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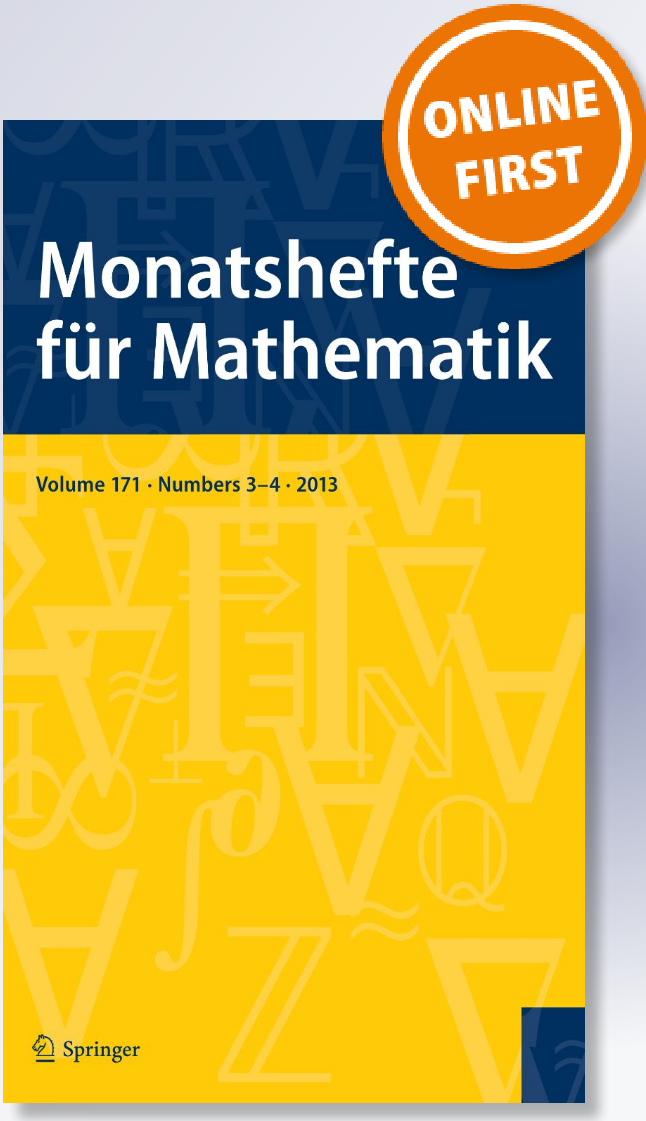

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A note on a system with radiation boundary conditions with non-symmetric linearisation

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Abstract We prove the existence of multiple solutions for a second order ODE system under radiation boundary conditions. The proof is based on the degree computation of $I - K$, where K is an appropriate fixed point operator. Under a suitable asymptotic Hartman-like assumption for the nonlinearity, we shall prove that the degree is 1 over large balls. Moreover, studying the interaction between the linearised system and the spectrum of the associated linear operator, we obtain a condition under which the degree is -1 over small balls. We thus generalize a result obtained in a previous work for the case in which the linearisation is symmetric.

Keywords Second order ODE systems · Radiation boundary conditions · Multiplicity · Topological degree

Mathematics Subject Classification 34B15

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1 Introduction

In a recent paper [1], an elliptic system of second order differential equations under indefinite Robin conditions was considered. The ODE version of this problem for a vector function $u(x) = (u_1(x), \ldots, u_N(x))$ reads

$$u''(x) = g_0(u(x)) + p(x) \tag{1}$$

with the boundary condition

$$u'(0) = a_0 u(0), \qquad u'(1) = a_1 u(1) \tag{2}$$

where $a_0, a_1 \in \mathbb{R}$, $p \in C([0, 1], \mathbb{R}^N)$ and $g_0 \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ with $g_0(0) = 0$. Existence and uniqueness/multiplicity results are deduced from [1] in terms of the interaction between the nonlinearity and the spectrum of the linear scalar operator $Lu := -u''$ under the boundary conditions (2). In order to formulate the statement in a more precise way, denote by $\lambda_1 < \lambda_2 < \cdots \to +\infty$ the eigenvalues of L and let $\lambda_0 := -\infty$; then it is verified that the assumption

