Cosmological singularity theorems for $f(R)$ gravity theories

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Abstract

In the present work some generalizations of the Hawking singularity theorems in the context of $f(R)$ theories are presented. The main assumptions are: the matter fields stress energy tensor satisfies the condition $\left(T_{ij}-\frac{g_{ij}}{2}T\right)k^i k^j \geq 0$ for any generic unit time like field k^i; the scalaron takes bounded positive values during its evolution and the resulting space time is globally hyperbolic. Then, if there exist a Cauchy hyper-surface Σ for which the expansion parameter θ of the geodesic congruence emanating orthogonally from Σ satisfies some specific bounds, then the resulting space time is geodesically incomplete. Some mathematical results of reference [62] are very important for proving this. The generalized theorems presented here apply directly for some specific models such as the Hu-Sawicki or Starobinsky ones [15], [21]. For other scenarios, some extra assumptions should be implemented in order to have a geodesically incomplete space time. The hypothesis considered in this text are sufficient, but not necessary. In other words, their negation does not imply that a singularity is absent.

1. Introduction

One of the most interesting phenomena in cosmology is the accelerated expansion of the universe [1]. Some models for explaining this behavior are the so called quintessence mechanisms [2]-[4]. These models include certain components which induce a repulsive term imitating the effect of a cosmological constant [5]. Another problem of importance in cosmology arises when studying the behavior of the galactic rotation curves and the mass discrepancy in clusters of galaxies [6]. The observations show that the mass increases linearly with the radius near the galaxy center. The estimated baryonic mass density seems not enough for explaining this, and the mass discrepancy is usually explained by postulating the existence of dark matter. This component is assumed to be neutral under electromagnetic interaction, cold and pressureless. Several dark matter scenarios include weakly interacting massive particles (WIMP) [7]-[10]. These components may be light or superheavy, but should interact with ordinary matter by a very weak interaction in order not to be detected by the current accelerator technology.

It is clear from the discussion given above that the confrontation between theory and experiments in GR is a fundamental task. A review can be found for instance in [11]. A further possibility to explain these experiments is that GR breaks down at a typical galaxy scale. Some experimental consequences about modified gravity theories were discussed long time ago in the book [12], and more recently in [13] and [34]. Some of these modified scenarios are $f(R)$ theories, and the present

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work is focused on them. Several theoretical predictions were studied for these models. For instance, corrections of the form $f(R) \sim R + \alpha R^2$ have been shown to cast an early inflationary period [14], while $f(R) \sim R + \alpha R^{-1}$ may induce a present accelerated period [15]. However, for functions $f(R)$ in powers of R, it has been shown that the resulting scale factor has a time dependence of the form $a(t) \sim t^{1/2}$ instead of $a(t) \sim t^{2/3}$ for small and large curvature [16]-[18]. Nevertheless, these theories describe a matter dominated period followed by an accelerated expansion, and although several models have been discarded, it was suggested in [19]-[50] that viable models do exist.

A serious problem related to curvature singularities was pointed out in [51]-[55] and [56]. As is well known, $f(R)$ theories in the metric formulation reduce, by means of a suitable conformal transformation, to GR plus an scalar field φ, known as the scalaron. This scalar is under the influence of a potential term $U(\varphi)$, and couples non trivially to the matter fields. The problem is that, for $f(R)$ theories which reduce to GR in the large R limit, the potential term corresponds to an unprotected curvature singularity. The presence of such singularities has disastrous observational consequences, in particular it may not allow the formation of relativistic stars such as neutron ones. However, there exist some models which seems to avoid these problems and may be realistic as well, such as the ones presented in [57].

Another relevant point is the presence of singularities in the cosmological context. For GR, one has the powerful singularity results of [60], and it may be desirable to generalize them to other gravity theories. The interest in this type of results is renewed by the appearance of the reference [74], and also by the older works [75]-[78]. As is well known, the standard singularity theorems [60] assume that that the matter content satisfies specific energy conditions. The results of reference [74] suggest that the singularity theorems are true even in situations in which such conditions are violated. These authors offer a kinematical argument for which, when the universe is initially expanding fast enough, the resulting space time is null and time like geodesically incomplete. Their result relies in averaging certain quantity, which reduces to the Hubble parameter in the isotropic and homogenous case.

The goal of the present paper is to characterize certain initial conditions for which the space is time like geodesically incomplete, by assuming that gravity is described by an $f(R)$ theory. We will argue below that our results are not equivalent, but complementary to the ones in [74]. The formulation of $f(R)$ theories we are concerned with is the metric one, and the resulting equations for the space time metric are of fourth order. This is precisely the formulation which gives rise to the scalaron degree of freedom. Our main goal is to present some singularity theorems which generalize the Hawking cosmological theorems, in the context of $f(R)$ theories. This generalization presents some technical complications since, for $f(R)$ scenarios, the standard condition for matter fields $\left(T_{ij} - \frac{g_{ij}}{2}T\right)k^i k^j \geq 0$ for any generic unit time like field does not imply the condition $R_{ij}k^i k^j \geq 0$, and the last condition is of essential importance when giving a proof of the Hawking theorems. In addition, the scalaron degree of freedom spoils the first condition. The aim of the present work is to sort out these problems and formulate some concrete generalization of the Hawking results. Some mathematical results proven in [62] are crucial for this achievement.

The present work may have an overlap with the existing literature. For instance, in [70] the finite future singularities emerging in alternative gravity dark energy models were studied both in the Jordan and Einstein frames. These reference shows that a singularity can be present even for a flat space time. Some viable modified gravity models were analyzed in detail and a cure to a singularity was found in terms of higher order curvature corrections. Also, in reference [71] the future evolution of the dark energy in modified gravity theories including $f(R)$ gravity, string-inspired scalar-Gauss-Bonnet and modified Gauss-Bonnet ones were studied. Several examples of modified gravity models which produce accelerating cosmologies ending at a finite-time future singularity were found and some scenarios to resolve the finite-time future singularity were presented. Furthermore, it is shown in those references that the non-minimal gravitational coupling can remove the finite-time future singularities or make the singularity stronger (or weaker) in modified gravity. Further features are described in [72].

The present work is organized as follows. In section 2 some generalities about $f(R)$ theories are reviewed, in particular the role of the scalaron. In section 3 the standard Hawking results are reviewed, and some important fact about strong energy conditions is reminded. Section 4 is devoted to some properties of the Raychaudhuri equation, which are then applied to obtain our singularity results. These results are applied to some known $f(R)$ theories and are all related to compact Cauchy hypersurfaces. Section 6 contains some results that relax the mentioned compactness assumption. Section 7 contains a discussion about the initial data formulation for these models. Section 8 contains our conclusions.

2. Generalities about $f(R)$ theories

The signature convention to be adopted in the following is $(-,+,+,+)$. As it was mentioned in the introduction, $f(R)$ models are generalizations of the standard GR. We collect here some of their basic features, further information can be found in the extensive references [19]-[50]. These theories start with the following lagrangian

$$S=\frac{1}{16\pi G_N}\int f(R)\sqrt{-g}d^4x+S_m, \tag{2.1}$$

where the matter action S_m for a given component takes the same form as in GR. This class of theories reduce to the Einstein GR when $f(x)=x$.

