On the Davenport-Mahler bound

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Abstract

We prove that the Davenport-Mahler bound holds for arbitrary graphs with vertices on the set of roots of a given univariate polynomial with complex coefficients.

Introduction

The Davenport-Mahler bound is a lower bound for the product of the lengths of the edges on a graph whose vertices are the complex roots of a given univariate polynomial $P \in \mathbb{C}[X]$, under certain assumptions. Its origins are the work of Mahler ([10]), where a lower bound for the minimum separation between two roots of P in terms of the discriminant of P is given, and the work of Davenport (see [2, Proposition 8]), where for the first time a lower bound for the joint product of many different distances between roots of P (which is not simply the product of a lower bound for each distance) is obtained. Roughly speaking, this bound makes evident an interaction between the involved distances, in the sense that if some of them are very small, the rest cannot be that small.

Throughout the literature, there are different versions of this bound. We include here the one from [5, Theorem 3.1] (see also [7, 12]). First, we remind the definitions of discriminant and Mahler measure (see also [1, 11]).

Definition 1 *Let $P \in \mathbb{C}[X]$, $P(X) = a_d \prod_{i=1}^d (X - v_i)$, the* discriminant *of P is*

$$\mathrm{Disc}(P) = a_d^{2d-2} \prod_{i<j} (v_i - v_j)^2.$$

Definition 2 *Let $P \in \mathbb{C}[X]$, $P(X) = a_d \prod_{i=1}^d (X - v_i)$, the* Mahler measure *of P is*

$$M(P) = |a_d| \prod_{i=1}^{d} \max\{1, |v_i|\}.$$

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Theorem 3 (Davenport-Mahler bound) *Let $P \in \mathbb{C}[X]$ be a polynomial of degree d. Let $G = (V, E)$ be a directed graph whose vertices $\{v_1, \ldots, v_k\}$ are a subset of the roots of P such that:*

1. *if $(v_i, v_j) \in E$, then $|v_i| \leq |v_j|$,*
2. *G is acyclic,*
3. *the in-degree of any vertex is at most 1.*

Then

$$\prod_{(v_i,v_j)\in E} |v_i - v_j| \;\geq\; |\mathrm{Disc}(P)|^{1/2}\, \mathrm{M}(P)^{-(d-1)} \left(\frac{d}{\sqrt{3}}\right)^{-\#E} d^{-d/2},$$

where $\mathrm{Disc}(P)$ *and* $\mathrm{M}(P)$ *are the discriminant and the Mahler measure of P.*

Note that when P is not a square-free polynomial, the bound becomes trivial since $\mathrm{Disc}(P)$ vanishes. One way to manage this situation is to consider the square-free part of P as in [6, Theorem 1]. Another way suggested by Eigenwillig ([4, Theorem 3.9]) is through the use of subdiscriminants, whose definition we recall below. This statement is nowadays known as the Generalized Davenport-Mahler bound.

Definition 4 *Let $P \in \mathbb{C}[X]$, $P(X) = a_d \prod_{i=1}^{d}(X - v_i)$, for $1 \leq r \leq d$, the $(d-r)$-*subdiscriminant *of P is*

$$\mathrm{sDisc}_{d-r}(P) = a_d^{2(r-1)} \sum_{\substack{I\subseteq\{1,\cdots,d\}\\ \#I=r}} \prod_{\substack{j,k\in I\\ j<k}} (v_j - v_k)^2.$$

Theorem 5 (Generalized Davenport-Mahler bound) *Let $P \in \mathbb{C}[X]$ be a polynomial of degree d with exactly r distinct complex roots. Let $G = (V, E)$ be a directed graph whose vertices $\{v_1, \ldots, v_k\}$ are a subset of the roots of P such that:*

1. *if $(v_i, v_j) \in E$, then $|v_i| \leq |v_j|$,*
2. *G is acyclic,*
3. *the in-degree of any vertex is at most 1.*

Then

$$\prod_{(v_i,v_j)\in E} |v_i - v_j| \;\geq\; |\mathrm{sDisc}_{d-r}(P)|^{1/2}\, \mathrm{M}(P)^{-(r-1)} \left(\frac{r}{\sqrt{3}}\right)^{-\#E} r^{-r/2} \left(\frac{1}{3}\right)^{\min\{d,2d-2r\}/6}.$$

It is clear that if P is a square-free polynomial, then $r = d$ and the bound by Eigenwillig is exactly the classical Davenport-Mahler bound. In the general case, as seen in [1, Remark 4.6], $\mathrm{sDisc}_{d-r}(P)$ is the first subdiscriminant of P which is different from zero.

