ITERATIVE ACTIONS OF NORMAL OPERATORS

A. ALDROUBI, C. CABRELLI, A. F. ÇAKMAK, U. MOLTER, A. PETROSYAN

Abstract. Let A be a normal operator in a Hilbert space $\mathcal{H}$, and let $\mathcal{G} \subset \mathcal{H}$ be a countable set of vectors. We investigate the relations between A, $\mathcal{G}$ and L that make the system of iterations $\{A^n g : g \in \mathcal{G},\ 0 \le n < L(g)\}$ complete, Bessel, a basis, or a frame for $\mathcal{H}$. The problem is motivated by the dynamical sampling problem and is connected to several topics in functional analysis, including, frame theory and spectral theory. It also has relations to topics in applied harmonic analysis including, wavelet theory and time-frequency analysis.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space, $A \in B(\mathcal{H})$ a bounded normal operator and $\mathcal{G}$ a countable (finite or countably infinite) collection of vectors in $\mathcal{H}$. Let L be a function $L : \mathcal{G} \to \mathbb{N}^*$, where $\mathbb{N}^* = \{1, 2, \dots\} \cup \{+\infty\}$. We are interested in the structure of the set of iterations of the operator A when acting on the vectors in $\mathcal{G}$ and are limited by the function L. More precisely, we are interested in the following two questions:

(I) Under what conditions on A, $\mathcal{G}$ and L is the iterated system of vectors

$$\{A^n g : g \in \mathcal{G},\ 0 \le n < L(g)\}$$

complete, Bessel, a basis, or a frame for $\mathcal{H}$?

(II) If $\{A^n g : g \in \mathcal{G},\ 0 \le n < L(g)\}$ is complete, Bessel, a basis, or a frame for $\mathcal{H}$ for some system of vectors $\mathcal{G}$ and a function $L : \mathcal{G} \to \mathbb{N}^*$, what can be deduced about the operator A?

We study these and other related questions and we give answers in many important and general cases. In particular, we show that there is a direct relation between the spectral properties of a normal operator and the properties of the systems of vectors generated by its iterative actions on a set of vectors. We are hoping that the questions above and the approach we use can be interesting for research in both, frame theory and operator theory.

For the particular case when $L(g) = \infty$ for every $g \in \mathcal{G}$, we show that, if the system of iterations $\{A^n g : g \in \mathcal{G},\ n \ge 0\}$ is complete and Bessel, then the spectral radius of A must be less than or equal to 1. Since A is normal, A must be a contraction in this case, i.e., $\|A\| \le 1$. Moreover, it's unitary part must be absolutely continuous (with respect to the Lebesgue measure on the circle). The converse of this is also true: for every normal contraction with absolutely continuous unitary part, there exists a set $\mathcal{G}$ such that $\{A^n g : g \in \mathcal{G},\ n \ge 0\}$ is a complete Bessel system. If $\{A^n g : g \in \mathcal{G},\ n \ge 0\}$ is a frame then its unitary part must be 0 and the converse is also true: for every normal contraction with no unitary part there exists a set $\mathcal{G}$ such that $\{A^n g : g \in \mathcal{G},\ n \ge 0\}$ is a (Parseval) frame.

The questions above, in their formulation have similarities with problems involving cyclical vectors in operator theory, and our analysis relies on the spectral theorem for normal operators with multiplicity [12]. There have been some attempts to generalize multiplicity theory to non-normal operators [33]. Although it cannot be generalized entirely, some aspects of it have been extended

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to general operators. In finite dimensions, the spectral theorem for normal operators, represents the underlying space as a sum of invariant subspaces. For general operators the decomposition into invariant subspaces leads to Jordan's theorem. In the infinite dimensional case, the extension leads to a decomposition into invariant subspaces, and one of the goals is to give conditions under which these subspaces $\{S_n\}$ form Riesz bases or equivalently unconditional bases, see [33, 39] and the references therein (this notion of Riesz basis is related but different from the one we use in this work as defined by (2)). The multiplicity of a spectral value for a normal operators has also been extended. For general operators, a global multiplicity (called multicyclicity) is particularly useful in the context of control theory: using multicyclicity theory for a completely non-unitary contraction A, a formula for $\min|\mathcal{G}|$ such that $\{A^n g : g \in \mathcal{G},\ n \geq 0\}$ is complete in $\mathcal{H}$ was obtained (see [34, 32] and the references therein). For a normal operator A, this number can be deduced from Theorem 3.1.

Our main goal in this paper is to find frames or other types of systems through the iterative action of a normal operator, and we use the full power of the spectral theorem with multiplicity.

For us, the motivation to study the iterative actions of normal operators comes from sampling theory and related topics [2, 9, 21, 29, 31, 5, 6, 18, 37, 38, 16, 15]. Specifically, the motivation derives from the so called *dynamical sampling problem* [3, 4, 13, 19, 25]: Let the initial state of a system be given by a vector f in a Hilbert space $\mathcal{H}$ and assume that the initial state is evolving under the action of a bounded operator $A \in \mathcal{B}(\mathcal{H})$ to the states

$$f_n = Af_{n-1},\ f_0 = f.$$

Given a finite or countably infinite set of vectors $\mathcal{G}$, the problem is then to find conditions on A, $\mathcal{G}$ and $L : \mathcal{G} \to \mathbb{N}^*$, that allow the recovery of the function $f \in \mathcal{H}$ from the set of samples

$$\{\langle A^n f, g\rangle : g \in \mathcal{G},\ 0 \leq n < L(g)\}. \tag{1}$$

Let X be the set $X = \{(g, k) : g \in \mathcal{G},\ 0 \leq k < L(g)\}$. Under appropriate conditions on A, L and $\mathcal{G}$, the sequence $\{\langle A^n f, g\rangle : (g, n) \in X\}$ belongs to $\ell^2(X)$. It is fundamental in applications that the reconstruction operator $R : \ell^2(X) \to \mathcal{H}$ given by $R(\langle A^n f, g\rangle) = f$ for all $f \in \mathcal{H}$, exists and is well defined. Moroever, it is very important that it is bounded. This is because very often, the samples $\{\langle A^n f, g\rangle : (g, n) \in X\}$ are corrupted by "noise" $\{\eta_{g,n} : (g, n) \in X\}$, and we require the reconstruction $\tilde{f} = R(\langle A^n f, g\rangle) + R(\eta_{g,n})$ to be close to f when the noise is small. When R exists and is bounded, it is said that f can be reconstructed in a stable way. If this property holds it is also said that the reconstruction is continuously dependent on the data.

It is not difficult to show that the problem above is related to the problem of finding whether the set of vectors $\{(A^*)^n g\}_{g\in\mathcal{G},\ 0\leq n<L(g)}$ (where A^* denote the adjoint of A) is complete or a frame (see definition in Section 2 below) for $\mathcal{H}$ as described in the following proposition.

Proposition ([3])**.** *For any vector $f \in \mathcal{H}$, f can be uniquely recovered from the samples (1) if and only if the system of vectors*

$$\{(A^*)^n g\}_{g\in\mathcal{G},\ 0\leq n<L(g)}$$

is complete in $\mathcal{H}$.

Moreover, f can be uniquely recovered in a stable way from the samples (1) if and only if the system of vectors

$$\{(A^*)^n g\}_{g\in\mathcal{G},\ 0\leq n<L(g)}$$

is a frame for $\mathcal{H}$.

1.1. **Contribution and organization.** In this paper we consider both: Problem (I) and (II) above, in the general separable Hilbert space setting, and for general normal operators. Problem (I) has already been studied in [3] for the special case when $A \in \mathcal{B}(\mathcal{H})$ is a self-adjoint operator that can be unitarily mapped to an infinite diagonalizable matrix in $\ell^2(\mathbb{N})$. Thus, all the results in [3] are subsumed by the corresponding theorems of this paper. The present paper contains new theorems that are not generalizations of those in [3]. In particular those related to Problem (II) and those that are connected to the action of a group of unitary operators.

In Section 3, we characterize all countable subsets $\mathcal{G} \subset \mathcal{H}$ such that $\{A^n g\}_{g\in\mathcal{G},\, 0\le n<\infty}$ is complete in $\mathcal{H}$ when the operator A is a normal reductive operator (Theorem 3.1). These results are also extended to the system of vectors $\{A^n g\}_{g\in\mathcal{G},\, 0\le n<L(g)}$, where L is any suitable function from $\mathcal{G}$ to $\mathbb{N}^*$. However, we also show that the system $\{A^n g\}_{g\in\mathcal{G},\, 0\le n<\infty}$ always fails to be a basis for $\mathcal{H}$ when A is a normal operator (Corollary 4.2). In fact, if the set $\mathcal{G} \subset \mathcal{H}$ is finite, and A is a reductive normal operator, then $\{A^n g\}_{g\in\mathcal{G},\, 0\le n<L(g)}$ cannot be a basis for $\mathcal{H}$ for any choice of the function L (Corollary 4.5). The obstruction to being a basis is the redundancy in the form of non-minimality of the set of vectors $\{A^n g\}_{g\in\mathcal{G},\, 0\le n<L(g)}$. Thus, the set $\{A^n g\}_{g\in\mathcal{G},\, 0\le n<\infty}$, (or $\{A^n g\}_{g\in\mathcal{G},\, 0\le n<L(g)}$) may be a frame, but cannot be a basis. It turns out that, in general, it is difficult for a system of vectors of the form $\{A^n g\}_{g\in\mathcal{G},\, 0\le n<\infty}$ to be a frame as is shown in Section 5. The difficulty is that the spectrum of A must be very special as can be seen from Theorem 5.1. Such frames however do exist, as shown by the constructions in [3]. Surprisingly, the difficulty becomes an obstruction if we normalize the system of iterations to become $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathcal{G},\, n\ge 0}$ when the operator A is self-adjoint as described in Section 6 (Theorem 6.2). In Section 7 we apply our results to systems that are generated by the unitary actions of a discrete group Γ on a set of vectors $\mathcal{G} \subset \mathcal{H}$ which is common in many constructions of wavelets and frames.

2. Notations and preliminaries

Let I be a countable indexing set, and $\mathcal{H}$ a separable Hilbert space. Recall that a system of vectors $\{f_i\}_{i\in I} \subset \mathcal{H}$ is a Riesz sequence in $\mathcal{H}$ if there exist two constants $m, M > 0$ such that

$$m\|c\|^2_{\ell^2(I)} \le \|\sum_{i\in I} c_i f_i\|^2_{\mathcal{H}} \le M\|c\|^2_{\ell^2(I)} \quad \text{for all } c \in \ell^2(I). \tag{2}$$

If, in addition, $\{f_i\}_{i\in I} \subset \mathcal{H}$ is complete then it is called a Riesz basis for $\mathcal{H}$.

