar{\nu}_t(i) + \sum_i i |\gamma_t(i)| \sum_i i^2 \bar{\nu}_t(i)}{\sum_i i \bar{\mu}_t(i) \sum_i i \bar{\nu}_t(i)} \leq \frac{2M}{\beta^2} \sum_i i^2 |\gamma_t(i)|.
\end{aligned}$$

Hence,

$$\begin{aligned}
|L(\bar{\mu}_t) - L(\bar{\nu}_t)| &\leq \frac{2M}{\beta^2} \sum_i i^2 |\gamma_t(i)|, \\
&\leq \frac{2M}{\beta^2} \left(\sup_i i^4 |\gamma_t(i)| \right) \sum_i i^{-2} \\
&\leq \frac{M\pi^2}{3\beta^2} \left(\sup_i i^8 |\gamma_t(i)|^2 \right)^{1/2} \\
&\leq \frac{M\pi^2}{3\beta^2} \left(\sum_i i^8 |\gamma_t(i)|^2 \right)^{1/2}.
\end{aligned} \tag{83}$$

On the other hand, using Cauchy Schwartz inequality,

$$\begin{aligned}
\sum_i \bar{\nu}(i) i^9 |\gamma_t(i)| &= \sum_i i^5 \bar{\nu}_t(i) i^4 |\gamma_t(i)| \\
&\leq \left(\sum_i i^{10} \bar{\nu}_t(i)^2 \right)^{1/2} \left(\sum_i i^8 \gamma_t(i)^2 \right)^{1/2} \\
&\leq \left(\sum_i i^5 \bar{\nu}_t(i) \right) \left(\sum_i i^8 \gamma_t(i)^2 \right)^{1/2} \\
&\leq M \left(\sum_i i^8 \gamma_t(i)^2 \right)^{1/2},
\end{aligned} \tag{84}$$

where in the second inequality we used the fact that for a finite series x having positive terms,

$$\sum_{i \in \mathbb{N}} x_i^2 \leq \left(\sum_{i \in \mathbb{N}} x_i \right)^2.$$

All the same, we have that

$$\begin{aligned}
\sum_i \bar{\nu}_t(i+1)(i+1) i^8 |\gamma_t(i)| &= \sum_i (i+1)^5 \bar{\nu}_t(i+1) i^4 |\gamma_t(i)| \\
&\leq \left(\sum_i (i+1)^{10} \bar{\nu}_t(i+1)^2 \right)^{1/2} \left(\sum_i i^8 \gamma_t(i)^2 \right)^{1/2} \\
&\leq \left(\sum_i i^{10} \bar{\nu}_t(i)^2 \right)^{1/2} \left(\sum_i i^8 \gamma_t(i)^2 \right)^{1/2} \\
&\leq M \left(\sum_i i^8 \gamma_t(i)^2 \right)^{1/2}.
\end{aligned} \tag{85}$$

Hence, using (83), (84) and (85) we obtain that

$$\begin{aligned}
B &\leq 2\lambda |L(\bar{\mu}_t) - L(\bar{\nu}_t)| \sum_i \bar{\nu}_t(i) i^9 |\gamma_t(i)| + \bar{\nu}_t(i+1) i^8 (i+1) |\gamma_t(i)| \\
&\leq \frac{4\lambda M^2 \pi^2}{3\beta^2} \left(\sum_i i^8 |\gamma_t(i)|^2 \right).
\end{aligned} \tag{86}$$

Finally, using (82) and (86) in (81), we obtain that for some positive constant C , for all $t \leq t_\beta^\mu \wedge t_\beta^\nu$,

$$\frac{d}{dt}\Gamma_t \leq C\Gamma_t.$$

Since $G(0) = 0$, G is a positive function and $t_\beta^\mu \wedge t_\beta^\nu > 0$, this shows using Gronwall's Lemma that $\Gamma_t = 0$ for all such t . Therefore, $t_\beta^\mu = t_\beta^\nu =: t_\beta$, and $\bar{\mu}_t$ and $\bar{\nu}_t$ coincide up to t_β . In other words there is at most one solution to (30) up to time t_β . Since this is true for all β , and since the only solution $\bar{\mu}$ is such that $t \rightarrow \langle \bar{\mu}_t, \chi \rangle$ is continuous, there is at most one solution up to the (positive, in view of (29)) instant

$$t_0 = \sup \{t_\beta; \beta > 0\}. \quad (87)$$

The proof of uniqueness is completed by noticing that whenever $t_0 < \infty$, the only solution $\bar{\mu}$ to (30) can be extended uniquely after t_0 , as follows:

$$\bar{\mu}_t = \mu_{t_0}(0)e^{-\lambda(t-t_0)}\delta_0, \quad t \geq t_0. \quad (88)$$

5.4. Convergence

Recall that $\zeta \in \mathcal{M}_f(\mathbb{N})$, $M > 1$ and $\alpha > 0$ are respectively defined by (28), (38) and (39). Fix $T > 0$.

Convergence before reaching a given positive threshold.

We first prove the following result.

Proposition 5.5. *For all $\phi \in \mathcal{B}_b$,*

$$\sup_{t \in [0, t_{2\alpha}]} |\langle \bar{\mu}_t^n, \phi \rangle - \langle \bar{\mu}_t, \phi \rangle| \xrightarrow[n \rightarrow \infty]{(\mathcal{P})} 0,$$

where $(\bar{\mu}_t)_{t \geq 0}$ is the unique solution of (30) on $\mathcal{C}([0, t_{2\alpha}], \mathcal{M}_F(\mathbb{N}))$.