One of the main differences between $f(R)$ models and the standard GR is that, for the former, the Palatini or metric formalism give the same theory. It is convenient to remind that the Palatini formalism involves an independent variation of the metric and the connection, while for the metric formalism the metric is the only object to be varied. For GR, both formalisms give the same system of equations, while for $f(R)$ theories both systems are inequivalent. The following exposition will be centered in the metric formalism. The corresponding generalized Einstein equations are of fourth order, and their explicit form is given by

$$f'(R)R_{ij}-\frac{1}{2}f(R)g_{ij}-[\nabla_i\nabla_j-g_{ij}g^{ab}\nabla_a\nabla_b]f'(R)=8\pi G_N T_{ij}. \tag{2.2}$$

When the matter fields are described by a lagrangian $\mathcal{L}_m$ the energy momentum tensor T_{ij} introduced of the left hand side is given by

$$T_{ij} = -2\frac{\delta\mathcal{L}_m}{\delta g^{ij}} + g_{ij}\mathcal{L}_m. \tag{2.3}$$

The equations (2.2) become of second order only in the specific situation $f(x) = x$, which corresponds to the GR limit. In other situations, the contraction of these equations with the inverse metric tensor $g^{\mu\nu}$ shows that

$$f'(R)R - 2f(R) + 3g^{ab}\nabla_a\nabla_b f'(R) = kT,$$

which is a differential equation for $f(R)$ unless $f(x) = x$. This is a non algebraic equation, and suggest that that $f(R)$ can be interpreted as an additional degree of freedom. In order to clarify this, it is convenient to introduce the following formulation of the theory

$$S = \frac{1}{16\pi G_N}\int (R\phi - V(\phi))\sqrt{-g}d^4x + S_m, \tag{2.4}$$

where ϕ is an auxiliary field, since it has no kinetic term. The lagrangian $\mathcal{L}(\phi, R) = R\phi - V(\phi)$ should be identified as $f(R)$. Therefore $f(R)$ is the Legendre transform of $V(\phi)$. From the equations of motion for ϕ and some standard properties of Legendre transforms it follows that

$$R = V'(\phi), \qquad \phi = f'(R), \tag{2.5}$$

The equations of motion resulting from (2.4) may be simplified by making the following conformal transformation (to the Einstein frame)

$$\phi \longrightarrow \varphi = \sqrt{\frac{3}{2k}}\log\phi, \qquad g_{ab} \to \widetilde{g}_{ab} = \phi g_{ab}, \qquad \sqrt{-g} = \phi^{-2}\sqrt{-\widetilde{g}}. \tag{2.6}$$

The inverse transformations are

$$\varphi \longrightarrow \phi = e^{\sqrt{\frac{2k}{3}}\varphi}, \qquad \widetilde{g}_{ab} \to g_{ab} = e^{-\sqrt{\frac{2k}{3}}\varphi}\widetilde{g}_{ab}, \qquad \sqrt{-\widetilde{g}} = e^{\sqrt{\frac{8k}{3}}\varphi}\sqrt{-g}. \tag{2.7}$$

By taking into account the well known transformation of the curvature scalar R under conformal transformation

$$\phi\widetilde{R} = R - \frac{3}{2}g^{\mu\nu}\partial_\mu\log\phi\partial_\nu\log\phi - 3\Box\log\phi, \tag{2.8}$$

and by introducing the last expression into (2.4), it is obtained after neglecting a total derivative term the following action

$$S'_g[\widetilde{g}_{ab}, \varphi] = \int\left[\frac{\widetilde{R}}{2k} - \frac{1}{2}\widetilde{g}^{ij}\partial_i\varphi\partial_j\varphi - U(\varphi)\right]\sqrt{-\widetilde{g}}d^4x + S_m. \tag{2.9}$$

Here

$$U(\varphi) = \frac{V(\phi)}{2\kappa\phi^2} = \frac{Rf'(R) - f(R)}{2\kappa f'(R)^2}, \tag{2.10}$$

and the matter lagrangian in S_m should be related to the new frame. An elementary calculation shows that the matter lagrangian $\widetilde{\mathcal{L}}_m$ in the new frame is related to $\mathcal{L}_m$ by

$$\widetilde{\mathcal{L}}_m(X, \widetilde{g}, \varphi) = \frac{1}{\phi^2}\mathcal{L}_m(X, \widetilde{g}\phi). \tag{2.11}$$

The field φ is known as the scalaron. Thus the $f(R)$ system has been reduced to GR coupled to the scalaron φ plus generic mater fields X interacting with the scalaron as well.

3. Strong energy conditions and cosmological singularity theorems

The aim of the present work is to study possible generalizations of the GR singularity theorems for the $f(R)$ models discussed above. As is well known in GR, an special role in studying singularity theorems is played by matter whose energy momentum tensor satisfy the condition

$$\Big(T_{ij}-\frac{g_{ij}}{2}T\Big)k^i k^j\geq 0, \tag{3.12}$$

where k^i is any generic time like field of unit length. For GR this implies the following inequality for the curvature tensor

$$R_{ij}k^i k^j\geq 0, \tag{3.13}$$

which is known as the strong energy condition. On the other hand, given such vector field k^i, it can be interpreted as a congruence of non intersecting world lines, not necessarily geodesics.

The following discussion is restricted to a generic globally hyperbolic space time $(M,\ g)$, with a particular choice of a Cauchy hyper-surface Σ where the initial data is formulated. Consider the geodesic congruence orthogonal to Σ. As is well known, the expansion scalar θ_γ of this congruence satisfies the Raychaudhuri equation

$$\frac{d\theta_\gamma}{d\tau}=-\frac{\theta_\gamma^2}{n-1}-\text{Ricc}(\gamma',\gamma')-2\sigma^2,\qquad \theta_\gamma(0)=\theta_{\gamma 0}. \tag{3.14}$$

Here n is the space time dimension, $\text{Ricc}(\gamma',\gamma')=R_{ab}\gamma'^a\gamma'^b$ and $\gamma:[0,\infty)\to M$ is any of the future complete unit speed geodesic of the congruence, which emanates orthogonally from Σ. The vectors γ'^a are assumed to be of unit length and the geodesics are parameterized by the proper time τ.

The expansion scalar θ_γ described by (3.14) has a fundamental geometrical interpretation. Given two points p and q in the space time $(M,\ g)$, they are named conjugate points to each other when there is a non zero Jacobi field which vanishes both at p and q. A necessary and sufficient condition for q to be conjugate to p is that, for the geodesic congruence emanating from p, the expansion $\theta_\gamma\to-\infty$ at q. This notion can be extended to space like surfaces, such as Σ. A point q on a geodesic γ of the geodesic congruence orthogonal to Σ is said to be conjugated to it, if there is a Jacobi field which does not vanish at Σ but vanish at q. Intuitively, the point q is conjugated to Σ when there are two infinitesimally close geodesics emanating from Σ which converge to q. The divergence of θ_γ corresponding to the geodesic congruence will also diverge at q in this situation [69], [60].

The conjugate points discussed above characterize the presence of a singularity. It is important to note that, in the signature $(-,+,+,+)$ we are working with, and for globally hyperbolic space times, there exist a time like curve joining two points p and q which maximizes the proper time. This curve is a geodesic without conjugate points between p and q, and is maximizing along all the continuous time like curves connecting p and q, not only the differentiable ones [69], [60]. In these terms it is possible to prove the following generic singularity theorem.

Proposition 1: Consider a globally hyperbolic space time (M, g) with a given Cauchy surface Σ. If any solution θ_γ of the Raychaudhuri equation (3.14) explodes before finite proper time $\tau_1 > 0$ (a condition which is to be developed further), then the space is future geodesically incomplete.

Recall that a Cauchy hyper-surface is achronal, and it is assumed that Σ corresponds to the proper time $\tau = 0$.

Proof: Assume that there exists a future directed time like curve λ with length $\tau > \tau_1$ in M. Let q a point of M lying on λ beyond the length τ_1. For any globally hyperbolic space time there exists a curve γ which maximizes the proper time between Σ and q [69], [60]. Let p the intersection between this curve γ and the Cauchy hyper-surface Σ. By its definition, the curve γ maximizes the proper time between p and q. Thus, it should be a geodesic without conjugate points between p and q. However, by the theorem assumptions, the expansion parameter θ_γ explodes at a finite proper time $\tau < \tau_1$ and thus there is a conjugate point between p and q. This contradiction shows that the curve λ does not exists and the space is future geodesically incomplete. (Q. E. D)

The generalization of the previous theorem to past directed curves presents no difficulties at all. It is important to remark that the Raychaudhuri equation (4.23) is purely geometric, and its deduction does not take into account wether the geometry is described by the Einstein equations or other type of gravity model. In addition, when the condition (3.13) is satisfied, the second term of the right hand side is positive. Thus

$$\frac{d\theta_\gamma}{d\tau} + \frac{\theta_\gamma^2}{n-1} \leq 0, \qquad \theta_\gamma(0) = \theta_{\gamma 0}, \tag{3.15}$$

which implies that

$$\frac{1}{\theta_\gamma} \geq \frac{1}{\theta_{\gamma 0}} + \frac{1}{n-1}\tau.$$

Clearly, this implies that when $\theta_{\gamma 0} > 0$ in all the Cauchy surface, then $\theta_\gamma \to -\infty$ at a finite proper time $\tau \leq (n-1)/C$. Thus, the following theorem follows [60].