A sequence $\{f_i\}_{i\in I} \subset \mathcal{H}$ is said to be a Bessel system in $\mathcal{H}$ if there exists a constant $\beta \ge 0$ such that

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 \le \beta\|f\|^2_{\mathcal{H}} \quad \text{for all } f \in \mathcal{H},$$

and it is said to be a frame for $\mathcal{H}$ if there exist two constants $\alpha, \beta > 0$ such that

$$\alpha\|f\|^2_{\mathcal{H}} \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le \beta\|f\|^2_{\mathcal{H}} \quad \text{for all } f \in \mathcal{H}.$$

If $\alpha = \beta$ we say that the system is a Parseval frame. The notion of frames introduced by Duffin and Schaeffer [14], generalizes the notion of Riesz bases. In particular, if $\{f_i\}_{i\in I} \subset \mathcal{H}$ is a frame for $\mathcal{H}$, then any vector $f \in \mathcal{H}$ has the representation $f = \sum_{i\in I}\langle f, f_i\rangle \tilde{f}_i$ where $\{\tilde{f}_i\}_{i\in I} \subset \mathcal{H}$ is a dual frame. It is well-known that a Riesz basis is a frame but the converse is not necessarily true because the vectors in a frame set $\{f_i\}_{i\in I} \subset \mathcal{H}$ may have linear dependencies. Further properties of frames, bases, and Bessel sequences can be found in [22, 10, 24].

One of the main tools that we use is the spectral theorem with multiplicity for normal operators described below.

Let μ be a non-negative regular Borel measure on $\mathbb{C}$ with compact support K. Denote by N_μ the operator

$$N_\mu f(z) = zf(z),\ z \in K$$

acting on functions $f \in L^2(\mu)$ (i.e. $f : \mathbb{C} \to \mathbb{C}$, measurable with $\int_{\mathbb{C}} |f(z)|^2 d\mu(z) < \infty$.)

For a Borel non-negative measure μ, we will denote by $[\mu]$ the class of Borel measures that are mutually absolutely continuous with μ.

Theorem 2.1 (Spectral theorem with multiplicity)**.** *For any normal operator A on $\mathcal{H}$ there are mutually singular compactly supported non-negative Borel measures μ_j, $1 \le j \le \infty$, such that A is equivalent to the operator*

$$N_{\mu_\infty}^{(\infty)} \oplus N_{\mu_1} \oplus N_{\mu_2}^{(2)} \oplus \cdots$$

i.e. there exists a unitary transformation

$$U : \mathcal{H} \to (L^2(\mu_\infty))^{(\infty)} \oplus L^2(\mu_1) \oplus (L^2(\mu_2))^{(2)} \oplus \cdots$$

such that

$$UAU^{-1} = N_{\mu_\infty}^{(\infty)} \oplus N_{\mu_1} \oplus N_{\mu_2}^{(2)} \oplus \cdots . \tag{3}$$

Moreover, if M is another normal operator with corresponding measures $\nu_\infty, \nu_1, \nu_2, \dots$ then M is unitarily equivalent to A if and only if $[\nu_j] = [\mu_j]$, $j = 1, \dots, \infty$.

A proof of the theorem can be found in [12] (Ch. IX, Theorem 10.16) and [11] (Theorem 9.14).

Since the measures μ_j are mutually singular, there are mutually disjoint Borel sets $\{\mathcal{E}_j\}$ such that μ_j is concentrated on $\mathcal{E}_j$ for every $1 \le j \le \infty$.

We will define the scalar measure μ, (usually called *the scalar spectral measure)* associated with the normal operator A to be

$$\mu := \sum_{1 \le j \le \infty} \mu_j. \tag{4}$$

The Borel function function $J : \mathbb{C} \to \mathbb{N}^* \cup \{0\}$ given by

$$J(z) = \begin{cases} j, & z \in \mathcal{E}_j \\ 0, & \textit{otherwise} \end{cases} \tag{5}$$

is called multiplicity function of the operator A.

From Theorem 2.1, every normal operator is uniquely determined, up to a unitary equivalence, by the pair $([\mu], J)$.

For $j \in \mathbb{N}$, define Ω_j to be the set $\{1, ..., j\}$ and Ω_∞ to be the set $\mathbb{N}$. Note that $\ell^2(\Omega_j) \cong \mathbb{C}^j$, for $j \in \mathbb{N}$, and $\ell^2(\Omega_\infty) = \ell^2(\mathbb{N})$. For $j = 0$ we define $\ell^2(\Omega_0)$ to be the trivial space $\{0\}$.

Let $\mathcal{W}$ be the Hilbert space

$$\mathcal{W} := (L^2(\mu_\infty))^{(\infty)} \oplus L^2(\mu_1) \oplus (L^2(\mu_2))^{(2)} \oplus \cdots$$

associated to the operator A and let $U : \mathcal{H} \to \mathcal{W}$ be the unitary operator given by Theorem 2.1. If $g \in \mathcal{H}$, we will denote by $\widetilde{g}$ the image of g under U. Since $\widetilde{g} \in \mathcal{W}$ we have $\widetilde{g} = (\widetilde{g}_j)_{j \in \mathbb{N}^*}$, where $\widetilde{g}_j$ is the restriction of $\widetilde{g}$ to $(L^2(\mu_j))^{(j)}$. Thus, for any $j \in \mathbb{N}^*$, $\widetilde{g}_j$ is a function from $\mathbb{C}$ to $\ell^2(\Omega_j)$ and

$$\sum_{j \in \mathbb{N}^*} \int_{\mathbb{C}} \|\widetilde{g}_j(z)\|^2_{\ell^2(\Omega_j)} d\mu_j(z) < \infty.$$

Let P_j be the projection defined for every $\widetilde{g} \in \mathcal{W}$ by $\widetilde{f} = P_j \widetilde{g}$ where $\widetilde{f}_j = \widetilde{g}_j$ and $\widetilde{f}_k = 0$ for $k \ne j$.

Let E be the spectral measure for the normal operator A. Then for every μ-measurable set $G \subseteq \mathbb{C}$ and vectors f, g in $\mathcal{H}$ we have the following formula

$$\langle E(G)f, g \rangle_{\mathcal{H}} = \int_G \left[\sum_{1 \le j \le \infty} \mathbb{1}_{\mathcal{E}_j}(z) \langle \widetilde{f}_j(z), \widetilde{g}_j(z) \rangle_{\ell^2(\Omega_j)} \right] d\mu(z),$$

which relates the spectral measure of A with the scalar spectral measure of A.

In [27] and [1] the spectral multiplicity of multiplication operator is computed.

As a generalization of self-adjoint operators, we will consider normal reductive operators. Reductive operators were first studied by P. Halmos [20] and J. Wermer [41].

Definition 2.2. *A closed subspace $\mathcal{V} \subseteq \mathcal{H}$ is called reducing for the operator A if both $\mathcal{V}$ and its orthogonal complement $\mathcal{V}^\perp$ are invariant subspaces of A.*

Notice that, $\mathcal{V} \subseteq \mathcal{H}$ being reducing subspace for A is equivalent to $\mathcal{V}$ being an invariant subspace both for A and its adjoint A^* and also equivalent to $AP_{\mathcal{V}} = P_{\mathcal{V}}A$ where $P_{\mathcal{V}}$ is the projection operator onto $\mathcal{V}$.

Definition 2.3. *An operator A is called reductive if every invariant subspace of A is reducing.*

It is not known whether every reductive operator is normal. In fact, every reductive operator being normal is equivalent to the veracity of the long standing *invariant subspace conjecture*, which states that every bounded operator on a separable Hilbert space has a non-trivial closed invariant subspace [17].

Proposition 2.4. [28] *A normal operator is reductive if and only if its restriction to every invariant subspace is normal.*

Proposition 2.5 ([41])**.** *Let A be a normal operator on the Hilbert space $\mathcal{H}$ and let μ_j be the measures in the representation (3) of A. Let μ be as in (4). Then A is reductive if and only if for any two vectors $f, g \in \mathcal{H}$*

$$\int_{\mathbb{C}} z^n \left[\sum_{1 \leq j \leq \infty} \mathbb{1}_{\mathcal{E}_j}(z) \langle \widetilde{g}_j(z), \widetilde{f}_j(z) \rangle_{\ell^2(\Omega_j)} \right] d\mu(z) = 0$$

for every $n \geq 0$ implies μ_j-a.e. $\langle \widetilde{g}_j(z), \widetilde{f}_j(z) \rangle_{\ell^2(\Omega_j)} = 0$ for ever $j \in \mathbb{N}^*$.

Note that the property in the above proposition is equivalent to the implication

$$\int_{\mathbb{C}} z^n h(z) d\mu(z) = 0 \text{ and } h \in L^1(\mu) \Rightarrow h = 0 \ \mu - \text{a.e.}$$

As proved in [41], being reductive is not entirely a property of the spectrum: it is possible to find two operators with the same spectrum such that one is reductive the other is not. However the following sufficient condition holds

Proposition 2.6 ([41])**.** *Let A be a normal operator on $\mathcal{H}$ whose spectrum $\sigma(A)$ has empty interior and $\mathbb{C} - \sigma(A)$ is connected. Then A is reductive.*

Corollary 2.7. *Every self-adjoint operator on a Hilbert space is reductive.*

The fact that self-adjoint operators are reductive is easily derived without the use of Proposition 2.6. However, to see how this fact follows form Proposition 2.6, simply note that for a self-adjoint operator A, $\sigma(A)$ is a compact subset of $\mathbb{R}$ hence, it has empty interior (as a subset of $\mathbb{C}$), and $\mathbb{C} - \sigma(A)$ is connected.

Also the following necessary condition for being reductive holds.

Proposition ([36])**.** *Let A be a normal operator. If the interior of $\sigma(A)$ is not empty then A is not reductive.*

3. Complete systems with iterations

This section is devoted to the characterization of completeness of the system $\{A^n g\}_{g \in \mathcal{G},\, n \geq 0}$ where A is a reductive normal operator on a Hilbert space $\mathcal{H}$ and $\mathcal{G}$ is a set of vectors in $\mathcal{H}$. This is done by "diagonalizing" the operator A using multiplicity theory for normal operators, and the properties of reductive operators.

Theorem 3.1. *Let A be a normal operator on a Hilbert space $\mathcal{H}$, and let $\mathcal{G}$ be a countable set of vectors in $\mathcal{H}$ such that $\{A^n g\}_{g \in \mathcal{G},\, n \geq 0}$ is complete in $\mathcal{H}$. Let $\mu_\infty, \mu_1, \mu_2, \dots$ be the measures in the representation (3) of the operator A. Then for every $1 \leq j \leq \infty$ and μ_j-a.e. z, the system of vectors $\{\widetilde{g}_j(z)\}_{g \in \mathcal{G}}$ is complete in $\ell^2\{\Omega_j\}$.*

If in addition to being normal, A is also reductive, then $\{A^n g\}_{g \in \mathcal{G},\, n \geq 0}$ being complete in $\mathcal{H}$ is equivalent to $\{\widetilde{g}_j(z)\}_{g \in \mathcal{G}}$ being complete in $\ell^2\{\Omega_j\}$ μ_j-a.e. z for every $1 \leq j \leq \infty$.

