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MATHEMATISCHE NACHRICHTEN

On null sequences for Banach operator ideals, trace duality and approximation properties

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Abstract

Let $\mathcal{A}$ be a Banach operator ideal and X be a Banach space. We undertake the study of the vector space of $\mathcal{A}$-null sequences of Carl and Stephani on X, $c_{0,\mathcal{A}}(X)$, from a unified point of view after we introduce a norm which makes it a Banach space. To give accurate results we consider local versions of the different types of accessibility of Banach operator ideals. We show that in the most common situations, when $\mathcal{A}$ is right-accessible for $(\ell_1; X)$, $c_{0,\mathcal{A}}(X)$ behaves much alike $c_0(X)$. When this is the case we give a geometric tensor product representation of $c_{0,\mathcal{A}}(X)$. On the other hand, we show an example where the representation fails. Also, via a trace duality formula, we characterize the dual space of $c_{0,\mathcal{A}}(X)$. We apply our results to study some problems related with the $\mathcal{K}_{\mathcal{A}}$-approximation property giving a trace condition which is used to solve the remaining case ($p = 1$) of a problem posed by Kim (2015). Namely, we prove that if a dual space has the $\mathcal{K}_1$-approximation property then the space has the $\mathcal{K}_{u1}$-approximation property.

KEYWORDS
Operator ideals, null sequences, compact sets, approximation properties

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1 | INTRODUCTION

The notions of null sequences and compact sets were shown to be closely related from the nowadays classical result of Grothendieck which characterizes relatively compact sets as those contained in the absolutely convex hull of a norm null sequence of vectors of the space. In the recent years strong forms of compactness have been studied, see for instance [1], [2], [3], [13], [17], [22], [29], [31], [32], [34]. Many of the results obtained can be revisited under the Carl–Stephani theory of $\mathcal{A}$-compact sets and $\mathcal{A}$-null sequences [4], where $\mathcal{A}$ denotes an arbitrary operator ideal. The key point in our theoretical approach is the fact that $\mathcal{A}$-compact sets are determined by $\mathcal{A}$-null sequences and vice versa [4, Theorem 1.1]. Let us introduce some definitions and notations.

As usual, $\mathcal{L}, \mathcal{F}, \overline{\mathcal{F}}$ and $\mathcal{K}$ are the ideals of bounded, finite rank, approximable and compact linear operators, respectively; all considered with the supremum norm $\|\cdot\|$. Also, X is a Banach space with closed unit ball B_X. Recall that a sequence $(x_n)_n$ in X is $\mathcal{A}$-*null* if there exist a Banach space Z, an operator $R \in \mathcal{A}(Z; X)$ and a null sequence $(z_n)_n \subset Z$ such that $x_n = Rz_n$ for all $n \in \mathbb{N}$ (see [4, Definition 1.1 and Lemma 1.2]). Notice that any $\mathcal{A}$-null sequence is, in particular, norm null. We denote by $c_0(X)$ the space of null sequences endowed with the supremum norm and by $c_{0,\mathcal{A}}(X)$ the linear space of $\mathcal{A}$-null sequences. Clearly, $c_0(X) = c_{0,\mathcal{K}}(X)$. In fact, $c_0(X) = c_{0,\mathcal{A}}(X)$ for every operator ideal $\mathcal{A}$ such that $\overline{\mathcal{F}} \subset \mathcal{A}$, as can be inferred from [20, Proposition 2.4].

The first objective of this work is to study, from a unified point of view, $c_{0,\mathcal{A}}(X)$ as a Banach space, here $\mathcal{A} = \left(\mathcal{A}, \|\cdot\|_{\mathcal{A}}\right)$ is a Banach operator ideal. To understand its structure we inspect to what extent classical results remain valid. Recall that the natural mapping: $(a_n)_n \otimes x \mapsto (a_n x)_n$ of $c_0 \otimes X$ into $c_0(X)$ yields, via the injective tensor norm ε, the well-known identification $c_0(X) = c_0 \widehat{\otimes}_\varepsilon X$ due to Grothendieck [15] (see, e.g., [33, Example 3.5]). This type of identities appeared recently in similar contexts. Oja proved [29, Theorem 4.1] that the space of p-null sequences of Delgado and Piñeiro [32] has a tensor representation via the Chevet–Saphar tensor norm d_p and with a similar proof J. M. Kim [17, Theorem 1.1] showed that the same is true for the space of unconditionally p-null sequences with the tensor norm w_p, $1 \leq p < \infty$. The link between these notions of nullity and null sequences given by operator ideals is as follows: p-null sequences correspond with $\mathcal{N}^p$-null sequences with $\mathcal{N}^p$ the ideal of right p-nuclear operators [22, Proposition 1.4 and Remark 1.3]. Also, unconditionally p-null sequences coincides with $\mathfrak{K}_{p'}$-null sequences [3, Corollary 4.2], where $\mathfrak{K}_{p'}$ is the ideal of the *classical* p'-compact operators of Fourie and Swart [11] and p' is the conjugate of p. Taking this into account, the results by Oja and Kim read as follows: $c_{0,\mathcal{N}^p}(X) = c_0 \widehat{\otimes}_{d_p} X$ and $c_{0,\mathfrak{K}_{p'}}(X) = c_0 \widehat{\otimes}_{w_p} X$, respectively. See Section 2 for a more informative explanation.

On the other hand, the trace duality allows other type of identifications and evidence the interplay existing between operator ideals and tensor norms. For example, $c_0(X) = \overline{\mathcal{F}}_{w^*}(\ell_1; X)$ isometrically, where $\overline{\mathcal{F}}_{w^*}$ stands for the weak*-weak continuous approximable linear operators. Also, $c_0(X)' = \ell_1(X') = \ell_1 \widehat{\otimes}_\pi X'$ (see [15] or, e.g., [33, Example 2.6]). Since ℓ_1 has the approximation property, $c_0(X)'$ admits the isometric identification as the ideal of nuclear operators $\mathcal{N}(c_0; X')$. In the p-null setting Delgado and Piñeiro showed [32, Proposition 3.1], via the trace duality, the identity $c_{0,\mathcal{N}^p}(X) = \Pi_{p'}(c_0; X')$, where $\Pi_{p'}$ stands for the ideal of p'-summing operators and p' is the conjugate of p.

The Carl–Stephani theory propitiates the conditions to understand these results under a more general framework. To advance our project we endow $c_{0,\mathcal{A}}(X)$ with a norm coming from the measure of $\mathcal{A}$-compact sets given in [22]. Following [4], a subset K of X is *relatively* $\mathcal{A}$*-compact* if there exist a Banach space Z, an operator $T \in \mathcal{A}(Z; X)$ and a compact set $M \subset Z$ such that $K \subset T(M)$. The $\mathcal{A}$*-compact measure* of K is defined as

$$m_{\mathcal{A}}(K; X) := \inf \left\{ \|T\|_{\mathcal{A}} \ : \ K \subset T(M),\ T \in \mathcal{A}(Z; X),\ M \subset B_Z \right\},$$

where the infimum is taken considering all Banach spaces Z, all operators $T \in \mathcal{A}(Z; X)$ and all compact sets $M \subset B_Z$ for which the inclusion $K \subset T(M)$ holds. We often use that we may consider only operators in $\mathcal{A}(\ell_1; X)$ (see [22, Proposition 1.8]).

Now, $c_{0,\mathcal{A}}(X)$ is a Banach space if considered with the norm

$$\left\|(x_n)_n\right\|_{c_{0,\mathcal{A}}} := m_{\mathcal{A}}(\{x_n\}_n; X).$$

This metric counterpart allows us to inspect a tensor representation for $c_{0,\mathcal{A}}(X)$. Recall that any Banach operator ideal has associated a finitely generated tensor norm (see Section 2 for details). The tensor representation valid for $c_0(X)$, $c_{0,\mathcal{N}^p}(X)$ and $c_{0,\mathfrak{K}_{p'}}(X)$ with the respective tensor norms ε, d_p and w_p extends to $c_{0,\mathcal{A}}(X) = c_0 \widehat{\otimes}_\alpha X$ when $\mathcal{A}$ is minimal and α is its associated tensor norm (Theorem 2.6), but it is not true in general as Example 2.14 shows. Nevertheless, considering *local* versions of accessibility of Banach operator ideals (Proposition 2.7) we prove that in the most common situations, when $\mathcal{A}$ is right-accessible for $(\ell_1; X)$, $c_{0,\mathcal{A}}(X)$ behaves much alike $c_0(X)$ (Theorem 2.11). The structure of the dual space of $c_{0,\mathcal{A}}(X)$ is studied in Section 3. We show that $c_{0,\mathcal{A}}(X)'$ is naturally determined by the adjoint ideal of $\mathcal{A}$ if $\mathcal{A}$ is right-accessible for (ℓ_1, X) (Theorem 3.1), and by the conjugate ideal of $\mathcal{A}$ in the general case (Theorem 3.5). Also, we show that the identification is given via a trace duality (Theorem 3.9).

The second objective of this work is the study of approximation properties related with $\mathcal{A}$-compact sets. This topic is treated in Section 4. Operators playing a crucial role here are those sending bounded sets into relatively $\mathcal{A}$-compact sets, namely the class of $\mathcal{A}$*-compact operators* [4] denoted by $\mathcal{K}_{\mathcal{A}}$ which, in fact, is a Banach operator ideal [22] (see Section 2 for more details). A Banach space X has the $\mathcal{K}_{\mathcal{A}}$*-approximation property* ($\mathcal{K}_{\mathcal{A}}$-AP for short) if its identity map Id_X can be approximated by finite-rank operators on $\mathcal{A}$-compact sets of X under $m_{\mathcal{A}}$. Recall that X has the *approximation property* (AP for short) if Id_X can be uniformly approximated by finite rank operators on compact sets. Thus, the $\mathcal{K}_{\mathcal{A}}$-AP, defined and studied in [28] and [22] (see also [20]), can be seen as the analogous to the AP where the system of compact sets is replaced with the system of $\mathcal{A}$-compact sets and the uniform norm with $m_{\mathcal{A}}$.