Proof. For all $n \in \mathbb{N}^*$, we define the stopping times

$$\tau_\alpha^n = \sup \{t \geq 0; \langle \mu_t^n, \chi \rangle \geq n\alpha\}$$

and denote for all t ,

$$\bar{\mu}_t^{n, \tau_\alpha^n} = \bar{\mu}_{t \wedge \tau_\alpha^n}^n.$$

Clearly, from Proposition 5.4 the sequence of stopped processes $\{\bar{\mu}^{n, \tau_\alpha^n}\}$ is also relatively compact for the topology of weak convergence, and we let $\bar{\mu}^*$ be a sub-sequential limit. Define

$$\tau_{2\alpha} = \sup \left\{ t \geq 0; \langle \bar{\mu}_t^*, \chi \rangle > 2\alpha \right\}.$$

Notice that the process $\bar{\mu}^*$ and hence the instant $\tau_{2\alpha}$, are *a priori* random. Fix $\phi \in \mathcal{B}_b$ throughout the proof. Let $n \in \mathbb{N}^*$. We have for all $t \in [0, T]$,

$$\begin{aligned} \int_0^{t \wedge \tau_{2\alpha} \wedge \tau_\alpha^n} \left\langle \Psi \left(\bar{\mu}_s^{n, \tau_\alpha^n} \right), \phi \right\rangle ds &= \left(\int_0^{t \wedge \tau_{2\alpha}} \left\langle \Psi \left(\bar{\mu}_s^{n, \tau_\alpha^n} \right), \phi \right\rangle ds \right) \mathbb{1}_{\{\tau_\alpha^n > \tau_{2\alpha} \wedge T\}} \\ &\quad + \left(\int_0^{t \wedge \tau_\alpha^n} \left\langle \Psi \left(\bar{\mu}_s^{n, \tau_\alpha^n} \right), \phi \right\rangle ds \right) \mathbb{1}_{\{\tau_\alpha^n \leq \tau_{2\alpha} \wedge T\}}. \end{aligned} \quad (89)$$

On the one hand, in view of Lemma A.5 of ([6]), the following map is continuous for the Skorokhod topology:

$$\begin{cases} \mathcal{D}([0, T], \mathcal{M}_{\alpha, M}) & \longrightarrow & \mathcal{D}([0, T], \mathbb{R}) \\ \mu & \longmapsto & \langle \mu, \chi \rangle, \end{cases}$$

and so does

$$\begin{cases} \mathcal{D}([0, T], \mathbb{R}) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \inf_{t \in [0, T]} x_t. \end{cases}$$

Therefore, from the Continuous Mapping Theorem ([4]), along the latter subsequence the following convergence in distribution holds:

$$\inf_{t \in [0, T]} \langle \bar{\mu}_{t \wedge \tau_{2\alpha}}^{n, \tau_{\alpha}^n}, \chi \rangle \Rightarrow \inf_{t \in [0, T]} \langle \bar{\mu}_{t \wedge \tau_{2\alpha}}^*, \chi \rangle.$$

Hence, from Fatou's Lemma,

$$1 = \mathbf{P} \left[\inf_{t \in [0, T]} \langle \bar{\mu}_{t \wedge \tau_{2\alpha}}^*, \chi \rangle > \alpha \right] \leq \lim_{n \rightarrow \infty} \mathbf{P} \left[\inf_{t \in [0, T]} \langle \bar{\mu}_{t \wedge \tau_{2\alpha}}^{n, \tau_{\alpha}^n}, \chi \rangle > \alpha \right] \leq \lim_{n \rightarrow \infty} \mathbf{P} [T \wedge \tau_{2\alpha} < \tau_{\alpha}^n]. \quad (90)$$

Therefore,

$$\left(\int_0^{t \wedge \tau_{\alpha}^n} \langle \Psi(\bar{\mu}^{n, \tau_{\alpha}^n})_s, \phi \rangle ds \right) \mathbb{1}_{\{\tau_{\alpha}^n \leq \tau_{2\alpha} \wedge T\}} \xrightarrow[n \rightarrow \infty]{(\mathcal{P})} 0. \quad (91)$$

Now, it follows again from Lemma A.5 of ([6]) that the following mappings are continuous for the Skorokhod topology:

$$\begin{cases} \mathcal{D}([0, T], \mathcal{M}_{\alpha, M}) & \longrightarrow & \mathcal{D}([0, T], \mathbb{R}) \\ \mu & \longmapsto & \langle \mu, \chi \rangle; \\ & & \langle \mu, \chi^2 \rangle; \\ & & \langle \mu, (\mathbf{1} + \chi) \phi \rangle; \\ & & \langle \mu, \chi \Delta \phi \rangle, \end{cases}$$

and it is a classical result that the followings map is also continuous:

$$\begin{cases} \mathcal{D}([0, T], \mathbb{R} \times \mathbb{R}^*) & \longrightarrow & \mathcal{D}([0, T], \mathbb{R}) \\ (x, y) & \longmapsto & \frac{x}{y}. \end{cases}$$

So from the Continuous Mapping Theorem, the map

$$\begin{cases} \mathcal{D}([0, T], \mathcal{M}_{\alpha, M}) & \longrightarrow & \mathcal{D}([0, T], \mathbb{R}) \\ \mu & \longmapsto & \langle \Psi(\mu)_t, \phi \rangle \end{cases}$$

is itself continuous, and it follows from the continuity of the map

$$\begin{cases} \mathcal{D}([0, T], \mathbb{R}) & \longrightarrow & \mathcal{C}([0, T], \mathbb{R}) \\ x & \longmapsto & \int_0^{\cdot} x_s ds, \end{cases}$$

together with (89), (90) and (91), that along the same sub-sequence

$$\int_0^{\cdot \wedge \tau_{2\alpha} \wedge \tau_{\alpha}^n} \langle \Psi(\bar{\mu}^{n, \tau_{\alpha}^n})_s, \phi \rangle ds \Longrightarrow \int_0^{\cdot \wedge \tau_{2\alpha}} \langle \Psi(\bar{\mu}^*)_s, \phi \rangle ds \quad \text{in } \mathcal{C}([0, T], \mathbb{R}). \quad (92)$$