Example 1. *Let A be a convolution operator on $\mathcal{H} = L^2(\mathbb{R})$ given by $Af = a * f$, where $a \in L^1(\mathbb{R})$ is a real valued, even function (hence the Fourier transform $\hat{a}$ of a is real valued even function), such that $\hat{a}$ is strictly decreasing on $[0, \infty)$. For example, A can be the discrete-time heat evolution operator given by the convolution with the Gaussian kernel $a(x) = \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}}$. Since $a \in L^1(\mathbb{R})$, $\hat{a}$ is continuous, and the spectrum of A is the compact interval $I = [0, \frac{1}{\sqrt{4\pi}}] \subset \mathbb{R}$. Hence as a subset of $\mathbb{C}$, I satisfies the assumption of Proposition 2.6 and thus A is reductive. Moreover, the facts that $\hat{a}$ is real valued, even function, strictly decreasing on $[0, \infty)$, imply that $\mu_j = 0$ for $j \neq 1, 2$. In fact, using* [1, Theorem 5]*, we get that $\mu_j = 0$ for $j \neq 2$. Then, using Theorem 3.1, for a set of functions $\mathcal{G} \subset L^2(\mathbb{R})$, the system of iterations $\{A^n g\}_{g \in \mathcal{G},\, n \geq 0}$ is complete in $L^2(\mathbb{R})$ if and only if $\{(\hat{g}(\xi), \hat{g}(-\xi))\}_{g \in \mathcal{G}}$ is complete in $\mathbb{R}^2$ for a.e. $x \in \mathbb{R}$.*

Definition 3.2. *For a given set $\mathcal{G}$, let $\mathcal{L}$ be the class of functions $L : \mathcal{G} \to \mathbb{N}^*$ such that*

$$cl(\operatorname{span}\{A^n g\}_{g \in \mathcal{G},\, 0 \leq n < L(g)}) = cl(\operatorname{span}\{A^n g\}_{g \in \mathcal{G},\, n \geq 0}). \tag{6}$$

Remark 3.3. Note that condition (6) is equivalent to

$$A^{L(h)} h \in cl(\operatorname{span}\{A^n g\}_{g \in \mathcal{G},\, 0 \leq n < L(g)}). \tag{7}$$

for every $h \in \mathcal{G}$ such that $L(h) < \infty$.

In particular, $\mathcal{L}$ contains the constant function $L(g) = \infty$ for every $g \in \mathcal{G}$. It also contains the function

$$l(g) = \min\left\{\left\{m \mid A^m g \in span\{g, Ag, \ldots, A^{m-1} g\}\right\}, \infty\right\} \qquad \text{for every } g \in \mathcal{G}. \tag{8}$$

When $l(g)$ is finite, it is called *the degree of the annihilator* of g.

Because of condition (6), the reduced system $\{A^n g\}_{g \in \mathcal{G},\, 0 \leq n < L(g)}$ will be complete in $\mathcal{H}$ if and only if $\{A^n g : g \in \mathcal{G}, n \geq 0\}$ is complete in $\mathcal{H}$. Therefore, Theorem 3.1 holds if we replace $\{A^n g\}_{g \in \mathcal{G}, n \geq 0}$ by $\{A^n g\}_{g \in \mathcal{G},\, 0 \leq n < L(g)}$ as long as $L \in \mathcal{L}$.

Although when $L \in \mathcal{L}$, $\{A^n g\}_{g \in \mathcal{G},\, n \geq 0}$ and $\{A^n g\}_{g \in \mathcal{G},\, 0 \leq n < L(g)}$ are either both complete or both incomplete, the system $\{A^n g\}_{g \in \mathcal{G},\, 0 \leq n < L(g)}$ may form a frame while $\{A^n g\}_{g \in \mathcal{G},\, n \geq 0}$ may not, since the possible extra vectors in $\{A^n g\}_{g \in \mathcal{G},\, n \geq 0}$ may inhibit the upper frame bound. This difference in behavior between the two systems makes it important to study $\{A^n g\}_{g \in \mathcal{G},\, 0 \leq n < L(g)}$ for $L \in \mathcal{L}$.

Example 2. *Let $\mathcal{H} = \ell^2(\mathbb{Z})$ and A be convolution operator with a kernel $a \in \ell^1(\mathbb{Z})$, i.e. $Af = a * f$. Let $\mathcal{G} = \{e_{mk}\}_{k \in \mathbb{Z}}$ for some $m > 1$ where $\{e_k\}_{k \in \mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$. The Fourier transform of a is defined as*

$$\hat{a}(\xi) = \sum_{k \in \mathbb{Z}} a(k) e^{-2\pi i \xi k},\ \xi \in [0, 1].$$

Denote

$$\mathcal{A}_m(\xi) = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \hat{a}(\frac{\xi}{m}) & \hat{a}(\frac{\xi+1}{m}) & \ldots & \hat{a}(\frac{\xi+m-1}{m}) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{a}^{(L-1)}(\frac{\xi}{m}) & \hat{a}^{(L-1)}(\frac{\xi+1}{m}) & \ldots & \hat{a}^{(L-1)}(\frac{\xi+m-1}{m}) \end{pmatrix}.$$

Let $\sigma(\xi)$ denote the smallest singular value of the matrix $\mathcal{A}_m(\xi)$. Let $L(g) = M$ for each $g \in \mathcal{G}$. From [4]*, the system $\{A^n g\}_{g \in \mathcal{G}, 0 \leq n < M}$ is complete in $\ell^2(\mathbb{Z})$ if and only if $\mathcal{A}_m(\xi)$ has a left inverse for a.e. $\xi \in [0, 1]$, or equivalently $\sigma(\xi) > 0$ for a.e. $\xi \in [0, 1]$, and it forms a frame if and only if $\sigma(\xi) \geq \alpha$ for a.e. $\xi \in [0, 1]$ for some $\alpha > 0$. Since $\mathcal{A}_m(\xi)$ is a Vandermonde matrix, iterations $n > m - 1$ will not affect the completeness of the system. Thus, we let $M = m$. In that case $\{A^n g\}_{g \in \mathcal{G}, 0 \leq n \leq m-1}$ is complete in $\ell^2(\mathbb{Z})$ if and only if $\det \mathcal{A}_m(\xi) \neq 0$ for a.e $\xi \in [0, 1]$, and it is a frame if and only if for a.e $\xi \in [0, 1]$, $|\det \mathcal{A}_m(\xi)| \geq \alpha\}$ for some $\alpha > 0$.*

Although there are infinitely many convolution operators that satisfy this last condition, many natural operators in practice do not. For example, an operator where a is real, even and $\hat{a}$ is strictly decreasing on $[0, \frac{1}{2}]$. For this case, it can be shown that the matrices $\mathcal{A}_m(0)$ and $\mathcal{A}_m(\frac{1}{2})$ are singular, while all *the other matrices* $\mathcal{A}_m(\xi)$ *are invertible. For this case any set of the form* $\mathcal{G} = \{e_{mk}\}_{k\in\mathbb{Z}} \cup \{e_{mlk+1}\}_{k\in\mathbb{Z}}$ *where* $l \geq 1$*, produces a system* $\{A^n g\}_{g\in\mathcal{G}, 0\leq n\leq m-1}$ *which is a frame for* $\mathcal{H} = \ell^2(\mathbb{Z})$*.*

The proof of Theorem 3.1 below, also shows that, for normal reductive operators, completeness in $\mathcal{H}$ is equivalent to the system $\left\{N^n_{\mu_j}\widetilde{g}_j\right\}_{g\in\mathcal{G},\, n\geq 0}$ being complete in $(L^2(\mu_j))^{(j)}$ for every $1 \leq j \leq \infty$, i.e. the completeness of $\{A^n g\}_{g\in\mathcal{G},\, n\geq 0}$ is equivalent to the completeness of its projections onto the mutually orthogonal subspaces $UP_jU^*\mathcal{H}$ of $\mathcal{H}$. This should be contrasted to the fact that, in general, completeness of a set of vectors $\{h_n\} \subset \mathcal{H}$ is not equivalent to the completeness of its projections on subspaces whose orthogonal sum is $\mathcal{H}$. We have

Theorem 3.4. *Let A be a normal reductive operator on a Hilbert space $\mathcal{H}$, and let $\mathcal{G}$ be a countable system of vectors in $\mathcal{H}$. Then, $\{A^n g\}_{g\in\mathcal{G},\, n\geq 0}$ is complete in $\mathcal{H}$ if and only if the system $\left\{N^n_{\mu_j}\widetilde{g}_j\right\}_{g\in\mathcal{G},\, n\geq 0}$ is complete in $(L^2(\mu_j))^{(j)}$ for every $1 \leq j \leq \infty$.*

Proof of Theorem 3.1. Since $\{A^n g\}_{g\in\mathcal{G},\, n\geq 0}$ is complete in $\mathcal{H}$,

$$U\{A^n g : g \in \mathcal{G}, n \geq 0\} = \{(N^n_{\mu_j}\widetilde{g}_j)_{j\in\mathbb{N}^*} : g \in \mathcal{G}, n \geq 0\}$$

is complete in $\mathcal{W} = U\mathcal{H}$. Hence, for every $1 \leq j \leq \infty$, the system $\mathcal{S} = \left\{N^n_{\mu_j}\widetilde{g}_j\right\}_{g\in\mathcal{G},\, n\geq 0}$ is complete in $(L^2(\mu_j))^{(j)}$.

To finish the proof of the first statement of the theorem we use the following lemma which is an adaptation of [27, Lemma 1].

Lemma 3.5. *Let $\mathcal{S}$ be a complete countable set of vectors in $(L^2(\mu_j))^{(j)}$, then for μ_j-almost every z $\{h(z) : h \in \mathcal{S}\}$ is complete in $\ell^2(\Omega_j)$.*

Since S is complete in $(L^2(\mu_j))^{(j)}$, Lemma 3.5 implies that $\{z^n\widetilde{g}_j(z)\}_{g\in\mathcal{G},\, n\geq 0}$ is complete in $\ell^2(\Omega_j)$ for each $j \in \mathbb{N}^*$. But $span\{z^n\widetilde{g}_j(z)\}_{g\in\mathcal{G},\, n\geq 0} = span\{\widetilde{g}_j(z)\}_{g\in\mathcal{G}}$. Thus, we have proved the first part of the theorem.

Now additionally assume that A is also reductive. Let

$$\widetilde{f} \in (L^2(\mu_\infty))^{(\infty)} \oplus L^2(\mu_1) \oplus (L^2(\mu_2))^{(2)} \oplus \cdots$$

and

$$\langle UA^n g, \widetilde{f}\rangle = \sum_{1\leq j\leq\infty} \int_{\mathbb{C}} z^n \langle \widetilde{g}_j(z), \widetilde{f}_j(z)\rangle_{\ell^2(\Omega_j)} d\mu_j(z) = 0$$

for every $g \in \mathcal{G}$ and every $0 \leq n < \infty$. Since the measures μ_j, $1 \leq j \leq \infty$, are mutually singular, we get that

$$\begin{aligned} \sum_{1\leq j\leq\infty} \int_{\mathbb{C}} z^n &< \langle \widetilde{g}_j(z), \widetilde{f}_j(z)\rangle_{\ell^2(\Omega_j)} d\mu_j(z) \\ &= \textstyle\int_{\mathbb{C}} z^n \left[\sum_{1\leq j\leq\infty} \mathbb{1}_{\mathcal{E}_j} \langle \widetilde{g}_j(z), \widetilde{f}_j(z)\rangle_{\ell^2(\Omega_j)}\right] d\mu(z) \end{aligned} \tag{9}$$

for every $g \in \mathcal{G}$ and every $n \geq 0$ with μ as in (4).

