

THE MINIMAL ANGLE CONDITION FOR QUADRILATERAL FINITE ELEMENTS OF ARBITRARY DEGREE

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ABSTRACT. We study $W^{1,p}$ Lagrange interpolation error estimates for general *quadrilateral* \mathcal{Q}_k finite elements with $k \geq 2$. For the most standard case of $p = 2$ it turns out that the constant C involved in the error estimate can be bounded in terms of the *minimal interior angle* of the quadrilateral. Moreover, the same holds for any p in the range $1 \leq p < 3$. On the other hand, for $3 \leq p$ we show that C also depends on the *maximal interior angle*. We provide some counterexamples showing that our results are sharp.

1. INTRODUCTION

This paper deals with error estimates in the $W^{1,p}$ norm for the \mathcal{Q}_k Lagrange interpolation on a general convex quadrilateral $K \subset \mathbb{R}^2$. Denoting the interpolant with Q_k the standard error estimate is usually found in the form

$$\|u - Q_k u\|_{0,p,K} + h |u - Q_k u|_{1,p,K} \leq C h^{k+1} |u|_{k+1,p,K}, \quad (1.1)$$

being h the diameter of K . Inequality (1.1) involves the L^p error estimate

$$\|u - Q_k u\|_{0,p,K} \leq C h^{k+1} |u|_{k+1,p,K}, \quad (1.2)$$

and the seminorm estimate

$$|u - Q_k u|_{1,p,K} \leq C h^k |u|_{k+1,p,K}. \quad (1.3)$$

A central matter of (1.1) concerns the dependence of C on basic geometric quantities of the underlying element K . It is known that the constant C in (1.2) remains uniformly bounded for arbitrary convex quadrilaterals (see Theorem 6.1). However this statement is false for the constant C in (1.3) (see for instance the counterexamples in the last section). The primary goal of this paper is to study the dependence of C in (1.3) on the *interior angles* of K . Although the role of the interior angles have been related to C in many previous works, none of them, to the best of the authors knowledge, have given a result as plain as the one offered in this paper. For instance, bounding the minimal and the maximal inner angle is considered a central matter in mesh generation algorithms since the early work by Ciarlet and Raviart [8], however no proof of sufficiency has been given as far (at least for an arbitrary degree of interpolation).

In order to present our results let us first introduce the following classical definition that we write for both triangles and quadrilaterals for further convenience.

Definition 1.1. *Let K (resp. T) be a convex quadrilateral (resp. a triangle). We say that K (resp. T) satisfies the minimum angle condition with constant $\psi_m \in \mathbb{R}$, or shortly $\text{mac}(\psi_m)$, if for any internal angle θ of K (resp. T) $0 < \psi_m \leq \theta$.*

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Our first result says that the constant in (1.3), for a fixed degree k and a fixed value of p with $1 \leq p < 3$, can be written as $C = C(\psi_m)$. As a consequence, the same can be stated about the constant in (1.1). This seems to be the most general result available for quadrilaterals in the case $k \geq 2$. In spite of the fact that much weaker geometrical conditions are known to be sufficient for the case $k = 1$, we show, by means of a counterexample, that they fail for a higher degree interpolation. This counterexample also warns that removing the minimal angle condition may indeed lead to a blowing-up constant in (1.3).

The *mac* is the most standard condition considered in textbooks for *triangular* finite elements. Actually, in that case, it is equivalent to the so called *regularity condition*, i.e. equivalent to the existence of a constant σ such that

$$h/\rho \leq \sigma, \quad (1.4)$$

where ρ denotes the diameter of the maximum circle contained in T . On the other hand, the term *anisotropic* or *narrow* is usually applied to elements that do not satisfy (1.4). Even when *triangles* can become narrow only if the minimal angle is approaching zero a very different situation occurs on quadrilaterals. Indeed, in that case the *mac* condition is less restrictive than (1.4) since arbitrarily narrow elements are allowed with a positive uniform bound for the minimal angle (for example, anisotropic rectangles always verify $mac(\pi/2)$). Anisotropic elements are important for instance in problems involving singular layers and the first works dealing with them arise during the seventies showing that (1.4) can be replaced (for triangles) by the weaker following condition

Definition 1.2. *Let K (resp. T) be a convex quadrilateral (resp. triangle), we say that K (resp. T) satisfies the maximum angle condition with constant $\psi_M \in \mathbb{R}$, or shortly $MAC(\psi_M)$, if for any internal angle θ of K (resp. T) $\theta \leq \psi_M < \pi$.*

Indeed, in [7, 9] it is proved that the *MAC* is sufficient to have optimal order error estimates for Lagrange interpolation on *triangles*. In the case of quadrilateral elements, (1.4) it is also a sufficient condition as it was shown by Jamet [10] for $k = 1$ and $p = 2$. This condition is less restrictive than that proposed in [8] where the authors require the existence of two positive constants μ_1, μ_2 such that

$$h/s \leq \mu_1 \quad (1.5)$$

where s is the length of the shortest side of K , and

$$|\cos(\theta)| \leq \mu_2 < 1 \quad (1.6)$$

for each inner angle θ of K . Observe that under the regularity condition (1.4) the quadrilateral can degenerate into a triangle (for instance if the shortest side tends to zero faster than their neighboring sides or if the maximum angle of the element approaches π), however this kind of quadrilateral cannot become too narrow. Condition (1.6) will play an important role in the sequel and therefore we introduce the following alternative definition.

Definition 1.3. *We say that a quadrilateral K satisfies the double angle condition with constants ψ_m, ψ_M , or shortly $DAC(\psi_m, \psi_M)$, if K verifies $mac(\psi_m)$ and $MAC(\psi_M)$ simultaneously, i.e., if all inner angles θ of K verify $0 < \psi_m \leq \theta \leq \psi_M < \pi$.*

The *DAC* allows anisotropic quadrilaterals (such as narrow rectangles) as well as families of quadrilaterals that may degenerate into triangles. To see that consider, for instance, a quadrilateral with vertices $(0, 0), (1, 0), (s, 1 - s)$ and $(0, 1 - s)$ and take $0 < s \rightarrow 0$.

For anisotropic quadrilaterals several papers have been written mainly in the *isoparametric* case with $k = 1$. In [13, 14] narrow quadrilaterals are studied and the authors require the two longest sides of the element to be opposite and almost parallel, the constant C obtained by them depends on an angle which *in some cases* is the minimum angle of the element. *Anisotropic*

error estimates for small perturbations of rectangles have been derived in [5, 6]. On the other hand, for $k = 1$, more general and subtle conditions can be found in the literature. For $k = 1$ and $p = 2$, it is proved in [2] that the optimal error estimate (1.3) can be obtained under the following weak condition

Definition 1.4. *Let K be a convex quadrilateral, and let d_1 and d_2 be the diagonals of K . We say that K satisfies the regular decomposition property with constants $N \in \mathbb{R}$ and $0 < \psi_M < \pi$, or shortly $RDP(N, \psi_M)$, if we can divide K into two triangles along one of its diagonals, that will be called always d_1 , in such a way that $|d_2|/|d_1| \leq N$ and both triangles have its maximum angle bounded by ψ_M .*

In Remarks 2.3 - 2.7 of [2] it is shown that the regular decomposition property RDP is certainly much weaker than those considered in previous works (including [5, 6, 8, 10, 13, 14]). We collect for further reference some elementary remarks

Remark 1.1. *If a quadrilateral K satisfies (1.4) then K verifies $RDP(\sigma, \psi)$ with $\psi = \psi(\sigma)$. Indeed, (1.4) implies that K verifies $mac(\theta)$ for some $\theta = \theta(\sigma) > 0$. Therefore there is at most one angle of K not bounded by $\pi - \theta$. Dividing K by the diagonal d_1 containing the vertex associated to that angle we get that K satisfies $RDP(\sigma, \pi - \theta)$.*

Remark 1.2. *If a quadrilateral K satisfies $MAC(\psi_M)$ then K verifies $RDP(1, \psi_M)$, as one can see by taking d_1 as the longest diagonal of K .*

Since, by definition, $DAC(\psi_m, \psi_M)$ implies $MAC(\psi_M)$ we have

Remark 1.3. *If a quadrilateral K satisfies $DAC(\psi_m, \psi_M)$ then K verifies $RDP(1, \psi_M)$.*

Remark 1.4. *If K verifies the $mac(\psi_m)$, then K either verifies*

- (1) $DAC(\psi_m, \pi - \frac{\psi_m}{2})$ or
- (2) the regularity condition (1.4) with $C = C(\psi_m)$.

Indeed, assume that (1) does not hold. Then K has an internal angle which is greater than $\pi - \psi_m/2$, it is easy to see that this angle is unique so we can call it θ . Divide K into two triangles T_1 and T_2 through the diagonal opposite to θ in such a way that θ becomes an internal angle of T_1 . Calling β_1 and β_2 to the other angles of T_1 it follows that $\beta_i < \psi_m/2$ with $i = 1, 2$. Let γ_i , $i = 1, 2$, the complementary angle of β_i (w.r.t. the corresponding internal angle of K). It is easy to see that $\gamma_i > \psi_m/2$, so T_2 is a triangle that have its three internal angles bounded away from 0. To be more precise, T_2 verifies $mac(\psi_m/2)$, therefore T_2 is a regular triangle in the sense of (1.4) with $C = C(\psi_m/2)$. From this fact follows easily that K is a regular quadrilateral in the same sense, i.e., in such a way that (1.4) holds (actually the same constant C can be used). \square

Remark 1.5. *Combining Remarks 1.1, 1.3 and 1.4 it is clear that $mac \implies RDP$.*

An strikingly result is that the RDP , in spite of being appropriate for $p = 2$, does not work for arbitrary values of p . Indeed in [3] and for $k = 1$ the results of [2] are extended for the error in $W^{1,p}$ with $1 \leq p < 3$. Moreover, it is shown, by means of a counterexample, that this range for p is sharp. As a consequence the stronger DAC (i.e. (1.6)) is proposed and shown that under this condition the error estimate hold for all $p \geq 3$. The second result given in the present paper is that, for p in this range, the DAC is also a sufficient for any $k \geq 2$.

For the reader's convenience we summarize in the following table simple and sufficient geometric conditions pointing out the role of p and k

	$1 \leq p < 3$	$3 \leq p$
$k = 1$	RDP	DAC
$k \geq 2$	mac	DAC

the first row of the table was proved in [3] while the new results are given in the second row.