One of the main applications of the Davenport-Mahler bound in both its classical and generalized version is its use in algorithmic complexity estimation as for instance in [3, 5, 8]. It has also been used in [6] to obtain separation bounds for roots of multivariate polynomial systems.

The main result in this paper is that the Generalized Davenport-Mahler bound holds for arbitrary graphs (undirected, no loops, no multiple edges) with vertices on the set of roots of P. More precisely:

Theorem 6 *Let $P \in \mathbb{C}[X]$ be a polynomial of degree d with exactly r distinct complex roots. Let $G = (V, E)$ be a graph whose vertices $\{v_1, \dots, v_k\}$ are a subset of the roots of P. Then*

$$\prod_{(v_i, v_j) \in E} |v_i - v_j| \ \geq \ |\mathrm{sDisc}_{d-r}(P)|^{1/2}\, \mathrm{M}(P)^{-(r-1)} \left(\frac{r}{\sqrt{3}}\right)^{-\#E} r^{-r/2} \left(\frac{1}{3}\right)^{\min\{d, 2d-2r\}/6}.$$

In order to prove Theorem 6, we revisit the classical proofs and the new ingredient is the use of divided differences to manage the cases where the assumptions in previous formulations do not hold; specially the one about the in-degree of any vertex being at most 1 (assumption *3*), which is the one that cannot be satisfied by simply redirecting edges.

Finally, after proving Theorem 6, we include some remarks and applications.

1 Proof of the results

First, we recall the definition of divided differences.

Definition 7 *For $f : \mathbb{C} \to \mathbb{C}$ and $v_1, \dots, v_n \in \mathbb{C}$ with $v_i \neq v_j$ if $1 \leq i < j \leq n$, the divided difference $f[v_1, \dots, v_n] \in \mathbb{C}$ is defined inductively in n by*

$$f[v_1] = f(v_1)$$

if $n = 1$ and

$$f[v_1, \dots, v_n] = \frac{f[v_1, \dots, v_{n-1}] - f[v_2, \dots, v_n]}{v_1 - v_n}$$

if $n > 1$.

For $F : \mathbb{C} \to \mathbb{C}^m$ given by $F(z) = (f_1(z), \dots, f_m(z))$ and $v_1, \dots, v_n \in \mathbb{C}$ with $v_i \neq v_j$ if $1 \leq i < j \leq n$, the divided difference $F[v_1, \dots, v_n]$ is defined as

$$F[v_1, \dots, v_n] = (f_1[v_1, \dots, v_n], \dots, f_m[v_1, \dots, v_n]) \in \mathbb{C}^m.$$

The only properties we will use concerning divided differences are stated in the next two lemmas. We refer the reader to [9, Chapter 6] for further properties of divided differences and their use in polynomial interpolation.

Lemma 8 *For $F : \mathbb{C} \to \mathbb{C}^m$ and $v_1, \dots, v_n \in \mathbb{C}$ with $v_i \neq v_j$ if $1 \leq i < j \leq n$, $F[v_1, \dots, v_n]$ is the linear combination of $F(v_1), \dots, F(v_n)$ given by*

$$F[v_1, \dots, v_n] = \sum_{h=1}^{n} \Big(\prod_{\substack{k=1 \\ k \neq h}}^{n} \frac{1}{v_h - v_k} \Big) F(v_h).$$

Proof: We proceed by induction on n. For $n = 1$ the identity is obvious. For the inductive step we proceed as follows:

$$
\begin{aligned}
F[v_1,\ldots,v_{n+1}] &= \\
\frac{F[v_1,\ldots,v_n]-F[v_2,\ldots,v_{n+1}]}{v_1-v_{n+1}} &= \\
\Big(\prod_{k=2}^{n+1}\frac{1}{v_1-v_k}\Big)F(v_1)+\frac{1}{v_1-v_{n+1}}\sum_{h=2}^{n}\Big(\prod_{\substack{k=1\\k\neq h}}^{n}\frac{1}{v_h-v_k}-\prod_{\substack{k=2\\k\neq h}}^{n+1}\frac{1}{v_h-v_k}\Big)F(v_h)+\Big(\prod_{k=1}^{n}\frac{1}{v_{n+1}-v_k}\Big)F(v_{n+1}) &= \\
\sum_{h=1}^{n+1}\Big(\prod_{\substack{k=1\\k\neq h}}^{n}\frac{1}{v_h-v_k}\Big)F(v_h).
\end{aligned}
$$