Using the fact that the operator A is reductive, from Proposition 2.5, we conclude that

$$\langle \widetilde{g}_j(z), \widetilde{f}_j(z)\rangle_{\ell^2(\Omega_j)} = 0 \;\; \mu_j\text{-a.e. } z.$$

Since, by assumption $\{\widetilde{g}_j(z)\}_{g\in\mathcal{G}}$ is complete in $\ell^2(\Omega_j)$ for μ_j-a.e z, we obtain

$$\widetilde{f}_j = 0 \;\; \mu_j\text{-a.e. } z \qquad \text{for every } j \in \mathbb{N}^*.$$

Thus $\widetilde{f} = 0$ μ-a.e., and therefore, $\{A^n g\}_{g\in\mathcal{G},\, n\geq 0}$ is complete in $\mathcal{H}$. $\square$

4. Minimality property and basis

The goal of this section is to study the conditions on the operator A and the set of vectors $\mathcal{G}$ such that the system $\{A^n g\}_{g\in\mathcal{G},\, 0\le n<L(g)}$ is minimal or a basis for $\mathcal{H}$. We start with the following proposition:

Proposition 4.1. *If A is a normal operator on $\mathcal{H}$ then, for any set of vectors $\mathcal{G}\subset\mathcal{H}$, the system of iterates $\{A^n g\}_{g\in\mathcal{G},n\ge 0}$ is not a complete and minimal system in $\mathcal{H}$.*

Note that Proposition 4.1 is trivial if the $\dim\mathcal{H}<\infty$ and becomes interesting only when $\dim\mathcal{H}=\infty$. As a corollary of Proposition 4.1 we get

Corollary 4.2. *If A is a normal operator on $\mathcal{H}$ then, for any set of vectors $\mathcal{G}\subset\mathcal{H}$, the system of iterates $\{A^n g\}_{g\in\mathcal{G},n\ge 0}$ is not a basis for $\mathcal{H}$.*

If we remove the completeness condition in the statement of Proposition 4.1 above, then the operator $Af=zf$ on the unit circle with arc length measure gives an orthogonal system when iterated on the vector $g\equiv 1$, i.e., for this case $\{z^n g\}_{n\ge 0}$ is minimal since it is an orthonormal system. However, if in addition to being normal, we assume that A is reductive then the statement of proposition 4.1 remains true without the completeness condition since, by Proposition 2.4, the restriction of A onto $cl(\operatorname{span}\{A^n g_{g\in\mathcal{G},n\ge 0})$ will be a normal operator and we will have a minimal complete system contradicting the claim of Proposition 4.1. Thus, we have the following corollary

Corollary 4.3. *If A is a reductive normal operator on $\mathcal{H}$, then, for any countable system of vectors $\mathcal{G}\subset\mathcal{H}$, the system of iterates $\{A^n g\}_{g\in\mathcal{G},n\ge 0}$ is not a minimal system.*

As another corollary of Proposition 4.1, we get

Corollary 4.4. *Let A be a reductive normal operator on $\mathcal{H}$, $\mathcal{G}$ a countable system of vectors in $\mathcal{H}$ and let $L\in\mathcal{L}$. If for some $h\in\mathcal{G}$, $L(h)=\infty$, then the system $\{A^n g\}_{g\in\mathcal{G},\, 0\le n<L(g)}$ is not a basis for $\mathcal{H}$.*

Proof. Let $V=cl(\operatorname{span}\{A^n h\}_{n\ge 0})$ where $L(h)=\infty$. V is a closed invariant subspace for A hence, by Proposition 2.4, the restriction of A on V is also normal, therefore, from Proposition 4.1, $\{A^n h\}_{n\ge 0}$ is not minimal. $\square$

In particular, since $\dim\mathcal{H}=\infty$ (the assumption in this paper), if $|\mathcal{G}|<\infty$, then there exists $g\in\mathcal{G}$ such that $L(g)=\infty$. Thus we have

Corollary 4.5. *Let A be a reductive normal operator. If $|\mathcal{G}|<\infty$, then for any $L\in\mathcal{L}$ the system $\{A^n g\}_{g\in\mathcal{G},\, 0\le n<L(g)}$ is never a basis for $\mathcal{H}$.*

Proof of Proposition 4.1. We prove that if $\{A^n g\}_{g\in\mathcal{G},n\ge 0}$ is complete in $\mathcal{H}$, then for any $m\ge 0$, $\{A^n g\}_{g\in\mathcal{G},n=0,m,m+1,\dots}$ is also complete in $\mathcal{H}$, which implies non-minimality.

Assume $\{A^n g\}_{g\in\mathcal{G},n\ge 0}$ is complete in $\mathcal{H}$. Let $\delta>0$ and $f\in\mathcal{H}$ be a vector such that $\widetilde{f}(z)=0$ for any $z\in\overline{\mathbb{D}}_\delta$ where $\overline{\mathbb{D}}_\delta$ is the closed unit disc of radius δ centered at 0. Then for a fixed m, $\frac{\widetilde{f}}{z^m}$ is in $U\mathcal{H}$ and hence can be approximated arbitrarily closely by finite linear combinations of the vectors in $\{z^n\widetilde{g}\}_{g\in\mathcal{G},n\ge 0}$. Let $\widetilde{f}^{(1)},\widetilde{f}^{(2)},\dots$ be a sequence in $U\mathcal{H}$ such that $\widetilde{f}^{(s)}\to\frac{\widetilde{f}}{z^m}$ in $U\mathcal{H}$ and $\widetilde{f}^{(s)}$ is a finite linear combinations of the vectors in $\{z^n\widetilde{g}\}_{g\in\mathcal{G},n\ge 0}$ for each s. Since z^m is bounded on the spectrum of A, it follows that $z^m\widetilde{f}^{(s)}\to\widetilde{f}$. Finally, we note that $z^m\widetilde{f}^{(s)}$ is a finite linear combination of the vectors $\{z^n\widetilde{g}\}_{g\in\mathcal{G},n\ge m}$.

For a general $f\in\mathcal{H}$, we have that

$$\lim_{\delta\to 0}\|\widetilde{f}-\widetilde{f}\mathbb{1}_{\overline{\mathbb{D}}^c_\delta}\|^2_{L^2(\mu)}=\sum_{j\in\mathbb{N}^*}\|\widetilde{f}_j(0)\|^2_{\ell^2(\Omega_j)}\mu(\{0\})=\mu_{J(0)}(\{0\})\|\widetilde{f}_{J(0)}(0)\|^2_{\ell^2(\Omega_{J(0)})} \tag{10}$$

where $J(0)$ is the value of the multiplicity function defined in (5) at point $z=0$.

From Theorem 3.1, for any $\epsilon > 0$ there exists a finite linear combination $\widetilde{h} \in U\mathcal{H}$ of vectors $\{\widetilde{g}\}_{g\in\mathcal{G}}$ such that $\mu_{J(0)}(\{0\})\|\widetilde{f}_{J(0)}(0)-\widetilde{h}_{J(0)}(0)\|_{\ell^2(\Omega_{J(0)})} < \frac{\epsilon}{2}$. Define $\widetilde{w} := \widetilde{f}-\widetilde{h}$. Using (10) for w, we can pick δ so small that $\|\widetilde{w} - \widetilde{w}\mathbb{1}_{\overline{\mathbb{D}}_\delta^c}\|^2_{L^2(\mu)} < \frac{\epsilon}{2}$. Let $\widetilde{u}$ be a finite linear combination of $\{z^n\widetilde{g}\}_{g\in\mathcal{G},n\geq m}$ such that $\|\widetilde{w}\mathbb{1}_{\overline{\mathbb{D}}_\delta^c} - \widetilde{u}\|^2_{L^2(\mu)} < \frac{\epsilon}{2}$. Then $\|\widetilde{w}-\widetilde{u}\|^2_{L^2(\mu)} < \epsilon$, i.e., $\|\widetilde{f}-\widetilde{h}-\widetilde{u}\|^2_{L^2(\mu)} < \epsilon$. Hence in this case we get that any vector $f \in \mathcal{H}$ is in the closure of the span of $\{\widetilde{g}\}_{g\in\mathcal{G}} \cup \{z^n\widetilde{g}\}_{g\in\mathcal{G},n\geq m} = \{z^n\widetilde{g}\}_{g\in\mathcal{G},n=0,m,m+1,\dots}$. □

If we remove the normality condition in Corollary 4.5, then for the unilateral shift operator S on $\ell^2(\mathbb{N})$ we have $S^n e_1 = e_n$, where e_n is the n-th canonical basis vector, i.e., in this case the iterated system is not only a Riesz basis, but an orthonormal basis.

Even though we cannot have bases for $\mathcal{H}$ by iterations of a countable system $\mathcal{G}$ by a normal operator, when the system $\{A^ng\}_{g\in\mathcal{G},\,0\leq n<L(g)}$ is complete, the non-minimality suggest that we may still have a situation in which the system is a frame leading us to the next section.

5. Complete Bessel systems and frames of iterations

It is shown in [3] that it is possible to construct frames from iteration $\{A^ng\}_{n\geq 0}$ of a single vector g for some special cases when the operator A is an infinite matrix acting on $\ell^2(\mathbb{N})$, has point spectrum and g is chosen appropriately [3]. However, it is also shown that generically, $\{A^ng\}_{n\geq 0}$ does not produce a frame for $\ell^2(\mathbb{N})$. Since a frame must be a Bessel system, we study the Bessel properties of $\{A^ng\}_{g\in\mathcal{G},\,n\geq 0}$ when A is normal. In addition, we find conditions that must be satisfied when the system $\{A^ng\}_{g\in\mathcal{G},\,n\geq 0}$ has the lower frame bound property for the case where $\mathcal{G}$ is finite.

Denote by $\mathbb{D}_r$ the open disk in $\mathbb{C}$ of radius r centered at the origin, by $\overline{\mathbb{D}}_r$ its clousure, and by S_r its boundary, that is $S_r = \overline{\mathbb{D}}_r \setminus \mathbb{D}_r$. For a set $E \subset \mathbb{C}$ we will use the notation $\mathbb{C}\setminus E$ or E^c for the complement of E. Then we have the following theorem.

Theorem 5.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be a normal operator, μ be its scalar spectral measure, and $\mathcal{G}$ a countable system of vectors in $\mathcal{H}$.*

(a) If $\{A^ng\}_{g\in\mathcal{G},\,n\geq 0}$ is complete in $\mathcal{H}$ and for every $g\in\mathcal{G}$ the system $\{A^ng\}_{n\geq 0}$ is Bessel in $\mathcal{H}$, then $\mu\left(\mathbb{C}\setminus\overline{\mathbb{D}}_1\right) = 0$ and $\mu|_{S_1}$ is absolutely continuous with respect to arc length measure (Lebesgue measure) on S_1.

(b) If $\{A^ng\}_{g\in\mathcal{G},\,n\geq 0}$ is frame in $\mathcal{H}$, then $\mu\left(\mathbb{C}\setminus\mathbb{D}_1\right) = 0$.