$$\liminf_{|u|\to\infty} \frac{\langle g_0(u), u\rangle}{|u|^2} > -\lambda_1. \tag{3}$$

implies that the problem has at least one solution. In particular, this includes the case when g_0 is superlinear, although the situation is exactly the opposite of the so-called *superlinear system*, in which the superlinear term lies on the left-hand side of the equation. The Dirichlet problem for this case was treated for example in [6], where a strong superlinearity condition is imposed in order to obtain infinitely many solutions. Differently, (3) can be understood as an asymptotic Hartman-like condition (see [9]) and implies that the set of solutions is bounded. However, unlike the Hartman assumption, when $\lambda_1 > 0$ it does not imply $\langle g_0(u) + p(x), u\rangle \geq 0$ for any $u \neq 0$. For example, this occurs in the definite case $a_0 > 0 > a_1$ and, more generally, when $a_0 > -1$ and $a_1 < \frac{a_0}{a_0+1}$. Moreover, the solution is unique if g_0 satisfies the strict monotonicity assumption

$$\langle g_0(u) - g_0(v) + \lambda_1(u - v), u - v\rangle > 0 \tag{4}$$

for all $u \neq v$. In contrast with this latter conclusion, multiple solutions are obtained under a complementary hypothesis, when the Jacobian matrix $Dg_0(0)$ is symmetric. In more precise terms, if the eigenvalues $\gamma_1 \leq \cdots \leq \gamma_N$ of $Dg_0(0)$ satisfy

$$\lambda_{\nu_k} < -\gamma_k < \lambda_{\nu_k+1} \tag{5}$$

for some $\nu_k \in \mathbb{N}_0$ with

$$\nu_1 + \cdots + \nu_N \text{ odd} \tag{6}$$

then the problem typically admits at least three solutions, provided that $\|p\|_\infty$ is small enough. It is worth noticing that if condition (4) holds, then $-\gamma_j \le \lambda_1$ for all j; thus, the assumption of the multiplicity result cannot be satisfied.

The proof is based on the degree computation of $I - K$, where K is an appropriate fixed point operator. Condition (3) allows to prove that if R is sufficiently large then K has no fixed points on $\partial B_R(0)$ and, furthermore, $deg(I - K, B_R(0), 0) = 1$, where 'deg' stands for the Leray–Schauder degree. On the other hand, condition (5) is employed to guarantee that, when $\rho > 0$ is small enough, $deg(I - K, B_\rho(0), 0) = -1$. This implies that the degree over $B_R(0)\backslash B_\rho(0)$ is 2, which yields the desired result.

It is remarked, however, that the degree computation over $B_\rho(0)$ in [1] is performed by means of the linearisation $\hat{K}$ of the operator K at $v = 0$ and relies strongly on the fact that $Dg_0(0)$ is symmetric, as it happens for example in the variational case $g_0 = \nabla G_0$. Indeed, by a lemma on symmetric bilinear forms proved in [10] (see also [5] for an application to a more abstract context), using condition (5) it is verified that the linear system

$$u''(x) = Dg_0(0)u(x) \tag{7}$$

has no nontrivial solutions satisfying the boundary condition (2) and, furthermore, $deg(I - \hat{K}, B_\rho(0), 0) = (-1)^{\nu_1+\cdots+\nu_N}$; thus, the conclusion follows from (6).