Hawking singularity theorem (future version): Consider an hyperbolic space time (M, g) with a Cauchy hyper-surface Σ for which $\theta_0(p) \geq C > 0$ for every point p at that surface. If the Ricci tensor satisfy the condition (3.13) then the space time is future geodesically incomplete. More precisely, all the future directed curves have length with absolute value no larger than $\tau_0 = (n-1)/C$.

The past version of this theorem is straightforward.

Hawking singularity theorem (past version): Consider an hyperbolic space time (M, g) with a Cauchy hyper-surface Σ for which $\theta_0(p) \leq C < 0$ for every point p at that surface. If the Ricci tensor of the geometry satisfies the condition (3.13) then space time is past geodesically incomplete. More precisely, all the past directed curves have length with absolute value no larger than $\tau_0 = (n-1)/C$.

It is important to remark that this theorem is valid when the conditions (3.13) are satisfied, independently on the gravity model describing the Ricci tensor R_{ij}. An example for which it applies is GR with matter satisfying the strong energy condition (3.12), since this automatically implies (3.13). This simple argument does not apply for $f(R)$ theories even when the matter content satisfies (3.12). In fact, the equation (2.2) does not imply (3.13). Thus, it is not trivial to understand for which situations $\theta_\gamma \to -\infty$ at finite time for $f(R)$ models.

In addition, it should be emphasized that several realistic matter content (2.11) of the theory are not conformally invariant. An example of a conformally invariant matter system is the electromagnetic one in four dimensions

$$L_m = -\frac{1}{4\pi} F_{\mu\nu} F^{\mu\nu}. \tag{3.16}$$

However, there exist other realistic systems for which conformal invariance is broken. An example is the stress tensor of a perfect fluid, which is given by

$$T_{ab} = (p+\rho) u_a u_b + P g_{ab}, \tag{3.17}$$

where the 4-velocity u_a such that $g^{ab} u_a u_b = -1$. As usual, P denotes the pressure of the fluid and ρ its energy density. The corresponding tensor in a conformally related frame $\widetilde{g}_{ab} = \phi g_{ab}$ is still of the form

$$\widetilde{T}_{ab} = (\widetilde{p}+\widetilde{\rho}) \widetilde{u}_a \widetilde{u}_b + \widetilde{P} \widetilde{g}_{ab}, \tag{3.18}$$

where the 4-velocity $\widetilde{u}_a$ such that $\widetilde{g}^{ab} \widetilde{u}_a \widetilde{u}_b = -1$. This can be deduced directly from the equations (2.2). It follows that

$$\widetilde{T}_{ab} = \frac{1}{\phi} T_{ab}, \tag{3.19}$$

which implies that

$$\widetilde{P} = \frac{P}{\phi^2}, \qquad \widetilde{\rho} = \frac{\rho}{\phi^2}. \tag{3.20}$$

The last relations show that for a barotropic equation of state $P = (\gamma - 1)\rho$ conformal invariance is respected. In the general case the state equation would be of the form $P = P(\rho)$ and conformal invariance will be broken.

Despite these drawbacks, there is an interesting observation to be made. By taking into account (2.11) it follows that the energy momentum for matter (2.3) in the Einstein frame is

$$\widetilde{T}_{ij} = \frac{1}{\phi} T_{ij}, \qquad \widetilde{T} \widetilde{g}_{ij} = (\widetilde{T}_{ab} \widetilde{g}^{ab}) \widetilde{g}_{ij} = \frac{1}{\phi} T g_{ij}, \qquad \widetilde{g}_{ij} k^i k^j = -\phi. \tag{3.21}$$

This also holds for the perfect fluid due to (3.19). Thus, when the matter energy momentum tensor satisfies the strong energy conditions (3.12) in the Jordan frame, it also satisfies them in the Einstein frame if $\phi > 0$. In other words

$$\left(\widetilde{T}_{ij} - \frac{\widetilde{g}_{ij}}{2} \widetilde{T} \right) \widetilde{k}^i \widetilde{k}^j \geq 0, \tag{3.22}$$

with $\widetilde{k}^i = \phi^{-1/2}k^i$ a generic time like field of unit length defined on $(\widetilde{M},\ \widetilde{g})$. This suggests the possibility of studying the time geodesically completeness of the conformal metric $\widetilde{g}$ defined in (2.6) and the properties of the scalaron φ first, in order to analyze these matters for the Jordan metric g later on.

The fact that (3.22) is satisfied in the Einstein frame is encouraging, since it suggests that the analysis of cosmological singularities for $f(R)$ models may be quite analogous as in GR. However, it should be kept in mind that the scalaron φ does not satisfy the strong energy condition (3.12). In fact, scalar fields usually do not satisfy this condition either. Thus, further work should be made in to avoid this problem. We turn the attention to some specific theorems which, under certain circumstances, insure that the Einstein metric $\widetilde{g}$ geodesically incomplete when the matter field satisfy (3.12). After this point is clarified, we will deal with the singularities of the Jordan metric g, which is our main goal.

4. Generalized singularity theorems for compact Cauchy hypersurfaces

4.1 General statements

The previous section shows that a fundamental tool for studying singularity theorems is the Raychaudhuri equation

$$\frac{d\theta_\gamma}{d\tau} = -\frac{\theta_\gamma^2}{n-1} - \mathrm{Ricc}(\gamma', \gamma') - 2\sigma^2, \qquad \theta_\gamma(0) = \theta_{\gamma 0}. \tag{4.23}$$

Some properties of its solutions may be deduced by making the change of variables $y(\tau) = -(\theta_\gamma + c_\gamma)e^{-2c_\gamma\tau/(n-1)}$. This transforms the last equation into

$$\frac{dy}{d\tau} = \frac{y^2}{q(\tau)} + p(\tau), \qquad y(0) = y_0, \tag{4.24}$$

with

$$q(\tau) = (n-1)e^{-2c_\gamma\tau/(n-1)}, \qquad p(\tau) = e^{-2c_\gamma\tau/(n-1)}\bigg(\mathrm{Ricc}(\gamma', \gamma') + 2\sigma^2 + \frac{c_\gamma^2}{n-1}\bigg). \tag{4.25}$$

Equations of the type (4.24) were considered in the literature [62] for a wide variety of functions $p(\tau)$ and $q(\tau)$, and some general knowledge about them is the following.

Proposition 2: Consider an equation of the form (4.24), and suppose that the functions $p(\tau)$ and $q(\tau)$ satisfy

$$\int_0^\infty \frac{d\tau}{q(\tau)} = \infty, \qquad \lim_{T\to\infty} \inf \int_0^T p(\tau)d\tau > -y_0, \tag{4.26}$$

with $q(\tau) > 0$ in $[0, \infty)$. Then, any of the solutions $y(\tau)$ of (4.24) with the initial condition $y(0) = y_0$ does not extend to the interval $[0, \infty)$. In other words, the solution $y(\tau)$ explodes for a time $\tau_0 < \infty$.