□

Lemma 9 *For* $p\in\mathbb{N}_0$, $f:\mathbb{C}\to\mathbb{C}$ *given by* $f(z)=z^p$, *and* $v_1,\ldots,v_n\in\mathbb{C}$ *with* $v_i\neq v_j$ *if* $1\leq i<j\leq n$,

$$
f[v_1,\ldots,v_n]=\begin{cases}\displaystyle\sum_{\substack{(t_1,\ldots,t_n)\in\mathbb{N}_0^n\\ t_1+\cdots+t_n=p-n+1}}\prod_{j=1}^{n}v_j^{t_j} & \text{if } n\leq p+1,\\ 0 & \text{if } n\geq p+2.\end{cases}
$$

Proof: We fix p and we proceed by induction on n. For $n=1$ the identity is obvious. For the inductive step, we consider three cases. First, if $n+1\leq p+1$, then $n\leq p+1$ and

$$
\begin{aligned}
f[v_1,\ldots,v_{n+1}] &= \\
\frac{f[v_1,\ldots,v_n]-f[v_2,\ldots,v_{n+1}]}{v_1-v_{n+1}} &= \\
\frac{1}{v_1-v_{n+1}}\Big(\sum_{\substack{(t_1,\ldots,t_n)\in\mathbb{N}_0^n\\ t_1+\cdots+t_n=p-n+1}}\prod_{j=1}^{n}v_j^{t_j}-\sum_{\substack{(t_2,\ldots,t_{n+1})\in\mathbb{N}_0^n\\ t_2+\cdots+t_{n+1}=p-n+1}}\prod_{j=2}^{n+1}v_j^{t_j}\Big) &= \\
\frac{1}{v_1-v_{n+1}}\Big(\sum_{1\leq t\leq p-n+1}(v_1^t-v_{n+1}^t)\sum_{\substack{(t_2,\ldots,t_n)\in\mathbb{N}_0^n\\ t_2+\cdots+t_n=p-n+1-t}}\prod_{j=2}^{n}v_j^{t_j}\Big) &= \\
\sum_{1\leq t\leq p-n+1}\Big(\sum_{\substack{(t_1,t_{n+1})\in\mathbb{N}_0^2\\ t_1+t_{n+1}=t-1}}v_1^{t_1}v_{n+1}^{t_{n+1}}\Big)\Big(\sum_{\substack{(t_2,\ldots,t_n)\in\mathbb{N}_0^n\\ t_2+\cdots+t_n=p-n+1-t}}\prod_{j=2}^{n}v_j^{t_j}\Big) &= \\
\sum_{\substack{(t_1,\ldots,t_{n+1})\in\mathbb{N}_0^n\\ t_1+\cdots+t_{n+1}=p-n}}\prod_{j=1}^{n+1}v_j^{t_j}.
\end{aligned}
$$

If $n+1 = p+2$, then $n = p+1$ and

$$\begin{aligned} f[v_1,\dots,v_{n+1}] &= \\ \frac{f[v_1,\dots,v_n] - f[v_2,\dots,v_{n+1}]}{v_1 - v_{n+1}} &= \\ \frac{1}{v_1 - v_{n+1}}(1-1) &= \\ 0. \end{aligned}$$

Finally, if $n+1 \geq p+3$, then $n \geq p+2$ and

$$\begin{aligned} f[v_1,\dots,v_{n+1}] &= \\ \frac{f[v_1,\dots,v_n] - f[v_2,\dots,v_{n+1}]}{v_1 - v_{n+1}} &= \\ \frac{1}{v_1 - v_{n+1}}(0-0) &= \\ 0. \end{aligned}$$

□

We will also use the following lemma.

Lemma 10 *For $d, r \in \mathbb{N}_0$ with $d \leq r-1$,*

$$\left(\sum_{i=d}^{r-1} \binom{i}{d}^2\right)^{1/2} \leq \binom{r-1}{d}\left(\frac{r+d}{2d+1}\right)^{1/2} \leq \left(\frac{r}{\sqrt{3}}\right)^d r^{1/2}.$$

Proof: For the first inequality, we fix $d \in \mathbb{N}_0$ and proceed by induction on $r \geq d+1$. For $r = d+1$ it is clear that the equality holds. For the inductive step:

$$\begin{aligned} \sum_{i=d}^{r} \binom{i}{d}^2 &\leq \\ \binom{r-1}{d}^2 \frac{r+d}{2d+1} + \binom{r}{d}^2 &= \\ \binom{r}{d}^2 \left(\frac{(r-d)^2}{r^2}\frac{r+d}{2d+1} + 1\right) &= \\ \binom{r}{d}^2 \left(\frac{r^2-d^2}{r^2}\frac{r-d}{2d+1} + 1\right) &\leq \\ \binom{r}{d}^2 \left(\frac{r-d}{2d+1} + 1\right) &= \\ \binom{r}{d}^2 \frac{r+d+1}{2d+1}. \end{aligned}$$