To finish this short review we recall that for $k = 1$ more results are available. In [11], H^1 error estimates are obtained for the \mathcal{Q}_1 isoparametric Lagrange interpolation under a weaker condition than the RDP . This condition can be regarded as a generalization of the last one and therefore called $GRDP$.

This paper is structured as follows: in Section 2 we introduce a family of reference elements appropriate for dealing with general convex quadrilaterals and some key results are provided. In Section 3 and Section 4 our family is related to different geometric conditions (such as RDP , MAC , DAC and mac) while some properties about the distribution of the interpolation nodes of the family are studied. Section 5 gives the general approach for bounding the interpolation error. Finally the main results as well as the counterexamples can be found in the last section of the paper.

2. THE FAMILY OF REFERENCE $K(a, b, \tilde{a}, \tilde{b})$

With K we denote an arbitrary *convex* quadrilateral with vertices V_1, V_2, V_3, V_4 enumerated in counterclockwise order. For positive numbers $a, b, \tilde{a}, \tilde{b}$, we use $K(a, b, \tilde{a}, \tilde{b})$ to represent a quadrilateral (always convex) with vertices $V_1 = (0, 0)$, $V_2 = (a, 0)$, $V_3 = (\tilde{a}, \tilde{b})$ and $V_4 = (0, b)$. In particular, $\hat{K} = K(1, 1, 1, 1)$ is the reference unit square and for any positive integer k we consider $(k+1)^2$ points $\{\widehat{M}_{ij}\}_{0 \leq i, j \leq k}$ of coordinates $\hat{x} = j/k$ and $\hat{y} = i/k$. For \hat{K} we write $\hat{V}_1 = \widehat{M}_{00}$, $\hat{V}_2 = \widehat{M}_{0k}$, $\hat{V}_3 = \widehat{M}_{kk}$ and $\hat{V}_4 = \widehat{M}_{k0}$.

We define as usual $F_K : \hat{K} \rightarrow K$ as $F_K(\hat{x}) = \sum_{i=1}^4 V_i \widehat{\phi}_i(\hat{x})$ being $\widehat{\phi}_i$ the bilinear basis function associated with the vertex \hat{V}_i , i.e., $\widehat{\phi}_i(\hat{V}_j) = \delta_i^j$. Then, in K we have $(k+1)^2$ points $\{M_{ij}\}_{0 \leq i, j \leq k}$ defined by

$$M_{ij} = F_K(\widehat{M}_{ij}).$$

For quadrilateral elements (isoparametric when $k = 1$ and subparametric otherwise), we have the basis function on K defined by $\phi_{ij}(X) = \widehat{\phi}_{ij}(F_K^{-1}(X))$ where $\widehat{\phi}_{ij} \in \widehat{\mathcal{Q}}_k(\hat{K})$ verifies $\widehat{\phi}_{ij}(\widehat{M}_{lr}) = \delta_{ij}^{lr}$ and therefore the \mathcal{Q}_k interpolation operator Q_k , on K is given by

$$Q_k u(X) = \widehat{Q}_k \widehat{u}(\widehat{X})$$

where $X = F_K(\widehat{X})$ and \widehat{Q}_k is the Lagrange interpolation of order k of $\widehat{u} = u \circ F_K$ on \hat{K} . Interpolation nodes of the form $\{M_{ij}\}_{1 \leq i, j \leq k-1}$ are called *interior* and any M_{ij} which is not an interior node is called an *edge* node. Also of interest is the triangle $T(a, b)$ of vertices $V_1^T = (0, 0)$, $V_2^T = (a, 0)$, $V_3^T = (0, b)$. The interpolation nodes M_{ij}^T of the Lagrange interpolation operator $\Pi_k \in \mathbb{P}_k$ of degree k are given by $M_{ij}^T = (aj/k, bi/k)$, $0 \leq i + j \leq k$. With C we denote a positive constant that may change from line to line. Sometimes we also use the notation $x \approx y$ for positive variables if they are comparable in the following sense $\frac{1}{C}x \leq y \leq Cx$.

For any element of the type $K(a, b, \tilde{a}, \tilde{b})$ considered in this work the following condition will become relevant in different contexts

$$\text{Condition } (\Delta 1): \quad \frac{\tilde{a}}{a}, \frac{\tilde{b}}{b} \leq C. \quad (2.1)$$

This condition takes sometimes the more restrictive form

$$\text{Condition } (D1): \quad \frac{\tilde{a}}{a}, \frac{\tilde{b}}{b} \leq 1. \quad (2.2)$$

In spite of the fact that both (2.1) and (2.2) look similar they characterize, under the supplementary geometric assumption (2.3) (see below), different classes of elements.

Calling d_1 to the diagonal joining V_2 and V_4 we see that d_1 divides K into two triangles, that we call T_1 and T_2 (see Figure 1). For the angle α of T_1 placed at V_4 , we introduce

$$\text{Condition (D2): } \frac{1}{\sin(\alpha)} \leq C, \quad (2.3)$$

which says that α is bounded away from 0 and π .

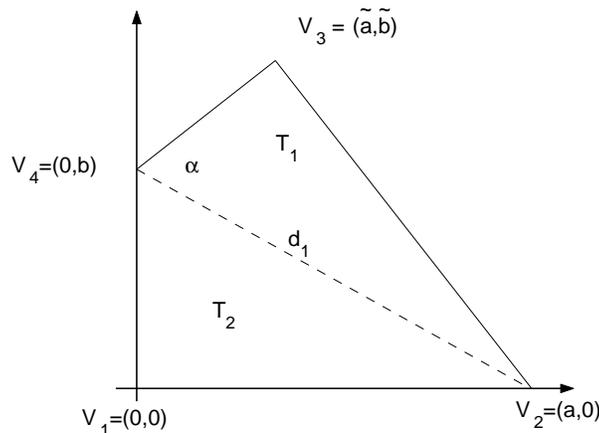


FIGURE 1. Notation for an element $K(a, b, \tilde{a}, \tilde{b})$.

Finally, in order to exploit some results given in previous works, we introduce yet another useful condition

$$\text{Condition } (\Delta 2): \quad |l| \leq C|s|. \quad (2.4)$$

where l is the segment $\overline{V_3V_4}$ joining V_3 and V_4 and s denotes the shortest side of $K(a, b, \tilde{a}, \tilde{b})$. That is, $(\Delta 2)$ amounts to say that the side l is comparable to the shortest side of K .

Not difficult to prove is the following

Lemma 2.1. *Let $K(a, b, \tilde{a}, \tilde{b})$ a general convex quadrilateral. Then, conditions $[\Delta 1, D2]$ and $[\Delta 2, D2]$ are equivalents.*

Proof. That $(\Delta 2)$ implies $(\Delta 1)$ follows easily as we have $\tilde{a} \leq |l| \leq C|s| \leq Ca$ and similarly $|b - \tilde{b}| \leq Cb$. Hence triangular inequality yields $\tilde{b} \leq Cb$. On the other hand assume $[\Delta 1, D2]$. Thanks to $(D2)$, and using the law of sines for the triangle $\Delta(V_2, V_3, V_4)$ we see that the angle β at V_3 is away from zero and π . Indeed $\frac{\sin \alpha}{\sin \beta} = \frac{|V_2V_3|}{|d_1|} \leq C$ due to $(\Delta 1)$. Since both α and β are away from zero and π the law of sines says that actually d_1 is comparable to $|V_2V_3|$. It implies, in turn, that $|l| \leq C \min\{|V_2V_3|, |d_1|\}$. Now consider two cases: if $\frac{b}{a}$ approaches zero then necessarily $\frac{|b-\tilde{b}|}{\tilde{a}} > C > 0$, otherwise the angle α can not obey $(D2)$. Hence we have simultaneously, $b < a$ and (due to $(\Delta 1)$) $|l| = \sqrt{(b - \tilde{b})^2 + \tilde{a}^2} \leq C|b - \tilde{b}| \leq Cb$ showing $(\Delta 2)$. To finish, let us assume that $\frac{b}{a} > C > 0$, in such a case if a and b are comparable we have nothing to prove since then they are also comparable to d_1 . Therefore we may assume that $\frac{a}{b}$ approaches zero. In that case $\frac{|b-\tilde{b}|}{\tilde{a}} \leq C$ since α can not get close to π nor to zero. Therefore we have $a < b$ and $|l| = \sqrt{(b - \tilde{b})^2 + \tilde{a}^2} \leq C\tilde{a} \leq Ca$ thanks to $(D1)$. The lemma follows. \square

The class of reference elements of the type $K(a, b, \tilde{a}, \tilde{b})$ is adequate for dealing with several geometrical conditions. For instance, as we show below, $[\Delta 1, D2]$ (or equivalently $[\Delta 2, D2]$) describe any element under the RDP and $[D1, D2]$ the family of elements under the DAC , etc.

In order to do so, the key tool is given by the following elementary lemma

Lemma 2.2. *Let K, \bar{K} be two convex quadrilateral elements, and let $L : K \rightarrow \bar{K}$ be an affine transformation $L(X) = BX + P$. Assume that $L(K) = \bar{K}$, $\|B\|, \|B^{-1}\| < C$ (in particular the condition number $\kappa(B) < C$). If \bar{Q}_k is the Q_k (isoparametric for $k = 1$ or subparametric for $k > 1$) interpolation on \bar{K} and $\bar{u} = u \circ L^{-1}$ then for any $p \geq 1$*

$$C_1 |\bar{u} - \bar{Q}_k \bar{u}|_{1,p,\bar{K}} \leq |u - Q_k u|_{1,p,K} \leq C_2 |\bar{u} - \bar{Q}_k \bar{u}|_{1,p,\bar{K}}$$

and

$$C_1 |\bar{u}|_{m,p,\bar{K}} \leq |u|_{m,p,K} \leq C_2 |\bar{u}|_{m,p,\bar{K}}$$

for any $m \geq 1$.