The converse of Theorem 5.1 is true in the following sense.

Theorem 5.2. *Let $A \in \mathcal{B}(\mathcal{H})$ be a normal operator, and μ be its scalar spectral measure.*

(a) If $\mu\left(\mathbb{C}\setminus\overline{\mathbb{D}}_1\right) = 0$ and $\mu|_{S_1}$ is absolutely continuous with respect to arc length measure on S_1, then there exists a countable set $\mathcal{G}\subset\mathcal{H}$ such that $\{A^ng\}_{g\in\mathcal{G},\,n\geq 0}$ is a complete Bessel system.

(b) If $\mu\left(\mathbb{C}\setminus\mathbb{D}_1\right) = 0$ then there exists a countable set $\mathcal{G}\subset\mathcal{H}$ such that $\{A^ng\}_{g\in\mathcal{G},\,n\geq 0}$ is a Parseval frame for $\mathcal{H}$.

Example 3. *Let A be the convolution operator as in Example 1. If there exists a complete Bessel system by iterations of A, then from Theorem 5.1 (a), $\hat{a}(0)\leq 1$. Conversely, if $\hat{a}(0)\leq 1$, then the conditions in Theorem 5.2 (b) are satisfied and hence there exists a set of vectors $\mathcal{G}\subset L^2(\mathbb{R})$ such that $\{A^ng\}_{g\in\mathcal{G},\,n\geq 0}$ is a Parseval frame in $L^2(\mathbb{R})$. From the proof of the theorem, to construct the set $\mathcal{G}$, we take an orthonormal basis $\mathcal{O}$ in cl $\left((1-|\hat{a}|^2)^{\frac{1}{2}}L^2(\mathbb{R})\right) = L^2(\mathbb{R})$, then $\mathcal{G} = (1-|\hat{a}|^2)\mathcal{O}$. Note that $\mathcal{G}$ is already complete in $L^2(\mathbb{R})$. A natural question will be, what is the smallest $\mathcal{G}$ (in terms of its span closure) such that $\{A^ng\}_{g\in\mathcal{G},\,n\geq 0}$ is a frame? We will see from Theorem 5.6, that $\mathcal{G}$ can not be finite for such a convolution operator since its spectrum is continuous.*

Using the previous two theorems, we get the following necessary and sufficient conditions for the system $\{A^ng\}_{g\in\mathcal{G},\,n\geq 0}$ to be a complete Bessel system in $\mathcal{H}$.

Corollary 5.3. *Let $A \in \mathcal{B}(\mathcal{H})$ be a normal operator, and μ be its scalar spectral measure. Then the following are equivalent.*

(1) There exists a countable set $\mathcal{G} \subset \mathcal{H}$ such that $\{A^n g\}_{g\in\mathcal{G},\, n\geq 0}$ is a complete Bessel system.
(2) $\mu(\mathbb{C}\setminus\mathbb{D}_1) = 0$ and $\mu|_{S_1}$ is absolutely continuous with respect to arc length measure on S_1.

For the case of iterates $\{A^n g\}_{g\in\mathcal{G},\, 0\leq n<L(g)}$, where $L \in \mathcal{L}$ as defined in Remark 3.2, one has the following theorem.

Theorem 5.4. *Let A be a normal operator on a Hilbert space $\mathcal{H}$ and $\mathcal{G}$ a system of vectors in $\mathcal{H}$, and assume $L \in \mathcal{L}$. If $\{A^n g\}_{g\in\mathcal{G},\, 0\leq n<L(g)}$ is a complete Bessel system for $\mathcal{H}$, then for each $g \in \mathcal{G}$ with $L(g) = \infty$, the set $\{x \in \overline{\mathbb{D}}_1^c | \, \widetilde{g}(x) \neq 0\}$ has μ-measure 0.*

When the system $\{A^n g\}_{g\in\mathcal{G},\, n\geq 0}$ has the lower frame bound property and $\mathcal{G}$ is finite, we have the following necessary condition.

Theorem 5.5. *Let $A \in \mathcal{B}(\mathcal{H})$ be a normal operator, and μ be its scalar spectral measure. If $|\mathcal{G}| < \infty$ and $\{A^n g\}_{g\in\mathcal{G},\, n\geq 0}$ satisfies the lower frame bound, then, for every $0 < \epsilon < 1$, $\mu\left(\overline{\mathbb{D}}_{1-\epsilon}^c\right) > 0$.*

As a corollary of 5.1, we get that

Theorem 5.6. *Let A be a bounded normal operator in an infinite dimensional Hilbert space $\mathcal{H}$. If the system of vectors $\{A^n g\}_{g\in\mathcal{G},\, n\geq 0}$ is a frame for some $\mathcal{G} \subset \mathcal{H}$ with $|\mathcal{G}| < \infty$, then $A = \sum_j \lambda_j P_j$ where P_j are projections such that $\operatorname{rank} P_j \leq |\mathcal{G}|$ (i.e. the global multiplicity of A is less than or equal to $|\mathcal{G}|$).*

Combining Theorem 5.6 with the result in [3], where A was assumed to be a diagonal operator on $\ell^2(\mathbb{N})$, we get the following characterization for a general normal operator $A \in \mathcal{B}(\mathcal{H})$, when $|\mathcal{G}| = 1$.

Theorem 5.7. *Let A be a bounded normal operator in an infinite dimensional Hilbert space $\mathcal{H}$. Then $\{A^n g\}_{n\geq 0}$ is a frame for $\mathcal{H}$ if and only if*
i) $A = \sum_j \lambda_j P_j$, where P_j are rank one orthogonal projections.
ii) $|\lambda_k| < 1$ for all k.
iii) $|\lambda_k| \to 1$.
iv) $\{\lambda_k\}$ satisfies Carleson's condition

$$\inf_n \prod_{k\neq n} \frac{|\lambda_n - \lambda_k|}{|1-\bar{\lambda}_n \lambda_k|} \geq \delta. \tag{11}$$

for some $\delta > 0$.
v) $0 < C_1 \leq \frac{\|P_j g\|}{\sqrt{1-|\lambda_k|^2}} \leq C_2 < \infty$, for some constants C_1, C_2.

Example 4. *Let $\mathcal{H} = \ell^2(\mathbb{N})$, A a semi-infinite diagonal matrix whose entries are given by $a_{jj} = \lambda_j = 1 - 2^{-j}$ for $j \in \mathbb{N}$, and let $g \in \ell^2(\mathbb{N})$ be given by $g(j) = \sqrt{1-\lambda_j^2}$. Then, the sequence $\lambda_j = 1-2^{-j}$ satisfies Carleson's condition (see e.g. [23]), and g satisfies condition (v). Thus, $\{A^n g\}_{n\geq 0}$ is a frame for $\ell^2(\mathbb{N})$.*

Remark 5.8. *The following problem is still open in full generality: Let $A = \sum_j \lambda_j P_j$ with $|\lambda_j| < 1$ for all j, $|\lambda_j| \to 1$, and $\sup_j \operatorname{rank} P_j < \infty$. Does there exist a set $\mathcal{G}$ with $|\mathcal{G}| < \infty$ such that $\{A^n g\}_{g\in\mathcal{G}, n\geq 0}$ is a frame for $\mathcal{H}$?*

For the special case defined by (8), we get the following necessary condition on the measure μ.

Theorem 5.9. *Suppose A is a normal operator, and $\{A^n g\}_{g\in\mathcal{G},\, n=0,1,\dots,l(g)}$ (where $l(g)$ is given by (8)) is a complete Bessel system for $\mathcal{H}$. Then*
(a) If $l(g) = \infty$ then $\{x \in \overline{\mathbb{D}}_1^c : \, \widetilde{g}(x) \neq 0\}$ has μ-measure 0.
(b) The restriction of μ on $\overline{\mathbb{D}}_1^c$ is concentrated on at most a countable set, i.e., either $\mu(\overline{\mathbb{D}}_1^c) = 0$, or there exists a countable set $E \subset \overline{\mathbb{D}}_1^c$ such that $\mu|_{\overline{\mathbb{D}}_1^c}\left(E^c \cap \overline{\mathbb{D}}_1^c\right) = 0$.
(c) $\mu|_{S_1}$ is a sum of a discrete and an absolutely continuous measure (with respect to arc length measure) on S_1.

In fact, if for every $g \in \mathcal{G}$, $l(g) < \infty$ then without the condition that the system is Bessel, but with the completeness condition alone, we get that the measure μ is concentrated on a countable subset of $\mathbb{C}$, as stated in the following theorem.

Theorem 5.10. *Let A be a normal operator and $\mathcal{G} \subset \mathcal{H}$ be a system of vectors such that, for every $g \in \mathcal{G}$, $l(g) < \infty$ and $\{A^n g\}_{g\in\mathcal{G},\, n=0,1,\dots,l(g)}$ is complete in $\mathcal{H}$. Then there exists a countable set $E \subset \mathbb{C}$ such that $\mu(E^c) = 0$. Moreover, every g is supported, with respect to the measure μ, on a finite set of cardinality not exceeding $l(g)$.*

5.1. **Proofs of Theorems in Section 5.**

Proof of Theorem 5.1. (a) Suppose $\mu(\overline{\mathbb{D}}_1^c) > 0$, then $\mu_k(\overline{\mathbb{D}}_1^c) > 0$ for some k, $0 \le k \le \infty$. Thus, there exists $\epsilon > 0$ such that $\mu_k(\overline{\mathbb{D}}_{1+\epsilon}^c) > 0$. Since the system of vectors $\{A^n g\}_{g\in\mathcal{G},\, n\ge 0}$ is complete in $\mathcal{H}$, it follows from Theorem 3.1 that there exists a $g \in \mathcal{G}$, such that $\mu_k(\overline{\mathbb{D}}_{1+\epsilon}^c \cap \operatorname{supp}(\widetilde{g}_k)) > 0$.