Proof: Assume on the contrary that $y(\tau)$ extend to the whole interval $[0,\infty)$. This will imply a contradiction. To see that, note that second assumption (4.26) implies the existence of a time τ_1 for which

$$\int_0^t p(\tau')d\tau' > -y_0, \qquad \text{for} \qquad \tau > \tau_1.$$

By integrating the equation (4.24) and taking into account the last inequality, it follows that

$$y(\tau) = \int_0^\tau \frac{y^2}{q} d\tau' + \int_0^\tau p d\tau' + y_0 > \int_0^\tau \frac{y^2}{q} d\tau'. \tag{4.27}$$

Now, let us introduce the quantity given by

$$R(\tau) = \int_0^\tau \frac{y^2(\tau')}{q(\tau')} d\tau'.$$

As $q(\tau)$ is positive in the half positive line and $\tau > 0$, it can directly be seen that this quantity is always positive. This definition and the inequality (4.27) shows that

$$\frac{R^2}{q} < \dot{R} = \frac{y^2}{q}, \tag{4.28}$$

for $\tau > \tau_1$. From here it is concluded that, for every $\tau_2 > \tau_1$, the following inequality takes place

$$\int_{\tau_2}^\tau \frac{d\tau'}{q} < \int_{\tau_2}^\tau \frac{\dot{R}}{R^2} d\tau' = \frac{1}{R(\tau_2)} - \frac{1}{R(\tau)} < \frac{1}{R(\tau_2)}.$$

However, from the first condition (4.26) it follows that the left hand side of the last expression is not bounded when $\tau \to \infty$. Thus, the last inequality makes sense only for times $\tau < \tau_0$, with τ_0 a fixed time. This shows that $y(\tau)$ can not extend to the whole interval $[0,\infty)$, but only to an interval inside $[0,\tau_0]$. (Q. E. D)

Note that the last proposition predicts the existence of a finite time $\tau_0 > 0$ for which the solution $y(\tau)$ explodes but it does not give an estimate about it. Now, the first condition (4.26) applied to the equation (4.25) simply give that $c_\gamma > 0$. The second (4.26) is translated after an elemental integration into the following proposition [62].

Generalized future singularity theorem: Consider a globally hyperbolic space time M with dimension $n > 2$ and let Σ a compact Cauchy surface for it. Suppose that for each future directed time like geodesic $\gamma : [0,\infty) \to M$ issuing orthogonally from Σ there exist a constant $c_\gamma > 0$ for which

$$\lim_{T\to\infty} \inf \int_0^T e^{-2c_\gamma \tau/(n-1)} r_\gamma(\tau) d\tau > \theta_{0\gamma} + \frac{c_\gamma}{2}, \tag{4.29}$$

where $r_\gamma(\tau) = \text{Ricc}(\gamma', \gamma')$ and $\theta_{0\gamma}$ the value of the expansion parameter at the intersection point between γ and Σ. Then θ_γ diverges at a finite time τ_γ and the space M is future geodesically incomplete.

Proof: The proof is a consequence of proposition 2. As discussed below (4.23), the Raychaudhuri equation may be converted by a change of variables $y(\tau) = -(\theta + c_\gamma)e^{-2c_\gamma \tau/(n-1)}$ into (4.24) with the functions $p(\tau)$ y $q(\tau)$ defined by (4.25). The function $q(\tau)$ clearly satisfies the condition of the proposition 2, when $c_\gamma > 0$. The function $p(\tau)$ will satisfy them if

$$\int_0^\tau p(\tau')d\tau' > -y_0,$$

which is translated by (4.25) into

$$\int_0^\tau e^{-2c_\gamma\tau'/(n-1)}\Big(\mathrm{Ricc}(\gamma',\gamma') + 2\sigma^2 + \frac{c_\gamma^2}{n-1}\Big)d\tau' > \theta_{0\gamma} + c_\gamma,$$

The last term is a simple integral whose maximal value is $c_\gamma/2$, and the second is strictly positive. Thus, it is clear that if the inequality

$$\lim_{T\to\infty}\inf\int_0^T e^{-2c_\gamma\tau/(n-1)}r_\gamma(\tau)d\tau > \theta_0 + \frac{c_\gamma}{2},$$

is fulfilled, with $r_\gamma(\tau) = \mathrm{Ricc}(\gamma',\gamma')$, then the proposition 2 applies. In these terms, the last condition for quantity $r_\gamma(\tau)$ becomes exactly (4.29). Now, the proposition 2 implies that for such space time θ_γ explodes at a finite future time τ_γ. When Σ is compact, these results imply incompleteness. Details of the last affirmation can be found in [62], [73]. (Q. E. D)

In many applications, the singularity theorems are formulated in terms of a past singularity, instead of a future one. By simple following the arguments of the previous propositions, it is elementary to find such version.

Generalized past singularity theorem: Consider a globally hyperbolic space time M with dimension $n > 2$ and with a given compact Cauchy hyper-surface surface Σ. Suppose that for each past directed time geodesic $\gamma : (-\infty, 0] \to M$ issuing orthogonally from Σ there exist a constant $c_\gamma < 0$ for which

$$\lim_{T\to-\infty}\inf\int_T^0 e^{-2c_\gamma\tau/(n-1)}r_\gamma(\tau)d\tau \geq -\theta_{0\gamma} - \frac{c_\gamma}{2}, \tag{4.30}$$

where $r_\gamma(\tau) = \mathrm{Ricc}(\gamma',\gamma')$ and $\theta_{0\gamma}$ the value of the expansion parameter at the intersection point between γ and Σ. Then θ_γ diverges at a finite past time and thus, the space M is past incomplete[1].

In the last theorem, the condition (4.30) implies the existence of a given fixed time $\tau_1 < 0$ for which

$$\int_\tau^0 e^{-2c_\gamma\tau/(n-1)}r_\gamma(\tau')d\tau' \geq -\theta_{0\gamma} - \frac{c_\gamma}{2},$$

for any $\tau < \tau_1$. In the following, these theorems will be applied to concrete $f(R)$ theories.

Note that if $r_\gamma(t)$ satisfies the strong energy condition, then the future version theorem implies that $c = 0$ and $\theta_0(p) \leq 0$ while the past version implies that $\theta_0(p) \geq 0$. This is the content of the standard singularity theorem of Hawking [60], although this theorem is not restricted to compact hyper-surfaces.

[1]Note that a given space time is geodesically incomplete when there exist at least one geodesic $\gamma : [0, a) \to M$ which can not be extended for all the values of the affine parameter a, that is, to $a \to \infty$. The propositions given above are in fact too restricted, since it shows that every time like geodesic is incomplete.

4.2 Application to $f(R)$ theories

The two generalized singularity theorems described above are not related to any specific gravity model. In the present section, these theorems will be applied to $f(R)$ models (2.9) with matter lagrangian $\mathcal{L}_m$ satisfying the strong energy condition (3.12)-(3.13). The energy momentum tensor of the scalaron field φ is given by

$$T_{ab} = \nabla_a\varphi\nabla_b\varphi - \frac{\widetilde{g}_{ab}}{2}\Big(\nabla_c\varphi\nabla^c\varphi + U(\varphi)\Big), \tag{4.31}$$

and such component generally does not satisfy the energy inequalities (3.12)-(3.13). The Einstein equations for the model (2.9) give the following expression for $r_\gamma(t) = \widetilde{R}_{ab}\gamma'^a\gamma'^b$

$$r_\gamma(t) = 8\pi\Big((\nabla_\gamma\varphi)^2 - \frac{U(\varphi)}{n-1}\Big) + \Big(\widetilde{T}^m_{ab} - \frac{\widetilde{g}_{ab}}{2}\widetilde{T}^m\Big)\gamma'^a\gamma'^b, \tag{4.32}$$

where $\widetilde{T}^m_{ab}$ is the stress energy momentum of matter in the Einstein frame. As it was discussed in formulas (3.16)-(3.20) a generic stress energy tensor T^m_{ab} for matter is usually not covariant under the change from the Jordan to the Einstein frame. However, the important point here is that, when the contribution of matter to $r(t)$ satisfies (3.12)-(3.13) in the Jordan frame, then it will satisfy (3.22) in the Einstein one. Therefore the contribution of $\widetilde{T}^m_{ab}$ to the right hand side of (4.32) will be positive. The contribution from the term $(\nabla_\gamma\varphi)^2$ is positive as well. The only negative part may come from the term with the potential $U(\varphi)$. By taking this into account it follows that

$$\int_0^T e^{-\frac{2c_\gamma t}{n-1}} r(t)dt > -\frac{K^2}{2c_\gamma}, \tag{4.33}$$

for all T and c_γ. Here

$$K = \sqrt{\frac{8\pi(n-1)U_{max}}{(n-2)}},$$

with U_{max} the maximum value that the potential takes during the evolution of φ, if this maximum exists. Then, the conditions of the generalized singularity theorem will be satisfied when

$$-\frac{K^2}{2c_\gamma} - \frac{c_\gamma}{2} > \theta_0.$$

The left hand of the last inequality is minimized when $c_\gamma = K$, and it follows that when the Cauchy surface Σ is compact and

$$\theta_0 < -K, \tag{4.34}$$

then the space is future geodesically incomplete. Results of this type were considered in [62] for GR with free scalar fields and for other types of matter as well.