This work is devoted to investigate the general situation, in which $Dg_0(0)$ is no longer symmetric and, in particular, condition (5) may make no sense. This situation is specially interesting because it covers the non-variational case. A similar problem under Dirichlet conditions was studied for example in [8] and more generally in [12] by studying the rotation of the associated vector field at 0 and ∞ under the assumption that g_0 is asymptotically linear and $p = 0$. In the symmetric case, asymptotically linear systems under Dirichlet conditions have been also considered in [7], where a multiplicity result was obtained for the problem $u''(x) = M(x, u(x))u(x)$ by means of a generalised shooting method. Under the assumption that the symmetric $N \times N$ matrix M converges uniformly to some $M_\infty(x)$ as $|u| \to \infty$, the proof is based on a comparison between the number of moments of verticality of the matrices $M_0 := M(\cdot, 0)$ and M_∞.

In this work, both assumptions (symmetry and asymptotic linearity) are dropped. We obtain a necessary and sufficient condition in terms of the eigenvalues $\{\lambda_j\}$ of the associated linear scalar operator, which ensures that (7)–(2) has no nontrivial solutions and, furthermore, that $deg(I - K, B\rho(0), 0) = deg(I - \hat{K}, B_\rho(0), 0) = -1$. As we shall see, when $Dg_0(0)$ is symmetric the condition is equivalent to (5)–(6), so the main theorem in this work may be regarded as a natural extension of the results of [1] to the non-symmetric case. We remark that the main theorem of this paper can be extended to the elliptic case by slightly strengthening condition (3) as in [1]; however, the ODE treated in this work has particular interest because it also admits, as shown below, a shooting-type approach.

To this end, let us firstly observe that, for any fixed matrix M, the determinant of $M + \lambda_j I$ is positive when j is large. Then the function

$$s(M) := \prod_{j=1}^{\infty} sgn(\det(M + \lambda_j I))$$

is in fact well defined as a finite product. Our main result reads:

Theorem 1.1 *Assume that* (3) *holds. Then problem* (1)–(2) *has at least one solution. If furthermore* $s(Dg_0(0)) < 0$*, then there exists* $r > 0$ *such that* (1)–(2) *has at least two solutions for* $\|p\|_\infty < r$.

Remark 1.2 Using the Sard–Smale theorem [11], it is seen that, for a residual set $\Sigma \subset B_r(0)$, the problem has at least three solutions for $p \in \Sigma$.

Remark 1.3 The result is readily extended to different boundary conditions, such as (homogeneous) Dirichlet and Neumann among others. For the periodic problem, the assumption takes a very precise and more restrictive form, because all the eigenvalues except the first one have multiplicity equal to 2. Thus, $s(M) = sgn(\det(M + \lambda_1 I))$ for arbitrary M, and hence the assumption for multiplicity simply reads:

$$sgn(\det(Dg_0(0) + \lambda_1 I)) < 0.$$

The paper is organized as follows. In the next section, we introduce some notation and present the basic facts concerning the spectrum of the associated linear operator. Moreover, we define a compact fixed point operator that shall be used in the following section for a proof of the main result. In Sect. 4, we sketch an alternative proof based on a shooting-type method. This requires the use of a lemma that has some interest in its own, since it implies that the degree of the operator $I - K$ coincides with the Brouwer degree of the shooting operator. An elementary example for the non-symmetric case is given in Sect. 5. Finally, further comments and some open problems are posed in the last section.

2 Preliminaries

In order to give a proof our main result, let us firstly observe that the scalar operator $Lu := -u''$ is symmetric (with respect to the L^2 inner product) over the space of H^2 functions that satisfy the boundary condition (2). By standard arguments, the existence of a sequence consisting of all the eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to +\infty$ of L and an associated orthonormal basis of $L^2((0,1),\mathbb{R})$ of eigenfunctions $\{\varphi_j\}_{j\in\mathbb{N}}$ is deduced. For convenience, we shall denote $\lambda_0 := -\infty$. Moreover, if φ and ψ are eigenfunctions associated to the same λ_j, then the Wronskian determinant $w(x) := \varphi(x)\psi'(x) - \varphi'(x)\psi(x)$ vanishes on the boundary, which implies that φ and ψ are linearly dependent, that is: λ_j is simple for all j. Because of the Robin condition, it is clear that φ_j does not vanish on the boundary; moreover, the first eigenfunction φ_1 is a global minimizer of the functional