Proof. By definition of the interpolation we have

$$Q_k u(X) = \hat{Q}_k \hat{u}(F_K^{-1}(X)) \quad \text{and} \quad \bar{Q}_k \bar{u}(\bar{X}) = \hat{Q}_k \hat{u}(F_{\bar{K}}^{-1}(\bar{X})).$$

Where \bar{X} denotes the variable on \bar{K} . Since L is an affine transformation, $F_{\bar{K}} = L \circ F_K$ and so $\bar{Q}_k \bar{u}(\bar{X}) = Q_k u(X)$. Then, the lemma follows easily by observing that $\|B\|, \|B^{-1}\| < C$. \square

Definition 2.1. *We say that two quadrilateral elements K, \bar{K} are C -equivalents (or simply equivalents) if and only if they can be mapped to each other by means of an affine transformation of the kind given in Lemma 2.2.*

Taking into account that each geometric condition defined as far is going to be mapped to an appropriate class of equivalent elements $K(a, b, \tilde{a}, \tilde{b})$ it is important to consider the map $F_K : \hat{K} \rightarrow K(a, b, \tilde{a}, \tilde{b})$,

$$F_K(\hat{x}, \hat{y}) = (a\hat{x}(1 - \hat{y}) + \tilde{a}\hat{x}\hat{y}, b\hat{y}(1 - \hat{x}) + \tilde{b}\hat{x}\hat{y}) = (x, y), \quad (2.5)$$

as well as its associated Jacobian

$$DF_K(\hat{x}, \hat{y}) = \begin{pmatrix} a + \hat{y}(\tilde{a} - a) & \hat{x}(\tilde{a} - a) \\ \hat{y}(\tilde{b} - b) & b + \hat{x}(\tilde{b} - b) \end{pmatrix}, \quad (2.6)$$

$$J_K := \det(DF_K)(\hat{x}, \hat{y}) = ab(1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1)). \quad (2.7)$$

Observe that since K is convex, we have $J_K > 0$ in the interior of \hat{K} . Indeed, since J_K is an affine function it is enough to verify that it is positive at some vertex of \hat{K} and non negative at the remaining ones. The positivity at $\hat{V}_1 = (0, 0)$ is trivial, as well as the non negativity at $\hat{V}_2 = (a, 0)$ and $\hat{V}_4 = (0, b)$. On the other hand, since K is convex, (\tilde{a}, \tilde{b}) lies above the segment joining V_2 and V_4 (for this segment $y(x) = -(b/a)(x - a)$ and $0 < \frac{\tilde{b} - y(\tilde{a})}{b} = \frac{\tilde{b}}{b} + \frac{\tilde{a}}{a} - 1$) therefore,

$$J_K(1, 1) = ab(\tilde{a}/a + \tilde{b}/b - 1) > 0. \quad (2.8)$$

Following again [2, 3] we introduce for $p \geq 1$ the next expression that becomes useful in the sequel

$$I_p = I_p(a, b, \tilde{a}, \tilde{b}) := \int_0^1 \int_0^1 \frac{1}{(1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1))^{p-1}} d\hat{x}d\hat{y}, \quad (2.9)$$

where the numbers $a, b, \tilde{a}, \tilde{b}$ are compatible with an element $K(a, b, \tilde{a}, \tilde{b})$.

Lemma 2.3. *If $K = K(a, b, \tilde{a}, \tilde{b})$ is convex and $(\Delta 1)$ given by (2.1) holds, then for any $p \geq 1$ and for any function basis ϕ there exists a positive constant C such that*

$$\left\| \frac{\partial \phi}{\partial x} \right\|_{0,p,K}^p \leq C \frac{b}{a^{p-1}} I_p \quad (2.10)$$

$$\left\| \frac{\partial \phi}{\partial y} \right\|_{0,p,K}^p \leq C \frac{a}{b^{p-1}} I_p. \quad (2.11)$$

Proof. Let $\hat{\phi}$ the function basis on \hat{K} corresponding to ϕ , then (from the chain rule) follows that

$$\begin{pmatrix} \left(\frac{\partial \phi}{\partial x} \circ F_K \right) (\hat{x}, \hat{y}) \\ \left(\frac{\partial \phi}{\partial y} \circ F_K \right) (\hat{x}, \hat{y}) \end{pmatrix} = \frac{1}{J_K(\hat{x}, \hat{y})} \begin{pmatrix} b + \hat{x}(\tilde{b} - b) & -\hat{y}(\tilde{b} - b) \\ -\hat{x}(\tilde{a} - a) & a + \hat{y}(\tilde{a} - a) \end{pmatrix} \begin{pmatrix} \frac{\partial \hat{\phi}}{\partial \hat{x}}(\hat{x}, \hat{y}) \\ \frac{\partial \hat{\phi}}{\partial \hat{y}}(\hat{x}, \hat{y}) \end{pmatrix}$$

where $J_K(\hat{x}, \hat{y}) = \det(DF_K) = ab(1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1))$.

Calling $R(\hat{x}, \hat{y}) = (1 + \hat{x}(\tilde{b}/b - 1)) \frac{\partial \hat{\phi}}{\partial \hat{x}}(\hat{x}, \hat{y}) - \hat{y}(\tilde{b}/b - 1) \frac{\partial \hat{\phi}}{\partial \hat{y}}(\hat{x}, \hat{y})$ and $S(\hat{x}, \hat{y}) = -\hat{x}(\tilde{a}/a - 1) \frac{\partial \hat{\phi}}{\partial \hat{x}}(\hat{x}, \hat{y}) + (1 + \hat{y}(\tilde{a}/a - 1)) \frac{\partial \hat{\phi}}{\partial \hat{y}}(\hat{x}, \hat{y})$ we have

$$\left(\frac{\partial \phi}{\partial x} \circ F_K \right) (\hat{x}, \hat{y}) = \frac{b}{J_K(\hat{x}, \hat{y})} R(\hat{x}, \hat{y}) \quad \text{and} \quad \left(\frac{\partial \phi}{\partial y} \circ F_K \right) (\hat{x}, \hat{y}) = \frac{a}{J_K(\hat{x}, \hat{y})} S(\hat{x}, \hat{y}).$$

By changing variables we get

$$\left\| \frac{\partial \phi}{\partial x} \right\|_{0,p,K}^p = \frac{b}{a^{p-1}} \int_0^1 \int_0^1 \frac{|R(\hat{x}, \hat{y})|^p}{(1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1))^{p-1}} d\hat{x}d\hat{y}$$

and

$$\left\| \frac{\partial \phi}{\partial y} \right\|_{0,p,K}^p = \frac{a}{b^{p-1}} \int_0^1 \int_0^1 \frac{|S(\hat{x}, \hat{y})|^p}{(1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1))^{p-1}} d\hat{x}d\hat{y},$$

and the proof concludes using that R and S are uniformly bounded since they are polynomials, $0 \leq \hat{x}, \hat{y} \leq 1$ and $0 \leq \tilde{a}/a, \tilde{b}/b \leq C$ by $(\Delta 1)$. \square

Previous result provides bounds for any basis function. As we show later basis functions associated to *internal nodes* require a particular treatment. In particular we have,

Lemma 2.4. *If $K = K(a, b, \tilde{a}, \tilde{b})$ is convex and $(\Delta 1)$ given by (2.1) holds then for any $p \geq 1$ and for any function basis ϕ associated to an internal node of K , there exists a positive constant C such that*

$$\left\| \frac{\partial \phi}{\partial x} \right\|_{0,p,K}^p \leq C \frac{b}{a^{p-1}} \left[|1 - \tilde{b}/b|^p I_p + \max\{1, (b/\tilde{b})^{p-1}/2\} \right] \quad (2.12)$$

$$\left\| \frac{\partial \phi}{\partial y} \right\|_{0,p,K}^p \leq C \frac{a}{b^{p-1}} \left[(\tilde{a}/a)^p I_p + \max\{1, (b/\tilde{b})^{p-1}/2\} \right]. \quad (2.13)$$

Proof. Since ϕ is associated to an internal node on K , it follows that $\hat{\phi}$ is associated to an internal node on \hat{K} so that there exists $P \in \mathcal{Q}_{k-2}(\hat{K})$ such that $\hat{\phi}(\hat{x}, \hat{y}) = \hat{x}(1 - \hat{x})\hat{y}(1 - \hat{y})P(\hat{x}, \hat{y})$. Then

$$\frac{\partial \hat{\phi}}{\partial \hat{x}} = \hat{y}(1 - \hat{y})A \quad \text{and} \quad \frac{\partial \hat{\phi}}{\partial \hat{y}} = \hat{x}(1 - \hat{x})B$$

where $A(\hat{x}, \hat{y}) = \frac{\partial}{\partial \hat{x}} [\hat{x}(1 - \hat{x})P(\hat{x}, \hat{y})]$ and $B(\hat{x}, \hat{y}) = \frac{\partial}{\partial \hat{y}} [\hat{y}(1 - \hat{y})P(\hat{x}, \hat{y})]$.

From the chain rule follows that

$$\begin{pmatrix} \left(\frac{\partial \phi}{\partial x} \circ F_K \right) (\hat{x}, \hat{y}) \\ \left(\frac{\partial \phi}{\partial y} \circ F_K \right) (\hat{x}, \hat{y}) \end{pmatrix} = \frac{1}{J_K(\hat{x}, \hat{y})} \begin{pmatrix} b + \hat{x}(\tilde{b} - b) & -\hat{y}(\tilde{b} - b) \\ -\hat{x}(\tilde{a} - a) & a + \hat{y}(\tilde{a} - a) \end{pmatrix} \begin{pmatrix} \hat{y}(1 - \hat{y})A(\hat{x}, \hat{y}) \\ \hat{x}(1 - \hat{x})B(\hat{x}, \hat{y}) \end{pmatrix}$$

where $J_K(\hat{x}, \hat{y}) = \det(DF_K) = ab(1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1))$.
 Calling $S = \hat{x}\hat{y}[(1 - \hat{x})B - (1 - \hat{y})A]$ and $R = \hat{y}A$ we have

$$\left(\frac{\partial\phi}{\partial x} \circ F_K\right)(\hat{x}, \hat{y}) = \frac{b}{J_K} \left[(1 - \tilde{b}/b)S + (1 - \hat{y})R \right]$$

and by a change of variables we get

$$\left\| \frac{\partial\phi}{\partial x} \right\|_{0,p,K}^p = \int_0^1 \int_0^1 \frac{b}{a^{p-1}} \frac{|(1 - \tilde{b}/b)S + (1 - \hat{y})R|^p}{(1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1))^{p-1}} d\hat{x}d\hat{y}.$$