Let $f \in \mathcal{H}$ be any vector such that $\widetilde{f} = P_k \widetilde{f}$, and $\widetilde{f}(z) = 0$ for $z \in \overline{\mathbb{D}}_{1+\epsilon}$. Then

$$\begin{aligned}
&|\langle f, A^n g\rangle| \\
&= \left| \sum_{0\le j\le\infty} \int_{\mathbb{C}} z^n \langle \widetilde{g}_j(z), \widetilde{f}_j(z)\rangle_{\ell^2(\Omega_j)} d\mu_j(z) \right| \\
&= \left| \int_{\overline{\mathbb{D}}_{1+\epsilon}^c \cap \operatorname{supp}(\widetilde{g}_k)} z^n \langle \widetilde{g}_k(z), \widetilde{f}_k(z)\rangle_{\ell^2(\Omega_k)} d\mu_k(z) \right|.
\end{aligned}$$

For each n, denote by $\lambda_n(f)$ the linear functional on the space $\mathcal{H}_0 := \{f \in \mathcal{H} : \widetilde{f} = P_k\widetilde{f},\ \widetilde{f}(z) = 0 \text{ for } z \in \overline{\mathbb{D}}_{1+\epsilon}\}$, defined by $\lambda_n(f) = \langle f, A^n g\rangle$. The norm of this functional (on $\mathcal{H}_0$) is

$$\begin{aligned}
\|\lambda_n\|_{op}^2 &= \int_{\overline{\mathbb{D}}_{1+\epsilon}^c \cap \operatorname{supp}(\widetilde{g}_k)} |z|^{2n} \|\widetilde{g}_k(z)\|^2_{\ell^2\{\Omega_k\}} d\mu_k(z) \\
&\ge (1+\epsilon)^{2n} \int_{\overline{\mathbb{D}}_{1+\epsilon}^c \cap \operatorname{supp}(\widetilde{g}_k)} \|\widetilde{g}_k(z)\|^2_{\ell^2\{\Omega_k\}} d\mu_k(z).
\end{aligned}$$

Since the right side of the last inequality tends to infinity as $n \to \infty$, so does $\|\lambda_n\|_{op}$. Thus, from the uniform boundedness principle there exists an $f \in \mathcal{H}_0$ such that

$$\lim_{n\to\infty} \left| \int_{\overline{\mathbb{D}}_{1+\epsilon}^c \cap \operatorname{supp}(\widetilde{g}_k)} z^n \langle \widetilde{g}_k(z), \widetilde{f}_k(z)\rangle_{\ell^2(\Omega_k)} d\mu_k(z) \right| = \infty.$$

For such f, $\lambda_n(f) = |\langle f, A^n g\rangle\| \to \infty$ as $n \to \infty$. Thus, we also have that

$$\sum_{n=0}^{\infty} |\langle f, A^n g\rangle|^2 = \infty$$

which is a contradiction to our assumption that $\{A^n g\}_{n\ge 0}$ is a Bessel system in $\mathcal{H}$.

To prove the second part of the statement, let $k \ge 1$ be fixed, and consider the Lebesgue decomposition of $\mu_k|_{S_1}$ given by $\mu_k|_{S_1} = \mu_k^{\mathrm{ac}} + \mu_k^s$ where μ_k^{ac} is absolutely continuous with respect to arc length measure on S_1, μ_k^s is singular and $\mu_k^{\mathrm{ac}} \perp \mu_k^s$. We want to show that $\mu_k^s \equiv 0$.

For a vector $a = (a_1, a_2, \dots, a_k) \in \ell^2(\Omega_k)$ define $Q_r a := a_r$. Fix $1 \le r \le k$ and $m \ge 1$. Let $f \in \mathcal{H}$ be the vector such that

i) $Q_r \widetilde{f}_k(e^{2\pi i t}) = e^{2\pi i m t}$, μ_k^s-a.e.
ii) $Q_r \widetilde{f}_k(e^{2\pi i t}) = 0$, μ_k^{ac}-a.e.
iii) $Q_s \widetilde{f}_j(z) = 0$ if $r \ne s$ or $k \ne j$
iv) $\widetilde{f}(z) = 0$ for $z \notin S_1$.

Then for such an f and a fixed $g \in \mathcal{G}$, from the assumption that $\{A^n g\}_{n\geq 0}$ is a Bessel system in $\mathcal{H}$, we have

$$\begin{aligned}
&\sum_{n\geq 0}\left|\int_{S_1} e^{2\pi i n t} Q_r \widetilde{g}_k(e^{2\pi i t})\overline{e^{2\pi i m t}}\, d\mu_k^s(e^{2\pi i t})\right|^2 \\
&= \sum_{n\geq 0}\left|\sum_{0\leq j\leq\infty}\int_{\mathbb{C}} z^n \langle \widetilde{g}_j(z), \widetilde{f}_j(z)\rangle_{\ell^2(\Omega_j)}\, d\mu_j(z)\right|^2 \\
&= \sum_{n=0}^{\infty} |\langle A^n g, f\rangle|^2 \leq C\|f\|^2 \leq C\mu(S_1).
\end{aligned}$$

Thus

$$\sum_{n\geq 0}\left|\int_{S_1} e^{2\pi i (n-m) t} Q_r \widetilde{g}_k(e^{2\pi i t})\, d\mu_k^s(e^{2\pi i t})\right|^2 \leq C\mu(S_1).$$

Since the last inequality holds for every $m \geq 1$, we have

$$\sum_{n\in\mathbb{Z}}\left|\int_{S_1} e^{2\pi i n t} Q_r \widetilde{g}_k(e^{2\pi i t})\, d\mu_k^s(e^{2\pi i t})\right|^2 \leq C\mu(S_1).$$

This means the Fourier-Stieltjes coefficients of the measure $Q_r\widetilde{g}_k(e^{2\pi i t})d\mu_k^s(e^{2\pi i t})$ are in $\ell^2(\mathbb{Z})$. Hence, from the uniqueness theorem of the Fourier Stieltjes coefficients ([26], p. 36) and the fact that any element of $\ell^2(\mathbb{Z})$ determines Fourier coefficients of an $L^2(S_1)$ function (with respect to arc length measure), $Q_r\widetilde{g}_k(e^{2\pi i t})d\mu_k^s(e^{2\pi i t})$ is absolutely continuous with respect to the arc length measure. But the measure μ_k^s is concentrated on a measure zero set as a singular measure, hence $Q_r\widetilde{g}_k(e^{2\pi i t})d\mu_k^s(e^{2\pi i t})$ is the zero measure. Since the system $\{A^n g\}_{g\in\mathcal{G},\, n\geq 0}$ is complete in $\mathcal{H}$, from Theorem 3.1 we obtain that $\mu_k^s = 0$ and hence μ_k is absolutely continuous with respect to the arc length measure on S_1. Thus μ is absolutely continuous with respect to the arc length measure on S_1.

(b) Suppose $\{A^n g\}_{g\in\mathcal{G},\, n\geq 0}$ is a frame with frame bounds α and β. Let $f \in \mathcal{H}$ be any vector such that $\widetilde{f} = 0$ on $\mathbb{C}\setminus S_1$. For such an f we have that $\|(A^*)^m f\| = \|A^m f\| = \|f\|$ for any $m \in \mathbb{Z}$. Thus, for any $m \in \mathbb{Z}$, we have

$$\begin{aligned}
\alpha\|f\| = \alpha\|(A^*)^m f\| &\leq \sum_{g\in\mathcal{G}}\sum_{n=0}^{\infty}|\langle (A^*)^m f, A^n g\rangle|^2 = \sum_{g\in\mathcal{G}}\sum_{n=0}^{\infty}|\langle f, A^{n+m} g\rangle|^2 \\
&= \sum_{g\in\mathcal{G}}\sum_{n=m}^{\infty}|\langle f, A^n g\rangle|^2 \leq \beta\|A^m f\| \leq \beta\|f\|.
\end{aligned} \tag{12}$$

Since (12) holds for every m, the right inequality implies $\sum_{n=m}^{\infty}\sum_{g\in\mathcal{G}}|\langle f, A^n g\rangle|^2 \to 0$ as $m \to \infty$. Hence, using the left inequality we conclude that $\|f\| = 0$. Since f is such that $\widetilde{f} = 0$ on $\mathbb{C}\setminus S_1$, but otherwise is arbitrary, it follows that $\mu(S_1) = 0$. But, from Part (a), we already know that $\mu(\mathbb{C}\setminus\overline{\mathbb{D}}_1) = 0$, hence $\mu(\mathbb{C}\setminus\mathbb{D}_1) = 0$. □

Proof of Theorem 5.2. Let $\mathcal{H}_1 = \{f \in \mathcal{H} : \widetilde{f}(z) = 0, z \notin S_1\}$ and $\mathcal{H}_2 = \{f \in \mathcal{H} : \widetilde{f}(z) = 0, z \notin \mathbb{D}_1\}$. Then $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Let $\mathcal{G}_i \subset \mathcal{H}_i$, be complete Bessel systems in $\mathcal{H}_i$, $i = 1, 2$, then, it is not difficult to see that $\mathcal{G}_1 \cup \mathcal{G}_2$ is a complete Bessel system in $\mathcal{H}$. We will proceed by constructing complete Bessel systems for $\mathcal{H}_1$ and $\mathcal{H}_2$. To construct a complete Bessel sequence for $\mathcal{H}_1$, we first consider the operator $N_{\mu_j|_{S_1}}$ on $L^2(\mu_j|_{S_1})$ for a fixed j, with $1 \leq j \leq \infty$, where μ_j is as in the decomposition of Theorem 2.1. Since for $f \in \mathcal{H}_1$, $\widetilde{f}(z) = 0$ for $z \notin S_1$, and since $\mu|_{S_1}$ (and hence also $\mu_j|_{S_1}$) is absolutely continuous with respect to the arc lengh measure σ, we have that on the circle S_1, $d\mu_j|_{S_1} = w_j d\sigma$ for some $w_j \in L^1(\sigma)$. Hence on the support E_j of w_j, μ_j and σ are mutually absolutely continuous, i.e., for ν_j defined by $d\nu_j = \mathbb{1}_{E_j} d\sigma$, μ_j and ν_j are mutually absolutely continuous.

Now consider the two functions p_j and q_j such that $p_j(z) = q_j(z) = 0$ for $z \notin S_1$, while on S_1, $p_j(e^{2\pi it}) = \mathbb{1}_{[0,\frac{1}{2}]}(t)$ and $q_j(e^{2\pi it}) = \mathbb{1}_{[\frac{1}{2},1]}(t)$. From the properties of the Fourier series on $L^2(S_1,\sigma)$, the sets $\{z^n p_j(z)\}_{n\geq 0}$ and $\{z^n q_j(z)\}_{n\geq 0}$ are Bessel systems in $L^2(S_1,\sigma)$ with bound 1. Thus, $\{z^n p_j(z)\}_{n\geq 0}$ and $\{z^n q_j(z)\}_{n\geq 0}$ are also Bessel systems in $L^2(S_1,\nu_j)$ with bound 1. Therefore, $\{z^n p_j(z)\}_{n\geq 0} \cup \{z^n q_j(z)\}_{n\geq 0}$ is a Bessel system for $\oplus_{j=1}^{\infty} L^2(S_1,\nu_j)$. By Proposition 2.6 and Theorem 2.1 the sytem $\{z^n p_j(z)\}_{n\geq 0} \cup \{z^n q_j(z)\}_{n\geq 0}$ is also complete in $\oplus_{j=1}^{\infty} L^2(S_1,\nu_j)$. Thus, $\{z^n p_j(z)\}_{n\geq 0} \cup \{z^n q_j(z)\}_{n\geq 0}$ is a complete Bessel system for $\oplus_{j=1}^{\infty} L^2(S_1,\nu_j)$.

Since μ_j and ν_j are mutually absolutely continuous, the multiplication operator z on $\oplus_{j=1}^{\infty} L^2(S_1,\nu_j)$ is unitarily equivalent to the multiplication operator z on $\oplus_{j=1}^{\infty} L^2(S_1,\mu_j)$ which we denote by V. Hence, $\{z^n V(p_j)(z)\}_{n\geq 0} \cup \{z^n V(q_j)(z)\}_{n\geq 0}$ is a complete Bessel system for $\oplus_{j=1}^{\infty} L^2(S_1,\mu_j)$. Finally, using Theorem 2.1, it follows that $\{A^n U^{-1} V(p_j(z))\}_{n\geq 0} \cup \{A^n U^{-1} V(q_j(z))\}_{n\geq 0}$ forms a complete Bessel system for $U\mathcal{H}_1 = \oplus_{j=1}^{\infty} L^2(S_1,\mu_j)$.