The results described above apply in presence of a maximum value for the potential U_{max}. There are several realistic models for which this assumption is justified. The $f(R)$ function is a free parameter of the model, but there are some physical constraints on it. The condition for the theory to be free of ghosts is that $f'(R) > 0$. The conditions for being free of tachyons is $f''(R) > 0$. In addition, there exists a large class of models in the literature for which, in the limit of large curvature $R \to \infty$

$$f(R) \to R, \qquad f'(R) \to 1, \qquad f''(R) \to 0. \tag{4.35}$$

These conditions implies that the scalaron φ have finite values as the curvature diverges. Some known examples are

$$f(R) = R - R_s\beta\alpha\Bigg\{1 - \frac{1}{\Big[1 + \Big(\frac{R}{R_s}\Big)^n\Big]^{\frac{1}{\beta}}}\Bigg\}. \tag{4.36}$$

These models encode a variety of scenarios, in particular the Starobinsky [15] or the Hu-Sawicki ones [21]. When $n = 1$ and $\beta \to \infty$ the last model reduces to [57]

$$f(R) = R - R_s\alpha \log\bigg(1 + \frac{R}{R_s}\bigg), \tag{4.37}$$

which satisfies all these constraints. Another interesting model is the following [58]

$$f(R) = R\bigg[1 - \frac{b}{1 + \log\frac{R}{R_s}}\bigg]. \tag{4.38}$$

The interest in the last model is that the effective Newton G_N^{eff} constant runs with the curvature in analogous fashion as the running of the QCD coupling constant $g(\mu)$ with the energy scale μ of a given process.

In order to apply the condition (4.34) to any of these $f(R)$ models it is needed to know if U_{max} exists. It is important to determine wether or not ϕ takes finite positive values during the universe evolution, since this field represents the conformal transformation between the Jordan and the Einstein frame. The last point is of great importance since, otherwise, the presence of a singularity in the Einstein metric may be due to a bad behavior or the conformal transformation and not due to a singularity in the Jordan metric.

Let us focus in the model (4.36) first. From (2.5) and (4.36) it is directly deduced that

$$\phi = f'(R) = 1 - n\alpha\frac{\Big(\frac{R}{R_s}\Big)^{n-1}}{\Big[1 + \Big(\frac{R}{R_*}\Big)^n\Big]^{\frac{1}{\beta}+1}}.$$

The maximum of the second term is given at the point

$$\bigg(\frac{R}{R_s}\bigg)^n = \frac{\beta(n-1)}{n+1+\beta},$$

and therefore the minimum value of $f'(R)$ is

$$\phi = f'_{min}(R) = 1 - \alpha n\frac{\Big(\frac{\beta(n-1)}{n+1+\beta}\Big)^{\frac{n-1}{n}}}{\Big[1 + \frac{\beta(n-1)}{n+1+\beta}\Big]^{\frac{1}{\beta}+1}}.$$

By choosing α appropriately, it may be shown that always $\phi > 0$. In addition, the value of $f'(R)$ is bounded from above. Thus ϕ is bounded both from above and below and takes always positive values. This means that the conformal transformation between the Jordan and the Einstein frame is always well defined. On the other hand (2.6) shows that

$$\varphi = \sqrt{\frac{3}{2k}}\log\phi,$$

and thus φ also takes bounded values. However, this fact does not ensure that $U(\varphi)$ is bounded, since it may have poles for finite scalaron values. Thus, a further check is needed in order to see that this is not the case. The expression for the potential (2.10) in terms of the function $f(R)$ is

$$U(\varphi) = \frac{Rf'(R) - f(R)}{2\kappa f'(R)^2}, \tag{4.39}$$

The denominator never goes to zero, since $f'(R) > 0$. The numerator is

$$Rf'(R) - f(R) = -nR_s\alpha \frac{\left(\frac{R}{R_s}\right)^n}{\left[1 + \left(\frac{R}{R_*}\right)^n\right]^{\frac{1}{\beta}+1}} + R_s\beta\alpha\left\{1 - \frac{1}{\left[1 + \left(\frac{R}{R_s}\right)^n\right]^{\frac{1}{\beta}}}\right\}.$$

This expression obviously has a maxima, and does not diverge for any value of the curvature when n is even. Thus, $U(\varphi)$ does not have poles and reaches a maximum value. Thus the condition (4.34) makes sense and, when is satisfied, it follows that the space is future geodesically incomplete.

Consider now the model (4.37). The scalar field ϕ is

$$\phi = f'(R) = 1 - \frac{\alpha}{1 + \frac{R}{R_s}}, \tag{4.40}$$

and is always positive when $\alpha < 1$. The numerator in (4.39) is

$$Rf'(R) - f(R) = -\frac{\alpha R}{1 + \frac{R}{R_s}} + R_s\alpha \log\left(1 + \frac{R}{R_s}\right).$$

This numerator is not bounded. Thus, it can not be guaranteed that the generalized future singularity theorem 2 of the previous section directly applies for this model.

It is interesting to note that, for the last model, when the curvature R is finite during the universe evolution, then the scalaron has positive finite values and the theorem indeed apply when $\theta_{0\gamma} < -K$. Thus, for non singular curvature values, the space may be geodesically incomplete. This makes perfect sense, since geodesic incompleteness does not necessarily implies a curvature singularity [59]. On the other hand, when $U_{max} \to \infty$ then the curvature R is exploiting. But it may be the case that the curvature $R \to \infty$ only when $\tau \to \infty$, and this is not a point in the space time manifold. The universe is regular in this case. We can not distinguish between these two situations with the tools presented here.

Finally, for the model (4.38) the scalar ϕ is

$$\phi = f'(R) = 1 - \frac{b}{1 + \log\frac{R}{R_s}} + \frac{b}{(1 + \log\frac{R}{R_s})^2}.$$

This scalar is not bounded from above, although it has a minimum. Again, the theorem does not apply in this situation without further ad hoc assumptions.

We finish this section by noticing that the past version of this result is that when $\theta_0 > K$ the space time is past geodesically incomplete. The deduction of this fact is completely analogous to the future case.

5. Relaxing the compactness assumption for the Cauchy surface

The singularity results described above correspond to compact Cauchy hyper-surfaces. However, the following variation of proposition 2 allows to circumvent this limitation [62].

Proposition 3: Consider an equation of the form (4.24), and suppose that there exists a fixed time t_1 for which the functions $p(\tau)$ and $q(\tau)$ satisfy

$$\int_0^\infty \frac{d\tau}{q(\tau)} = \infty, \qquad \inf_{T\geq 0}\int_0^T p(\tau)d\tau + y_0 = \alpha > 0, \tag{5.41}$$

with $q(\tau) > 0$ in $[0, \infty)$. Then, any of the solutions $y(\tau)$ of (4.24) with the initial condition $y(0) = y_0$ exploits inside the interval $[0, \tau]$, where τ is the time defined by

$$\int_0^\tau \frac{dt}{q(t)} = \frac{2}{\alpha}.$$

Proof: By integrating the equation (4.24) it follows that

$$y(\tau) = \int_0^\tau \frac{y^2}{q}d\tau' + \int_0^\tau p d\tau' + y_0 > \int_0^\tau \frac{y^2}{q}d\tau' + \alpha > \alpha, \tag{5.42}$$

the last equality follows from the positivity of $q(t)$. Now, let us introduce the quantity given by

$$R(\tau) = \int_0^\tau \frac{y^2(\tau')}{q(\tau')}d\tau'.$$

From (5.42) it is clear that

$$R(\tau) = \alpha^2 \int_0^\tau \frac{1}{q(\tau')}d\tau'. \tag{5.43}$$

As $q(\tau)$ is positive in the half positive line and $\tau > 0$, it can directly be seen that this quantity is always positive. On the other hand, an analogous reasoning to (5.42) shows that

$$\frac{R^2}{q} < \dot{R} = \frac{y^2}{q}, \tag{5.44}$$

for $\tau > 0$. From here it is concluded, by fixing $\tau_2 > 0$, that

$$\int_{\tau_2}^\tau \frac{d\tau'}{q} < \int_{\tau_2}^\tau \frac{\dot{R}}{R^2}d\tau' = \frac{1}{R(\tau_2)} - \frac{1}{R(\tau)} < \frac{1}{R(\tau_2)}.$$