On the other hand, whenever $\bar{\mu}_0^n \in \mathcal{M}_{\alpha, M}$ we clearly have that $\bar{\mu}_{t \wedge \tau_{2\alpha}}^{n, \tau_{\alpha}^n} \in \mathcal{M}_{\alpha, M}$ for all t . Therefore, as a consequence of Lemmas 5.1, 5.2 and 5.3 together with (29), (26), (41) and (44), we have a.s. for

all $n \in \mathbb{N}^*$ and $t \geq 0$,

$$\begin{aligned} \langle \bar{\mu}_{t \wedge \tau_{2\alpha}}^{n, \tau_{2\alpha}^n}, \phi \rangle &= \langle \bar{\mu}_0^n, \phi \rangle + \left(\int_0^{t \wedge \tau_{2\alpha}^n \wedge \tau_{2\alpha}} \langle \Psi(\bar{\mu}^{n, \tau_{2\alpha}^n})_s, \phi \rangle ds + o_t^{n, \alpha} \right) \mathbb{1}_{\{\bar{\mu}_0^n \in \mathcal{M}_{\alpha, M}\}} \\ &\quad + \left(\frac{1}{n} \int_0^{t \wedge \tau_{2\alpha}^n \wedge \tau_{2\alpha}} \mathcal{Q}^n \Pi_\phi(\mu_s^n) ds \right) \mathbb{1}_{\{\bar{\mu}_0^n \notin \mathcal{M}_{\alpha, M}\}} + \bar{M}^n(\phi)_{t \wedge \tau_{2\alpha}^n \wedge \tau_{2\alpha}}, \end{aligned} \quad (93)$$

where $o^{n, \alpha}$ is a process converging to 0 in probability and uniformly over compact sets.

Now, applying Doob's inequality to the stopped martingale $\bar{M}^n(\phi)_{\cdot \wedge \tau_{2\alpha}^n \wedge \tau_{2\alpha}}$ and using (27) as in (80) yields that

$$\sup_{t \in [0, T]} |\bar{M}^n(\phi)_{t \wedge \tau_{2\alpha}^n}| \xrightarrow[n \rightarrow \infty]{(\mathcal{P})} 0. \quad (94)$$

Moreover, for all n, t and all ε we have that

$$\begin{aligned} &\mathbf{P} \left[\sup_{t \in [0, T]} \left(\frac{1}{n} \int_0^{t \wedge \tau_{2\alpha}^n \wedge \tau_{2\alpha}} \mathcal{Q}^n \Pi_\phi(\mu_s^n) ds \right) \mathbb{1}_{\{\bar{\mu}_0^n \notin \mathcal{M}_{\alpha, M}\}} > \varepsilon \right] \\ &= \mathbf{P} \left[\sup_{t \in [0, T]} \frac{1}{n} \int_0^{t \wedge \tau_{2\alpha}^n \wedge \tau_{2\alpha}} \mathcal{Q}^n \Pi_\phi(\mu_s^n) ds > \varepsilon, \bar{\mu}_0^n \notin \mathcal{M}_{\alpha, M} \right] \\ &\leq \mathbf{P} [\bar{\mu}_0^n \notin \mathcal{M}_{\alpha, M}] \\ &\leq \mathbf{P} [\langle \bar{\mu}_0^n, \chi \rangle \leq \alpha] + \mathbf{P} [\langle \bar{\mu}_0^n, \chi^6 \rangle \geq M] \\ &\leq \mathbf{P} \left[\langle \bar{\mu}_0^n, \chi \rangle < \langle \bar{\mu}_0, \chi \rangle - \frac{\langle \bar{\mu}_0, \chi \rangle - \alpha}{2} \right] + \mathbf{P} \left[\langle \bar{\mu}_0^n, \chi^6 \rangle > \langle \bar{\mu}_0, \chi^6 \rangle + \frac{M - \langle \bar{\mu}_0, \chi^6 \rangle}{2} \right] \xrightarrow[n \rightarrow \infty]{} 0, \end{aligned} \quad (95)$$

in view of (28) and (29). Plugging (28) together with (92), (94) and (95) into (93), and using Skorokhod Representation Theorem implies that on some probability space, almost surely

$$\langle \bar{\mu}_{t \wedge \tau_{2\alpha}}^*, \phi \rangle = \langle \bar{\mu}_0, \phi \rangle + \int_0^{t \wedge \tau_{2\alpha}} \langle \Psi(\bar{\mu}^*)_s, \phi \rangle ds, \quad t \geq 0.$$

In other words, $\bar{\mu}^*$ is a $\mathcal{C}([0, T], \mathcal{M}_F(\mathbb{N}))$ -valued process having initial deterministic value ζ , and solving (30) on $[0, T \wedge \tau_{2\alpha}]$. As the solution of the latter, if any, is unique, we conclude (i) that $\tau_{2\alpha}$ is deterministic, hence (ii) that there exists a solution $\bar{\mu}$ to (30) on $[0, T \wedge \tau_{2\alpha}]$, with which $\bar{\mu}^*$ coincides almost surely and (iii) that $\tau_{2\alpha} = t_{2\alpha}$, where

$$t_{2\alpha} = \sup \left\{ t \geq 0; \langle \bar{\mu}_t, \chi \rangle > 2\alpha \right\}. \quad (96)$$

In particular, as the tightness of $\{\bar{\mu}^{n, \tau_{2\alpha}^n}\}$ clearly implies that of $\left\{ \int_0^{\cdot \wedge \tau_{2\alpha} \wedge \tau_{2\alpha}^n} \langle \Psi(\bar{\mu}^{n, \tau_{2\alpha}^n})_s, \phi \rangle ds \right\}$, we deduce from (92) that

$$\int_0^{\cdot \wedge t_{2\alpha} \wedge \tau_{2\alpha}^n} \langle \Psi(\bar{\mu}^{n, \tau_{2\alpha}^n})_s, \phi \rangle ds \implies \int_0^{\cdot \wedge t_{2\alpha}} \langle \Psi(\bar{\mu})_s, \phi \rangle ds \quad \text{in } \mathcal{C}([0, T], \mathbb{R}).$$