The existence of complete Bessel system in $\mathcal{H}_2$ (moreover, a Parseval frame) follows from Part (b) of Theorem 5.2 which we prove next.

(b) Let D be the operator $(I - AA^*)^{-\frac{1}{2}}$. Let $\mathcal{O}$ be an orthonormal basis for $cl(D\mathcal{H})$, and define $\mathcal{G} = \{g = Dh : h \in \mathcal{O}\}$. Then

$$\begin{aligned}\sum_{n=0}^{m}\sum_{h\in\mathcal{O}} |\langle f, A^n Dh\rangle|^2 &= \sum_{n=0}^{m}\sum_{h\in\mathcal{O}} |\langle D(A^*)^n f, h\rangle|^2 = \sum_{n=0}^{m} \|D(A^*)^n f\|^2 \\ &= \sum_{n=0}^{m} \langle D^2(A^*)^n f, (A^*)^n f\rangle = \sum_{n=0}^{m} \langle (I - AA^*)(A^*)^n f, (A^*)^n f\rangle \\ &= \|f\|^2 - \|(A^*)^{m+1} f\|.\end{aligned}$$

Using Lebesgue's Dominated Convergence Theorem, $\|(A^*)^m f\|^2 = \int_{\overline{\mathbb{D}}_1} |z|^{2m} \|\widetilde{f}(z)\|^2 d\mu(z) \to 0$ as $m \to \infty$ since $|z|^{2m} \to 0$, $\mu - a.e.$ on $\overline{\mathbb{D}}_1$. Hence, from the identity above we get that

$$\sum_{n=0}^{\infty}\sum_{h\in\mathcal{I}} |\langle f, A^n Dh\rangle|^2 = \|f\|^2.$$

Therefore the system of vectors $\mathcal{G} = \{g = Dh : h \in \mathcal{O}\}$ is a tight frame for $\mathcal{H}$. □

The proof of Theorem 5.4 is a direct consequence of the proof of (a) in the above theorem.

Proof of Theorem 5.5. Suppose $\mu(\overline{\mathbb{D}}_{1-\epsilon}^c) = 0$ for some $0 < \epsilon < 1$. Because $|\mathcal{G}| < \infty$ and $\dim(\mathcal{H}) = \infty$, the system $\{A^n g\}_{g\in\mathcal{G},\, n=0,1,\dots,M}$ is not complete in $\mathcal{H}$ for $M < \infty$. From the Hahn-Banach theorem, there exists a vector $h \in \mathcal{H}$ with $\|h\| = 1$ such that $\langle A^n g, h\rangle = 0$ for every $g \in \mathcal{G}$, and $n = 0,\dots,M$. Then

$$\begin{aligned}&\sum_{g\in\mathcal{G}}\sum_{n=0}^{\infty} |\langle h, A^n g\rangle| \\ &= \sum_{g\in\mathcal{G}}\sum_{n=M+1}^{\infty} \Big| \sum_{0\leq j\leq\infty} \int_{\mathbb{C}} z^n \langle \widetilde{h}_j(z), \widetilde{g}_j(z)\rangle_{\ell^2(\Omega_j)} d\mu_j(z)\Big|^2 \\ &\leq \sum_{g\in\mathcal{G}}\sum_{n=M+1}^{\infty} \Big| \int_{\overline{\mathbb{D}}_{1-\epsilon}} z^n \sum_{0\leq j\leq\infty} \mathbb{1}_{\mathcal{E}_j}(z) \langle \widetilde{h}_j(z), \widetilde{g}_j(z)\rangle_{\ell^2(\Omega_j)} d\mu(z)\Big|^2 \\ &\leq \sum_{g\in\mathcal{G}}\sum_{n=M+1}^{\infty} (1-\epsilon)^{2n} \Big(\sum_{0\leq j\leq\infty} \int_{\mathbb{C}} |\langle \widetilde{h}_j(z), \widetilde{g}_j(z)\rangle_{\ell^2(\Omega_j)}| d\mu_j(z)\Big)^2.\end{aligned}$$

Applying Hölder's inequality several times, we get

$$\begin{aligned}
&\sum_{0\le j\le\infty}\int_{\mathbb{C}}|\langle\widetilde{h}_j(z),\widetilde{g}_j(z)\rangle_{\ell^2(\Omega_j)}|d\mu_j(z)\\
\le\;& \sum_{0\le j\le\infty}\int_{\mathbb{C}}\|\widetilde{h}_j(z)\|_{\ell^2(\Omega_j)}\|\widetilde{g}_j(z)\|_{\ell^2(\Omega_j)}d\mu_j(z)\\
\le\;& \sum_{0\le j\le\infty}\Big(\int_{\mathbb{C}}\|\widetilde{h}_j(z)\|^2_{\ell^2(\Omega_j)}d\mu_j(z)\Big)^{\frac12}\Big(\int_{\mathbb{C}}\|\widetilde{g}_j(z)\|^2_{\ell^2(\Omega_j)}d\mu_j(z)\Big)^{\frac12}\\
\le\;& \|h\|\|g\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{g\in\mathcal{G}}\sum_{n=0}^{\infty}|\langle h,A^n g\rangle|^2\le\sum_{g\in\mathcal{G}}\sum_{n=M+1}^{\infty}(1-\epsilon)^{2n}\|h\|^2\|g\|^2\\
=&\frac{(1-\epsilon)^{2(M+1)}}{1-(1-\epsilon)^2}\|h\|^2\sum_{g\in\mathcal{G}}\|g\|^2\ \to 0\quad\text{as}\quad M\to\infty.
\end{aligned}$$

Therefore the left frame inequality does not hold, and we have a contradiction. □

Proof of Theorem 5.6. Define the subspace V_ρ of $\mathcal{H}$ to be $V_\rho=\{f:\operatorname{supp}\tilde{f}\subseteq\overline{\mathbb{D}}_\rho\}$. The restriction of A to V_ρ is a normal operator with its spectrum equal to the part of the spectrum of A inside $\overline{\mathbb{D}}_\rho$. Let $\widetilde{\mathcal{G}}=U\mathcal{G}$ where U is as in Theorem 2.1. Let $\widetilde{\mathcal{G}}_\rho=\{\mathbb{1}_{\overline{\mathbb{D}}_\rho}\widetilde{g}:\widetilde{g}\in\widetilde{\mathcal{G}}\}$. Since $\{A^n g\}_{g\in\mathcal{G},n\ge0}$ is a frame by assumption, $\{z^n\widetilde{w}\}_{\widetilde{w}\in\widetilde{\mathcal{G}}_\rho}$ is a frame for UV_ρ. Thus, since $\rho<1$, Theorem 5.5 implies that V_ρ is finite dimensional. Hence the restriction of the spectrum of A to $\overline{\mathbb{D}}_\rho$ for any $\rho<1$ is a finite set of points. We also know from Theorem 5.1 (b) that $\mu(\mathbb{D}_1^c)=0$. Thus, UAU^{-1} has the form $\Lambda=\sum_j\lambda_jP_j$. □

Proof of Theorem 5.9. (a) Follows from Theorem 5.4.

(b) If $l(g)<\infty$ then $A^{l(g)}g-\sum_{k=0}^{l(g)-1}c_kA^kg=0$ for some complex numbers c_k. Call Q the polynomial $Q(z):=z^{l(g)}-\sum_{k=0}^{l(g)-1}c_kz^k$. We have $Q(A)g=0$ and therefore $0=U(Q(A)g)(z)=Q(z)\widetilde{g}(z)$ μ–a.e. z.

Let E_g be the set of roots of Q. Hence $\widetilde{g}(z)=0$ μ a.e. in $(\mathbb{C}\setminus E_g)$. This together with part (a) of the theorem gives us that, for all $g\in\mathcal{G}$,

$$\widetilde{g}(z)=0\quad\text{a.e. }\mu\quad\text{in}\ \bigcap_{g\in\mathcal{G}_F}(\overline{\mathbb{D}}_1^c\setminus E_g)\tag{13}$$

where, $\mathcal{G}_F=\{g\in\mathcal{G}:l(g)<\infty\}$.

The set $E:=\bigcup_{g\in\mathcal{G}_F}E_g$ is countable and $\bigcap_{g\in\mathcal{G}_F}(\overline{\mathbb{D}}_1^c\setminus E_g)=\overline{\mathbb{D}}_1^c\setminus E$. So (13) holds on $\overline{\mathbb{D}}_1^c\setminus E$. It follows that for each $j\in\mathbb{N}^*$, span $\{\widetilde{g}_j(z)\}_{g\in\mathcal{G}}$ is not complete in $\ell^2(\Omega_j)$ μ_j – a.e. $z\in\overline{\mathbb{D}}_1^c\setminus E$ and therefore $\mu_j(\overline{\mathbb{D}}_1^c\setminus E)=0$. We conclude that $\mu(\overline{\mathbb{D}}_1^c\setminus E)=0$.

(c) Let $\Delta:=\{x\in S_1:x\in\operatorname{supp}g,\ g\in\mathcal{G},\ l(g)<\infty\}$. From the proof of (b) Δ is countable. Then, since the projection of a Bessel system is Bessel, and the projection of a complete set is complete, following the proof of Theorem 5.1(a) we can see that μ is absolutely continuous on $S_1\setminus\Delta$. □

6. Self-adjoint operators

The class of self-adjoint operators is an important subclass of normal reductive operators which has some interesting properties that we study in this section. In particular, we prove that for self-adjoint operators the normalized system $\left\{\frac{A^ng}{\|A^ng\|}\right\}_{g\in\mathcal{G},\ n\ge0}$ is never a frame. The proof of this fact relies on the following theorem.

Theorem 6.1. *Every unit norm frame is a finite union of Riesz basis sequences.*

Theorem 6.1 was conjectured by Feichtinger and is equivalent to the Kadison-Singer theorem [8, 7] which was proved recently in [30].

Theorem 6.2. *If A is a self-adjoint operator on $\mathcal{H}$ then the system* $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathcal{G},\ n\geq 0}$ *is not a frame for $\mathcal{H}$.*

Remark 6.3. *An open problem is whether the theorem remains true for general normal operators. The theorem does not hold if the operator is not normal. For example, the shift operator S on $\ell^2(\mathbb{N})$ defined by $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$, is not normal, and $\{S^n e_1\}$ where $e_1 = (1, 0, \dots)$ is an orthonormal basis for $\ell^2(\mathbb{N})$.*

Remark 6.4. It may be that the system $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathcal{G},\ n\geq 0}$ is not a frame for $\mathcal{H}$ because it is overly redundant due to the fact that we are iterating $\{A^n g\}_{g\in\mathcal{G}}$ for all $n \geq 0$. We may reduce the redundancy by letting $0 \leq n < L(g)$ where $L \in \mathcal{L}$ as defined in Remark 3.2. For example if $\{g\}_{g\in\mathcal{G}}$ is an orthonormal basis for $\mathcal{H}$, then trivially, we can choose $L(g) = 1$ and the system $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathcal{G}, 0\leq n<L(g)}$ is an orthonormal basis for $\mathcal{H}$. However, if $\mathcal{G}$ is finite, $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathcal{G}, 0\leq n<L(g)}$ cannot be a frame for $\mathcal{H}$ as in the corollary below.