Using the fact that S and R are uniformly bounded we see that

$$\left\| \frac{\partial\phi}{\partial x} \right\|_{0,p,K}^p \leq C \frac{b}{a^{p-1}} \int_0^1 \int_0^1 \frac{|1 - \tilde{b}/b|^p + (1 - \hat{y})^p}{(1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1))^{p-1}} d\hat{x}d\hat{y}.$$

Then

$$\left\| \frac{\partial\phi}{\partial x} \right\|_{0,p,K}^p \leq C \frac{b}{a^{p-1}} \left[|1 - \tilde{b}/b|^p I_p + \int_0^1 \int_0^1 \frac{(1 - \hat{y})^p}{(1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1))^{p-1}} d\hat{x}d\hat{y} \right].$$

From the convexity of K we have $\tilde{a}/a + \tilde{b}/b - 1 > 0$, hence

$$1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1) > 1 + \hat{x}(\tilde{b}/b - 1) - \hat{y}\tilde{b}/b$$

Assume now that $\tilde{b}/b < 1$. Since $0 \leq \hat{x} \leq 1$ we conclude $1 + \hat{x}(\tilde{b}/b - 1) \geq \tilde{b}/b$ and finally

$$1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1) \geq \tilde{b}/b(1 - \hat{y}).$$

Therefore

$$\int_0^1 \int_0^1 \frac{(1 - \hat{y})^p}{(1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1))^{p-1}} d\hat{x}d\hat{y} \leq \frac{1}{2} \left(\frac{b}{\tilde{b}} \right)^{p-1}.$$

On the other hand, if $\tilde{b}/b \geq 1$

$$1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1) \geq 1 + \hat{y}(\tilde{a}/a - 1) \geq 1 - \hat{y}$$

hence

$$\int_0^1 \int_0^1 \frac{(1 - \hat{y})^p}{(1 + \hat{x}(\tilde{b}/b - 1) + \hat{y}(\tilde{a}/a - 1))^{p-1}} d\hat{x}d\hat{y} \leq 1,$$

and (2.12) follows.

Finally, the estimate for $\left\| \frac{\partial\phi}{\partial y} \right\|_{0,p,K}^p$ can be obtained in a similar way from the expression

$$\left(\frac{\partial\phi}{\partial y} \circ F_K\right)(\hat{x}, \hat{y}) = \frac{a}{J_K} [\tilde{a}/a S + (1 - \hat{y})\bar{R}]$$

where $\bar{R} = \hat{x}(1 - \hat{x})B + \hat{x}R$. \square

In order to clarify in advance the role of the term $\frac{b}{\tilde{b}}$ in the previous lemma let us notice the following

Lemma 2.5. *For any arbitrary convex quadrilateral $K(a, b, \tilde{a}, \tilde{b})$ under condition $[\Delta 1, D2]$ (equiv. $[\Delta 2, D2]$) there exists another equivalent element (in the sense of Definition 2.1) obeying $[\Delta 1, D2]$ (equiv. $[\Delta 2, D2]$, see Lemma 2.1) with the same constants and for which $\frac{\tilde{b}}{b} \geq \frac{1}{2}$.*

Proof. Consider the triangle $V_2V_3V_4$ and the angles α and β at V_4 and V_2 respectively. If for the original $K(a, b, \tilde{a}, \tilde{b})$, $\frac{\tilde{b}}{b} \geq \frac{1}{2}$, then we have nothing to prove. Otherwise $\frac{\tilde{b}}{b} < \frac{1}{2}$ and hence we see that $\alpha \leq \beta$. On the other hand, since both are interior angles of a triangle, $\beta \leq \pi - \alpha$, therefore using (D2) we see that β is away from 0 and π and therefore under a rigid movement we can transform our element into $K(b, a, \tilde{b}, \tilde{a})$. The resulting element satisfies the required conditions $[\Delta 1, D2]$ with the same constants than those $K(a, b, \tilde{a}, \tilde{b})$ and the lemma follows thanks to the fact that $\frac{\tilde{a}}{a} \geq \frac{1}{2}$ (see (2.8)). \square

Remark 2.1. Observe that previous lemma also applies to elements $K(a, b, \tilde{a}, \tilde{b})$ under $[D1, D2]$ since $D1 \implies \Delta 1$.

3. $K(a, b, \tilde{a}, \tilde{b})$ AND DIFFERENT GEOMETRIC CONDITIONS

In this section we explore in detail how to use the family $K(a, b, \tilde{a}, \tilde{b})$. The following lemma is useful in the rest of this section.

Lemma 3.1. Let L be the linear transformation associated with a matrix B . Given two vectors w_1 and w_2 , let α be the angle between them and let $\bar{\alpha}$ be the angle between $L(w_1)$ and $L(w_2)$. Calling $\kappa(B)$ the condition number of B we have

$$\frac{2}{\kappa(B)\pi}\alpha \leq \bar{\alpha} \leq \pi \left(1 - \frac{2}{\kappa(B)\pi}\right) + \alpha \frac{2}{\kappa(B)\pi}.$$

Proof. The proof is elementary and can be found in [1]. \square

3.1. The RDP and the family $K(a, b, \tilde{a}, \tilde{b})$. In order to characterize the elements under the RDP we begin with the following elementary result

Lemma 3.2. Let K be an element of the type $K(a, b, \tilde{a}, \tilde{b})$ and assume $[\Delta 1, D2]$ (equivalently $[\Delta 2, D2]$). Then K verifies the RDP with constants depending only on those given in conditions $[\Delta 1, D2]$.

Proof. Follows straightforwardly taking $d_1 = \overline{V_2V_4}$ as the dividing diagonal. \square

Now we are ready for the following characterization

Theorem 3.1. Let K be a general convex quadrilateral. Then K verifies the RDP if and only if K is equivalent to some $K(a, b, \tilde{a}, \tilde{b})$ under $[\Delta 2, D2]$ (equiv. $[\Delta 1, D2]$).

Proof. First we assume that K is equivalent to some $K(a, b, \tilde{a}, \tilde{b})$ under $[\Delta 2, D2]$. From Lemma 3.2 we know that $K(a, b, \tilde{a}, \tilde{b})$ verifies $RDP(N, \psi_M)$ with constants bounded in terms of those given in $[\Delta 2, D2]$. Since $K = L(K(a, b, \tilde{a}, \tilde{b}))$ for certain affine mapping $Lx = Bx + P$ of the type considered in the Definition 2.1 we see from Lemma 3.1 and taking into account that such an L preserves lengths (up to a constant depending on $\|B\|, \|B^{-1}\| < C$) that K verifies the RDP with constants depending on L as well as the RDP constants associated to $K(a, b, \tilde{a}, \tilde{b})$. To show the other implication we follow [2]. Assume that K satisfies the RDP and divide it along d_1 into T_1 and T_2 in such a way that all their interior angles are bounded by ψ_M , while $\frac{|d_2|}{|d_1|} \leq N$. We choose the notation in such a way that the shortest side of K , called s , is one of the sides of T_1 and call β the angle of T_2 opposite to d_1 . After a rigid movement we can assume that the vertex V_1 corresponding to β is placed at the origin and that K is contained in the upper half-plane. We can also assume that V_2 is placed at the point $(a, 0)$ with $a > 0$, being the side $\overline{V_1V_2}$ opposite to the shortest side of K . Define now $b = |\overline{V_1V_4}|\sin(\beta)$. Then, we have that V_4 is placed at $(\cot(\beta)b, b)$. Let us notice that β is away from π , since $\beta < \psi_M$. Moreover, it is also away from 0 as one can see by means of the law of sines and taking into

account that d_1 (the side of T_2 opposite to β) is comparable to the largest side of T_2 (due to the fact that $|d_2| \leq N|d_1|$ and recalling that the diameter of K agrees with the length of the longest diagonal). Then the linear mapping L associated to the matrix $B = \begin{pmatrix} 1 & \cot(\beta) \\ 0 & 1 \end{pmatrix}$ performs the

desired transformation $L(K(a, b, \tilde{a}, \tilde{b})) = K$ while it fulfills the requirements of Definition 2.1 as $\|B\|, \|B^{-1}\| < C$ (with C depending on ψ_M, N). In particular $\kappa(B) < \frac{2}{\sin^2(\beta)}$. On the other hand, calling $L(\tilde{a}, \tilde{b}) = V_3$ we observe that $(\Delta 2)$ holds since L preserves lengths (up to constants depending on $\|B\|, \|B^{-1}\|$ and $\overline{V_3 V_4} = L(s)$). On the other hand, since T_1 verifies $MAC(\psi_M)$ and s is the shortest side of T_1 then the angle of T_1 placed at the common vertex of d_1 and s is away from 0 and π . Therefore $(D2)$ holds thanks to Lemma 3.1. The theorem follows. \square

From now on (see Lemma 2.2) we assume that any element verifying the RDP is of the kind $K(a, b, \tilde{a}, \tilde{b})$ under $[\Delta 2, D2]$ (equiv. $[\Delta 1, D2]$). In [3] it is proved that the RDP is sufficient to get optimal order error estimates in $W^{1,p}$ for Q_1 whenever $1 \leq p < 3$. In the last section we give a counterexample showing in particular that this result does not hold for $k \geq 2$.

The next result, borrowed from [3], help us to shorten our exposition playing also a role in the construction of a counterexample.

Lemma 3.3. *Let $K = K(a, b, \tilde{a}, \tilde{b})$ a convex quadrilateral. Assume $[\Delta 2, D2]$ (equiv. $[\Delta 1, D2]$), then for any $1 \leq p < 3$*

$$\max \left\{ \frac{a}{b^{p-1}}, \frac{b}{a^{p-1}} \right\} I_p \leq C \frac{h}{|l|^{p-1}} \quad (3.14)$$

with C a constant depending only on those given in $[\Delta 2, D2]$ and p .