Using once again the Representation Theorem we obtain that on some probability space,

$$\int_0^{\cdot \wedge t_{2\alpha} \wedge \tau_{2\alpha}^n} \langle \Psi(\bar{\mu}^{n, \tau_{2\alpha}^n})_s, \phi \rangle ds \xrightarrow[n \rightarrow \infty]{} \int_0^{\cdot \wedge t_{2\alpha}} \langle \Psi(\bar{\mu})_s, \phi \rangle ds \quad \text{a.s. in } \mathcal{C}([0, T], \mathbb{R})$$

which, as the Skorokhod topology and the uniform topology coincide on $\mathcal{C}([0, T], \mathbb{R})$ (see [4], p.112), implies that

$$\sup_{t \in [0, T]} \left| \int_0^{t \wedge t_{2\alpha} \wedge \tau_\alpha^n} \langle \Psi(\bar{\mu}^{n, \tau_\alpha^n})_s, \phi \rangle ds - \int_0^{t \wedge t_{2\alpha}} \langle \Psi(\bar{\mu})_s, \phi \rangle ds \right| \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.}$$

and therefore,

$$\sup_{t \in [0, T]} \left| \int_0^{t \wedge t_{2\alpha} \wedge \tau_\alpha^n} \langle \Psi(\bar{\mu}^{n, \tau_\alpha^n})_s, \phi \rangle ds - \int_0^{t \wedge t_{2\alpha}} \langle \Psi(\bar{\mu})_s, \phi \rangle ds \right| \xrightarrow[n \rightarrow \infty]{(\mathcal{P})} 0.$$

As the latter holds true for all $T > 0$, we obtain that

$$\sup_{t \in [0, t_{2\alpha}]} \left| \int_0^{t \wedge \tau_\alpha^n} \langle \Psi(\bar{\mu}^{n, \tau_\alpha^n})_s, \phi \rangle ds - \int_0^t \langle \Psi(\bar{\mu})_s, \phi \rangle ds \right| \xrightarrow[n \rightarrow \infty]{(\mathcal{P})} 0. \quad (97)$$

Consequently, from (93) we obtain that for all $\varepsilon > 0$,

$$\begin{aligned} & \mathbf{P} \left[\sup_{t \in [0, t_{2\alpha}]} |\langle \bar{\mu}_t^n, \phi \rangle - \langle \bar{\mu}_t, \phi \rangle| > \varepsilon \right] \\ & \leq \mathbf{P} \left[\left\{ \sup_{t \in [0, t_{2\alpha}]} \left| \int_0^{t \wedge \tau_\alpha^n} \langle \Psi(\bar{\mu}^{n, \tau_\alpha^n})_s, \phi \rangle ds - \int_0^t \langle \Psi(\bar{\mu})_s, \phi \rangle ds \right| > \frac{\varepsilon}{3} \right\} \cap \{\tau_\alpha^n > t_{2\alpha}\} \right] \\ & \quad + \mathbf{P} \left[\sup_{t \in [0, t_{2\alpha}]} |\bar{M}^n(\phi)_{t \wedge \tau_\alpha^n} + o_t^{n, \alpha}| > \frac{\varepsilon}{3} \right] + \mathbf{P} \left[\left| \langle \bar{\mu}_0^{n, \tau_\alpha^n}, \phi \rangle - \langle \bar{\mu}_0, \phi \rangle \right| > \frac{\varepsilon}{3} \right] + \mathbf{P}[\tau_\alpha^n \leq t_{2\alpha}]. \end{aligned}$$

The first term on the r.h.s. vanishes for large n thanks to (97), the second one from Doob's inequality, the third one in view of (28) and the last one from (90). This concludes the proof. \square

Existence of the solution on \mathbb{R}_+ . A consequence of Proposition 5.5 is the existence of a solution $(\bar{\mu}_t)_{t \geq 0}$ of (30) until $t_{2\alpha}$. As a matter of fact, it appears clearly that the latter result can be extended to any $0 < \beta < \alpha$. Therefore, by its continuity the solution $(\bar{\mu}_t)_{t \geq 0}$ can be extended at least until t_0 , the hitting time of 0 defined by (87). The existence of the solution $(\bar{\mu}_t)_{t \geq 0}$ after t_0 then follows from the explicit form (88).

Asymptotics of the mass at the origin. We now focus on the mass concentrated at 0 in the hydrodynamic limit. Let us first give the following result,

Lemma 5.6. *For all $t \geq t_{2\alpha}$,*

$$\lambda \int_{t_{2\alpha}}^{t \wedge t_0} \bar{\mu}_s(1) \left(\frac{\langle \bar{\mu}_s, \chi^2 \rangle}{\langle \bar{\mu}_s, \chi \rangle} - 1 \right) ds \leq \langle \bar{\mu}_{t_{2\alpha}}, \chi \rangle,$$

where $(\bar{\mu}_t)_{t \geq 0}$ is the only solution to (30) on \mathbb{R}_+ and $t_{2\alpha}$ is defined by (96).