$$I(\varphi) := \int_0^1 \frac{\varphi'(x)^2}{2}\,dx + a_0\frac{\varphi(0)^2}{2} - a_1\frac{\varphi(1)^2}{2}$$

subject to the restriction $\|\varphi\|_{L^2} = 1$. This shows that φ_1 does not vanish in $(0, 1)$ because, otherwise, $|\varphi_1|$ would be another minimizer of I. The fact that φ_1 cannot have multiple roots implies that φ_1 and $|\varphi_1|$ are linearly independent, a contradiction. Observe, incidentally, that $\lambda_j = 0$ for some j if and only if $a_1 = \frac{a_0}{a_0+1}$. In this case, we conclude that if $a_0 > -1$ (or equivalently, $a_1 < 1$) then the (linear) eigenfunction corresponding to $\lambda_j = 0$ does not vanish and, by a comparison argument, it is deduced that $j = 1$.

On the other hand, since $I(\varphi) \geq I(\varphi_1) = \frac{\lambda_1}{2}$, it is deduced that

$$\lambda_1 \|\varphi\|_{L^2}^2 \leq -\int_0^1 \varphi''(x)\varphi(x)\, dx \tag{8}$$

for any smooth function φ satisfying (2). In particular, for fixed $\eta > 0$ we may add $(\eta - \lambda_1)\|\varphi\|_{L^2}^2$ at both sides in order to obtain the following estimate:

$$\|\varphi\|_{L^2} \leq \frac{1}{\eta}\|\varphi'' + (\lambda_1 - \eta)\varphi\|_{L^2}. \tag{9}$$

This obviously yields

$$\begin{aligned}\|\varphi''\|_{L^2} &\leq \|\varphi'' + (\lambda_1 - \eta)\varphi\|_{L^2} + |\lambda_1 - \eta|\|\varphi\|_{L^2} \\ &\leq \left(1 + \left|1 - \frac{\lambda_1}{\eta}\right|\right) \|\varphi'' + (\lambda_1 - \eta)\varphi\|_{L^2}\end{aligned}$$

whence an inequality analogous to (9) is deduced for the uniform norm

$$\|\varphi\|_\infty \leq c\|\varphi'' + (\lambda_1 - \eta)\varphi\|_\infty \tag{10}$$

for some appropriate constant c. If we define

$$X := \{\varphi \in C^2([0, 1], \mathbb{R}) : \varphi \text{ satisfies } (2)\},$$

then (10) just expresses the obvious fact that the operator $L_\eta : X \to C([0, 1])$ given by $L_\eta\varphi := -\varphi'' - (\lambda_1 - \eta)\varphi$ is an isomorphism. Furthermore, L_η is strictly positive in the L^2 sense, because

$$\int_0^1 L_\eta\varphi(x)\varphi(x)\, dx \geq \eta\|\varphi\|_{L^2}^2.$$

In particular, this implies that the Green function associated to L_η is positive and, as a consequence, the following (strict) maximum principle is deduced: if $\varphi \in X$ satisfies $L_\eta\varphi(x) > 0$ for all x, then $\varphi(x) > 0$ for all x.

The proof of our main theorem shall be based on the computation of the degree of $I - K$, where the operator $K : C([0, 1], \mathbb{R}^N) \to C([0, 1], \mathbb{R}^N)$ is defined by $Kv := u$, the unique solution of the linear problem

$$\begin{cases} -L_\eta u(x) = g_0(v(x)) + (\lambda_1 - \eta)v(x) \\ u'(0) = a_0 u(0), \qquad u'(1) = a_1 u(1). \end{cases} \tag{11}$$

It is clear that K is compact and that u is a solution (1)–(2) if and only if u is a fixed point of K.