Proof. See the proof of Lemma 3.5 in [3, pag. 140] together with eqs. (15) and (16) in that paper. In the mentioned lemma it is precisely the expression (3.14) what is proved. $K(a, b, \tilde{a}, \tilde{b})$ and I_p have the same meaning that in the present work (see also [3, pag. 136], where the invoked hypotheses (H1),(H2),(H3) and (H4) are introduced and derived from the RDP condition.) \square

Remark 3.1. *Although it is not written here we know from [3] that the constant C in (3.14) may behave like $1/(3-p)$. In order to get an uniform bound for $3 \leq p$ it is necessary to restrict the class of the underlying reference elements $K = K(a, b, \tilde{a}, \tilde{b})$. Later we show that (3.14) holds for any $1 \leq p$ if we work with the family $K(a, b, \tilde{a}, \tilde{b})$ associated to the DAC.*

3.2. The regularity condition $h/\rho < \sigma$ and the family $K(a, b, \tilde{a}, \tilde{b})$. For dealing with regular elements we need to introduce a new geometrical condition associated to the class $K(a, b, \tilde{a}, \tilde{b})$,

$$\text{Condition (D3):} \quad a \underset{C}{\sim} b. \quad (3.15)$$

Theorem 3.2. *Let K be a general convex quadrilateral. Then K is regular (in the sense of (1.4)) if and only it is equivalent to some $K(a, b, \tilde{a}, \tilde{b})$ under $[\Delta 2, D2, D3]$ (equiv. $[\Delta 1, D2, D3]$).*

Proof. Thanks to Remark 1.1 we know that elements satisfying the regularity condition (1.4) satisfy also the RDP . Therefore from Theorem 3.1 we see that K can be mapped, by means of an affine transformation $LX = BX + P$ (see Definition 2.1) into an element $K(a, b, \tilde{a}, \tilde{b})$. This reference element should be regular since $\|B\|, \|B^{-1}\| < C$ and now it is easy to see that $(\Delta 1)$ implies $(D3)$. To prove the other implication take an element $K(a, b, \tilde{a}, \tilde{b})$ under $[\Delta 2, D2, D3]$ (equiv. $[\Delta 1, D2, D3]$). Thanks to $(D3)$ we have that $T(a, b)$ is a regular triangle. Now it is easy to see that this together with $[\Delta 1, D2]$ implies that $K(a, b, \tilde{a}, \tilde{b})$ verifies (1.4). As a consequence, any element of the form $L(K(a, b, \tilde{a}, \tilde{b}))$ is regular for any affine mapping of the kind considered in Definition 2.1. \square

3.3. The DAC and the family $K(a, b, \tilde{a}, \tilde{b})$. As it is mentioned before *DAC* implies the *RDP*, and as consequence Theorem 3.1 says that any element under the *DAC* can be mapped into an element $K(a, b, \tilde{a}, \tilde{b})$ for which $[\Delta 1, D2]$ holds. Nevertheless, the following result, partially borrowed from [3], states that actually we may assume $\frac{\tilde{a}}{a}, \frac{\tilde{b}}{b} \leq 1$, and this, as we show later, not only simplifies the treatment of the error but also allows to deal with the case $1 \leq p$.

Theorem 3.3. *Let K be a general convex quadrilateral. Then K satisfies the $DAC(\psi_m, \psi_M)$ if and only if it is equivalent to an element $K(a, b, \tilde{a}, \tilde{b})$ under $[D1, D2]$.*

Proof. Notice that it is always possible to select two neighboring sides l_1, l_2 of K such that the parallelogram defined by these sides contains the element K . Call V_1 the common vertex of l_1, l_2 and β the angle at V_1 . After a rigid movement we may assume that $V_1 = (0, 0)$ and that l_2 lies along the x axis (with nonnegative coordinates $(a, 0)$). Moreover we can also assume that l_1 belongs to the upper half plane. Notice that l_1 is the side joining V_1V_4 and following the proof of Theorem 3.1 take $b = |l_1| \sin(\beta)$ in such a way that V_4 can be written as $V_4 = (b \cot(\beta), b)$. Then the linear mapping L associated to the matrix B defined in such theorem performs the desired transformation. Indeed, since $\|B\|, \|B^{-1}\| < \frac{\sqrt{2}}{\sin(\beta)}$ with β away from 0 and π (due to the fact that K is under the *DAC*) we know that L is of the class considered in Definition 2.1. On the other hand, calling $L(\tilde{a}, \tilde{b}) = V_3$ we observe that (D1) holds while thanks to the fact that the angle at V_3 is away from 0 and π the same holds for the angle at (\tilde{a}, \tilde{b}) meaning that at least one of the remaining angles of the triangle of vertices $(0, b), (\tilde{a}, \tilde{b}), (a, 0)$ does not approach zero nor π . Performing a rigid movement if necessary we may assume that this is the one at $(0, b)$ and hence (D2) follows. Reciprocally, assume that $K(a, b, \tilde{a}, \tilde{b})$ verifies $[D1, D2]$ and it is equivalent to K . Notice that the maximal and minimal angle of $K(a, b, \tilde{a}, \tilde{b})$ are away from 0 and π (in terms of the constants given by $[D1, D2]$). Indeed, since at V_1 we have a right angle we only need to check the remaining vertices. The angle at V_4 is bounded above by $\pi/2$ due to (D1) and below by α . Let us focus now on the angle at vertex V_3 . It should be bounded below by $\pi/2$ due to (D1). On the other hand, it can not approach 0 due to (D2). Finally, the angle at V_2 is greater than α and also bounded above by $\pi/2$. The proof concludes by using that K is equivalent to $K(a, b, \tilde{a}, \tilde{b})$ (in the sense of Definition 2.1) and Lemma 3.1. \square

There is a property that can be derived from $[D1, D2]$.

Lemma 3.4. *Let $K(a, b, \tilde{a}, \tilde{b})$ be a general element under $[D1, D2]$, then $a \leq Cb$.*

Proof. The proof is elementary since $\tan \alpha \leq \frac{b}{a}$. \square

An advantage of the *DAC* is that it simplifies the treatment of I_p . Indeed from (D1) we get

$$\frac{1}{(1 + \hat{x}(\frac{\tilde{b}}{b} - 1) + \hat{y}(\frac{\tilde{a}}{a} - 1))^{p-1}} \leq \frac{1}{(\frac{\tilde{b}}{b} + \frac{\tilde{a}}{a} - 1)^{p-1}},$$

since $0 \leq \hat{x}, \hat{y} \leq 1$. On the other hand, calling $y(x) = -(b/a)(x - a)$ to the equation of the straight line joining V_2 and V_4 we have

$$\frac{\tilde{b} - y(\tilde{a})}{b} = \frac{\tilde{a} - y^{-1}(\tilde{b})}{a} = \frac{\tilde{b}}{b} + \frac{\tilde{a}}{a} - 1,$$

and since $|l| \sin(\alpha) \leq \tilde{b} - y(\tilde{a})$ and $|l| \sin(\alpha) \leq \tilde{a} - y^{-1}(\tilde{b})$ we get

$$I_p \leq \min\{a^{p-1}, b^{p-1}\} \frac{1}{|l|^{p-1} \sin(\alpha)^{p-1}}. \quad (3.16)$$

As a consequence we obtain for *DAC* the following equivalent of Lemma 3.3, that holds for any $1 \leq p$.

Lemma 3.5. *If $K = K(a, b, \tilde{a}, \tilde{b})$ is a general convex quadrilateral under $[D1, D2]$ then, for any $1 \leq p$,*

$$\max \left\{ \frac{a}{b^{p-1}}, \frac{b}{a^{p-1}} \right\} I_p \leq C \frac{h}{|l|^{p-1}} \quad (3.17)$$

with a constant C depending only on those of $[D1, D2]$.

3.4. The mac and the family $K(a, b, \tilde{a}, \tilde{b})$. We finish this section with a characterization of the elements under the minimal angle condition. A direct consequence of previous subsections an Remark 1.4 is the following

Theorem 3.4. *Let K be a general convex quadrilateral. Then K satisfies the mac if and only if K is equivalent to some $K(a, b, \tilde{a}, \tilde{b})$ for which holds either $[D1, D2]$ or $[\Delta1, D2, D3]$ (equiv. $[\Delta2, D2, D3]$).*

Proof. Follows straightforwardly from Theorems 3.2 and 3.3 together with Remark 1.4. \square

4. TRIANGLES UNDER THE MAC INSIDE $K(a, b, \tilde{a}, \tilde{b})$.

Let $K = K(a, b, \tilde{a}, \tilde{b})$ be a general convex quadrilateral and consider the associated triangle $T = T(a, b)$. Let us recall that M_{ij} (resp. M_{ij}^T) are the interpolation nodes of Q_k on K (resp. Π_k on T). We are interested, loosely speaking, in the problem of finding for each M_{ij} a close enough M_{ij}^T . Notice that in general M_{ij} does not agree with M_{ij}^T , except for $i = 0$ or $j = 0$. For any other node M_{ij} (i.e. for $i \neq 0 \neq j$) we consider a suitable triangle having M_{ij} as one of its vertices and with the remaining vertices belonging to the set of (edge) interpolation nodes of Π_k . We choose it in the following way: if M_{ij} is an *edge* node on the top of $K(a, b, \tilde{a}, \tilde{b})$ (i.e. $i = k, 1 \leq j \leq k$) we consider the triangle $T_{kj} = \Delta(M_{kj} M_{k0}^T M_{k-j,j}^T)$ if M_{ij} is an *edge* node on the right edge (i.e. $j = k, 1 \leq i \leq k$) we chose $T_{ik} = \Delta(M_{ik} M_{0k}^T M_{i,k-i}^T)$ and finally if M_{ij} is *interior* (i.e. $1 \leq i, j \leq k-1$) we define a triangle $T_{ij} = \Delta(M_{ij} M_{0j}^T M_{i0}^T)$ (see Figure 2). The

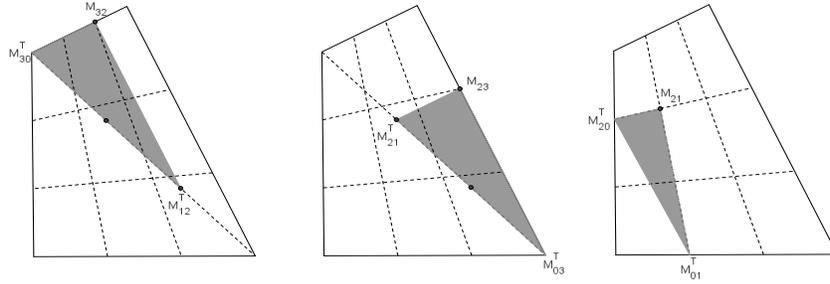


FIGURE 2. Representation of T_{32} , T_{23} and T_{21} in a Q_3 element *left: T_{32} , center: T_{23} , right: T_{21} .*

geometry of these triangles are important in the sequel. In particular notice that T_{kj} and T_{ik} are similar to the triangle $\Delta(V_2 V_3 V_4)$ therefore we have immediately

Lemma 4.1. *Let $K(a, b, \tilde{a}, \tilde{b})$ be a general convex quadrilateral under $[\Delta1, D2]$ (equiv. $[\Delta2, D2]$), then for any $T = T_{kj}$ (resp. $T = T_{ik}$) defined above we have that the side $\overline{M_{kj} M_{k0}^T}$ (resp. $\overline{M_{ik} M_{i,k-i}^T}$) is comparable to $l = \sqrt{V_3 V_4}$ and the angle of T at M_{k0} (resp. $M_{i,k-i}^T$) is the angle α of condition (D2). In particular T verifies the MAC.*

For interior nodes we have the following

Lemma 4.2. *Let $K = K(a, b, \tilde{a}, \tilde{b})$ be a convex quadrilateral which satisfy either $[D1, D2]$ or $[\Delta1, D2, D3]$ (equiv. $[\Delta2, D2, D3]$). If $1 \leq i, j \leq k-1$ then*

(a)

$$|\overline{M_{i0}M_{ij}}| \underset{C}{\simeq} a \quad \text{and} \quad |\overline{M_{0j}M_{ij}}| \underset{C}{\simeq} b$$

(in particular $|\overline{M_{i0}M_{0j}}| \underset{C}{\simeq} h$).