We focus our attention on the recent works of J. M. Kim [18] and [19]. In [18], the author introduced the $\mathcal{K}_{up}$*-approximation property* ($\mathcal{K}_{up}$-AP for short) related with unconditionally p-compact sets, $1 \leq p < \infty$, (see Section 4 for details). This property is, in terms of the metric Carl–Stephani theory, the $\mathcal{K}_{\mathcal{A}}$-AP for $\mathcal{A} = \mathfrak{K}_{p'}^{max}$ the maximal hull of $\mathfrak{K}_{p'}$ with p' is the conjugate of p (Remark 4.6). The importance of this approximation property is the dual relation it satisfies with the κ_p*-approximation property*

(κ_p-AP for short) of Delgado, Piñeiro and Serrano [8, Defintion 1.1]. The κ_p-AP coincides with the $\mathcal{K}_{\mathcal{N}^p}$-AP (as, for instance, [22, Remark 1.7] shows). The main result in [18] reads as follows ([18, Theorem 1.1]).

Theorem (J. M. Kim) *Let $1 < p \leq \infty$. If the dual space X' of a Banach space X has the $\mathcal{K}_{up}$-AP, then X has the $\mathcal{K}_{\mathcal{N}^p}$-AP, and if X' has the $\mathcal{K}_{\mathcal{N}^p}$-AP, then X has the $\mathcal{K}_{up}$-AP.*

In [18], the author wonders if the result is true for $p = 1$. In [19] the duality between the $\mathcal{K}_1$- and the $\mathcal{K}_{u1}$-AP's is treated. Here, the author shows that the first statement of the above theorem reamisn valid when $p = 1$ [19, Theorem 1.1]. Also, the second assertion is shown to be true whenever X is Asplund [19, Theorem 1.3]. Our results on the $\mathcal{K}_{\mathcal{A}}$-AP allows us to prove that this additional assumption on X, the Asplundness, can be removed (Theorem 4.7). In order to tackle our objective we proceed in a natural way. Following Grothendieck's prototype result [15] we first characterize the dual space of $(\mathcal{L}(X;Y), \tau_{s\mathcal{A}})$, the space of bounded linear operators between Banach spaces X, Y, endowed with the topology of uniform convergence on $\mathcal{A}$-compact subsets of X under $m_{\mathcal{A}}$ (Theorem 4.2). Finally, we give a trace condition for the $\mathcal{K}_{\mathcal{A}}$-AP (Theorem 4.4). Similar approaches to understand dual relations between approximation properties given by operator ideals can be found in [5] and [18].

Our notation is standard. We consider Banach spaces X, Y over the same, either real or complex, field $\mathbb{K}$. We denote by X' the topological dual of X and by $\widehat{x}$ an element in X canonically embedded in the bidual space X''. For the sequence spaces c_0, ℓ_1 and ℓ_∞ we denote respectively the nth-canonical element by e_n, e'_n and $\widehat{e}_n$. Also, operators in $\mathcal{F}(X;Y)$ are regarded as elements of the algebraic tensor product $X' \otimes Y$ and tensors in $X \otimes Y$ as operators in $\mathcal{F}(X';Y)$. The tensor product $X \otimes Y$ endowed with a tensor norm α is denoted by $X \otimes_\alpha Y$ and $X \widehat{\otimes}_\alpha Y$ stands for its completion. Every tensor norm considered in this paper is assumed to be finitely generated (see [6, 12.4] for definition). As usual, we denote by α^t, α' and $/\alpha$ the transpose, the dual and the left injective associate tensor norm of α, respectively (see [6, 12.3], [6, 15.2] and [6, 20.7] for definitions). Given a Banach operator ideal $\mathcal{A}$, we denote by $\mathcal{A}^{min}$, $\mathcal{A}^d$, $\mathcal{A}^{max}$, $\mathcal{A}^{sur}$, $\mathcal{A}^{inj}$ the minimal kernel, the dual ideal, the maximal, surjective and injective hulls of $\mathcal{A}$, respectively, all considered with their usual norms (see [30, Ch. 8] for definitions).

All other relevant terminology and preliminaries are given in corresponding sections. For the theory of operator ideals and normed tensor products we refer the reader to the books of Defant and Floret [6], of Diestel, Fourie and Swart [9] and of Ryan [33]. For further reading on operator ideals we refer the reader to the books of Pietsch [30] and of Diestel, Jarchow and Tonge [10]. For approximation properties we refer the reader to the books of Lindenstrauss and Tzafriri [25] and to the books [6], [33].

2 | TENSOR REPRESENTATION OF $\mathcal{A}$-NULL SEQUENCES

As a first insight to the Banach space of $\mathcal{A}$-null sequences on X we give the following results. Proposition 2.2 collects the basics of the theory relating null sequences, compact sets and compact operators given by operator ideals as well as the key role that the space ℓ_1 plays. In general, we use this result without further mentioning.

Proposition 2.1. *Let $\mathcal{A}$ be a Banach operator ideal and X be a Banach space. Then c_0 is complemented in $c_{0,\mathcal{A}}(X)$. As a consequence, $c_{0,\mathcal{A}}(X)$ is not a dual space.*

Proof. *Fix $x \in X$ with $\|x\| = 1$ and take $x' \in B_{X'}$ such that $x'(x) = 1$. Consider the linear operators $T_1 : c_0 \longrightarrow c_{0,\mathcal{A}}(X)$ and $T_2 : c_{0,\mathcal{A}}(X) \longrightarrow c_0$ defined by $T_1((\alpha_n)_n) = (\alpha_n x)_n$ and $T_2((x_n)_n) = (x'(x_n))_n$. Routine arguments show that $\|T_1\| = \|T_2\| = 1$ and that $T_2 T_1 = Id_{c_0}$, which implies that $T_1 T_2$ is a projection of $c_{0,\mathcal{A}}(X)$ with range equal to c_0. Then, c_0 is complemented in $c_{0,\mathcal{A}}(X)$. On the other hand, Lindenstrauss [24] observed that a Banach space is complemented in some dual space if and only if it is complemented in its own second dual. Then, the second statement follows.* □

Another main ingredient in this theory is the space of $\mathcal{A}$-compact operators, denoted by $\mathcal{K}_{\mathcal{A}}$. For Banach spaces X and Y,

$$\mathcal{K}_{\mathcal{A}}(X;Y) := \{T \in \mathcal{L}(X;Y) : \ T(B_X) \subset Y \ \text{ is relatively } \ \mathcal{A}\text{-compact}\},$$

which is a Banach operator ideal under the norm $\|T\|_{\mathcal{K}_{\mathcal{A}}} = m_{\mathcal{A}}(T(B_X);Y)$ (see [22]). Recall that $\mathcal{K}_{\mathcal{A}}$ is a surjective ideal and can be seen as a composition ideal as $\mathcal{K}_{\mathcal{A}} = \mathcal{A}^{sur} \circ \mathcal{K} = (\mathcal{A} \circ \overline{\mathcal{F}})^{sur}$ (see [4, Theorem 2.1] and [22, Proposition 2.1]). Since ℓ_1 has the lifting metric property, $\mathcal{K}_{\mathcal{A}}(\ell_1;X) = \mathcal{A} \circ \overline{\mathcal{F}}(\ell_1;X)$.

Proposition 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach operator ideals and X be a Banach space. The following statements are equivalent.*

(i) *$c_{0,\mathcal{A}}(X) = c_{0,\mathcal{B}}(X)$ isometrically.*

(ii) *Relatively $\mathcal{A}$-compact and $\mathcal{B}$-compact sets of X coincide and $m_{\mathcal{A}}(\,\cdot\,;X) = m_{\mathcal{B}}(\,\cdot\,;X)$.*

(iii) *$\mathcal{K}_{\mathcal{A}}(Y;X)=\mathcal{K}_{\mathcal{B}}(Y;X)$ isometrically, for all Banach spaces Y.*

(iv) *$\mathcal{A}\circ\overline{\mathcal{F}}(\ell_1;X)=\mathcal{B}\circ\overline{\mathcal{F}}(\ell_1;X)$ isometrically.*

Proof. *On the one hand $\mathcal{A}$-compact sets of X are those sets contained in the convex hull of sequences in $c_{0,\mathcal{A}}(X)$ [4, Theorem 1.1]. On the other hand, by [22, Propostion 1.4], a sequence $(x_n)_n$ in $c_0(X)$ is $\mathcal{A}$-null if $\{x_n\}_n$ is $\mathcal{A}$-compact. Then, (i) and (ii) are equivalent (the isometric part of the statements is straightforward). Clearly, (ii) implies (iii) and (iii) implies (iv) follows from the paragraph below Proposition 2.1. Finally, by [22, Corollary 1.9], an $\mathcal{A}$-compact set may be regarded as an $\mathcal{A}\circ\overline{\mathcal{F}}$-compact set with equal measure and by [22, Proposition 1.8], this type of sets are determined by the image of operators defined on ℓ_1. Thus, (iv) implies (ii) and the proof is complete.* □

Given an operator ideal $\mathcal{A}$, we denote by $\mathcal{A}_{w^*}(X';Y)$ the subspace of all w^*-w continuous operators of $\mathcal{A}$ from X' to Y. Clearly, $\mathcal{F}_{w^*}(X';Y)=X\otimes Y$. Also, we denote by $\pi_n:\ell_1\to\ell_1$ projection to the linear space generated by the first n-coordinates of the canonical unit vectors of ℓ_1.

Lemma 2.3. *Let $\mathcal{A}$ be a Banach operator ideal, X be a Banach space and T be in $\mathcal{L}(\ell_1;X)$. Then $T\in\mathcal{K}_{\mathcal{A}\,w^*}(\ell_1;X)$ if and only if $(Te_n')_n$ is an $\mathcal{A}$-null sequence in X. Moreover, $\|T\|_{\mathcal{K}_{\mathcal{A}}}=m_{\mathcal{A}}(\{Te_n'\}_n;X)$.*

Proof. *First note that $T(B_{\ell_1})$ is the absolutely convex hull of $\{Te_n'\}_n$. Then, $\{Te_n'\}_n$ is relatively $\mathcal{A}$-compact if and only if T is $\mathcal{A}$-compact and, from the definitions, $\|T\|_{\mathcal{K}_{\mathcal{A}}}=m_{\mathcal{A}}(\{Te_n'\}_n;X)$. Also, $T\in\mathcal{K}_{w^*}(\ell_1;X)$ if and only if $(Te_n')_n\in c_0(X)$. By [22, Proposition 1.4], $\mathcal{A}$-null sequences are those norm null sequences contained in $\mathcal{A}$-compact sets, thus the proof is complete.* □

Lemma 2.4. *Let $\mathcal{A}$ be a Banach operator ideal and X be a Banach space. Then*

$$\overline{\mathcal{F}_{w^*}(\ell_1;X)}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}}=\mathcal{K}_{\mathcal{A}\,w^*}(\ell_1;X)$$

holds isometrically. Moreover, if $T\in\mathcal{K}_{\mathcal{A}\,w^*}(\ell_1;X)$ then $\lim_{n\to\infty}\|T-T\pi_n\|_{\mathcal{K}_{\mathcal{A}}}=0$.