As mentioned in the introduction, the degree computation requires, in the first place, to establish conditions guaranteeing that a linear problem

$$u''(x) = Mu(x) \tag{12}$$

has no nontrivial solutions satisfying (2). For convenience, we may fix $J = J(M)$ such that if $j > J$ then

$$\det(M + \lambda_j I) = \lambda_j^N \det\left(\frac{M}{\lambda_j} + I\right) > 0.$$

Thus we may define the function

$$s(M) := \prod_{j=1}^{J} sgn(\det(M + \lambda_j I)). \tag{13}$$

It shall be proved that uniqueness of solutions of (12)–(2) is equivalent to the condition $s(M) \neq 0$, namely, that $M+\lambda_j I$ is invertible for all j. In particular, when $M = Dg_0(0)$ is symmetric, this condition is clearly equivalent to (5); moreover, writing $M = CDC^1$ with D diagonal, it is verified that

$$\det(M + \lambda_j I) < 0 \iff \#\{k : \nu_k \geq j\} \text{ is odd}$$

or, in other words,

$$sgn(\det(M + \lambda_j I)) = (-1)^{\#\{k:\nu_k \geq j\}}.$$

Furthermore, it follows from basic combinatorics that, if J is large enough, then

$$\sum_{j=1}^{J} \#\{k : \nu_k \geq j\} = \nu_1 + \cdots + \nu_N.$$

Thus, condition $s(Dg_0(0)) < 0$ in Theorem 1.1 may be regarded as a natural extension of (5)–(6) to the non-symmetric case.

3 Proof of the main result

In order to prove Theorem 1.1, we shall compute the degree of $I - K$ over large and small balls, where the compact operator $K : v \mapsto u$ is defined from problem (11). To this end, let us firstly prove an auxiliary lemma concerning the linear operator K_M defined by $K_M v := u$ the unique solution of

$$\begin{cases} u''(x) + (\lambda_1 - \eta)u(x) = Mv(x) + (\lambda_1 - \eta)v(x) \\ u'(0) = a_0 u(0), \qquad u'(1) = a_1 u(1). \end{cases} \tag{14}$$

for an arbitrary matrix $M \in \mathbb{R}^{N\times N}$.

Lemma 3.1 *Assume that $s(M) \neq 0$. Then K_M has no nontrivial fixed points and, moreover,*

$$deg(I - K_M, B_\rho(0), 0) = s(M)$$

for all $\rho > 0$.

Proof Write $u(x) = \sum_{j=1}^{\infty} \varphi_j(x)u^j$, where the column vector $u^j \in \mathbb{R}^N$ is given by $(u^j)_k = \int_0^1 u_k(x)\varphi_j(x)\,dx$ for $k = 1, \ldots, N$. Then u is a fixed point of K_M if and only if

$$\sum_{j=1}^{\infty} -\lambda_j \varphi_j u^j = \sum_{j=1}^{\infty} \varphi_j M u^j,$$

that is,

$$(M + \lambda_j I)u^j = 0$$

for all j. This implies that $u^j = 0$ for all j. Next, observe that

$$(K_M u)^j = \frac{M + (\lambda_1 - \eta)I}{\lambda_1 - \lambda_j - \eta} u^j$$

and hence

$$(u - K_M u)^j = \frac{M + \lambda_j I}{\lambda_j + \eta - \lambda_1} u^j := M_j u^j.$$

It is readily verified that

$$\left\| \sum_{j>k} \varphi_j (K_M u)^j \right\|_\infty \to 0$$

uniformly for $\|u\|_\infty \le \rho$ as $k \to \infty$; thus, $deg(I - K_M, B_\rho(0), 0)$ is simply computed as

$$\prod_{j \le k} sgn[\det(M_j)]$$

for some k large enough. Since $\lambda_j + \eta - \lambda_1 > 0$ for all j, the latter quantity coincides with $s(M)$ and so completes the proof. □

As a first application, we shall prove that the degree of $I - K$ over a large ball is equal to 1. In particular, this implies the existence of at least one solution under the sole assumption (3).

Lemma 3.2 *Assume that* (3) *holds. Then* $deg(I - K, B_R(0), 0) = 1$ *for* $R \gg 0$.