By properly taking into account the inequality (5.43) it follows that

$$\Big(\int_{\tau_2}^\tau \frac{d\tau'}{q}\Big)\Big(\int_0^{\tau_2} \frac{d\tau'}{q}\Big) \leq \frac{1}{\alpha^2},$$

for $0 \leq t_2 \leq t$. Now, the intermediate value theorem allows to find a value t_2 for which the left hand side is

$$\Big(\int_0^\tau \frac{d\tau'}{q(\tau')}\Big)^2 \leq \frac{1}{\alpha^2},$$

from where the proposition follows. (Q. E. D)

In terms of the previous proposition, it is possible to prove the following theorem, for which the initial Cauchy surface is not assumed to be compact.

Generalized future singularity theorem 2: Consider a globally hyperbolic space time M with dimension $n>2$ and with a given Cauchy hyper-surface Σ for it. Suppose that for each future directed time geodesic $\gamma:[0,\infty)\to M$ issuing orthogonally from Σ there exist two constants $c_\gamma>0$ and $\beta_\gamma\geq C>0$ and a time τ_1 for which

$$\inf_{T\geq 0}\int_0^T e^{-2c_\gamma\tau/(n-1)}r_\gamma(\tau)d\tau=\beta_\gamma+\theta_{0\gamma}+c_\gamma, \tag{5.45}$$

for $\tau>\tau_1$, where $r_\gamma(\tau)=\text{Ricc}(\gamma',\gamma')$. Then θ_γ diverges at a finite proper time $\tau\leq(n-1)/C$ and thus, the space M is future incomplete.

Proof: Assume that the hypothesis of the theorem are true. Then

$$\inf\int_0^T e^{-2c_\gamma\tau'/(n-1)}\text{Ricc}(\gamma',\gamma')d\tau'=\beta_\gamma+\theta_{0\gamma}+c_\gamma.$$

From the last expression it follows that

$$\inf\int_0^T e^{-2c_\gamma\tau'/(n-1)}\bigg(\text{Ricc}(\gamma',\gamma')+2\sigma^2+\frac{c_\gamma^2}{n-1}\bigg)d\tau'=\alpha_\gamma+\theta_{0\gamma}+c_\gamma, \tag{5.46}$$

where $\alpha_\gamma>\beta_\gamma>C>0$. This inequality follows since the two added terms are positive or zero. As discussed below (4.23), the Raychaudhuri equation may be converted into (4.24) by making the variable redefinition $y(\tau)=-(\theta+c_\gamma)e^{-2c_\gamma\tau/(n-1)}$ with the functions $p(\tau)$ y $q(\tau)$ defined by (4.25). The function $q(\tau)$ clearly satisfies the hypothesis of the proposition 2, when $c_\gamma>0$. The function $p(\tau)$ also is inside these assumptions of the theorem when

$$y_0+\inf\int_0^T p(\tau')d\tau=\alpha_\gamma>0,$$

which is exactly equivalent to (5.46) under this variable change. Under these circumstances, the proposition 3 applies. Now, this proposition implies that the value of θ_γ will explode at a time inside the interval $[0,\tau_\gamma]$ being τ_γ the solution of the equation

$$\int_0^{\tau_\gamma}\frac{e^{2c_\gamma t/(n-1)}}{(n-1)}dt=\frac{1}{\alpha_\gamma},$$

in deriving the last formula (4.25) has been taken into account. By calculating explicitly the last integral it follows that

$$\tau_\gamma=\frac{n-1}{2c_\gamma}\log(1+\frac{3c_\gamma}{\alpha_\gamma}).$$

The last formula can be expressed as

$$\tau_\gamma=\frac{2(n-1)}{3\alpha_\gamma}\frac{1}{x}\log(1+x),$$

with $x = 3c_\gamma/\alpha_\gamma$ a positive variable. The maximum value of the function $f(x) = x^{-1}\log(1+x)$ for positive values of x is $f(0) = 1$. Thus

$$\tau_\gamma \leq \frac{2(n-1)}{3\alpha_\gamma} \leq \frac{2(n-1)}{3C}.$$

The last part of this inequality does not depend on the choice of the curve γ, thus any θ_γ explodes at a time less than $\tau_0 = 2(n-1)/C$. A direct application of proposition 1 and the previous results shows that the space is geodesically incomplete. (Q. E. D)

Note that, the smaller C is, the larger the bound for the explosion time will be. In fact, the singularity theorems of the previous sections correspond to the case $C \to 0$. For completeness, we mention that the past version of this theorem follows by replacing $\int_0^T$ by $\int_T^0$, $\theta_{0\gamma}$ by $-\theta_{0\gamma}$ and by considering constants $c_\gamma < 0$. The constants α_γ are, as before, positive.

5.1 Singularities for $f(R)$ models

In order to apply the generalized future singularity theorem 2 to $f(R)$ models, it should be noted that, when $e^{-2c_\gamma\tau/(n-1)}r_\gamma(\tau)$ is integrable, it follows from (4.30) that

$$\int_0^T e^{-2c_\gamma\tau/(n-1)}r_\gamma(\tau)d\tau \geq \int_0^\infty e^{-2c_\gamma\tau/(n-1)}r_{\gamma-}(\tau)d\tau,$$

for any value of $T > 0$. Here $r_{-\gamma}(\tau) = \text{Min}(0, r_\gamma(t))$. Therefore, if

$$\int_0^\infty e^{-2c_\gamma\tau/(n-1)}r_{\gamma-}(\tau)d\tau = \beta_\gamma + \theta_{0\gamma} + c_\gamma,$$

is satisfied with $\beta_\gamma > C > 0$, then the hypothesis of the generalized future theorem 2 will be fulfilled. By further taking into account (4.31) and (4.32), it follows from the last condition that

$$\int_0^\infty e^{-\frac{2c_\gamma t}{n-1}}r_{\gamma-}(t)dt > -\frac{K^2}{2c_\gamma}, \tag{5.47}$$

with

$$K = \sqrt{\frac{8\pi(n-1)U_{max}}{(n-2)}},$$

U_{max} being the maximum value of the potential during the evolution of φ, if this maximum exists. Then, the conditions of the generalized singularity theorem will be satisfied when

$$-\frac{K^2}{2c_\gamma} - c_\gamma = \theta_{0\gamma} + \beta_\gamma.$$

The left hand is minimized when $\sqrt{2}c_\gamma = K$, thus the theorem will hold when

$$-\sqrt{2}K - \beta'_\gamma > \theta_{0\gamma},$$

with $\beta'_\gamma > C > 0$. The previous theorem insures that the blowup proper time is less than $\tau_0 = \frac{2(n-1)}{3C}$.

Analogous considerations hold for past singularities. The manifold will be past geodesically incomplete when

$$\sqrt{2}K + \beta'_\gamma < \theta_{0\gamma}.$$

where $\beta'_\gamma \geq C > 0$ for this case as well. The past blowup proper time τ_0 satisfy $0 > \tau_0 > -\frac{2(n-1)}{3C}$.

The application of this result to the previous discussed theories is straightforward. The results are completely analogous, but the compactness assumption for the Cauchy surface is removed.