Proof. *Notice that $(T\pi_n)_n$ is in $\mathcal{F}_{w^*}(\ell_1;X)$ for any T in $\mathcal{K}_{\mathcal{A}\,w^*}(\ell_1;X)$. On the other hand,*

$$\|T-T\pi_n\|_{\mathcal{K}_{\mathcal{A}}}\leq m_{\mathcal{A}}(T(I-\pi_n)(B_{\ell_1});X)=m_{\mathcal{A}}(\{Te_j'\}_{j\geq n};X).$$

By Lemma 2.3, $(Te_j')_j$ is $\mathcal{A}$-null and then $\lim_{n\to\infty}m_{\mathcal{A}}(\{Te_j'\}_{j\geq n};X)=0$ (see for instance [35, Lemma 4]). Thus, the proof is complete. □

A combination of the above lemmas yields the following characterization of the Banach space of $\mathcal{A}$-null sequences.

Proposition 2.5. *Let $\mathcal{A}$ be a Banach operator ideal and X be a Banach space. Then*

$$c_{0,\mathcal{A}}(X)=\overline{\mathcal{F}_{w^*}(\ell_1;X)}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}}$$

holds isometrically. The identification is given by $(x_n)_n\mapsto\sum_{n=1}^{\infty}e_n\otimes x_n$.

A finitely generated tensor norm α and a Banach operator ideal $\mathcal{A}$ are *associated* if $\mathcal{A}(M;N)=M'\otimes_\alpha N$ holds for every finite dimensional spaces M and N [6, 17.1]. Observe that $\mathcal{A}^{max}$, $\mathcal{A}$ and $\mathcal{A}^{min}$ are associated with the same finitely generated tensor norm. In particular, two maximal (or minimal) ideals coincide if the are associated with the same finitely generated tensor norm. Recall that the ideal $\overline{\mathcal{F}}$ is associated with the tensor norm ε. Also, the minimal ideal $\mathcal{N}^p$ is associated with the Chevet–Saphar tensor norm d_p (see for instance [33, p. 140]) and the minimal ideal $\mathfrak{K}_{p'}$ is associated with the tensor norm w_p as it can be inferred from [12, Proposition 4.1].

Theorem 2.6. *Let $\mathcal{A}$ be a Banach operator ideal with associated tensor norm α and let X be Banach space. Then,*

(a) *$\mathcal{K}_{\mathcal{A}}$ has associated tensor norm $/\alpha$.*

(b) *$\mathcal{K}_{\mathcal{A}^{min}}(\ell_1;X)=\ell_\infty\widehat{\otimes}_\alpha X$ holds isometrically.*

(c) *$c_{0,\mathcal{A}^{min}}(X)=c_0\widehat{\otimes}_\alpha X$ holds isometrically. The identification is given by $(x_n)_n\mapsto\sum_{n=1}^{\infty}e_n\otimes x_n$.*

Proof. *To prove (a) note that the operator ideal $\mathcal{A}\circ\overline{\mathcal{F}}$ also has associated tensor norm α. Now, denote by β the associated tensor norm of $\mathcal{K}_{\mathcal{A}}$. Since $\mathcal{K}_{\mathcal{A}}$ is surjective, β is left injective (see for instance [13, Lemma 3.2]). On the other hand, for every* *n and every finite dimensional space N we have the isometric identities*

$$\ell_\infty^n \otimes_\alpha N = \mathcal{A} \circ \overline{\mathcal{F}}(\ell_1^n; N) = \mathcal{K}_{\mathcal{A}}(\ell_1^n; N) = \ell_\infty^n \otimes_\beta N.$$

Using a left version of [6, Proposition 20.9], we conclude that $\beta = /\alpha$.

To prove (b) note that by [22, Proposition 2.1], $\mathcal{K}_{\mathcal{A}^{min}} = (\mathcal{A}^{min})^{sur}$. Then, by [6, Corollary 9.8], $\mathcal{K}_{\mathcal{A}^{min}}(\ell_1; X) = \mathcal{A}^{min}(\ell_1; X)$. Now, since ℓ_1' has the AP, by [6, Corollary 22.2.1], $\mathcal{A}^{min}(\ell_1; X) = \ell_\infty \widehat{\otimes}_\alpha X$. Hence,

$$\mathcal{K}_{\mathcal{A}^{min}}(\ell_1; X) = \mathcal{A}^{min}(\ell_1; X) = \ell_\infty \widehat{\otimes}_\alpha X,$$

as desired. Finally, by the embedding lemma [6, Lemma 13.3] and (b), we have

$$c_0 \widehat{\otimes}_\alpha X \subset \ell_\infty \widehat{\otimes}_\alpha X = \mathcal{K}_{\mathcal{A}^{min}}(\ell_1; X).$$

These isometric identifications and the identity $c_0 \otimes X = \mathcal{F}_{w^*}(\ell_1; X)$ allows us to see that

$$c_0 \widehat{\otimes}_\alpha X = \overline{\mathcal{F}_{w^*}(\ell_1; X)}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}^{min}}}}.$$

Then, (c) follows by Proposition 2.5. □

Notice that when $\mathcal{A}$ is minimal, Theorem 2.6 (c) gives a representation of $c_{0,\mathcal{A}}(X)$ as a Banach tensor product. Also, this result can be seen as an extension of the characterization given, with a different approach, by Oja [29] for p-null sequences and J. M. Kim [17] for unconditionally p-null sequences. To be more precise we introduce some definitions.

Fixed $1 \le p < \infty$, $\ell_p(X)$ and $\ell_p^u(X)$ denote the spaces of p-summable and unconditionally p-summable sequences in X endowed with their natural norms. For $(x_n)_n \in \ell_p(X)$ we denote by p-co$\{x_n\}$ the absolutely p-convex hull of $(x_n)_n$ defined as

$$p\text{-co}\{x_n\} := \left\{ \sum_{n=1}^{\infty} \alpha_n x_n \,:\, (\alpha_n)_n \in B_{\ell_{p'}} \right\},$$

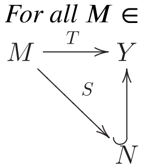
where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\ell_{p'} = c_0$ if $p = 1$. A sequence $(x_n)_n$ in X is *p-null* [32, Definition 2.1] if, given $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ and $(z_k)_k \in \varepsilon B_{\ell_p(X)}$ such that $x_n \in p$-co$\{z_k\}$ for all $n \ge n_0$. By [22, Proposition 1.5], p-null sequences and $\mathcal{N}^p$-null sequences coincide. When $\ell_p(X)$ is replaced with $\ell_p^u(X)$ the unconditionally p-null sequences of J. M. Kim [17] are obtained. By [3, Corollary 4.2], unconditionally p-null sequences and $\mathfrak{K}_{p'}$-null sequences coincide. Therefore, once these concepts are described under the framework of the metric Carl–Stephani theory, an application of Theorem 2.6 (c) for $\mathcal{N}^p$ covers the result due to Oja [29, Theorem 4.1] for p-null sequences, $c_{0,\mathcal{N}^p}(X) = c_0 \widehat{\otimes}_{d_p} X$. With $\mathfrak{K}_{p'}$, the result by Kim [17, Theorem 1.1], $c_{0,\mathfrak{K}_{p'}}(X) = c_0 \widehat{\otimes}_{w_p} X$, is covered. Also, Theorem 2.6 extends the representations given in [29, Theorem 4.1] and [17, Theorem 1.1], by showing that a general element $u \in c_0 \widehat{\otimes}_\alpha X$ can be written as $u = \sum_{n=1}^\infty e_n \otimes x_n$, where $(x_n)_n \subset X$ is $\mathcal{A}^{min}$-null. In addition we have $\|u\|_{c_0 \widehat{\otimes}_\alpha X} = \|(x_n)_n\|_{c_{0,\mathcal{A}^{min}}}$.

Our next objective is to find out to what extent the identity $c_{0,\mathcal{A}}(X) = c_0 \widehat{\otimes}_\alpha X$ holds. As it can be seen along the monograph by Defant and Floret [6], accessibility of operator ideals facilitates the study of different type of characterizations via tensor products, problems related with trace duality and approximation properties. Recall that the minimal kernel of $\mathcal{A}$ is the composition ideal $\mathcal{A}^{min} = \overline{\mathcal{F}} \circ \mathcal{A} \circ \overline{\mathcal{F}}$. When the identity $\mathcal{A}^{min} = \mathcal{A} \circ \overline{\mathcal{F}}$ is satisfied, $\mathcal{A}$ is said to be right-accessible and if $\overline{\mathcal{F}} \circ \mathcal{A} = \mathcal{A}^{min}$, $\mathcal{A}$ is said to be left-accessible (see [6, Propostion 25.2]). Close related with these notions we have the concepts of accesible and totally accesible Banach operator ideal (see [6, 21.2]). In what follows we consider local versions of these notions by fixing an specific pair of Banach spaces (X, Y). With the same proofs of [6, Propostion 25.2] we have the next localized results. As usual, FIN(X) and COFIN(X) denote respectively the set of all finite dimensional and all co-finite dimensional subspaces of X.

Proposition 2.7. *Let $\mathcal{A}$ be a Banach operator ideal and X, Y be Banach spaces.*

(a) *The following statements are equivalent.*

(i) *For all $M \in$ FIN(X), $T \in \mathcal{L}(M; Y)$ and $\varepsilon > 0$ there exist $N \in$ FIN(Y) and $S \in \mathcal{L}(M; N)$ such that T factors*



and $\|S\|_{\mathcal{A}} \le (1+\varepsilon)\,\|T\|_{\mathcal{A}}$.

(ii) $\mathcal{A}^{min}(X;Y) = \mathcal{A} \circ \overline{\mathcal{F}}(X;Y)$ *isometrically.*

(iii) $\mathcal{A}^{min}(M;Y) = \mathcal{A}(M;Y)$ *isometrically, for all* $M \in \mathrm{FIN}(X)$.