Proof Fix $\varepsilon > 0$ and $C > 0$ such that

$$\langle g_0(u), u\rangle \ge (\varepsilon - \lambda_1)|u|^2 - C$$

for all $u \in \mathbb{R}^N$. We claim that there exists a constant $R > 0$ depending only on ε and C such that if u is a solution of (1)–(2) with $p = 0$ then $\|u\|_\infty < R$. Indeed, multiply the equation by u and integrate to obtain

$$\lambda_1 \|u\|_{L^2}^2 + \int_0^1 \langle u''(x), u(x)\rangle \, dx \ge \varepsilon \|u\|_{L^2}^2 - C.$$

Thus, from (8) we deduce that

$$\|u\|_{L^2}^2 \le \frac{C}{\varepsilon}$$

and consequently

$$-\int_0^1 \langle u''(x), u(x)\rangle \, dx \le mC,$$

where $m = \max\left\{\frac{\lambda_1}{\varepsilon}, 1\right\}$. Integrating by parts, it follows that

$$\|u'\|_{L^2}^2 \le mC + a_1 \frac{|u(1)|^2}{2} - a_0 \frac{|u(0)|^2}{2} = mC + \psi(1) - \psi(0),$$

where $\psi(x) := [(a_1 - a_0)x + a_0]\frac{|u(x)|^2}{2}$. Writing

$$\begin{aligned}\psi(1) - \psi(0) &= \int_0^1 \psi'(x)\, dx \\ &= \frac{a_1 - a_0}{2}\|u\|_{L^2}^2 + \int_0^1 [(a_1 - a_0)x + a_0]\langle u(x), u'(x)\rangle \, dx\end{aligned}$$

and using the inequality $|\langle u(x), u'(x)\rangle| \leq \frac{\gamma}{4}|u(x)|^2 + \frac{|u'(x)|^2}{\gamma}$ for some constant $\gamma > \max\{|a_0|, |a_1|\}$, it follows that

$$\|u'\|_{L^2}^2 \leq mC + C_1\|u\|_{L^2}^2 + C_2\|u'\|_{L^2}^2 \leq mC + \frac{C_1 C}{\varepsilon} + C_2\|u'\|_{L^2}^2$$

where $C_2 < 1$. This yields a bound for $\|u\|_{H^1}$ and the claim follows from the embedding $H^1((0,1), \mathbb{R}^n) \hookrightarrow C([0,1], \mathbb{R}^n)$.

In particular, the same *a priori* estimate is obtained if $g_0(u)$ is replaced by $tg_0(u) + (1-t)(\varepsilon - \lambda_1)u$ for $t \in [0,1]$ which, in turn, implies that

$$deg(I - K, B_R(0), 0) = deg(I - K_M, B_R(0), 0)$$

with $M := (\varepsilon - \lambda_1)I$. The conclusion is deduced then from Lemma 3.1. □

The following lemma shows that, over small balls, the degree of $I - K$ coincides with the degree of its linearisation. The proof is standard, we include it here for the sake of completeness.

Lemma 3.3 *Assume that $s(M) \neq 0$, where $M := Dg_0(0)$. Then K has no fixed points on $\partial B_\rho(0)$ and*

$$deg(I - K, B_\rho(0), 0) = deg(I - K_M, B_\rho(0), 0)$$

provided that $\rho > 0$ is small enough.

Proof Let $\varepsilon > 0$ to be specified and fix ρ such that

$$|g_0(u) - Dg_0(0)u| \leq \varepsilon|u|$$

for $|u| \leq \rho$. Then

$$\|Ku - K_M u\|_\infty \leq c\|g_0(u) - Dg_0(0)u\|_\infty \leq c\varepsilon\rho$$

if $\|u\|_\infty \leq \rho$. We know from Lemma 3.1 that K_M has no nontrivial fixed points, so by compactness we deduce the existence of a constant $\theta > 0$ such that $\|u - K_M u\|_\infty \geq \theta\rho$ for $u \in \partial B_\rho(0)$. Hence, taking $\varepsilon < \frac{\theta}{c}$ it is seen that

$$\|tKu + (1-t)K_M u - u\| \geq \theta\rho - c\varepsilon\rho > 0$$

for $t \in [0, 1]$ and $u \in \partial B_\rho(0)$, so the result follows. □

Proof of Theorem 1.1 From the previous lemmas and the excision property of the degree, we know, for some $R > \rho > 0$, that

$$deg(I - K, B_\rho(0), 0) = -1, \qquad deg(I - K, B_R(0)\backslash B_\rho(0), 0) = 2.$$