(b) α_{ij} is bounded away from 0 and π where α_{ij} is the angle between $\overline{M_{i0}M_{ij}}$ and $\overline{M_{i0}M_{0j}}$. In particular for $1 \leq i, j \leq k-1$, any triangle $T = \Delta(M_{i0}, M_{0j}, M_{ij})$ verifies the MAC.

Proof.

(a) To prove that the measure of the segment $\overline{M_{i0}M_{ij}}$ is comparable to a it is sufficient to prove that the measure of the segment $\overline{M_{i0}M_{ik}}$ is comparable to a since these segments are mutually proportional.

For $0 < \hat{y} = i/k < 1$ have

$$|\overline{M_{i0}M_{ik}}|^2 = \|F_K(1, \hat{y}) - F_K(0, \hat{y})\|^2 = \left\| (a(1 - \hat{y}) + \tilde{a}\hat{y}, (\tilde{b} - b)\hat{y}) \right\|^2$$

therefore

$$a^2(1 - \hat{y})^2 \leq |\overline{M_{i0}M_{ik}}|^2 \leq a^2(1 - \hat{y})^2 + 2a\tilde{a}\hat{y}(1 - \hat{y}) + \hat{y}^2 l^2.$$

Using $(\Delta1)$ (or $(D1)$) and that l is comparable to the shortest side the statement is proved. Similarly, to the appropriate $0 < \hat{x} < 1$ we have

$$|\overline{M_{0j}M_{kj}}|^2 = \|F_K(\hat{x}, 1) - F_K(\hat{x}, 0)\|^2 = \left\| ((\tilde{a} - a)\hat{x}, b(1 - \hat{x}) + \tilde{b}\hat{x}) \right\|^2$$

therefore for a suitable constant C , we get

$$b^2(1 - \hat{x})^2 \leq |\overline{M_{0j}M_{kj}}|^2 \leq 2 \left[(\tilde{a} - a)^2 \hat{x}^2 + b^2 + (\tilde{b} - b)^2 \hat{x}^2 \right] \leq C(a^2 + b^2) \quad (4.18)$$

and the proof concludes by using $(D3)$ in one case or Lemma 3.4 in the other. Now we immediately have, by using $\Delta1$ or $D1$, that $|\overline{M_{i0}M_{0j}}| \underset{C}{\simeq} h$.

(b) Calling μ the matrix with rows $w_1 = M_{i0} - M_{ik}$ and $w_2 = M_{i0} - M_{0j}$ we see that

$$\frac{1}{\sin \alpha_{ij}} = \frac{\|w_1\| \|w_2\|}{|\det \mu|}.$$

Thanks to the previous item we know that the numerator can be bounded in terms of ab . We claim that $0 < ab < C|\det \mu|$. Indeed, a direct calculation gives for $y = \frac{i}{k}$ and $x = \frac{j}{k}$

$$|\det \mu| = aby \left| 1 + \left(\frac{\tilde{a}}{a} - 1 \right) y + \left(\frac{\tilde{b}}{b} - 1 \right) x \right|$$

and since $1 \leq i \leq k-1$ all we need to show is that term inside the modulus $m = 1 + \left(\frac{\tilde{a}}{a} - 1 \right) y + \left(\frac{\tilde{b}}{b} - 1 \right) x$ stays away from zero. Now, if $\frac{\tilde{b}}{b} - 1 \geq 0$ then $m > 1 - y$ and we are done. On the other hand, if $\frac{\tilde{b}}{b} - 1 < 0$ then we write

$$m \geq \left(\frac{\tilde{b}}{b} + \frac{\tilde{a}}{a} - 1 \right) y + \frac{\tilde{b}}{b}(1 - y) > \frac{\tilde{b}}{b}(1 - y)$$

where the first inequality follows taking $x = 1$ and the second one by (2.8). Using Lemma 2.5 (see also Remark 2.1) the proof is complete. \square

5. THE ERROR TREATMENT

Lemma 5.1. *Let $K(a, b, \tilde{a}, \tilde{b})$ be a convex quadrilateral and assume $(\Delta 1)$. Let $T = T(a, b)$, then for any polynomial $q \in \mathbb{P}_k$, there exists a constant depending only on k and on the C given in $(\Delta 1)$ such that*

$$\|q\|_{0,p,K(a,b,\tilde{a},\tilde{b})} \leq C\|q\|_{0,p,T}. \quad (5.1)$$

Proof. The proof is standard. Let us introduce an small rectangle K_s and a large rectangle K_l as follows

$$K_s := K(a/2, b/2, a/2, b/2) \subset T \subset K(a, b, \tilde{a}, \tilde{b}) \subset K_l := K(\bar{a}, \bar{b}, \bar{a}, \bar{b})$$

where $\bar{a} = \max\{a, \tilde{a}\}$, $\bar{b} = \max\{b, \tilde{b}\}$. All we need is to show that

$$\|q\|_{0,p,K_l} \leq C\|q\|_{0,p,K_s}. \quad (5.2)$$

Thanks to $(\Delta 1)$ we have that the quotients $\frac{\tilde{a}}{a}, \frac{\tilde{b}}{b}$ are bounded in terms of a generic constant C . For the sake of clarity we rename this time the constant and write \bar{C} . Consider now the reference sets

$$\hat{K}_{\bar{C}} = K\left(\frac{1}{2\bar{C}}, \frac{1}{2\bar{C}}, \frac{1}{2\bar{C}}, \frac{1}{2\bar{C}}\right) \subset \hat{K} = K(1, 1, 1, 1).$$

Using equivalence of norms in the finite dimensional space \mathbb{P}_k we get

$$\|\hat{q}\|_{0,p,\hat{K}} \leq C\|\hat{q}\|_{0,p,\hat{K}_{\bar{C}}},$$

for any $\hat{q} \in \mathbb{P}_k$ and where C depends only on k and \bar{C} . Now (5.2) follows by changing variables with a linear map $L : \hat{K} \rightarrow K_l$ taking into account that for such an L , $L(\hat{K}_{\bar{C}}) \subset K_s$. \square

Write $K = K(a, b, \tilde{a}, \tilde{b})$ and let Π_k be the Lagrange interpolation operator of order k on the triangle $T = T(a, b)$ and let $p \geq 1$ then we can write

$$|u - Q_k u|_{1,p,K} \leq |u - \Pi_k u|_{1,p,K} + |\Pi_k u - Q_k u|_{1,p,K}.$$

Since $\Pi_k u - Q_k u$ belongs to the \mathcal{Q}_k quadrilateral finite element space and vanishes at M_{0j} and M_{i0} for all $0 \leq i, j \leq k$, it follows that

$$(\Pi_k u - Q_k u)(X) = \sum_{i,j \neq 0} (\Pi_k u - u)(M_{ij}) \phi_{ij}(X)$$

where ϕ_{ij} is the basis function associated to M_{ij} . Therefore

$$|u - Q_k u|_{1,p,K} \leq |u - \Pi_k u|_{1,p,K} + \sum_{i,j \neq 0} |(\Pi_k u - u)(M_{ij})| |\phi_{ij}|_{1,p,K}. \quad (5.3)$$

Taking into account that T verifies the MAC (actually $MAC(\pi/2)$) we have [6, 7, 9] that

$$\|u - \Pi_k u\|_{0,p,T} \leq Ch^{k+1} |u|_{k+1,p,T}, \quad (5.4)$$

and

$$|u - \Pi_k u|_{1,p,T} \leq Ch^k |u|_{k+1,p,T}. \quad (5.5)$$

The next lemma extends this approximation result to K .

Lemma 5.2. *Let $K = K(a, b, \tilde{a}, \tilde{b})$ be a convex quadrilateral and assume $(\Delta 1)$. Let $T = T(a, b)$ and $\Pi_k u$ the \mathbb{P}_k Lagrange interpolation operator on T . Then for any $1 \leq p$*

$$|u - \Pi_k u|_{1,p,K} \leq Ch^k |u|_{k+1,p,K}. \quad (5.6)$$

Proof. Let $u \in W^{k+1,p}(K)$ and $\mathcal{P}_k u \in \mathbb{P}_k$ defined as

$$\int_K D^\alpha u = \int_K D^\alpha \mathcal{P}_k u \quad (|\alpha| \leq k).$$

Since K is convex, we have

$$|u - \mathcal{P}_k u|_{1,p,K} \leq Ch^k |u|_{k+1,p,K}, \quad (5.7)$$

as one can see by applying repeatedly the Poincaré inequality. Writing

$$|u - \Pi_k u|_{1,p,K} \leq |u - \mathcal{P}_k u|_{1,p,K} + |\mathcal{P}_k u - \Pi_k u|_{1,p,K},$$

we observe that the first term is fine. For the second one we consider an arbitrary first derivative of $\mathcal{P}_k u - \Pi_k u$ and call it $D(\mathcal{P}_k u - \Pi_k u) \in \mathbb{P}_{k-1}$. Using Lemma 5.1

$$\|D(\mathcal{P}_k u - \Pi_k u)\|_{0,p,K} \leq C \|D(\mathcal{P}_k u - \Pi_k u)\|_{0,p,T} \leq C [|\mathcal{P}_k u - u|_{1,p,K} + |u - \Pi_k u|_{1,p,T}]$$

and the lemma follows from (5.5) and (5.7). \square

Lemma 5.3. *Let $K = K(a, b, \tilde{a}, \tilde{b})$ be a convex quadrilateral.*

- (a) *Assume that $1 \leq p < 3$ and that K satisfies $[\Delta 1, D2]$ (equiv. $[\Delta 2, D2]$) then for any basis function ϕ ,*

$$|\phi|_{1,p,K} \leq C \frac{h^{1/p}}{|l|^{1/q}},$$

where q is the conjugate exponent of p (the constant C may behave as $\frac{1}{3-p}$, see Remark 3.1).