Proof. Let $t \geq t_{2\alpha}$. Plainly, applying (30) to $\phi := \chi$ leads to

$$\begin{aligned} \langle \bar{\mu}_t, \chi \rangle &= \langle \bar{\mu}_{t_{2\alpha}}, \chi \rangle - \lambda \left\{ \int_{t_{2\alpha}}^t \langle \bar{\mu}_s, \chi \rangle ds + \int_{t_{2\alpha}}^{t \wedge t_0} \langle \bar{\mu}_s, \chi^2 \rangle ds \right. \\ & \quad \left. + \int_{t_{2\alpha}}^{t \wedge t_0} \langle \bar{\mu}_s, \chi \Delta \chi \rangle \left(\frac{\langle \bar{\mu}_s, \chi^2 \rangle}{\langle \bar{\mu}_s, \chi \rangle} - 1 \right) ds \right\}. \end{aligned}$$

Therefore, as $\langle \bar{\mu}_t, \chi \rangle \geq 0$ we obtain that

$$\langle \bar{\mu}_{t_{2\alpha}}, \chi \rangle \geq \lambda \int_{t_{2\alpha}}^{t \wedge t_0} \langle \bar{\mu}_s, \chi \Delta \chi \rangle \left(\frac{\langle \bar{\mu}_s, \chi^2 \rangle}{\langle \bar{\mu}_s, \chi \rangle} - 1 \right) ds.$$

The proof is completed by noticing that for all s ,

$$\langle \bar{\mu}_s, \chi \Delta \chi \rangle = \sum_{i \in \mathbb{N}} \bar{\mu}_s(i) i (i - (i - 1)) = \sum_{i \in \mathbb{N}} \bar{\mu}_s(i) i \geq \bar{\mu}_s(1).$$

□

Now, let for all $n \in \mathbb{N}^*$ and all $t \geq t_{2\alpha}$,

$$\begin{aligned} X_t^{n,\alpha} &= \text{Card} \left\{ \text{U-vertices of degree 0 at } t_{2\alpha} \text{ that have become A before time } t \right\}; \\ Y_t^{n,\alpha} &= \text{Card} \left\{ \text{U-vertices of degree } \geq 1 \text{ at } t_{2\alpha}, \right. \\ &\quad \left. \text{having become of degree 0 at time } t \text{ and being still U at } t \right\}. \end{aligned}$$

Denote also $\bar{X}_t^{n,\alpha} = \frac{1}{n} X_t^{n,\alpha}$ and $\bar{Y}_t^{n,\alpha} = \frac{1}{n} Y_t^{n,\alpha}$ for all n and t . Then, clearly

$$\bar{\mu}_t^n(0) = \bar{\mu}_{t_{2\alpha}}^n(0) - \bar{X}_t^{n,\alpha} + \bar{Y}_t^{n,\alpha}, \quad t \geq t_{2\alpha}. \quad (98)$$

As the U-vertices of degree 0 are independent of the rest of the graph and eventually all become A-vertices, it is clear that for any n , the process $X^{n,\alpha}$ is Markov on \mathbb{N} . It has rcll paths, its generator clearly reads

$$\tilde{\mathcal{Q}}F(x) = \lambda (\mu_{t_{2\alpha}}^n(0) - x)$$

for all functions $F : \mathbb{R} \rightarrow \mathbb{R}$ and all $x \in \mathbb{N}$, so it is routine to check that for all $n \in \mathbb{N}^*$, for some square integrable martingale $\bar{M}^{n,\alpha}$, for all $t \geq t_{2\alpha}$,

$$\bar{X}_t^{n,\alpha} = \lambda \int_{t_{2\alpha}}^t (\bar{\mu}_{t_{2\alpha}}^n(0) - \bar{X}_s^{n,\alpha}) ds + \bar{M}_t^{n,\alpha}. \quad (99)$$

Therefore, with (98), we obtain that

$$\begin{aligned} \bar{\mu}_t^n(0) &= \bar{\mu}_{t_{2\alpha}}^n(0) - \lambda \int_{t_{2\alpha}}^t (\bar{\mu}_{t_{2\alpha}}^n(0) - \bar{X}_s^{n,\alpha}) ds - \bar{M}_t^{n,\alpha} + \bar{Y}_t^{n,\alpha} \\ &= \bar{\mu}_{t_{2\alpha}}^n(0) - \lambda \int_{t_{2\alpha}}^t (\bar{\mu}_s^n(0) - \bar{Y}_s^{n,\alpha}) ds - \bar{M}_t^{n,\alpha} + \bar{Y}_t^{n,\alpha}. \end{aligned} \quad (100)$$

It also readily follows once again by Doob's inequality that the martingale term vanishes uniformly in L^2 , and in particular, that for all $t \geq t_{2\alpha}$,

$$\sup_{s \in [t_{2\alpha}, t]} |\bar{M}_s^{n,\alpha}| \xrightarrow[n \rightarrow \infty]{(P)} 0. \quad (101)$$

On the other hand, applying (30) to $\phi \equiv \mathbb{1}_0$, and observing that for all $i \in \mathbb{N}$,

$$\chi(i) \Delta \mathbb{1}_0(i) = -\mathbb{1}_1(i),$$

we get that

$$\bar{\mu}_t(0) = \bar{\mu}_{t_{2\alpha}}(0) - \lambda \left\{ \int_{t_{2\alpha}}^t \bar{\mu}_s(0) ds - \int_{t_{2\alpha}}^{t \wedge t_0} \bar{\mu}_s(1) \left(\frac{\langle \bar{\mu}_s, \chi^2 \rangle}{\langle \bar{\mu}_s, \chi \rangle} - 1 \right) du \right\}, \quad t \geq t_{2\alpha}.$$

Combining this with (100) leads for all n to

$$\begin{aligned} |\bar{\mu}_t^n(0) - \bar{\mu}_t(0)| &\leq |\bar{\mu}_{t_{2\alpha}}^n(0) - \bar{\mu}_{t_{2\alpha}}(0)| + \lambda \int_{t_{2\alpha}}^t |\bar{\mu}_s^n(0) - \bar{\mu}_s(0)| ds + |\bar{M}_t^{n,\alpha}| \\ &\quad + \bar{Y}_t^{n,\alpha} + \lambda \int_{t_{2\alpha}}^t \bar{Y}_s^{n,\alpha} ds + \lambda \int_{t_{2\alpha}}^t \bar{\mu}_s(1) \left(\frac{\langle \bar{\mu}_s, \chi^2 \rangle}{\langle \bar{\mu}_s, \chi \rangle} - 1 \right) ds. \end{aligned} \quad (102)$$