This implies the existence of $\hat{r} > 0$ such that the equation $u - Ku = P$ has at least two solutions when $\|P\|_\infty < \hat{r}$.

Now observe that, as mentioned, it follows from (10) that the application $\Phi(p) := P$, where P is the unique solution of the problem

$$P'' + (\lambda_1 - \eta)P = p \qquad P'(0) = a_0 P(0), \quad P'(1) = a_1 P(1)$$

is an isomorphism. Thus, the result is deduced from the fact that $u - Ku = \Phi(p)$ if and only if u is a solution of (1)–(2). □

4 Alternative proof by the shooting method

In this section, we introduce a shooting type operator that allows to give a different proof of Theorem 1.1. This requires to use the following lemma, that may have some interest in its own since it reveals the connection between the Leray-Schauder degree of the previous operator $I - K$ and the Brouwer degree of the shooting operator.

Lemma 4.1 *Let $M \in \mathbb{R}^{N\times N}$ satisfy $s(M) \neq 0$ and consider the linear application $T_M : \mathbb{R}^N \to \mathbb{R}^N$ defined by $T_M(v) := u'(1) - a_1 u(1)$, where u is the unique solution of the initial value problem*

$$u''(x) = Mu(x), \qquad u'(0) = a_0 u(0) = a_0 v.$$

Then T_M is an isomorphism and $deg_B(T_M, B_R(0), 0) = sgn(\det(T_M)) = s(M)$ for any $R > 0$.

Proof Write $M = C^{-1}UC$, where U is an upper triangular matrix, then setting $w = Cu$ it is verified that $T_M = C^{-1}T_U C$. Thus, we may assume that M is upper triangular. Next, consider the diagonal matrix D with entries $D_{ii} := M_{ii}$ and the homotopy given by $M_t := M + t(D - M)$. It is clear that $\det(M_t + \lambda_j I)$ is constant and, consequently, $s(M_t) = s(M)$ for all $t \in [0, 1]$. This proves that the corresponding mapping T_{M_t} is injective for all t and hence $deg(T_M, B_R(0), 0) = deg(T_{M_1}, B_R, 0)$. In other words, we may assume that M is diagonal, so the system is uncoupled, namely

$$u_k''(x) = M_{kk}u_k(x), \qquad k = 1, \ldots N.$$

Let $\nu_k := \max\{j \geq 0 : \lambda_j < M_{kk}\}$, then from [2, Lemma 2.1] we know that

$$sgn(u_k'(1) - a_1 u_k(1)) = (-1)^{\nu_k}$$

and the result follows. □

We are in condition of defining a shooting operator as follows. In the first place, since (3) holds we may assume, for some $R \gg 0$, that $g_0(u) = (\varepsilon - \lambda_1)u$ for $|u| \geq R$. By standard results, this implies that, for each $v \in \mathbb{R}^N$, the unique solution u_v of (1) with initial values $u'(0) = a_0 u(0) = a_0 v$ is defined up to $x = 1$. Thus, the mapping

$$S(v) := u_v'(1) - a_1 u_v(1)$$

is well defined and the zeros of S correspond to solutions of (1)–(2). Using the previous lemma with $M = (\varepsilon - \lambda_1)I$, it follows that $deg(S, B_R(0), 0) = 1$. On the other hand, for $p = 0$, by linearisation and using the same lemma with $M = Dg_0(0)$ it is deduced that if $s(M) < 0$ then $deg(S, B_\rho(0), 0) = -1$ for ρ small enough. Thus, the proof follows for small p by a continuity argument. Details are left to the reader.