- (b) *Assume that $1 \leq p < 3$ and that K satisfies either $[\Delta 1, D2, D3]$ (equiv. $[\Delta 2, D2, D3]$) then*

$$|\phi|_{1,p,K} \leq C \frac{h^{1/p}}{a^{1/q}},$$

where ϕ is an internal basis function.

- (c) *For any $1 \leq p$, assume that K satisfies $[D1, D2]$ then for any internal basis function*

$$|\phi|_{1,p,K} \leq C \frac{h^{1/p}}{a^{1/q}},$$

- (d) *For any $1 \leq p$, assume that K satisfies $[D1, D2]$ then for any edge basis function*

$$|\phi|_{1,p,K} \leq C \frac{h^{1/p}}{|l|^{1/q}}.$$

Proof. Part (a) follows from Lemma 2.3 and Lemma 3.3. On the other hand, by Lemma 2.4 and Lemma 2.5 we notice that to show (b) it is sufficient to prove that

$$\frac{b}{a^{p-1}} |1 - \tilde{b}/b|^p I_p, \quad \frac{a}{b^{p-1}} (\tilde{a}/a)^p I_p \leq C \frac{h}{a^{p-1}}. \quad (5.8)$$

Using that $\tilde{a} \leq |l|$ and $|b - \tilde{b}| \leq |l|$, together with $(\Delta 2)$ and Lemma 3.3 we have

$$\frac{b}{a^{p-1}} |1 - \tilde{b}/b|^p I_p \leq C \frac{|l|^p}{b^p} \frac{b}{a^{p-1}} I_p \leq C \frac{h}{b^{p-1}} \leq C \frac{h}{a^{p-1}}$$

where the last inequality follows from $(D3)$. Similarly,

$$\frac{a}{b^{p-1}} (\tilde{a}/a)^p I_p \leq C \frac{h}{a^{p-1}}.$$

Item (c) follows similarly to item (b) using Lemma 3.5 instead of Lemma 3.3 and Lemma 3.4 instead of $(D3)$. Finally, the last item (d) follows straightforwardly from Lemma 2.3 and Lemma 3.5. \square

Let us now recall the following

Lemma 5.4. *Let T be a triangle with diameter h_T and e be any of its sides. For any $p \geq 1$ we have*

$$\|u\|_{0,p,e} \leq 2^{1/q} \left(\frac{|e|}{|T|} \right)^{1/p} \left\{ \|u\|_{0,p,T} + h_T |u|_{1,p,T} \right\}$$

where q is the dual exponent of p .

Proof. See for instance [12]. \square

Now we are ready to get bounds for $|(u - \Pi_k u)(M_{ij})|$. In order to do that we consider the triangle T_{ij} associated with M_{ij} defined in Section 4.

Lemma 5.5. *Let $K = K(a, b, \tilde{a}, \tilde{b})$ be a convex quadrilateral satisfying either $[D1, D2]$ or $[\Delta 1, D2, D3]$ (equiv. $[\Delta 2, D2, D3]$). For any $p \geq 1$ we consider its dual exponent q . We have,*

(a) (Edge nodes) Assume either $i = k$ and $1 \leq j \leq k$ or $j = k$ and $1 \leq i \leq k$ then

$$|(u - \Pi_k u)(M_{ij})| \leq C \frac{|l|^{1/q}}{h^{1/p}} \left[|u - \Pi_k u|_{1,p,T} + h_T |u - \Pi_k u|_{2,p,T} \right],$$

where $T = T_{ij}$.

(b) (Interior nodes) If $1 \leq i, j \leq k - 1$ then

$$|(u - \Pi_k u)(M_{ij})| \leq C \frac{a^{1/q}}{h^{1/p}} \left[|u - \Pi_k u|_{1,p,T} + h_T |u - \Pi_k u|_{2,p,T} \right],$$

where $T = T_{ij}$.

Proof.

(a) We write the case $i = k$ and $1 \leq j \leq k$ since the other one follows identically. Calling e the side of $T = T_{kj}$ given by $e = \overline{M_{k0}M_{kj}}$ we get (by using Hölder's inequality, Lemma 5.4 and the fact that $(u - \Pi_k u)(M_{k0}) = 0$)

$$\begin{aligned} |(u - \Pi_k u)(M_{kj})| &\leq \int_e |\partial_e(u - \Pi_k u)| \, dx \\ &\leq |e|^{1/q} \|\partial_e(u - \Pi_k u)\|_{0,p,e} \\ &\leq 2^{1/q} \frac{|e|}{|T|^{1/p}} \left[\|\partial_e(u - \Pi_k u)\|_{0,p,T} + h_T |\partial_e(u - \Pi_k u)|_{1,p,T} \right]. \end{aligned} \tag{5.9}$$

The item follows now by Lemma 4.1 that implies $\frac{|e|}{|T|^{1/p}} \leq C \frac{|l|^{1/q}}{h^{1/p}}$.

(b) With the same ideas, consider $(u - \Pi_k u)(M_{i0}) = 0$ call $e = \overline{M_{i0}M_{ij}}$ and use now Lemma 4.2. \square

Lemma 5.6. *Let K be a general convex quadrilateral and $1 \leq i, j \leq k$,*

- (1) *If $1 \leq p < 3$ and K satisfies $[\Delta 1, D2, D3]$ (equiv. $[\Delta 2, D2, D3]$), then (5.10) holds.*
- (2) *If $1 \leq p$ and K satisfies $[D1, D2]$, then (5.10) holds.*

$$|(u - \Pi_k u)(M_{ij})| |\phi_{ij}|_{1,p,K} \leq Ch^k |u|_{k+1,p,K}, \tag{5.10}$$

where ϕ_{ij} is the function basis associated to M_{ij} .

Proof. The proof is essentially a combination of Lemmas 5.3 and 5.5 together with the error estimation for triangles (5.5), recalling that each T_{ij} satisfies the maximum angle condition (Lemmas 4.1 and 4.2). \square

6. MAIN THEOREM

The L^p error estimate for a general convex quadrilateral was done in [3] for $k = 1$ and any p . The argument used there works exactly in the same way for an arbitrary k .

Theorem 6.1. *Let K be an arbitrary convex quadrilateral with diameter h . For any $1 \leq k$ and $1 \leq p$. There exists a constant C independent of K such that*

$$\|u - Q_k u\|_{0,p,K} \leq Ch^{k+1} |u|_{k+1,p,K}. \quad (6.1)$$

Proof. See equation (41) in Theorem 6.1 of [3], as well as Lemma 6.1 in the same paper. \square

Now we can present our main result.

Theorem 6.2. *Let K be a convex quadrilateral with diameter h and $2 \leq k$ an integer:*

- (1) *If K satisfies $DAC(\psi_m, \psi_M)$, hence (6.2) holds for any $1 \leq p$ with $C = C(\psi_m, \psi_M)$.*
- (2) *If K satisfies $mac(\psi_m)$, hence (6.2) holds for any $1 \leq p < 3$ with $C = C(\psi_m)$.*

$$\|u - Q_k u\|_{0,p,K} + h |u - Q_k u|_{1,p,K} \leq Ch^{k+1} |u|_{k+1,p,K}. \quad (6.2)$$

Proof. Since the L^p estimate holds for any convex quadrilateral, it is enough to prove

$$|u - Q_k u|_{1,p,K} \leq Ch^k |u|_{k+1,p,K}. \quad (6.3)$$

Moreover, in order to prove (1) (resp. (2)) and thanks to Theorem 3.3 (resp. Theorem 3.4) we can assume that $K = K(a, b, \tilde{a}, \tilde{b})$ under $[D1, D2]$ (resp. either $[\Delta 1, D2, D3]$ or $[D1, D2]$). Therefore (6.3) follows from (5.3) combined with (5.6) and (5.10). \square

To finish we present two counterexamples. In the first one we focus on the case $1 \leq p < 3$ showing a collection of elements with uniform RDP parameters (actually $RDP(\sqrt{5}, 3/4\pi)$) for which the constant in the $W^{1,p}$ interpolation error blows up. The family does not obey mac although all the elements are under the $MAC(\frac{3\pi}{4})$. In particular, the counterexample shows that the estimate may fail if an angle approaches zero. For the sake of simplicity we choose $k = 2$.