But by its very definition, we have that for all $s \geq t_{2\alpha}$ and all n ,

$$\bar{Y}_s^{n,\alpha} \leq \bar{\mu}_{t_{2\alpha}}^n(1) \leq \langle \bar{\mu}_{t_{2\alpha}}^n, \chi \rangle.$$

Plugging this together with Lemma 5.6 in (102) yields to

$$\begin{aligned} |\bar{\mu}_t^n(0) - \bar{\mu}_t(0)| &\leq |\bar{\mu}_{t_{2\alpha}}^n(0) - \bar{\mu}_{t_{2\alpha}}(0)| + \lambda \int_{t_{2\alpha}}^t |\bar{\mu}_s^n(0) - \bar{\mu}_s(0)| ds + |\bar{M}_t^{n,\alpha}| \\ &\quad + (1 + \lambda(t - t_{2\alpha})) \langle \bar{\mu}_{t_{2\alpha}}^n, \chi \rangle + \langle \bar{\mu}_{t_{2\alpha}}, \chi \rangle. \end{aligned}$$

Therefore, from Gronwall's Lemma we conclude that for all $n \in \mathbb{N}^*$ and all $t \geq t_{2\alpha}$,

$$|\bar{\mu}_t^n(0) - \bar{\mu}_t(0)| \leq \left(|\bar{\mu}_{t_{2\alpha}}^n(0) - \bar{\mu}_{t_{2\alpha}}(0)| + |\bar{M}_t^{n,\alpha}| + (1 + \lambda(t - t_{2\alpha})) \langle \bar{\mu}_{t_{2\alpha}}^n, \chi \rangle + 2\alpha \right) e^{\lambda(t - t_{2\alpha})}. \quad (103)$$

Proof of Theorem 4.1. We are now in position to prove Theorem 4.1. Fix $T > 0$. First consider a function $\phi \in \mathcal{B}_b$ such that $\phi(0) \neq 0$. Let $\varepsilon > 0$. We can chose α small enough in (39), and small enough positive numbers δ , η and ξ so that

$$\left(\delta + \eta + (1 + \lambda(T - t_{2\alpha})^+) (\alpha + \xi) + 2\alpha \right) e^{\lambda(T - t_{2\alpha})^+} < \frac{\varepsilon}{2 \|\phi(0)\|}; \quad (104)$$

$$4\alpha + \xi < \frac{\varepsilon}{2 \|\phi\|}. \quad (105)$$

First, if $T \leq t_{2\alpha}$, Proposition 5.5 trivially implies that

$$\mathbf{P} \left[\sup_{t \in [0, T]} |\langle \bar{\mu}_t^n, \phi \rangle - \langle \bar{\mu}_t, \phi \rangle| > \varepsilon \right] \xrightarrow{n \rightarrow \infty} 0.$$

If $T > t_{2\alpha}$, define the following events for all $n \in \mathbb{N}^*$,

$$\begin{aligned} \Omega_{\alpha, \delta}^n &= \left\{ \sup_{t \in (0, t_{2\alpha}]} |\bar{\mu}_t^n(0) - \bar{\mu}_t(0)| \leq \delta \right\}; \\ \Omega_{\alpha, \eta}^n &= \left\{ \sup_{t \in (t_{2\alpha}, T]} |\bar{M}_t^{n,\alpha}| \leq \eta \right\}; \\ \Omega_{\alpha, \xi}^n &= \left\{ \sup_{t \in (0, t_{2\alpha}]} |\langle \bar{\mu}_t^n, \chi \rangle - \langle \bar{\mu}_t, \chi \rangle| \leq \xi \right\}, \end{aligned}$$

where $\bar{M}^{n,\alpha}$ is the martingale defined by (99). We have for all n ,

$$\begin{aligned}
& \mathbf{P} \left[\sup_{t \in (t_{2\alpha}, T]} |\langle \bar{\mu}_t^n, \phi \rangle - \langle \bar{\mu}_t, \phi \rangle| > \varepsilon \right] \\
& \leq \mathbf{P} \left[\sup_{t \in (t_{2\alpha}, T]} |\phi(0)| |\bar{\mu}_t^n(0) - \bar{\mu}_t(0)| > \frac{\varepsilon}{2} \right] + \mathbf{P} \left[\sup_{t \in (t_{2\alpha}, T]} \|\phi\| |\langle \bar{\mu}_t^n, \mathbb{1}_{\mathbb{N}^*} \rangle - \langle \bar{\mu}_t, \mathbb{1}_{\mathbb{N}^*} \rangle| > \frac{\varepsilon}{2} \right] \\
& \leq \mathbf{P} \left[\left\{ \sup_{t \in (t_{2\alpha}, T]} |\bar{\mu}_t^n(0) - \bar{\mu}_t(0)| > \frac{\varepsilon}{2 \|\phi(0)\|} \right\} \cap \Omega_{\alpha, \delta}^n \cap \Omega_{\alpha, \eta}^n \cap \Omega_{\alpha, \xi}^n \right] \\
& \quad + \mathbf{P} [(\Omega_{\alpha, \delta}^n)^c] + \mathbf{P} [(\Omega_{\alpha, \eta}^n)^c] + 2\mathbf{P} [(\Omega_{\alpha, \xi}^n)^c] \\
& \quad + \mathbf{P} \left[\left\{ \sup_{t \in (t_{2\alpha}, T]} |\langle \bar{\mu}_t^n, \mathbb{1}_{\mathbb{N}^*} \rangle - \langle \bar{\mu}_t, \mathbb{1}_{\mathbb{N}^*} \rangle| > \frac{\varepsilon}{2 \|\phi\|} \right\} \cap \Omega_{\alpha, \xi}^n \right]. \quad (106)
\end{aligned}$$