(b) *The following statements are equivalent.*

(i) *For all* $N \in \mathrm{FIN}(Y)$, $T \in \mathcal{L}(X;N)$ *and* $\varepsilon > 0$ *there exist* $L \in \mathrm{COFIN}(X)$ *and* $S \in \mathcal{L}(X/L;N)$ *such that* T *factors*

$$\begin{array}{ccc} X & \xrightarrow{T} & N \\ \downarrow & \nearrow_{S} & \\ X/L & & \end{array} \qquad \text{and} \qquad \|S\|_{\mathcal{A}} \leq (1+\varepsilon)\,\|T\|_{\mathcal{A}}\,.$$

(ii) $\mathcal{A}^{min}(X;Y) = \overline{\mathcal{F}} \circ \mathcal{A}(X;Y)$ *isometrically.*

(iii) $\mathcal{A}^{min}(X;N) = \mathcal{A}(X;N)$ *isometrically, for all* $N \in \mathrm{FIN}(Y)$.

We say that $\mathcal{A}$ is *right-accessible* (resp. *left-accessible*) *for* (X,Y) *if any of the statements of Proposition* 2.7 (a) (resp. (b)) is satisfied. In an analogous way, we also may consider the local notions of *accessible* and *totally accessible* Banach operator ideal *for* (X,Y). The next lemmas present basic results on local accessibility.

Lemma 2.8. *Let* $\mathcal{A}$ *be a Banach operator ideal and* X, Y *be Banach spaces.*

(a) $\mathcal{A}$ *is right-accessible for* (X,Y) *if and only if* $\mathcal{A}$ *is right-accessible for* (X'',Y).

(b) $\mathcal{A}$ *is left-accessible for* (X,Y) *if and only if* $\mathcal{A}$ *is left-accessible for* (X,Y'').

Proof. *To prove (a) we only show the "only if" part. Take* $M \in \mathrm{FIN}(X'')$, $T \in \mathcal{L}(M;Y)$ *and* $\varepsilon > 0$. *Using the principle of local reflexivity, there exists an injective linear operator* $u : M \to u(M) \subset X$ *such that* $\|u\|\,\left\|u^{-1}\right\| \leq 1+\varepsilon$. *Let* $\widetilde{M} = u(M) \in \mathrm{FIN}(X)$, *then* $\mathcal{A}^{min}\big(\widetilde{M};Y\big) = \mathcal{A}\big(\widetilde{M};Y\big)$ *(see (a) (iii) of Proposition 2.7). Also,*

$$\begin{aligned} \|T\|_{\mathcal{A}^{min}(M;Y)} &= \|Tu^{-1}u\|_{\mathcal{A}^{min}(M;Y)} \leq \|Tu^{-1}\|_{\mathcal{A}^{min}(\widetilde{M};Y)}\,\|u\| \\ &= \|Tu^{-1}\|_{\mathcal{A}(\widetilde{M};Y)}\,\|u\| \leq \|T\|_{\mathcal{A}(M;Y)}\,\|u^{-1}\|\,\|u\|\,. \end{aligned}$$

Thus, $\|T\|_{\mathcal{A}^{min}(M;Y)} \leq (1+\varepsilon)\,\|T\|_{\mathcal{A}(M;Y)}$ *and statement (a) follows. The proof of (b) is analogous.* □

Lemma 2.9. *Let* $\mathcal{A}$ *be a Banach operator ideal and* X, Y *be Banach spaces. If* $\mathcal{A}$ *is right-accessible for* (X',Y') *then* $\mathcal{A}^d$ *is left-accessible for* (Y,X).

Proof. *We first prove that condition (b) (ii) of Proposition 2.7 is satisfied for* $\mathcal{A}^d$ *and* (Y,X''). *Take* $T \in \overline{\mathcal{F}} \circ \mathcal{A}^d(Y;X'')$. *Then, by Lemma 2.8 (a),* $T' \in \mathcal{A} \circ \overline{\mathcal{F}}(X''';Y') = \mathcal{A}^{min}(X''';Y')$ *and therefore,* $T \in (\mathcal{A}^{min})^d(Y;X'')$, *with* $\left\|T'\right\|_{\mathcal{A}^{min}} \leq \|T\|_{\overline{\mathcal{F}} \circ \mathcal{A}^d}$. *Since the range space is a dual space, by [6, Corollary 22.8 (1)],* $(\mathcal{A}^{min})^d(Y;X'') = (\mathcal{A}^d)^{min}(Y;X'')$ *isometrically. Thus,* $\overline{\mathcal{F}} \circ \mathcal{A}^d(Y;X'') \subset (\mathcal{A}^d)^{min}(Y;X'')$. *The other inclusion, with norm non greater than 1, always holds. Now, the result follows by Lemma 2.8 (b).* □

Lemma 2.10. *Let* $\mathcal{A}$ *be a Banach operator ideal and* X, Y *be Banach spaces.*

(a) $\mathcal{A}$ *is right-accessible for* (X,Y) *if and only if* $\mathcal{A} \circ \mathcal{K}$ *is right-accessible for* (X,Y).

(b) $\mathcal{A}$ *is left-accessible for* (X,Y) *if and only if* $\mathcal{K} \circ \mathcal{A}$ *is left-accessible for* (X,Y).

Proof. *The proof is straightforward from the identities* $\mathcal{K} \circ \overline{\mathcal{F}} = \overline{\mathcal{F}}$ *and* $\overline{\mathcal{F}} \circ \mathcal{K} = \overline{\mathcal{F}}$ *and the respective definitions.* □

Theorem 2.11. *Let* $\mathcal{A}$ *be a Banach operator ideal with associated tensor norm* α *and let* X *be a Banach space. The following statements are equivalent.*

(i) $c_{0,\mathcal{A}}(X) = c_0 \widehat{\otimes}_\alpha X$ *isometrically.*

(ii) $\mathcal{K}_{\mathcal{A}}(Y;X) = \mathcal{K}_{\mathcal{A}^{min}}(Y;X)$ *isometrically, for all Banach space* Y.

(iii) $\mathcal{A}^{sur}$ *is right-accessible for* (Y,X) *for all Banach space* Y.

(iv) $\mathcal{A}^{sur}$ *is right-accessible for* (ℓ_1,X).

(v) $\mathcal{A}$ *is right-accessible for* (ℓ_1,X).

Proof. *To see that (i) implies (ii), first note that* $\mathcal{K}_{\mathcal{A}^{min}} = (\mathcal{A}^{min})^{sur}$. *By Theorem 2.6, we know that* $c_{0,\mathcal{A}}(X) = c_{0,\mathcal{A}^{min}}(X)$ *which, by Proposition 2.2 is equivalent to the identity* $\mathcal{K}_{\mathcal{A}}(Y; X) = \mathcal{K}_{\mathcal{A}^{min}}(Y; X)$ *for every Banach space* Y. *Now suppose that (ii) holds. Since, by [6, Ex. 21.1 (e)],* $(\mathcal{A}^{min})^{sur}$ *is right-accessible, we have that* $\mathcal{K}_{\mathcal{A}}$ *is right-accessible for* (Y, X) *for all Banach spaces* Y. *Since* $\mathcal{K}_{\mathcal{A}} = \mathcal{A}^{sur} \circ \mathcal{K}$, *and application of Lemma 2.10 gives (iii). That (iii) implies (iv) is clear. Now, suppose that* $\mathcal{A}^{sur}$ *is right-accessible for* (ℓ_1, X) *and use that* $\mathcal{A}$ *and* $\mathcal{A}^{sur}$ *produce the same compact sets [4, p. 79]. Then, by Proposition 2.2,*

$$\mathcal{A} \circ \overline{\mathcal{F}}(\ell_1; X) = \mathcal{A}^{sur} \circ \overline{\mathcal{F}}(\ell_1; X) = \mathcal{A}^{sur\,min}(\ell_1; X).$$

On the other hand, we claim that $\mathcal{A}^{min}(\ell_1; X) = \mathcal{A}^{sur\,min}(\ell_1; X)$. *This follows from the inclusions*

$$\mathcal{A}^{min}(\ell_1; X) \subset \mathcal{A}^{sur\,min}(\ell_1; X) \subset \mathcal{A}^{min\,sur}(\ell_1; X) = \mathcal{A}^{min}(\ell_1; X).$$

Thus, $\mathcal{A} \circ \overline{\mathcal{F}}(\ell_1; X) = \mathcal{A}^{min}(\ell_1; X)$ *and (v) follows. Finally, let us see that (v) implies (i). As* $\mathcal{A}$ *is right-accessible for* (ℓ_1, X), $\mathcal{A} \circ \overline{\mathcal{F}}(\ell_1; X) = \mathcal{A}^{min}(\ell_1; X) = \mathcal{A}^{min} \circ \overline{\mathcal{F}}(\ell_1; X)$. *By Proposition 2.2,* $c_{0,\mathcal{A}}(X) = c_{0,\mathcal{A}^{min}}(X)$. *Using Theorem 2.6 (c), we have* $c_{0,\mathcal{A}}(X) = c_0 \widehat{\otimes}_\alpha X$ *isometrically and the proof is complete.* □

Theorem 2.11 shows that in order to obtain a geometric tensor product representation for $c_{0,\mathcal{A}}(X)$, regardless the Banach space X, a condition of right-accessibility on the ideal is required. As it is usual in this type of problems, we may obtain the same kind of description by imposing some additional hypothesis on X, regardless the Banach operator ideal $\mathcal{A}$ as it is shown in the next proposition. Nevertheless, it turns out that in some situations one or the other restriction is necessary, as Example 2.14 shows.

Proposition 2.12. *Let* $\mathcal{A}$ *be a Banach operator ideal with associated tensor norm* α *and let* X *be a Banach space. If* $\mathcal{A}$ *is right-accessible or* X *has the MAP, then* $c_{0,\mathcal{A}}(X) = c_0 \widehat{\otimes}_\alpha X$ *isometrically.*

Proof. *Notice that by [6, Proposition 25.2 (1)], if* X *has the MAP, then every Banach operator ideal* $\mathcal{A}$ *is right-accessible for* (Y, X) *for all Banach spaces* Y. *Now, the result follows as an application of Theorem 2.11.* □

To exhibit a normed operator ideal which is not right-accessible, we resort to Pisier's operator ideal $\left(\mathcal{A}_P, \|\cdot\|_{\mathcal{A}_P}\right)$, whose construction appeals to Pisier's space [6, Theorem 31.6]. Notice that a careful reading to [6, Theorem 31.6] and [6, Corollary 31.6] allows us to localize both results as follows.

Theorem 2.13. (Defant–Floret–Pisier) *Let* $\mathcal{A}_P$ *be the Pisier Banach operator ideal and* P *be the Pisier space. Then*

- **(a)** $\mathcal{A}_P$ *is neither left- nor right-accessible for* (P, P).
- **(b)** $\mathcal{A}_P^{inj}$ *is not left-accessible for* (P, P).