Counterexample 6.1. *(Case $1 \leq p < 3$) For $0 < s < 1/2$ take $K = K(1, s, s, 2s)$ and*

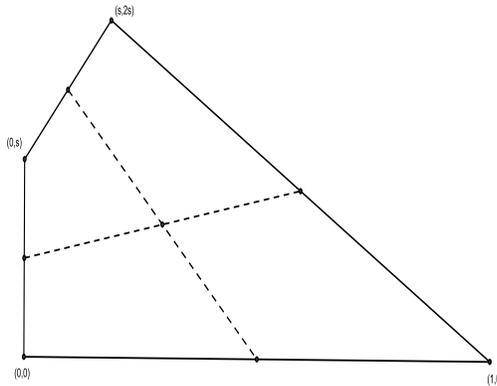


FIGURE 3. Representation of the quadrilateral $K(1, s, s, 2s)$ and its nodes as a \mathcal{Q}_2 element.

consider the function $u(x, y) = x(x - 1/2)(x - 1)$ which does not belong to the \mathcal{Q}_2 space. Since $u(M_{0l}) = 0 = u(M_{10})$ for $0 \leq l \leq 2$ we have

$$Q_2 u = u(M_{11})\phi_{11} + u(M_{12})\phi_{12} + u(M_{22})\phi_{22} + u(M_{21})\phi_{21}$$

and therefore

$$\frac{1}{s-1} \frac{\partial Q_2 u}{\partial y} = \frac{(s-3)(s+1)}{2^6} \frac{\partial \phi_{11}}{\partial y} + s \left(\frac{s+1}{2^3} \frac{\partial \phi_{12}}{\partial y} + (s-1/2) \frac{\partial \phi_{22}}{\partial y} + \frac{(s-2)}{2^3} \frac{\partial \phi_{21}}{\partial y} \right)$$

Then, for a suitable constant C independent of s , we have

$$\left\| \frac{\partial \phi_{11}}{\partial y} \right\|_{0,p,K} \leq C \left[\left\| \frac{\partial Q_2 u}{\partial y} \right\|_{0,p,K} + s \left(\left\| \frac{\partial \phi_{12}}{\partial y} \right\|_{0,p,K} + \left\| \frac{\partial \phi_{22}}{\partial y} \right\|_{0,p,K} + \left\| \frac{\partial \phi_{21}}{\partial y} \right\|_{0,p,K} \right) \right].$$

Using item (a) of Lemma 5.3 and taking into account that $h \underset{C}{\sim} 1$ and $|l| \underset{C}{\sim} s$ we have

$$\left\| \frac{\partial \phi_{11}}{\partial y} \right\|_{0,p,K} \leq 3C \left(\left\| \frac{\partial Q_2 u}{\partial y} \right\|_{0,p,K} + s^{1/p} \right).$$

Assume that (6.3) holds for this family. In that case we would have

$$\left\| \frac{\partial Q_2 u}{\partial y} \right\|_{0,p,K} = \left\| \frac{\partial(Q_2 u - u)}{\partial y} \right\|_{0,p,K} \leq |Q_2 u - u|_{1,p,K} \leq \bar{C} h^2 |u|_{3,p,K}$$

where $h^2 \underset{C}{\sim} 1$ and $|u|_{3,p,K} \underset{C}{\sim} |K|^{1/p} \underset{C}{\sim} s^{1/p}$. Consequently

$$\left\| \frac{\partial \phi_{11}}{\partial y} \right\|_{0,p,K} \leq C s^{1/p}. \quad (6.4)$$

On the other hand, a straightforward computation shows that

$$\left(\frac{\partial \phi_{11}}{\partial y} \circ F_K \right) (\hat{x}, \hat{y}) = \frac{2^4 \hat{x} [(s-1)\hat{y}(\hat{x}-\hat{y}) + (1-\hat{x})(1-2\hat{y})]}{s[1+\hat{x}+(s-1)\hat{y}]}.$$

Therefore

$$\left\| \frac{\partial \phi_{11}}{\partial y} \right\|_{0,p,K}^p = \int_{[0,1]^2} \frac{2^{4p} \hat{x}^p |(s-1)\hat{y}(\hat{x}-\hat{y}) + (1-\hat{x})(1-2\hat{y})|^p}{s^{p-1} [1+\hat{x}+(s-1)\hat{y}]^{p-1}} d\hat{x}d\hat{y}$$

Let $R = [0, 1/8] \times [1/4, 3/8] \subset [0, 1]^2$. It is easy to check that on R we have

$$(s-1)\hat{y}(\hat{x}-\hat{y}) + (1-\hat{x})(1-2\hat{y}) > (s-1)\hat{y}(\hat{x}-\hat{y}) > 0 \quad (6.5)$$

which together with the fact $1+\hat{x}+(s-1)\hat{y} \leq 1+\hat{x}$ allow us to obtain

$$\left\| \frac{\partial \phi_{11}}{\partial y} \right\|_{0,p,K}^p \geq \frac{2^{4p}(1-s)^p}{s^{p-1}} \int_R \frac{\hat{x}^p \hat{y}^p (\hat{x}-\hat{y})^p}{[1+\hat{x}]^{p-1}} d\hat{x}d\hat{y}. \quad (6.6)$$

Since the function $\hat{y}^p(\hat{x}-\hat{y})^p$ is bounded below by a positive constant on R and the function $\hat{x}^p/(1+\hat{x})^{p-1}$ is integrable over this domain follows that

$$\left\| \frac{\partial \phi_{11}}{\partial y} \right\|_{0,p,K} \geq C \frac{1}{s^{1/q}} \quad (6.7)$$

where q is the dual exponent of p . Finally, combining (6.4) with (6.7) and taking $s \rightarrow 0$ we are lead to a contradiction and as a consequence the error estimate can not hold with a uniform constant C .

Remark 6.1. Recall that for $k = 1$ and $1 \leq p < 3$ the constant in the interpolation estimate can be bounded in terms of the constants given in the RDP condition [3]. Actually, the interior node (available in $k = 2$) plays a fundamental role in the counterexample. This could lead the reader to the conclusion that removing internal nodes may help to weaken the conditions under which the estimate (1.3) holds. Regretfully this is not possible. Indeed from [4] we know that the accuracy of serendipity elements can be seriously deteriorated even for regular elements. The

reason of that is the failure of the inclusion of \mathbb{P}_k in the interpolation space. Our proof relies strongly on this property (see for instance the derivation of (5.3)).

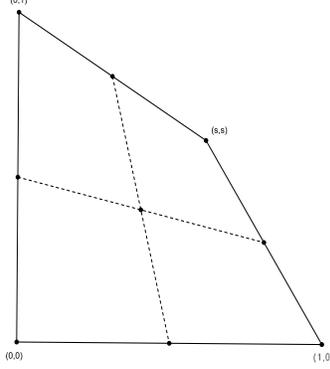


FIGURE 4. Representation of the quadrilateral $K(1, 1, s, s)$ and its nodes as a Q_2 element.

Counterexample 6.2. (Case $3 \leq p$) Consider the family $K(1, 1, s, s)$, with $\frac{1}{2} < s \leq 5/8$, and the function $u(x, y) = x(x - 1/4)(x - 3/4)(x - 3/8)(x - 1)$. Observe that the maximum angle of $K = K(1, 1, s, s)$ approaches π as $s \rightarrow \frac{1}{2}$ while K verifies $\text{mac}(\pi/4)$ for any value of s in the selected range. Arguing as in previous counterexample we have

$$Q_2 u = u(M_{11})\phi_{11} + u(M_{12})\phi_{12} + u(M_{21})\phi_{21} + u(M_{22})\phi_{22},$$

hence

$$\frac{\partial(Q_2 u)}{\partial y} = u(M_{11})\frac{\partial\phi_{11}}{\partial y} + u(M_{12})\frac{\partial\phi_{12}}{\partial y} + u(M_{21})\frac{\partial\phi_{21}}{\partial y} + u(M_{22})\frac{\partial\phi_{22}}{\partial y}.$$

Observe that $u(M_{11}), u(M_{12})$ and $u(M_{21})$ are polynomial expressions in the variable s having $1/2$ as a single root. Therefore we can write

$$u(M_{ij}) = (s - 1/2)q_{ij}(s)$$

for each $(i, j) \in I = \{(1, 1), (1, 2), (2, 1)\}$ where q_{ij} is a polynomial. On the other hand $|u(M_{22})| > C > 0$ if $\frac{1}{2} < s \leq 5/8$. Therefore

$$\left\| \frac{\partial\phi_{22}}{\partial y} \right\|_{0,p,K} \leq C \left[\left\| \frac{\partial Q_2 u}{\partial y} \right\|_{0,p,K} + (s - 1/2) \sum_{(i,j) \in I} \left\| \frac{\partial\phi_{ij}}{\partial y} \right\|_{0,p,K} \right]. \quad (6.8)$$

If the error estimates holds then

$$\left\| \frac{\partial Q_2 u}{\partial y} \right\|_{0,p,K} = \left\| \frac{\partial Q_2 u - u}{\partial y} \right\|_{0,p,K} \leq |Q_2 u - u|_{1,p,K} \leq C|u|_{3,p,K},$$

since $h \underset{C}{\sim} 1$ and as a consequence

$$\left\| \frac{\partial Q_2 u}{\partial y} \right\|_{0,p,K} \leq C. \quad (6.9)$$

On the other hand for $1/2 < s \leq 5/8$ we readily notice that $\sin(\alpha) \underset{C}{\sim} (s - 1/2)$. Then combining this with (3.16) and Lemma 2.3 we get

$$\sum_{(i,j) \in I} \left\| \frac{\partial\phi_{ij}}{\partial y} \right\|_{0,p,K} \leq C \frac{1}{(s - 1/2)^{1/q}} \quad (6.10)$$

where q is the dual exponent to p . Finally, (6.10) combined with (6.8) with (6.9) give us

$$\left\| \frac{\partial \phi_{22}}{\partial y} \right\|_{0,p,K} \leq C \quad (6.11)$$

for some positive constant. However, a straightforward calculation yields

$$\left(\frac{\partial \phi_{22}}{\partial y} \circ F_K \right) (\hat{x}, \hat{y}) = \frac{2\hat{x} [(s-1)\hat{y}(\hat{x}-\hat{y}) + (2\hat{x}-1)(4\hat{y}-1)]}{1+(s-1)(\hat{x}+\hat{y})}$$

hence

$$\left\| \frac{\partial \phi_{22}}{\partial y} \right\|_{0,p,K}^p = \int_0^1 \int_0^1 \frac{(2\hat{x} [(s-1)\hat{y}(\hat{x}-\hat{y}) + (2\hat{x}-1)(4\hat{y}-1)])^p}{(1+(s-1)(\hat{x}+\hat{y}))^{p-1}} d\hat{x}d\hat{y}.$$

Let T be the triangle with vertices $(3/4, 3/4)$, $(3/4, 1)$ and $(1, 1)$. It is easy to check that

$$\left\| \frac{\partial \phi_{22}}{\partial y} \right\|_{0,p,K}^p \geq C \int_T \frac{1}{(1+(s-1)(\hat{x}+\hat{y}))^{p-1}} d\hat{x}d\hat{y},$$

and integrating explicitly for $p > 3$ we get

$$\left\| \frac{\partial \phi_{22}}{\partial y} \right\|_{0,p,K}^p \geq C \frac{(2s-1)^{3-p}/2 + (3s-1)^{3-p}/2^{4-p} - (7s-3)^{3-p}/4^{3-p}}{(s-1)^2(2-p)(3-p)}$$

and hence $\left\| \frac{\partial \phi_{22}}{\partial y} \right\|_{0,p,K}^p \rightarrow \infty$ if $s \rightarrow 1/2$. Since this fact contradicts (6.11) we conclude that the error estimate does not hold. The case $p = 3$ follows similarly.

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