Clearly, from (103) and (104), for all $n \in \mathbb{N}^*$,

$$\mathbf{P} \left[\left\{ \sup_{t \in (t_{2\alpha}, T]} |\bar{\mu}_t^n(0) - \bar{\mu}_t(0)| > \frac{\varepsilon}{2 \|\phi(0)\|} \right\} \cap \Omega_{\alpha, \delta}^n \cap \Omega_{\alpha, \eta}^n \cap \Omega_{\alpha, \xi}^n \right] = 0. \quad (107)$$

On another hand, applying (101) and Proposition 5.5 respectively to $\phi = \mathbb{1}_0$ and $\phi = \chi$ yields that

$$\mathbf{P} [(\Omega_{\alpha, \delta}^n)^c] + \mathbf{P} [(\Omega_{\alpha, \eta}^n)^c] + 2\mathbf{P} [(\Omega_{\alpha, \xi}^n)^c] \xrightarrow{n \rightarrow \infty} 0. \quad (108)$$

Finally, notice that for all $t \geq t_{2\alpha}$ and for all n ,

$$|\langle \bar{\mu}_t^n, \mathbb{1}_{\mathbb{N}^*} \rangle - \langle \bar{\mu}_t, \mathbb{1}_{\mathbb{N}^*} \rangle| \leq \langle \bar{\mu}_{t_{2\alpha}}^n, \mathbb{1}_{\mathbb{N}^*} \rangle + \langle \bar{\mu}_{t_{2\alpha}}, \mathbb{1}_{\mathbb{N}^*} \rangle \leq \langle \bar{\mu}_{t_{2\alpha}}^n, \chi \rangle + \langle \bar{\mu}_{t_{2\alpha}}, \chi \rangle \leq \langle \bar{\mu}_{t_{2\alpha}}^n, \chi \rangle + 2\alpha,$$

and therefore with (105),

$$\mathbf{P} \left[\left\{ \sup_{t \in (t_{2\alpha}, T]} |\langle \bar{\mu}_t^n, \mathbb{1}_{\mathbb{N}^*} \rangle - \langle \bar{\mu}_t, \mathbb{1}_{\mathbb{N}^*} \rangle| > \frac{\varepsilon}{2 \|\phi\|} \right\} \cap \Omega_{\alpha, \xi}^n \right] = 0.$$

This together with (107) and (108) in (106), concludes the proof for all $\phi \in \mathcal{B}_b$ such that $\phi(0) \neq 0$.

Only the case where $\phi(0) = 0$ remains to be treated. For this, we fix $\varepsilon > 0$ and let $\alpha > 0$ small enough in (39), and $\xi > 0$ small enough so that

$$4\alpha + \xi < \varepsilon.$$

Here again, if $T \leq t_{2\alpha}$ then Proposition 5.5 implies that

$$\mathbf{P} \left[\sup_{t \in [0, T]} |\langle \bar{\mu}_t^n, \phi \rangle - \langle \bar{\mu}_t, \phi \rangle| > \varepsilon \right] \xrightarrow{n \rightarrow \infty} 0.$$

If $T > t_{2\alpha}$, just write that for all n ,

$$\begin{aligned}
& \mathbf{P} \left[\sup_{t \in (t_{2\alpha}, T]} |\langle \bar{\mu}_t^n, \phi \rangle - \langle \bar{\mu}_t, \phi \rangle| > \varepsilon \right] \\
& \leq \mathbf{P} \left[\left\{ \sup_{t \in (t_{2\alpha}, T]} |\langle \bar{\mu}_t^n, \phi \rangle - \langle \bar{\mu}_t, \phi \rangle| > \varepsilon \right\} \cap \Omega_{\alpha, \xi}^n \right] + \mathbf{P} [(\Omega_{\alpha, \xi}^n)^c]. \quad (109)
\end{aligned}$$

But then, we have for all $t \geq t_{2\alpha}$ that

$$|\langle \bar{\mu}_t^n, \phi \rangle - \langle \bar{\mu}_t, \phi \rangle| \leq \| \phi \| \left(\langle \bar{\mu}_{t_{2\alpha}}^n, \chi \rangle + 2\alpha \right),$$

and hence

$$\mathbf{P} \left[\left\{ \sup_{t \in (t_{2\alpha}, T]} |\langle \bar{\mu}_t^n, \phi \rangle - \langle \bar{\mu}_t, \phi \rangle| > \varepsilon \right\} \cap \Omega_{\alpha, \xi}^n \right] = 0$$

which, together with (108) in (109), concludes the proof. \square

5.5. Proof of Corollary 4.4

We conclude with the proof of Corollary 4.4. Recall the definitions (32) and (33). Let for all $t \geq 0$,

$$c_\zeta^t = \lambda \int_0^t \langle \bar{\mu}_s, \mathbf{1} \rangle ds.$$

Let $\varepsilon > 0$. Using simple manipulations of the limiting differential system, we have that for all $t \geq 0$, $\langle \bar{\mu}_t, \mathbf{1} \rangle \leq \exp(-\lambda t)$. Similarly, applying (26) to $\phi \equiv \mathbf{1}$ and taking expectations yields that for any $t \geq 0$ and $n \in \mathbb{N}^*$,

$$\frac{d}{dt} \mathbf{E} [\langle \mu_t^n, \mathbf{1} \rangle] \leq -\lambda \mathbf{E} [\langle \mu_t^n, \mathbf{1} \rangle].$$

Consequently, there exists $S > 0$ such that for all $n \in \mathbb{N}^*$,

$$\max \left\{ \int_S^\infty \langle \bar{\mu}_t, \mathbf{1} \rangle dt; \int_S^\infty \mathbf{E} [\langle \bar{\mu}_t^n, \mathbf{1} \rangle] dt \right\} \leq \frac{\varepsilon}{4\lambda}. \quad (110)$$