Example 2.14. *Let* $\mathcal{A}_P$ *be the Pisier Banach operator ideal and* P' *be the dual space of the Pisier space* P. *Let* $\mathcal{A} = \mathcal{A}_P^d$ *and* α *its associated tensor norm. Then* $c_{0,\mathcal{A}}(P') \neq c_0 \widehat{\otimes}_\alpha P'$.

Proof. *Suppose that* $c_{0,\mathcal{A}}(P') = c_0 \widehat{\otimes}_\alpha P'$, *then by Theorem 2.11 (i)* $\Rightarrow$ *(iii), we obtain that* $(\mathcal{A}_P^d)^{sur}$ *is right-accessible for* (Y, P') *for every Banach space* Y *and, in particular, for* $Y = P'$. *By [30, Theorem 8.5.8] we have* $(\mathcal{A}_P^d)^{sur} = (\mathcal{A}_P^{inj})^d$. *An application of Lemma 2.9 gives that* $(\mathcal{A}_P^{inj})^{dd}$ *is left-accessible for* (P, P). *Being maximal,* $\mathcal{A}_P^{inj} = (\mathcal{A}_P^{inj})^{dd}$. *Thus,* $(\mathcal{A}_P^{inj})$ *is left-accessible for* (P, P), *contradicting Theorem 2.13.* □

3 | THE DUAL SPACE OF $c_{0,\mathcal{A}}(X)$ AND THE TRACE DUALITY

The main goal of this section is to characterize the dual space of $c_{0,\mathcal{A}}(X)$ via a trace duality formula. If $\mathcal{A}$ is right-accessible for (ℓ_1, X), the characterization given below is immediate from the results obtained in Section 2. In what follows, $\mathcal{A}^*$ is the adjoint ideal of the Banach operator ideal $\mathcal{A}$, that is the maximal operator ideal associated to the tensor norm $\alpha^* := (\alpha^t)' = (\alpha')^t$ (see [6, 17.9] for more details).

Theorem 3.1. *Let* $\mathcal{A}$ *be a Banach operator ideal and* X *be a Banach space. If* $\mathcal{A}$ *is right-accessible for* (ℓ_1, X), *then*

$$c_{0,\mathcal{A}}(X)' = \mathcal{A}^*(X; \ell_1)$$

holds isometrically. The identification is given by $T \longrightarrow \phi_T((x_n)_n) = \sum_{n=1}^{\infty} \hat{e}_n(T x_n)$.

Proof. *Let α be the associated tensor norm of $\mathcal{A}$. By Theorem 2.11, $c_{0,\mathcal{A}}(X) = c_0\widehat{\otimes}_\alpha X = X\widehat{\otimes}_{\alpha^t} c_0$ isometrically. Applying the representation theorem for maximal operator ideals we have $\mathcal{A}^*(X, c_0') = (X \otimes_{\alpha'} c_0)'$ (see e.g. [6, Theorem 17.5]). Thus, the identity follows and Theorem 2.6 (c), completes the proof.* □

Having in mind the identities $c_0(X) = c_0\widehat{\otimes}_\varepsilon X$ and $c_0(X)' = \ell_1(X') = X'\widehat{\otimes}_\pi \ell_1$, it is natural to ask in what cases the dual space of $c_{0,\mathcal{A}}(X)$ is, for some tensor norm, the completion of $X'\otimes \ell_1$. The following result gives a partial positive answer to this matter. Recall that a finitely generated tensor norm α has the Radon–Nikodým property if $X' \otimes_\alpha \ell_1 = (X \otimes_{\alpha'} c_0)'$ holds isometrically for all Banach spaces X (see [6, Definition 33.2]).

Proposition 3.2. *Let $\mathcal{A}$ be a maximal accessible Banach operator ideal with associated tensor norm α. Then there exists a tensor norm β such that $c_{0,\mathcal{A}}(X)' = X'\widehat{\otimes}_\beta \ell_1$ for all Banach spaces X if and only if $c_{0,\mathcal{A}}(X)' = X'\widehat{\otimes}_{\alpha^*}\ell_1$ and α^* has the Radon–Nikodým property.*

Proof. *Note that being $\mathcal{A}$ right-accessible, by Theorem 2.11, $c_{0,\mathcal{A}}(X) = X\widehat{\otimes}_{\alpha^t} c_0$ for every Banach space X. Since α^* has the Radon–Nikodým property, the "if" part follows directly.*

For the converse, suppose that $c_{0,\mathcal{A}}(X)' = X'\widehat{\otimes}_\beta \ell_1$ for every X. Using that ℓ_1 has the AP, by Theorem 3.1, we obtain that $\mathcal{B}^{min}(X;\ell_1) = X'\widehat{\otimes}_\beta \ell_1 = \mathcal{A}^*(X;\ell_1)$; where $\mathcal{B}$ is an operator ideal associated to β. Since the latter identities hold for all X, $\mathcal{B}^{min} \circ \overline{\mathcal{F}}(X;\ell_1) = \mathcal{A}^* \circ \overline{\mathcal{F}}(X;\ell_1)$. Now, since $\mathcal{A}$ is accessible, by [6, Corrolary 21.3], $\mathcal{A}^*$ is also accessible (hence right-accessible) and, using again that ℓ_1 has the AP, we have

$$X'\widehat{\otimes}_{\alpha^*}\ell_1 = (\mathcal{A}^*)^{min}(X;\ell_1) = \mathcal{A}^* \circ \overline{\mathcal{F}}(X;\ell_1) = X'\widehat{\otimes}_\beta \ell_1.$$

Thus, $c_{0,\mathcal{A}}(X)' = X'\widehat{\otimes}_{\alpha^*}\ell_1$ for every Banach space X and α^* has the Radon–Nikodým property. □

As a consequence, since d_p^*, $1 < p \le \infty$ and w_p^*, $1 \le p < \infty$ have the Radon–Nikodým property [6, Theorem 33.5], we see that the dual spaces of the spaces of p-null and unconditionally p-null sequences are tensor spaces. Indeed, we have

$$c_{0,\mathcal{N}^p}(X)' = X'\widehat{\otimes}_{d_p^*}\ell_1 \quad \text{and} \quad c_{0,\mathfrak{K}_{p'}}(X)' = X'\widehat{\otimes}_{w_p^*}\ell_1.$$

To characterize the dual space of $c_{0,\mathcal{A}}(X)$ for a general Banach operator ideal $\mathcal{A}$ we appeal to its conjugated operator ideal, $\mathcal{A}^\Delta$ introduced and studied by Gordon, Lewis and Retheford [14]. Recall that $\mathcal{A}^\Delta(X;Y)$ is the class of all operators $T \in \mathcal{L}(X;Y)$ for which there is a $\rho > 0$ such that for any $S \in \mathcal{F}(Y;X)$,

$$|\mathrm{tr}(ST)| \le \rho \, \|S\|_{\mathcal{A}},$$

where tr denotes the trace function. With $\|T\|_{\mathcal{A}^\Delta} = \inf\{\rho > 0\}$ where ρ ranges over all constants satisfying the above inequality, $\mathcal{A}^\Delta$ is always a Banach operator ideal. In view of Proposition 2.5, we characterize the dual space of $\overline{\mathcal{F}_{w^*}(X';Y)}^{\|\cdot\|_{\mathcal{K}_\mathcal{A}}}$.

Proposition 3.3. *Let $\mathcal{A}$ be a Banach operator ideal, and X, Y be Banach spaces. Then*

$$\left(\overline{\mathcal{F}_{w^*}(X';Y)}^{\|\cdot\|_{\mathcal{K}_\mathcal{A}}}\right)' = \mathcal{K}_\mathcal{A}^\Delta(Y;X')$$

holds isometrically. The identification is given by $T \mapsto \phi_T(S) = \mathrm{tr}(ST) = \mathrm{tr}(TS)$ for $S \in \mathcal{F}_{w^*}(X';Y)$.

Proof. *One of the inclusions is straightforward. Indeed, any $T \in \mathcal{K}_\mathcal{A}^\Delta(Y;X')$ gives, via the trace function, a $\|\cdot\|_{\mathcal{K}_\mathcal{A}}$-continuous linear functional on $\mathcal{F}_{w^*}(X';Y)$, which by density can be continuously extended to the closure of its domain.*

Conversely, take ϕ in the dual of $\overline{\mathcal{F}_{w^*}(X';Y)}^{\|\cdot\|_{\mathcal{K}_\mathcal{A}}}$ and define the operator $T : Y \to X'$ as $(Ty)(x) = \phi(\widehat{x} \otimes y)$, which is clearly linear. To show that T is in $\mathcal{K}_\mathcal{A}^\Delta(Y;X')$, take $\varepsilon > 0$ and $S \in \mathcal{F}(X';Y)$, $S = \sum_{j=1}^m z_j \otimes y_j$ for some $z_1, \dots, z_m$ in X'' and $y_1, \dots, y_m$ in Y. Denote $Z = \mathrm{span}\{z_1, \dots, z_m\}$ and $W = \mathrm{span}\{Ty_1, \dots, Ty_m\}$.

Since $Z \subset X''$ and $W \subset X'$ are finite dimensional subspaces, applying the principle of local reflexivity, we may find a linear operator $U : Z \to X$ such that

$$\|U\| \le (1+\varepsilon) \quad \textit{and} \quad z_j(Ty_i) = (Ty_i)(Uz_j) \quad \textit{for } i = 1, \dots, m; \ j = 1, \dots, m.$$

Then we have,

$$\operatorname{tr}(TS) = \sum_{j=1}^{m} z_j(Ty_j) = \sum_{j=1}^{m}(Ty_j)(Uz_j) = \phi\left(\sum_{j=1}^{m} \widehat{Uz_j} \otimes y_j\right),$$

and hence $|\operatorname{tr}(TS)| \leq \|\phi\| \left\|\sum_{j=1}^{m} \widehat{Uz_j} \otimes y_j\right\|_{\mathcal{K}_{\mathcal{A}}}$.