5 Example

For $N = 2$, consider the system

$$\begin{cases} u_1''(x) = u_1(x)^3 + au_2(x) + p_1(x) \\ u_2''(x) = u_2(x)^3 + bu_1(x) + p_2(x) \end{cases}$$

under condition (2), with $ab \neq \lambda_j^2$ for all $j \in \mathbb{N}$. Here, g_0 is superlinear since

$$\langle g_0(u), u \rangle = u_1^4 + u_2^4 + (a+b)u_1 u_2 \geq \frac{u_1^4 + u_2^4}{2} - C$$

for some constant C; thus, condition (3) is fulfilled. Moreover,

$$Dg_0(0,0) = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

so the linearisation at the origin is non-symmetric when $a \neq b$. Finally,

$$det(Dg_0(0,0) + \lambda_j I) = \lambda_j^2 - ab \neq 0,$$

which implies that $s(Dg_0(0,0))$ is different from zero. Hence, the existence of generically three solutions follows from Theorem 1.1 when $\|p\|_\infty$ is small, provided that

$$\#\{j \in \mathbb{N} : \lambda_j^2 < ab\}$$

is odd.

6 Open problems

1. Find a suitable extension of the main result for the non-autonomous case $g_0 = g_0(x, u)$.
2. Is it possible to say something when $s(Dg_0(0))) > 0$? For the scalar case with $a_0, a_1 > 0$, it was shown in [2] that if $\lambda_1 < 0$, $g_0'(0) < 0$ and

$$\lambda_k < -g_0'(0) < \lambda_{k+1} \tag{15}$$

for some even $k > 0$, then the problem admits at least five solutions for small p. It was observed, moreover, that the condition $g_0'(0) < 0$ was redundant, because for $a_0, a_1 > 0$ it is verified that $\lambda_2 > 0$. The methods in this paper allow to improve this result in the following way. Assume only that (15) holds for some even $k > 0$. Then $\alpha := \sigma\varphi_1 > 0$ is a strict lower solution for $\sigma > 0$ small enough for the problem with $p = 0$, because $-\lambda_1 \geq \lambda_k > g_0'(0)$. Moreover, if $\theta \gg 0$ then $\beta := \theta\varphi_1$ is a strict upper solution since

$$\beta\beta'' = -\lambda_1\beta^2 \leq g_0(\beta)\beta + C - \varepsilon\beta^2 < g_0(\beta)\beta.$$

Using the maximum principle (see Sect. 2) and taking $\eta > 0$ sufficiently large it is seen that any solution of the truncated problem

$$L_\eta u(x) = g_0(P(x, u(x))) + (\lambda_1 - \eta)P(x, u(x))$$

satisfying (2), where

$$P(x,u) := \begin{cases} u & \alpha(x) \leq u \leq \beta(x) \\ \beta(x) & u > \beta(x) \\ \alpha(x) & u < \alpha(x) \end{cases}$$

lies strictly between α and β. This implies that the degree of $I - K$ over the set $U_{\alpha,\beta} := \{u : \alpha < u < \beta\}$ is equal to 1, and the same is true for $U_{-\beta,-\alpha}$ and $U_{-\beta,\beta}$. Because k is even, the degree is also equal to 1 over small balls. This shows that the problem has (generically) five solutions for p small. If one applies the shooting method instead, it is seen that the number of solutions is indeed 5. It would be interesting to see whether or not this result can be extended for the non-scalar case.
3. *Exact multiplicity* As shown in [3], for the scalar case it is seen that if g_0 is strictly increasing with $g_0'(0) < -\lambda_1$ and $g_0'(u) > \frac{g_0(u)}{u}$ for all $u \neq 0$, then the problem has exactly three solutions for $p > 0$ small. This result gives a positive answer to a question posed in [4], although it is not clear how it could be extended to the case $N > 1$.

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