6. Singularities for the Jordan metric

6.1 The vacuum case

As it was shown in (2.9), there exists a conformal transformation for which the $f(R)$ models reduce to ordinary GR plus an scalar field φ, the scalaron, together with matter coupled non trivially to the scalar degree of freedom by a term of the form (2.11). In this context, it is of great importance to analyze whether or not the resulting system is well behaved. For instance, it may be of interest to understand in which situations, given some suitable boundary conditions, the scalaron φ is uniquely defined. Otherwise, the conformal transformation to the original Jordan frame would be ambiguous. The present section deals with these type of issues.

It is convenient to study first the vacuum model, that is, the case for which $\mathcal{L}_m = 0$. In this case, as previously stated, the theory is reduced in the Einstein frame to GR coupled to an scalar field φ. Surprisingly, it was only recently that the properties of such system were firmly established. These properties are fully proved in [63] and we refer the reader to that reference for further details. However, the main results will be exposed here without proof.

Let $(M,\, g)$ be a time oriented Lorentz manifold, together with an scalar field φ which can be considered as an smooth function on M. Let Σ an smooth space like surface and let h_{ij} and k_{ij} the metric and the second fundamental form on Σ induced by the space time metric g_{ij}. The future directed normal unit vector to Σ will be denoted by N and the Levi-Civita covariant derivative induced on Σ by h will be denoted by D_μ. In these terms, the following equalities take place for the Einstein tensor $G_{ij} = R_{ij} - g_{ij}R$ of the space time manifold $(M,\, g)$

$$G(N_p, N_p) = \frac{1}{2}\Big[S - k_{ij}k^{ij} + (\mathrm{Tr}_g k)^2\Big](p), \tag{6.48}$$

$$G(N_p, v) = \frac{1}{2}\Big[D^j k_{ji} - D_i \mathrm{Tr}_g k\Big]v^i. \tag{6.49}$$

Here $p \in \Sigma$ and $v \in T_p\Sigma$. In addition, S denotes the scalar curvature of the three dimensional space like hyper-surface $(\Sigma,\, h)$.

The relations given above are quite general and the scalaron does not play any role in its deduction. But if it is further assumed that the system in consideration is the Einstein one coupled to an scalar field φ, then it is deduced from (6.48) the so called hamiltonian constraint

$$\frac{1}{2}\Big[S - k_{ij}k^{ij} + (\mathrm{Tr}_g k)^2\Big] = \rho, \tag{6.50}$$

with ρ defined as

$$\rho = \frac{1}{2}\Big[(N\varphi)^2 + D_i\varphi D^i\varphi\Big] + U(\varphi).$$

On the other hand, the constraint (6.49) give the so called momentum constraint

$$D^j k_{ji} - D_i \mathrm{Tr}_g k = j(\varphi), \tag{6.51}$$

with $j(\varphi) = N(\varphi)D^i\varphi$.

Given the equations (6.48)-(6.51) the initial formulation for the vacuum problem for the coupled scalar-Einstein model goes as follows. The initial data consist on an n dimensional manifold Σ endowed with a riemannian metric g_0 on it, together with a covariant tensor k_{ij}, and two smooth functions φ_0 and φ_1 on Σ, satisfying

$$r - k_{ij}k^{ij} + (\mathrm{Tr}_g k)^2 = (\varphi_1)^2 + D_i\varphi_0 D^i\varphi_0 + 2U(\varphi_0). \tag{6.52}$$

$$D^j k_{ji} - D_i \mathrm{Tr}_g k = \varphi_1 D^i \varphi_0. \tag{6.53}$$

Here D^j is the Levi-Civita connection on Σ and r the scalar curvature of $(g_0,\, \Sigma)$, and the indices are raised and lowered with the help of g_0. Given this data, the problem is to find an $n+1$ manifold M endowed with a Lorenzian metric g and a $C^\infty(M)$ map φ such that the Einstein equations are satisfied, together with an embedding $i : \Sigma \to M$ such that $i^* : g \to g_0$ and $\varphi\, o\, i = \varphi_0$. If in addition N is a future directed normal time like vector to Σ and K is the second form of $i(\Sigma)$, then $i^*(K) = k$ and $(N\varphi)\ o\ i = \varphi_1$. The triple $(M,\, g,\, \varphi)$ is known as the development of the data. If furthermore $i(\Sigma)$ is a Cauchy hyper-surface in $(M,\, g)$ then $(M,\, g,\, \varphi)$ is called globally hyperbolic development. The first question that arise is wether or not such developments exist. The answer is affirmative, as shown in the following proposition [63].

Proposition 4: There always exists a global hyperbolic development for the data $(\Sigma,\, k,\, g_0,\, \varphi_0,\, \varphi_1)$ satisfying the constraints (6.52)-(6.53) described above.

The following proposition shows that two different developments of a given data are an extension of a common development [63].

Proposition 5: Consider a given data $(\Sigma,\, k,\, g_0,\, \varphi_0,\, \varphi_1)$ and two hyperbolic developments $(M_a,\, g_a,\, \varphi_a)$ and $(M_b,\, g_b,\, \varphi_b)$ with corresponding embeddings $i_a : \Sigma \to M_a$ and $i_b : \Sigma \to M_b$. Then there exist sa global hyperbolic development $(M,\, g,\, \varphi)$ with a corresponding embedding $i : \Sigma \to M$ and an smooth orientation preserving maps $\psi_a : M \to M_a$ and $\psi_b : M \to M_b$, which are diffeomorphisms onto their images, such that $\psi_a^* g_a = g$, $\psi_a^* \varphi_a = \varphi$ and $\psi_b^* g_b = g$, $\psi_b^* \varphi_b = \varphi$. In addition $\psi_a\ oi = i_a$ and $\psi_b\ oi = i_b$.

A fundamental notion is then the notion of an maximal hyperbolic development. An hyperbolic development $(M,\, g,\, \varphi)$ is called maximal if, for any other global hyperbolic development $(M',\, g',\, \varphi')$, there is an embedding $i' : \Sigma \to M'$ and an smooth orientation preserving maps $\psi : M' \to M$ such that $\psi^* g = g'$, $\psi^* \varphi = \varphi'$ and $\psi \; o \; i' = i$. The following proposition shows that maximal hyperbolic developments always exist [63].

Proposition 6: Given a data $(\Sigma,\, k,\, g_0,\, \varphi_0,\, \varphi_1)$ there exists a maximal global hyperbolic development, which is unique up to an isometry.

The last proposition we would like to mention is related to the Cauchy stability of the problem [63].

Proposition 7: Let $(M = \Sigma \times I,\, g,\, \varphi)$ a background solution of the Einstein-scalar system. By denoting by $(\Sigma,\, k,\, g_0,\, \varphi_0,\, \varphi_1)$ the data induced on $\{0\} \times \Sigma$ by the full solution, consider a sequence $(k_j,\, g_{0j},\, \varphi_{j0},\, \varphi_{1j})$ of initial conditions converging to $(\Sigma,\, k,\, g_0,\, \varphi_0,\, \varphi_1)$ in the Sobolev norm H^{l+1}, with $2l > n+2$ with $n+1$ the space time dimension, and satisfying the corresponding constraint equations. Then there are two values t_{1j} and t_{2j} such that on $M_j = \Sigma \times (t_{1j}, t_{2j})$ there exists a Lorentzian metric h_j and an scalar φ_j which satisfies the combined Einstein scalar field equations, and such that the initial data is $(k_j,\, g_{0j},\, \varphi_{j0},\, \varphi_{1j})$. The surface $\tau \times \Sigma$ is a Cauchy one when $\tau \in (t_{1j}, t_{2j})$. Furthermore, when $\tau \in I$, the data on such Cauchy hyper-surface induced by (h_j, φ_j) converges to the one induced by $(g,\, \varphi)$ for large j.

The last statement is the most subtle of this theorem. In any case, all these results shows that the Einstein-scalar field system is a well conditioned one. However, the analysis is more involved when matter is added. This problem is to be discussed below.

6.2 The addition of matter

The theorems described above apply for a vacuum solution of the theory only. The theorems are not generalized in straightforward manner when matter is added, since the addition of the lagrangian (2.11) changes the equations of motion for φ. This is unless the matter term is conformally invariant. In four dimensions, this is the case for the Maxwell electromagnetic lagrangian (3.16). When such type matter is added, there should not be a problem coupling the scalar field to any other type of matter for which local existence and uniqueness has been proven. Nevertheless, this has to be checked in each individual case.