Let us see that $\left\|\sum_{j=1}^{m} \widehat{Uz_j} \otimes y_j\right\|_{\mathcal{K}_{\mathcal{A}}} \leq (1+\varepsilon)\|S\|_{\mathcal{K}_{\mathcal{A}}}$. *First, notice that as* $\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{A}}^{dd}$ *[22, Corollary 2.4], we have* $\|S\|_{\mathcal{K}_{\mathcal{A}}} = \|S''\|_{\mathcal{K}_{\mathcal{A}}}$. *Now, consider the operator* $R : Z' \to Y$ *given by* $R = \sum_{j=1}^{m} \hat{z}_j \otimes y_j$ *which satisfies* $RU' = \sum_{j=1}^{m} \widehat{Uz_j} \otimes y_j$. *Then,*

$$\left\|\sum_{j=1}^{m} \widehat{Uz_j} \otimes y_j\right\|_{\mathcal{K}_{\mathcal{A}}} \leq (1+\varepsilon)\|R\|_{\mathcal{K}_{\mathcal{A}}}.$$

Also, with the inclusion $\imath : Z \to X''$, *we have* $S'' = J_Y R\imath'$. *Since* $\mathcal{K}_{\mathcal{A}}$ *is regular [22, Theorem 2.2] and surjective, as* $\imath'$ *is a metric surjection we have* $\|R\|_{\mathcal{K}_{\mathcal{A}}} = \|J_Y R\imath'\|_{\mathcal{K}_{\mathcal{A}}} = \|S''\|_{\mathcal{K}_{\mathcal{A}}}$, *and the result follows.* □

Note that the proof of Proposition 3.3 essentially uses that the ideal $\mathcal{K}_{\mathcal{A}}$ is surjective, regular and that $\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{A}}^{dd}$. A similar proof works for any ideal satisfying these conditions, among which we have all maximal surjective ideals. As a consequence, Proposition 3.3 admits the following partial extension.

Corollary 3.4. *Let* $\mathcal{A}$ *be a surjective, regular Banach operator ideal satisfying* $\mathcal{A} \subset \mathcal{A}^{dd}$, *and let* X, Y *be Banach spaces. Then*

$$\left(\overline{\mathcal{F}_{w^*}(X';Y)}^{\|\cdot\|_{\mathcal{A}}}\right)' = \mathcal{A}^{\Delta}\left(Y;X'\right)$$

holds isometrically. The identification is given by $T \mapsto \phi_T(S) = \operatorname{tr}(ST) = \operatorname{tr}(TS)$ *for* $S \in \mathcal{F}_{w^*}(X';Y)$.

Theorem 3.5. *Let* $\mathcal{A}$ *be a Banach operator ideal and* X *be a Banach space. Then*

$$c_{0,\mathcal{A}}(X)' = \mathcal{K}_{\mathcal{A}}^{\Delta}(X;\ell_1) = \left(\mathcal{A} \circ \overline{\mathcal{F}}\right)^{\Delta}(X;\ell_1)$$

holds isometrically. The identification is given by $T \longrightarrow \phi_T(x_n) = \sum_{n=1}^{\infty} \hat{e}_n(Tx_n)$.

Proof. *The first identity is a direct combination of Proposition 2.5 and Proposition 3.3. For the second identity, use that* $\mathcal{K}_{\mathcal{A}}(\ell_1;X) = \mathcal{A} \circ \overline{\mathcal{F}}(\ell_1;X)$. □

Remark 3.6. *Theorem 3.5 extends Theorem 3.1 since when* $\mathcal{A}$ *is right-accessible for* $(\ell_1;X)$, $\left(\mathcal{A} \circ \overline{\mathcal{F}}\right)^{\Delta}(X;\ell_1) = \mathcal{A}^*(X;\ell_1)$. *Indeed, for any Banach operator ideal* $\mathcal{A}$ *we always have* $\mathcal{A}^{\Delta} \subset \mathcal{A}^*$ *and* $\mathcal{A}^* = (\mathcal{A}^{min})^*$. *Thus, the norm one inclusion* $(\mathcal{A}^{min})^{\Delta} \subset \mathcal{A}^*$ *holds.*

Now, suppose that $\mathcal{A}$ *is right-accessible for* $(\ell_1;X)$, *then* $\mathcal{A} \circ \overline{\mathcal{F}}(\ell_1;X) = \mathcal{A}^{min}(\ell_1;X)$ *and* $\left(\mathcal{A} \circ \overline{\mathcal{F}}\right)^{\Delta}(X;\ell_1) = (\mathcal{A}^{min})^{\Delta}(X;\ell_1) \subset \mathcal{A}^*(X;\ell_1)$. *To prove what remains, note that since* $\mathcal{A}^{min}$ *is right-accessible and* ℓ_1' *has the MAP, the local version of [6, Ex. 21.1.(c)] gives that* $\mathcal{A}^{min}$ *is totally-accessible for* $(\ell_1;X)$ *for all* X. *Now, for* $T \in \mathcal{A}^*(X;\ell_1)$, $\varepsilon > 0$ *and* $R \in \mathcal{F}(\ell_1;X)$, *the local version of [6, Ex.17.4] implies that* $|\operatorname{tr}(TR)| \leq (1+\varepsilon)\|T\|_{\mathcal{A}^*}\|R\|_{\mathcal{A}^{min}}$. *Therefore, the isometric result follows.*

Our purpose now is to show that the duality formula for $c_{0,\mathcal{A}}(X)$ and its dual space $c_{0,\mathcal{A}}(X)'$ given in Theorem 3.5 is, in fact, a trace duality. First we give a useful description of $\overline{\mathcal{F}}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}}$ as a composition ideal.

Lemma 3.7. *Let* $\mathcal{A}$ *be a Banach operator ideal. Then*

$$\overline{\mathcal{F}}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}} = \mathcal{K}_{\mathcal{A}} \circ \overline{\mathcal{F}}$$

holds isometrically.

Proof. *It is clear that the norm one inclusion* $\mathcal{K}_{\mathcal{A}} \circ \overline{\mathcal{F}} \subset \overline{\mathcal{F}}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}}$ *holds. Now, take* $T \in \overline{\mathcal{F}}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}}(Y;X)$ *and fix* $\varepsilon > 0$. *Choose a sequence* $(T_n)_n \subset \mathcal{F}(Y;X)$ *such that* $T = \sum_{n=1}^{\infty} T_n$ *in* $\mathcal{K}_{\mathcal{A}}(Y;X)$ *and* $\sum_{n=1}^{\infty}\|T_n\|_{\mathcal{K}_{\mathcal{A}}} \leq (1+\varepsilon)\|T\|_{\mathcal{K}_{\mathcal{A}}}$. *Since* $\mathcal{K}_{\mathcal{A}}$ *is surjective and* $\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{A}} \circ \mathcal{K}$ *isometrically, by [16, Lemma 2.4 (b)], for each* n *we may find a Banach space* Z_n *and operators* $S_n \in \mathcal{F}(X;Z_n)$

and $R_n \in \mathcal{K}_{\mathcal{A}}(Z_n;Y)$ *such that* $T_n = R_n S_n$ *and* $\|R_n\|_{\mathcal{K}_{\mathcal{A}}} \|S_n\| \le (1+\varepsilon)\|T_n\|_{\mathcal{K}_{\mathcal{A}}}$. *Moreover, we may choose the operators so that* $\|S_n\| \le 1$ *with* $\|S_n\| \searrow 0$ *and* $\|R_n\|_{\mathcal{K}_{\mathcal{A}}} \le (1+\varepsilon)\|T_n\|_{\mathcal{K}_{\mathcal{A}}}$.

Take $Z = \{z = (z_n)_n : z_n \in Z_n$ *and* $\|z\|_Z := \sup_{n\in\mathbb{N}} \|z_n\| < \infty\}$ *and denote, for each n,* ι_n *and* π_n *the canonical inclusion and projection associated to* Z *and* Z_n*, respectively. Now, define the operators* $R : Z \to Y$ *and* $S : X \to Z$ *as*

$$R = \sum_{n=1}^{\infty} R_n \pi_n, \qquad S = \sum_{n=1}^{\infty} \iota_n S_n.$$

Then we have $R \in \mathcal{K}_{\mathcal{A}}(Z;Y)$, $S \in \overline{\mathcal{F}}(X;Z)$, $T = RS$ *and* $\|RS\|_{\mathcal{K}_{\mathcal{A}} \circ \overline{\mathcal{F}}} \le (1+\varepsilon)^2 \|T\|_{\mathcal{K}_{\mathcal{A}}}$. □

The next result should be related with [6, Proposition 17.19.1] and [6, Proposition 25.4.1]. Recall that for Banach operator ideals $\mathcal{A}$ and $\mathcal{B}$, the quotient ideal $\mathcal{B} \circ \mathcal{A}^{-1}$ consists of all the operators T such that $TS \in \mathcal{B}$ whenever $S \in \mathcal{A}$ (see [6, 25.6]).

Proposition 3.8. *Let* $\mathcal{A}$ *be Banach operator ideal. Then*

$$\mathcal{K}_{\mathcal{A}}^{\Delta} \circ \mathcal{K}_{\mathcal{A}} \subset \mathcal{I}.$$

As a consequence, $\mathcal{K}_{\mathcal{A}}^{\Delta} \circ \mathcal{K}_{\mathcal{A}} \circ \overline{\mathcal{F}} \subset \mathcal{N}$ *and* $\overline{\mathcal{F}} \circ \mathcal{K}_{\mathcal{A}}^{\Delta} \circ \mathcal{K}_{\mathcal{A}} \subset \mathcal{N}$.