Corollary 6.5. *Let $\{g\}_{g\in\mathcal{G}} \subset \mathcal{H}$ and assume that $|\mathcal{G}| < \infty$ and $L \in \mathcal{L}$. Then for a self-adjoint operator A,* $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathcal{G}, 0\leq n<L(g)}$ *is not a frame for $\mathcal{H}$.*

Proof of Theorem 6.2. Suppose it is a frame. Using Feichtinger's theorem, we decompose the set $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathcal{G},\ n\geq 0}$ into a finite union of Riesz sequences. Choose a vector $h \in \mathcal{G}$. Thus the subsystem $\left\{\frac{A^n h}{\|A^n h\|}\right\}_{n\geq 0}$ can be decomposed into a union of Riesz sequences and therefore a union of minimal sets. Since there are finitely many sequences, the powers of A in one of these sequences must contain infinite number of even numbers $\{2n_k\}$ (in particular, the system $\{A^{2n_k} h\}_{k=1,\dots}$ is a minimal set) such that

$$\sum_{k\geq 1} \frac{1}{n_k} = \infty. \tag{14}$$

If we consider the operator A^2, then its spectrum is a subset of $[0, \infty)$. In order to finish the proof of the Theorem, we use the following Lemma whose proof is a corollary of the Müntz-Szász theorem [35].

Lemma 6.6. *Let μ be a regular Borel measure on $[0, \infty)$ with a compact support and n_k, $k = 0, 1, \dots,$ be a sequence of natural numbers such that $n_0 = 0$ and*

$$\sum_{k\geq 1} \frac{1}{n_k} = \infty.$$

For a function $\phi \in L^1(\mu)$, if

$$\int_0^\infty x^{n_k} \phi(x) d\mu(x) = 0 \textit{ for every } k,$$

then $\phi = 0$ μ a.e..

Let $V = cl(span\{(A^2)^n h\}_{n\geq 0})$, and let B be the restriction of A^2 on V. Since B is positive definite, its spectrum $\sigma(B) \subset [0, b]$ for some $b \geq 0$. Let μ be the measure defined in (4) associated with B. By Theorem 3.1, $\mu_j = 0$ for all $j \neq 1$ (i.e., $\mu = \mu_1$), and $\widetilde{h}(x) \neq 0$ a.e. μ.

Let $n_k, k \geq 1$ be the sequence of integers chosen above such that $\{A^{2n_k} h\}_{k=1,\dots}$ is a minimal set and (14) holds. Set $n_0 = 0$. Note that both sequences $\{n_k\}_{k\geq 0}$, and $\{n_k\}_{k=0,m,m+1,\dots}$ satisfy the condition of the Lemma 6.6, hence $\int_0^b x^{n_k} \widetilde{h}(x) \overline{\widetilde{f}(x)} d\mu(x) = 0$ for all $k \geq 0$ implies that $\widetilde{f} = 0$

a.e. μ, as well as $\int_0^b x^{n_k}\widetilde{h}(x)\overline{\widetilde{f}(x)}d\mu(x) = 0$ for all $k = 0, m, m+1, \dots$ implies that $\widetilde{f} = 0$ a.e. μ. Thus, $V = cl(span\{(A^2)^n h\}_{n\geq 0}) = cl(span\{(A^2)^{n_k} h\}_{k=0,m,m+1,\dots}) = cl(span\{(A^2)^{n_k} h\}_{k\geq 0})$ which contradicts the minimality condition. □

Proof of Corollary 6.5. suppose $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathcal{G},\, 0\leq n<L(g)}$ is a frame for $\mathcal{H}$. Because $\dim \mathcal{H} = \infty$, the set $\mathcal{G}_\infty = \{g \in \mathcal{G} |\ L(g) = \infty\}$ is non-empty. Then the system $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathcal{G}_\infty, 0\leq n<L(g)=\infty}$ is a frame for its closure since we get it by removing finite number of vectors from a frame. The closure is an invariant subspace and A restricted to it remains self-adjoint which contradicts Theorem 6.2. □

7. Applications to groups of unitary operators

In this section, we apply some of our results to discrete groups of unitary operators. These often occur in wavelet, time frequency and frame constructions.

As a corollary of the spectral theorem of normal operators, a normal operator is unitary if and only if its spectrum is a subset of the unit circle. We will need the following Proposition from Wermer [41].

Proposition 7.1 ([41])**.** *For a unitary operator T the following are equivalent*

(1) T is not reductive
(2) The arc length measure is absolutely continuous with respect to the spectral measure of T.

Let π be a unitary representation of a discrete group Γ on Hilbert space $\mathcal{H}$. The order $o(\gamma)$ of an element $\gamma \in \Gamma$ is the smallest natural number m such that $\gamma^m = 1$. If no such number exists then we say $o(\gamma) = \infty$. The same way we define the order of an operator $\pi(\gamma)$.

Notice that if $o(\pi(\gamma)) < \infty$ then it is reductive and its spectrum is a subset of the set of $o(\pi(\gamma))$-th roots of unity.

Theorem 7.2. *Let π be a unitary representation of a discrete group Γ on Hilbert space $\mathcal{H}$ and suppose there exists a set of vectors $\mathcal{G} \subseteq \mathcal{H}$, such that $\{\pi(\gamma)g :\ \gamma \in \Gamma, g \in \mathcal{G}\}$ is a minimal system. Then, for every $\gamma \in \Gamma$ with $o(\gamma) = \infty$, $\pi(\gamma)$ is non-reductive and hence the arc length measure on S_1 is absolutely continuous with respect to the spectral measure of $\pi(\gamma)$.*

Proof. The minimality condition implies that π is injective and hence $o(\gamma) = o(\pi(\gamma))$. Let $\gamma \in \Gamma$ be such that $o(\gamma) = \infty$, then from the minimality assumption, $\{\pi(\gamma)^n g :\ \gamma \in \Gamma, g \in \mathcal{G}\}$ is a minimal subsystem. Thus, from Corollary 4.3, $\pi(\gamma)$ is non-reductive. The rest follows from Proposition 7.1 above. □

Theorem 7.3. *Let π be a unitary representation of a discrete group Γ on Hilbert space $\mathcal{H}$ and suppose there exists a set of vectors $\mathcal{G} \subseteq \mathcal{H}$, such that $\Gamma\{\mathcal{G}\} = \{\pi(\gamma)g :\ \gamma \in \Gamma, g \in \mathcal{G}\}$ is complete in $\mathcal{H}$ and, for every $g \in \mathcal{G}$, $\Gamma\{g\} = \{\pi(\gamma)g :\ \gamma \in \Gamma\}$ is a Bessel system in $\mathcal{H}$. Then for every $\gamma \in \Gamma$ with $o(\gamma) = \infty$, the measure μ associate with $\pi(\gamma)$ is absolutely continuous with respect to the arc length measure on S_1.*

Proof. Suppose $o(\gamma) = \infty$. The assumption that the system $\{\pi(\gamma)g :\ \gamma \in \Gamma\}$ is Bessel implies that the kernel of the representation π must be finite, otherwise any vector in the system will be repeated infinitely many times, prohibiting the Bessel property from holding. Thus $o(\gamma) = \infty$ implies $o(\pi(\gamma)) = \infty$.

Pick any vector $\pi(h)g$ where $h \in \Gamma, g \in \mathcal{G}$. Then $\{\pi(\gamma)^n \pi(h)g\}_{n\geq 0}$ is a subsystem of $\Gamma\{g\}$ since $\pi(\gamma)^n\pi(h) \neq \pi(\gamma)^m\pi(h)$ if $n \neq m$. Hence, using the fact that $\{\pi(\gamma)^n\pi(h)g\}_{n\geq 0}$ is a Bessel sequence, from the proof of Theorem 5.1(a) we get that, for every $j \in \mathbb{N}^*$, the measure μ_j in the (3) representation of $\pi(\gamma)$ is absolutely continuous on $\operatorname{supp}[(\pi(h)g)\breve{}_j]$. Since $\{\pi(h)g :\ h \in \Gamma, g \in \mathcal{G}\}$ is complete in $\mathcal{H}$, from Theorem 3.1, μ is concentrated on the set $\cup_{0\leq j\leq\infty}\operatorname{supp}[(\pi(h)g)\breve{}_j]$ thus we get that the spectrum of $\pi(\gamma)$ is absolutely continuous with respect to arc length measure. □

In fact, it was shown in [40] (lemma 4.19) that the assumptions in the previous theorem hold if and only if π is a subrepresentation of the left regular representation of Γ with some multiplicity. And as a corollary of that, if the conditions of Theorem 7.3 hold, it is possible to find another set $\mathcal{G}' \subset \mathcal{H}$ such that $\{\pi(\gamma)g : \gamma \in \Gamma, g \in \mathcal{G}'\}$ is a Parseval frame for $\mathcal{H}$.

8. Concluding remarks

In this paper we consider a system of iterations of the form $\{A^n g \mid g \in \mathcal{G},\ 0 \leq n < L(g)\}$ where A is a normal bounded operator in a Hilbert space $\mathcal{H}$, $\mathcal{G} \subset \mathcal{H}$ is countable set of vectors, and where L is a function defined in Definition 3.2. The goal is to find relations between the operator A, the set $\mathcal{G} \subset \mathcal{H}$, and the function L that makes the system of iterations $\{A^n g \mid g \in \mathcal{G},\ 0 \leq n < L(g)\}$ complete, Bessel, a basis, or a frame. Although we have exhibited some of these relations for the case of normal operators and reductive normal operators, there are many open questions. For example, a necessary condition as to when the system of iterations $\{A^n g \ \ n \geq 0\}$ is a frame for $\mathcal{H}$ is derived, but the necessary and sufficient conditions is only answered for $\mathcal{H} \in \ell^2(\mathbb{N})$ and when A is essentially a compact self-adjoint operator (see [3]). In fact, some of the problems in this work are connected to the still open invariant subspace problem in Hilbert spaces. The connection between sampling theory and frame theory, and some of the problems in operator algebras, and spectral theory makes the Dynamical Sampling problem, which is the underlying problem driving this work, a fertile ground for interaction between these areas of mathematics.

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(Akram Aldroubi) Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240-0001 USA
E-mail address: aldroubi@math.vanderbilt.edu

(Carlos Cabrelli) Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina and IMAS, UBA-CONICET, Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina
E-mail address: cabrelli@dm.uba.ar

(Ahmet Faruk Çakmak) Department of Mathematical Engineering, Yildiz Technical Univ., Davutpaşa Campus, Esenler, İstanbul, 80750, Turkey, Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240-0001 USA
E-mail address: acakmak@yildiz.edu.tr

(Ursula Molter) Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina and IMAS, UBA-CONICET, Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina
E-mail address: umolter@dm.uba.ar

(Armenak Petrosyan) Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240-0001 USA
E-mail address: armenak.petrosyan@vanderbilt.edu