For the non conformal case, such as the stress energy tensor of a perfect fluid (3.17), the problem is more difficult, but it may be studied following the suggestions in [64]-[66]. Following these references, one may study the theory in the O' Hanlon formulation with the following harmonic coordinate choice

$$F^i_\phi = F^i - H^i = 0 \qquad \text{with} \qquad F^i := g^{pq}\Gamma^i_{pq}, \quad H^i = \frac{1}{\phi}\nabla^i \phi. \tag{6.54}$$

The equation of motions for the free case reduce to

$$R_{ij} = \frac{1}{\phi}\left[T_{ij} - \frac{1}{2}T g_{ij}\right], \tag{6.55}$$

where the energy momentum tensor is

$$T_{ij} = \nabla_i \nabla_j \phi - g_{ij} g^{pq} \nabla_p \nabla_q \phi - \frac{1}{2} V(\phi) g_{ij}, \tag{6.56}$$

The Ricci tensor in this gauge takes the simpler form

$$R_{ij} = R^{\phi}_{ij} + \frac{1}{2}\left[g_{ip}\partial_j \left(F^p_{\phi} + H^p \right) + g_{jp}\partial_i \left(F^p_{\phi} + H^p \right) \right] \tag{6.57}$$

with

$$R^{\phi}_{ij} = -\frac{1}{2} g^{pq} \partial^2_{pq} g_{ij} + A_{ij}(g, \partial g), \tag{6.58}$$

where only first order derivatives appear in the functions A_{ij}. Assuming that $F^i_{\phi} = 0$ and taking the expression of H^i into account, we obtain the following representation

$$R_{ij} = -\frac{1}{2} g^{pq} \partial^2_{pq} g_{ij} + \frac{1}{\phi} \partial^2_{ij}\phi + B_{ij}(g, \phi, \partial g, \partial \phi) \tag{6.59}$$

where the functions B_{ij} depend on the metric g, the scalar field ϕ and their first order derivatives. In addition

$$\frac{1}{\phi}\left[T_{ij} - \frac{1}{2} T g_{ij} \right] = \frac{1}{\phi}\partial^2_{ij}\phi + C_{ij}(g, \phi, \partial g, \partial \phi)$$

Again, in the functions C_{ij}, only first order derivatives are involved. The last two formulas show that the equations of motion are

$$g^{pq}\partial^2_{pq} g_{ij} = D_{ij}(g, \phi, \partial g, \partial \phi). \tag{6.60}$$

The initial data for these equations should satisfy the two constraints

$$F_{\phi} = 0, \qquad G^{0i} = \frac{1}{\phi} T^{0i} \quad i = 0, \ldots, 3, \tag{6.61}$$

the last one is the hamiltonian constraint. On the other hand

$$\nabla^i \left(\phi G_{ij} - T_{ij} \right) = \left(\nabla^i \phi \right) R_{ij} - \frac{1}{2}\phi_j \left(R - \frac{dV}{d\phi} \right) + \phi \nabla^i G_{ij} - \left(\nabla^i \nabla_i \nabla_j - \nabla_j \nabla^i \nabla_i \right) \phi \tag{6.62}$$

Now, by taking into account that $\nabla^i G_{ij} = 0$, the equality

$$(\nabla^i \phi) R_{ij} = -\left(\nabla^i \nabla_i \nabla_j - \nabla_j \nabla^i \nabla_i \right) \phi,$$

and that $R = V'(\phi)$ it is obtained that

$$\nabla^i \left(\phi G_{ij} - T_{ij} \right) = 0. \tag{6.63}$$

This can be generalized to the case when matter fields are present, such as electromagnetic, perfect fluids and dust. In these case the authors of [64]-[66] suggest that the Leray and Choquet-Bruhat theorems [67]-[68] can be implemented to show that this is a well posed problem. It may be of interest to make an explicit proof of this important fact in the future.

6.3 The singularity in the Jordan frame

The propositions of the previous sections have shown two main results for the metric formulation of $f(R)$ theories. The first is that, when the space time is assumed to be globally hyperbolic and the scalaron φ takes bounded values during its evolution then, for some suitable conditions on a Cauchy surface Σ, the Einstein metric $\widetilde{g}_{\mu\nu}$ is time like geodesically incomplete. On the other hand, when these conditions are fulfilled, the scalaron and the metric evolutions are both well posed. In particular, the values of φ and $\widetilde{g}$ at future or past times are uniquely determined by the initial data on the Cauchy surface. The metric in the Jordan frame is given by (2.6) and it follows that

$$\widetilde{g}_{ab} = e^{\sqrt{\frac{2k}{3}}\varphi} g_{ab}.$$

Therefore the following inequality takes place

$$e^{-\sqrt{\frac{2k}{3}}\varphi_{max}}\widetilde{g}_{ab} < g_{ab} < e^{-\sqrt{\frac{2k}{3}}\varphi_{min}}\widetilde{g}_{ab}.$$

Here φ_{min} and φ_{max} are the minimum and the maximum of φ at the full evolution. We gave above some examples for which these extreme values do exist. The singularity theorems described in the previous section show that Einstein metric $\widetilde{g}$ can not be extended beyond certain finite time τ_0 and therefore g_{ab} can not be extended beyond certain time τ' defined by

$$e^{-\sqrt{\frac{k}{6}}\varphi_{max}}\tau_0 < \tau' < e^{-\sqrt{\frac{k}{6}}\varphi_{min}}\tau_0.$$

In other words, the Jordan metric is also past or future geodesically incomplete.

It is usually emphasized in the literature that the scalaron φ is not a surjective function of the curvature R. In other words there may exist different values of the curvature R_1 and R_2 giving rise to the same value for φ. However, this is not related to the uniqueness results described here. These results shows instead that, once the initial data is properly formulated on the Cauchy surface, the evolution of φ and $\widetilde{g}$ is completely determined. Note that, in addition, the information about the curvature is given by both quantities $\widetilde{g}$ and φ, not by φ alone.

7. Discussion

In the present work, the cosmological singularity theorems of Hawking where extended to $f(R)$ models, by assuming that the matter coupled to these models satisfies the condition $\left(T_{ij} - \frac{g_{ij}}{2}T\right)k^i k^j \geq 0$ for any generic unit time like field. The difficulty relies in that, for $f(R)$ theories, this condition does not imply that $R_{ij}\gamma^i\gamma^j \geq 0$ for every unit time like vector field. The last is a key property in proving the standard singularity theorems. Furthermore, the additional degree of freedom, the scalaron, violates the condition $\left(T_{ij} - \frac{g_{ij}}{2}T\right)k^i k^j \geq 0$. These complications were sorted out by the use of certain generic results about non Lipschitz differential equations.

The application of theorems presented here is straightforward for certain $f(R)$ models such as the Hu-Sawicki or the Starobinsky ones [15], [21]. For these scenarios, the corresponding scalaron potential has a maximum, which is one of the requirements for the singularity to occur. For other models, this condition is not automatically satisfied and should be checked individually, by considering different types of matter couplings. However, the negation of the hypothesis of these propositions presented here does not necessarily imply that a singularity is absent. In other word, we do not claim that we have exhausted all the possible situations for which a singularity takes place, but just a class of them.

In our opinion, the results presented here complements those of [74]. Those results are based on certain weighted averages of a parameter, which reduces to the Hubble constant when the space time is isotropic and homogenous. Instead, our results are based on certain theorems in [62], which are obtained by generalizing the so called average null energy conditions (ANEC) appropriately. Other reference also dealing with these types of averages are [79]-[81]. These conditions are formulated in terms of the matter content stress energy momentum tensor, and its translated into conditions for the Ricci tensor. So it is likely that the results presented here are complementary to those in [74], and enlarge the list of situations in which a singularity is unavoidable.

An interesting task may be to relax the global hyperbolicity condition for the underlying space time. In addition, it may be of interest as well to generalize the Penrose singularity theorems [61] to these scenarios. We leave these matters for a future investigation.

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