Proof. *We use again that* $\mathcal{K}_{\mathcal{A}}$ *is surjective and that* $\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{\mathcal{A}} \circ \mathcal{K}$. *Combining [16, Lemma 2.4] and [16, Proposition 2.5] we have the identity*

$$\mathcal{K}_{\mathcal{A}}^{\Delta} = \mathcal{K}^{\Delta} \circ \mathcal{K}_{\mathcal{A}}^{-1}.$$

Since $\mathcal{K}^{\Delta} = \mathcal{I}$ *[14, p. 93], then* $\mathcal{K}_{\mathcal{A}}^{\Delta} = \mathcal{I} \circ \mathcal{K}_{\mathcal{A}}^{-1}$ *implying that* $\mathcal{K}_{\mathcal{A}}^{\Delta} \circ \mathcal{K}_{\mathcal{A}} \subset \mathcal{I}$. *The other inclusions follow from the identities* $\mathcal{I} \circ \overline{\mathcal{F}} = \overline{\mathcal{F}} \circ \mathcal{I} = \mathcal{N}$. □

Theorem 3.9. (Trace duality) *Let* $\mathcal{A}$ *be a Banach operator ideal and* X *be a Banach space. Let* $\phi \in c_{0,\mathcal{A}}(X)'$ *with associated map* $T \in \mathcal{K}_{\mathcal{A}}^{\Delta}(X;\ell_1)$ *and* $(x_n)_n \in c_{0,\mathcal{A}}(X)$ *with associated map* $S \in \overline{\mathcal{F}_{w^*}(\ell_1;X)}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}}$. *Then* TS *belongs to* $\mathcal{N}_{w^*}(\ell_1;\ell_1)$ *and*

$$\langle \phi, (x_n)_n \rangle = \mathrm{tr}(TS).$$

Proof. *First note that by Proposition 2.5,* $S = \sum_{n=1}^{\infty} e_n' \otimes x_n$. *By Lemma 3.7,* S *belongs to* $\mathcal{K}_{\mathcal{A}} \circ \overline{\mathcal{F}}_{w^*}(\ell_1;X)$. *On the other hand, since* $T \in \mathcal{K}_{\mathcal{A}}^{\Delta}(X;\ell_1)$*, by Proposition 3.8,* TS *is in* $\mathcal{N}_{w^*}(\ell_1;\ell_1)$. *Thus, the trace map is well defined and continuous. Now, a density argument completes the proof.* □

4 | ON THE DUAL OF $(\mathcal{L}(X;Y), \tau_{s\mathcal{A}})$ AND APPLICATIONS TO THE $\mathcal{K}_{\mathcal{A}}$-AP

For X, Y Banach spaces and τ a locally convex vector topology, the study of the dual space of $(\mathcal{L}(X;Y), \tau)$ has been useful to determine the presence of some approximation properties. It was Grothendieck [15] who first gave a representation of the continuous linear functionals on $(\mathcal{L}(X;Y), \tau_c)$ and used it to show the relation of the AP with the denseness of finite rank operators. As usual, τ_c denotes de topology of uniform convergence on compact sets. There is a recent inclination to study approximation properties related to (Banach) operator ideals, as it can be seen, among others, in [5], [7], [8], [18], [19], [20], [22], [23], [27], [28], [34]. Our interest lies in the study of the $\mathcal{A}$-*approximation property* ($\mathcal{A}$-AP for short) which is defined taking into account the geometry of $\mathcal{A}$. Recall that a Banach space X has the $\mathcal{A}$-AP if for every Banach space Y, $\overline{\mathcal{F}(Y;X)}^{\|\cdot\|_{\mathcal{A}}} = \mathcal{A}(Y;X)$ (see [21, Definition 4.3] and [28]). More precisely we deal with the $\mathcal{K}_{\mathcal{A}}$-AP.

As an immediate consequence of Lemma 3.7, we extend [22, Proposition 3.4] to arbitrary Banach operator ideals. There, a reformulation of the $\mathcal{K}_{\mathcal{A}}$-AP is given for $\mathcal{A}$ right-accessible, (see also the comments after [13, Proposition 3.9]).

Proposition 4.1. *Let* $\mathcal{A}$ *be a Banach operator ideal. A Banach space* X *has the* $\mathcal{K}_{\mathcal{A}}$*-AP if and only if for all Banach spaces* Y,

$$\mathcal{K}_{\mathcal{A}}(Y;X) = \mathcal{K}_{\mathcal{A}} \circ \overline{\mathcal{F}}(Y;X)$$

holds isometrically.

In order to give a trace condition for the $\mathcal{K}_{\mathcal{A}}$-AP, we begin with a characterization of the dual space of $(\mathcal{L}(X;Y), \tau_{s\mathcal{A}})$ where $\tau_{s\mathcal{A}}$ denotes the topology of uniform convergence on $\mathcal{A}$-compact sets under $m_{\mathcal{A}}$. Indeed, $\tau_{s\mathcal{A}}$ is the topology generated by the seminorms

$$q_K(T) = m_{\mathcal{A}}(T(K); Y)$$

where K runs trough all relatively $\mathcal{A}$-compact sets.

Theorem 4.2. *Let $\mathcal{A}$ be a Banach operator ideal, X, Y be Banach spaces and ψ be a linear functional on $\mathcal{L}(X;Y)$. The following are equivalent.*

(i) $\psi \in (\mathcal{L}(X;Y), \tau_{s\mathcal{A}})'$.
(ii) *There exist a sequence $(x_n)_n \in c_{0,\mathcal{A}}(X)$ and an operator R in $\mathcal{K}_{\mathcal{A}}^{\Delta}(Y;\ell_1)$ such that $\psi(T) = \sum_{n=1}^{\infty} \hat{e}_n(RTx_n)$.*
(iii) *There exist operators S in $\overline{\mathcal{F}_{w^*}(\ell_1;X)}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}}$, R in $\mathcal{K}_{\mathcal{A}}^{\Delta}(Y;\ell_1)$ and U in $\overline{\mathcal{F}}(\ell_1;\ell_1)$ such that $\psi(T) = \operatorname{tr}(URTS)$.*
(iv) *There exist a Banach space Z, an operator $S \in \mathcal{K}_{\mathcal{A}}(Z;X)$ and a functional $\xi \in (\mathcal{K}_{\mathcal{A}}(Z;Y))'$ such that $\psi(T) = \xi(TS)$.*

Proof. *The proof of (i) implies (ii) essentially follows its classical prototype [25, Proposition 1.e.3], we briefly sketch a proof. Take $\psi \in (\mathcal{L}(X;Y), \tau_{s\mathcal{A}})'$. There exists a sequence $(x_n)_n \in c_{0,\mathcal{A}}(X)$ such that $|\psi(T)| \le m_{\mathcal{A}}(\{Tx_n\}_n; Y)$. Consider $\Psi : (\mathcal{L}(X;Y), \tau_{s\mathcal{A}}) \to c_{0,\mathcal{A}}(Y)$ the map given by $\Psi(T) = (Tx_n)_n$. It is clear that Ψ is linear and continuous. Now, consider the map ϕ defined on $\Psi(\mathcal{L}(X;Y)) \subset c_{0,\mathcal{A}}(Y)$ as*

$$\phi((y_n)_n) = \psi(T) \quad \text{if} \quad (y_n)_n = (Tx_n)_n.$$

Routine arguments show that ϕ is well defined, linear and continuous. Then, by the Hahn–Banach Theorem, it has an extension to $c_{0,\mathcal{A}}(Y)$ which we still call ϕ. Applying Theorem 3.5, ϕ has an associated map $R \in \mathcal{K}_{\mathcal{A}}^{\Delta}(Y;\ell_1)$ such that

$$\psi(T) = \phi((Tx_n)_n) = \sum_{n=1}^{\infty} \hat{e}_n(RTx_n).$$

To see that (ii) implies (iii) suppose that $\psi(T) = \sum_{n=1}^{\infty} \hat{e}_n(RTx_n)$ with $(x_n)_n \in c_{0,\mathcal{A}}(X)$ and $R \in \mathcal{K}_{\mathcal{A}}^{\Delta}(Y;\ell_1)$, and take $(\beta_n)_n \in c_0$ such that $(x_n/\beta_n)_n$ belongs to $c_{0,\mathcal{A}}(X)$. With $\tilde{x}_n = x_n/\beta_n$, define $S : \ell_1 \to X$ by $S = \sum_{n=1}^{\infty} \hat{e}_n \otimes \tilde{x}_n$. Now, by Proposition 2.5, $S \in \overline{\mathcal{F}_{w^*}(\ell_1;X)}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}}$ and

$$\psi(T) = \sum_{n=1}^{\infty} \beta_n \hat{e}_n(RTSe_n').$$

Now, consider $U : \ell_1 \to \ell_1$ the diagonal operator satisfying that $Ue_n = \beta_n e_n$. Clearly, $U \in \overline{\mathcal{F}}(\ell_1;\ell_1)$ and, by Theorem 3.9, the composition $URST$ is in $\mathcal{N}(\ell_1;\ell_1)$. Then, we have

$$\psi(T) = \sum_{n=1}^{\infty} \hat{e}_n(URTSe_n') = \operatorname{tr}(URTS).$$

Now suppose that (iii) holds. Note that $UR \in \mathcal{K}_{\mathcal{A}}^{\Delta}(Y;\ell_1)$ then, by Theorem 3.9, we may define a continuous functional $\xi \in \left(\overline{\mathcal{F}_{w^*}(\ell_1;Y)}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}}\right)'$ as $\xi(V) = \operatorname{tr}(URV)$. Since $\overline{\mathcal{F}_{w^*}(\ell_1;Y)}^{\|\cdot\|_{\mathcal{K}_{\mathcal{A}}}} \subset \mathcal{K}_{\mathcal{A}}(\ell_1;Y)$, by the Hahn–Banach Theorem, there is an extension of ξ to $\mathcal{K}_{\mathcal{A}}(\ell_1;Y)$ which we still call ξ. Thus, $\psi(T) = \operatorname{tr}(URTS) = \xi(TS)$, proving (iv) for $Z = \ell_1$.

Finally, to see that (iv) implies (i), note that if $\psi(T) = \xi(TS)$ with $S \in \mathcal{K}_{\mathcal{A}}(Z;X)$ and $\xi \in (\mathcal{K}_{\mathcal{A}}(Z;Y))'$, then

$$|\psi(T)| = |\xi(TS)| \le \|\xi\| \, \|TS\|_{\mathcal{K}_{\mathcal{A}}} = \|\xi\| \, m_{\mathcal{A}}(TS(B_Z); X).$$

Since $S(B_Z)$ is a relatively $\mathcal{A}$-compact set in X, $q(T) = m_{\mathcal{A}}(TS(B_Z); X)$ is a $\tau_{s\mathcal{A}}$-continuous seminorm on $\mathcal{L}(X;Y)$ and ψ is $\tau_{s\mathcal{A}}$-continuous, concluding the proof. □

When $\mathcal{A}$ is right-accessible, by Remark 3.6, the identity $\left(\mathcal{A} \circ \overline{\mathcal{F}}\right)^{\Delta}(Y;\ell_1) = \mathcal{A}^*(Y;\ell_1)$ holds. If, in addition, $\mathcal{A}$ is maximal, then $\mathcal{A}^*$ is left-accessible [6, Corollary 21.3]. With this in mind, the equivalences (i)–(iii) of Theorem 4.2 read as follows.

Corollary 4.3. *Let $\mathcal{A}$ be a maximal, right-accessible Banach operator ideal, X, Y be Banach spaces and ψ be a linear functional on $\mathcal{L}(X;Y)$. The following are equivalent.*

(i) $\psi \in (\mathcal{L}(X;Y), \tau_{s\mathcal{A}})'$.