Observe now that for all n , (μ^n, J^n) is a Markov jump process on $\mathcal{M}_F(\mathbb{N}) \times \mathbb{N}$, whose infinitesimal generator can be readily deduced from (25). Applying Dynkin's lemma to the test function

$$F : \begin{cases} \mathcal{M}_F(\mathbb{N}) \times \mathbb{R} & \rightarrow \mathbb{R} \\ (\mu, x) & \mapsto x \end{cases}$$

clearly entails that for all $n \in \mathbb{N}^*$ and $t \geq 0$,

$$\frac{J_t^n}{n} = \lambda \int_0^t \langle \bar{\mu}_s^n, \mathbf{1} \rangle ds + \frac{N_t^n}{n},$$

where N^n is a \mathcal{F}_t^n -martingale such that, uniformly over compact time sets

$$\frac{N^n}{n} \xrightarrow[n \rightarrow \infty]{} \mathbf{0} \text{ in } L^2,$$

as can be proven using Doob's inequality. We obtain that for n large enough,

$$\begin{aligned} \mathbf{E} [|\bar{J}^n - c_\zeta|] &\leq \mathbf{E} \left[\left| \frac{J_S^n}{n} - c_\zeta^S \right| \right] + \lambda \int_S^\infty \langle \bar{\mu}_t, \mathbf{1} \rangle dt + \lambda \mathbf{E} \left[\int_S^\infty \langle \bar{\mu}_t^n, \mathbf{1} \rangle dt \right] \\ &\leq \mathbf{E} \left[\left| \lambda \int_0^S \langle \bar{\mu}_t^n, \mathbf{1} \rangle dt - c_\zeta^S \right| \right] + \left(\mathbf{E} \left[\left(\frac{N_S^n}{n} \right)^2 \right] \right)^{1/2} + \varepsilon/2 \\ &\leq \varepsilon. \end{aligned}$$

In the latter, the second inequality follows from (110), the third one from the martingale convergence and from Theorem 4.1 applied to $\phi \equiv \mathbf{1}$, and by observing that the family of r.v.'s $\left\{ \int_0^S \langle \bar{\mu}_t^n, \mathbf{1} \rangle dt; n \in \mathbb{N}^* \right\}$ is bounded by S , and hence uniformly integrable. \square

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Appendix A. Combinatorial results

Let us fix the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We first introduce the definition, and several basic properties which are used in section 5.1, for the so called *hypergeometrical distribution*:

Definition Appendix A.1. Let n, N and p be three integers such that $p \geq 1$ and $N \geq 1$. Let $P := (P(1), \dots, P(p)) \in \mathbb{N}^p$ such that $\sum_{i=1}^p P(i) = N$. We say that the measure-valued random variable $Y \in \mathcal{M}_F(\mathbb{N})$ follows a multivariate hypergeometrical distribution of parameters (n, N, p, P) if for all $y \in \mathcal{M}_F(\mathbb{N})$ of support in $\llbracket 1, p \rrbracket$ such that $y(i) \leq P(i)$ for all i and $\sum_{i=1}^p y(i) = n$,

$$\mathbf{P}[Y = y] = \frac{\prod_{i=1}^p \binom{P(i)}{y(i)}}{\binom{N}{n}}.$$

The following main characteristics are well-known and easily calculated:

$$\mathbf{E}[Y(i)] = \frac{nP(i)}{N}, \quad i \in \mathbb{N} \cap [1, n-1] \quad (\text{A.1})$$

$$\begin{aligned} \mathbf{Cov}(Y(i), Y(j)) &= \frac{nP(i)P(j)}{N^2} \frac{N-n}{N-1}, \quad i, j \in \mathbb{N} \cap [1, n-1], \\ \mathbf{E}[Y(i)^3] &= n(n-1)(n-2) \frac{P(i)^3}{N^3} + 3n(n-1) \frac{P(i)^2}{N^2} + \frac{nP(i)}{N}, \quad i \in \mathbb{N} \cap [1, n-1]. \end{aligned} \quad (\text{A.2})$$

In particular, we readily deduce from the latter that

$$\begin{aligned} \mathbf{E}[\langle Y, \chi \rangle^2] &= \sum_i \sum_j ij \mathbf{E}[y(i)y(j)] \\ &= \sum_i \sum_j ij \left\{ \mathbf{Cov}(y(i), y(j)) - \mathbf{E}[y(i)] \mathbf{E}[y(j)] \right\} \\ &= \sum_i \sum_j ij \left\{ \frac{nP(i)P(j)}{N^2} \frac{N-n}{N-1} - \frac{n^2 P(i)P(j)}{N^2} \right\} \\ &= \sum_i \sum_j ij \frac{n(n-1)P(i)P(j)}{N(N-1)}. \end{aligned} \quad (\text{A.3})$$

On another hand, as there are at most p integers i such that $y(i) > 0$, a simple computation gives

$$\mathbf{E}[\langle Y, \chi \rangle^3] = \sum_{(i,j,\ell) \in [1, n-1] \cap \mathbb{N}} ij\ell \mathbf{E}[y(i)y(j)y(\ell)] \leq 3n^2 \sum_i i^3 \mathbf{E}[Y(i)^3].$$

So with (A.2), we obtain

$$\mathbf{E}[\langle Y, \chi \rangle^3] \leq \frac{3n^5}{N^3} \sum_{i \in [1, n-1] \cap \mathbb{N}} i^3 P(i)^3 + \frac{3n^4}{N^2} \sum_{i \in [1, n-1]} i^3 P(i)^2 + \frac{3n^3}{N} \sum_{i \in [1, n-1]} i^3 P(i). \quad (\text{A.4})$$