(ii) *There exist a sequence $(x_n)_n \in c_{0,\mathcal{A}}(X)$ and an operator R in $\mathcal{A}^*(Y;\ell_1)$ such that $\psi(T) = \sum_{n=1}^{\infty} \hat{e}_n(RTx_n)$.*

(iii) *There exist operators $S \in \mathcal{A}_{w^*}^{min}(\ell_1;X)$ and $R \in (\mathcal{A}^*)^{min}(Y;\ell_1)$ such that $\psi(T) = \mathrm{tr}(RTS)$.*

As a by-product of the above results we give a trace condition for the $\mathcal{K}_{\mathcal{A}}$-AP. Theorem 4.4 covers [8, Theorem 3.1].

Theorem 4.4. *Let $\mathcal{A}$ be a Banach operator ideal and X be a Banach space. Then X has the $\mathcal{K}_{\mathcal{A}}$-AP if and only if for every $\mathcal{A}$-null sequence $(x_n)_n$ in X and every operator R in $\mathcal{K}_{\mathcal{A}}^{\Delta}(X;\ell_1)$ such that $\sum_{n=1}^{\infty} \hat{e}_n(Rx)x_n = 0$ for all $x \in X$, then $\sum_{n=1}^{\infty} \hat{e}_n(Rx_n) = 0$.*

Proof. *By [22, Proposition 3.1], X has the $\mathcal{K}_{\mathcal{A}}$-AP if and only if $Id_X \in \overline{\mathcal{F}(X;X)}^{\tau_{s\mathcal{A}}}$. Or equivalently, if $\psi(Id_X) = 0$ for any $\psi \in (\mathcal{L}(X;X);\tau_{s\mathcal{A}})'$ such that $\psi(x' \otimes x) = 0$ for all $x' \in X'$ and $x \in X$. Thus, the result follows by the characterization of ψ given in Theorem 4.2 (ii).* □

The next lemma is an extension of a key tool often used in the course to prove results inferring approximation properties on a dual space to the space.

Lemma 4.5. *Let $\mathcal{A}, \mathcal{B}$ be Banach operator ideals and X be a Banach space. Assume that for every $\phi \in (\mathcal{L}(X;X), \tau_{s\mathcal{A}})'$ there is a functional $\psi \in (\mathcal{L}(X';X'), \tau_{s\mathcal{B}})'$ such that $\psi(T') = \phi(T)$ for all $T \in \mathcal{L}(X;X)$. If X' has the $\mathcal{K}_{\mathcal{B}}$-AP, then X has the $\mathcal{K}_{\mathcal{A}}$-AP.*

Proof. *Take $\phi \in (\mathcal{L}(X;X), \tau_{s\mathcal{A}})'$ such that ϕ vanishes on the finite rank operators and let us see that $\phi(Id_X) = 0$. Take $\psi \in (\mathcal{L}(X';X'), \tau_{s\mathcal{B}})'$ such that $\psi(T') = \phi(T)$ for all $T \in \mathcal{L}(X;X)$. Then, $\psi(S) = 0$ for every $S \in \mathcal{F}_{w^*}(X';X') = X \otimes X'$. Now, it is easy to check that, for any $\mathcal{B}$, $X \otimes X'$ is $\tau_{s\mathcal{B}}$-dense in $\mathcal{F}(X';X')$ (see e.g. [22, Lemma 3.6]). As X' has the $\mathcal{K}_{\mathcal{B}}$-AP, $\psi(Id_{X'}) = 0$ and*

$$\phi(Id_X) = \psi(Id'_X) = \psi(Id_{X'}) = 0.$$

Thus, $Id_X \in \overline{\mathcal{F}(X;X)}^{\tau_{s\mathcal{A}}}$ meaning that X has the $\mathcal{K}_{\mathcal{A}}$-AP [22, Proposition 3.1]. □

Now, we are in a position to show that the hypothesis of being Asplund for the Banach space X in [19, Theorem 1.3] can be removed. In this way, [19, Theorem 1.1] and Theorem 4.7 complete the case $p = 1$ of [18, Theorem 1.1]. Before proceeding recall that, as already mentioned, $\mathcal{K}_{up}$ and $\mathcal{K}_{\mathfrak{K}_{p'}}$ are isometrically isomorphic. For our purposes, let us spare some lines giving a description of $\mathcal{K}_{up}$ in terms of a maximal accessible ideal.

Remark 4.6. *The identity $\mathcal{K}_{up} = \mathcal{K}_{\mathfrak{K}_{p'}^{max}}$ holds isometrically. Indeed, on the one hand, [22, Proposition 2.1] states that any $\mathcal{A}$-compact operator ideal $\mathcal{K}_{\mathcal{A}}$ satisfies the isometric identity $\mathcal{K}_{\mathcal{A}} = \left(\mathcal{A} \circ \overline{\mathcal{F}}\right)^{sur}$. Using this and that $\mathfrak{K}_{p'}$ is minimal [6, 22.3], we get that $\mathcal{K}_{up} = \mathcal{K}_{\mathfrak{K}_{p'}} = (\mathfrak{K}_{p'} \circ \overline{\mathcal{F}})^{sur} = (\mathfrak{K}_{p'})^{sur}$. On the other hand, $\mathfrak{K}_{p'} = (\mathfrak{K}_{p'}^{max})^{min} = \mathfrak{K}_{p'}^{max} \circ \overline{\mathcal{F}}$, as $\mathfrak{K}_{p'}^{max}$ is accessible [6, Theorem 21.5]. Another application of [22, Proposition 2.1] gives that $\mathfrak{K}_{p'}^{sur} = \mathcal{K}_{\mathfrak{K}_{p'}^{max}}$ and the assertion follows.*

Theorem 4.7. *Let X be a Banach space. If X' has the $\mathcal{K}_1$-AP then X has the $\mathcal{K}_{u1}$-AP.*

Proof. *Let us show that conditions of Lemma 4.5 are fulfilled. Take, $\phi \in (\mathcal{L}(X;X), \tau_{s\mathcal{K}_{u1}})'$ and note that $\mathcal{K}_{u1} = \mathcal{K}_{\mathfrak{K}_{\infty}^{max}}$ with $\mathfrak{K}_{\infty}^{max}$ an accessible ideal [6, Theorem 21.5]. By Corollary 4.3 (i) ⇒ (iii), there exist operators $S \in \mathfrak{K}_{\infty w^*}(\ell_1;X)$ and $R \in \left(\mathfrak{K}_{\infty}^*\right)^{min}(X;\ell_1)$ such that $\phi(T) = \mathrm{tr}(RTS)$. Now, we define ψ on $\mathcal{L}(X';X')$ by $\psi(U) = \mathrm{tr}(S'UR')$. Let us show that ψ is well defined and $\tau_{s\mathcal{K}_1}$-continuous. It is clear that if this is the case, $\psi(T') = \phi(T)$ for all T.*

First, we claim that S' is in $\mathcal{K}_1^*(X';\ell_\infty)$ and R' is in $\mathcal{K}_1^{min}(\ell_\infty;X')$. Indeed, as S factors compactly through c_0, S' factors compactly through ℓ_1 and then S' is in $\mathfrak{K}_1(X';\ell_\infty)$ with $\mathfrak{K}_1$ associated to the tensor norm w_1 [6, 18.2]. Following the tensor norm identities listed in [6, 12.7] we have $w_1 = d_\infty = ((d_\infty)^t)^t = g_\infty^t$. Now, we resume several tensor norm equalities by appealing to [33, Theorem 7.20], which provides us with the identity $g'_\infty = /d_1$. Now, taking duals and transposes we get $w_1 = g_\infty^t = ((/d_1)')^t = (/d_1)^*$. As in addition, $\mathcal{K}_1$ and $/d_1$ are associated [13, Theorem 3.3], giving that $\mathcal{K}_1^*$ and w_1 are associated. Then, the minimal kernel of the operator ideals associated with w_1 coincide. That is $\mathfrak{K}_1 = (\mathcal{K}_1^*)^{min}$ and, in particular, S' belongs to $\mathcal{K}_1^*(X';\ell_\infty)$.

Regarding R', we have to show that R is in $\mathcal{K}_1^{min\,d}(X;\ell_1)$. As ℓ_1 is a dual space, $\mathcal{K}_1^{min\,d}(X;\ell_1) = \mathcal{K}_1^{d\,min}(X;\ell_1)$ [6, Corollary 22.8.1]. As $R \in (\mathfrak{K}_\infty^*)^{min}(X;\ell_1)$ it is enough to show that $(\mathfrak{K}_\infty^*)^{min} = \mathcal{K}_1^{d\,min}$. This is equivalent to prove that the associated tensor norms to $\mathfrak{K}_\infty^*$ and $\mathcal{K}_1^d$ coincide. By [6, 18.2], $\mathfrak{K}_\infty^*$ is associated to w_∞^*. Using again that $\mathcal{K}_1$ and $/d_1$ are associated we see *that $\mathcal{K}_1^d$ and $(/d_1)^t$ are associated. With [6, Proposition 20.14] and the tensor norm identities given above, $w_\infty^* = w_1' = (/d_1)^t$ and the claim is proved.*

Now, by [13, Remark 3.8], $\mathcal{K}_1^{min}(\ell_\infty; X') = \ell'_\infty \widehat{\otimes}_{/d_1} X'$ and since ℓ'_∞ has the AP, the identity $\ell'_\infty \widehat{\otimes}_{/d_1} X' = \mathcal{K}_1(\ell_\infty; X')$ holds [13, Theorem 3.11]. Then we have $\mathcal{K}_1^{min}(\ell_\infty; X') = \mathcal{K}_1(\ell_\infty; X')$. On the other hand, as $\mathcal{K}_1$ is right-accessible, [26, Theorem 3.1] asserts that $\mathcal{K}_1^* \circ \mathcal{K}_1 \subset \mathcal{I}$. Therefore, [6, Proposition 25.4.2] gives the norm one inclusion $\mathcal{K}_1^* \circ \mathcal{K}_1^{min} \subset \mathcal{I}^{min} = \mathcal{N}$. Then, for all $V \in \mathcal{K}_1(\ell_\infty; X')$, $S'V \in \mathcal{N}(\ell_\infty; \ell_\infty)$ and the AP of ℓ_∞ allows us to define the continuous functional $\xi \in (\mathcal{K}_1(\ell_\infty; X'))'$ by $\xi(V) = \mathrm{tr}(S'V)$. As UR' belongs to $\mathcal{K}_1(\ell_\infty; X')$ for every $U \in \mathcal{L}(X'; X')$, we have that $\psi(U) = \mathrm{tr}(S'UR') = \xi(UR')$ and ψ is well defined.

Finally, an application of Theorem 4.2 (iv) ⇒ (i), gives that ψ is $\tau_{s\mathcal{K}_1}$-continuous. Now Lemma 4.5 applies which completes the proof. □

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