For the second inequality, it can be easily seen first that the inequality holds for $d = 0, 1, 2$. For $d \geq 3$, since $d! \geq \sqrt{3}^d$, we have that:

$$\binom{r-1}{d} = \frac{(r-1)\ (r-2)\cdots(r-d)}{d!} \leq \left(\frac{r}{\sqrt{3}}\right)^d$$

and since

$$\frac{r+d}{2d+1} \leq r,$$

the inequality holds. $\square$

Finally, before proving our main result, we recall [4, Lemma 3.8].

Lemma 11 *If $m_1, \dots, m_r \in \mathbb{N}$ and $\sum_{i=1}^r m_i = d$, then*

$$\prod_{i=1}^r m_i \leq 3^{\min\{d, 2d-2r\}/3}.$$

We can now give the proof of our main result.

Proof of Theorem 6: Let $P(X) = a_d \prod_{j=1}^r (X - v_j)^{m_j} \in \mathbb{C}[X]$ with $v_i \neq v_j$ if $1 \leq i < j \leq r$, $m_i \in \mathbb{N}$ for $1 \leq i \leq r$. It is easy to see that the result holds if $r = 1$, so from now we suppose $r \geq 2$. Without loss of generality, we suppose also that $V = \{v_1, \dots, v_r\}$ and that the roots of P are numbered in such a way that

$$|v_1| \leq \cdots \leq |v_r|.$$

We give a direction to each edge in E: if e is an edge joining v_i and v_j with $i < j$, we consider $e = (v_i, v_j)$ as the oriented edge going from v_i to v_j. Note that now $G = (V, E)$ satisfies conditions *1* and *2* in Theorems 3 and 5. We consider the edges in E listed by

$$e_1 = (v_{\alpha(1)}, v_{\beta(1)}), \dots, e_{\#E} = (v_{\alpha(\#E)}, v_{\beta(\#E)}).$$

Finally, for $1 \leq j \leq r$, let $d_j \in \mathbb{N}_0$ be the in-degree of the vertex v_j. Note that $d_1 = 0$ since there is no edge finishing in v_1, and $d_j \leq r - 1$ for $1 \leq j \leq r$.
As seen in [4, Proposition 3.7],

$$|\mathrm{sDisc}_{d-r}(P)|^{1/2} = |a_d|^{r-1} \left(\prod_{j=1}^r m_j\right)^{1/2} \prod_{1 \leq i < j \leq r} |v_i - v_j|. \quad (1)$$

On the other hand,

$$\prod_{1 \leq i < j \leq r} |v_i - v_j| = |\det W| \quad (2)$$

where W is the Vandermonde matrix

$$W = \begin{pmatrix} 1 & v_1 & \dots & v_1^{r-1} \\ 1 & v_2 & \dots & v_2^{r-1} \\ \vdots & \vdots & & \vdots \\ 1 & v_r & \dots & v_r^{r-1} \end{pmatrix} \in \mathbb{C}^{r \times r}.$$

We consider $F : \mathbb{C} \to \mathbb{C}^r$, $F(z) = (1, z, \ldots, z^{r-1})$ and define a sequence of matrices $W_r, W_{r-1}, \ldots, W_1$ in $\mathbb{C}^{r\times r}$. First, we define $W_r = W$. Then, for fixed $j = r, \ldots, 2$, once W_j is defined, we only modify its j-th row (if any) in order to define W_{j-1}, as follows: we take the (possibly empty) sublist of edges $e_{k_1}, \ldots, e_{k_{d_j}}$ finishing in v_j and take as the j-th row of W_{j-1} the divided difference

$$F[v_{\alpha(k_1)}, \ldots, v_{\alpha(k_{d_j})}, v_j] =$$

$$\sum_{i=1}^{d_j} \Big(\prod_{\substack{\ell=1\\ \ell\neq i}}^{d_j} \frac{1}{v_{\alpha(k_i)} - v_{\alpha(k_\ell)}} \Big) \frac{1}{v_{\alpha(k_i)} - v_j} F(v_{\alpha(k_i)}) \;+\; \prod_{\ell=1}^{d_j} \frac{1}{v_j - v_{\alpha(k_\ell)}} F(v_j)$$

by Lemma 8. Note that the j-th row of W_j equals the j-th row of W, which is $F(v_j)$; and since for $1 \le i \le d_j$, $\alpha(k_i) < \beta(k_i) = j$, the $\alpha(k_i)$-th row of W_j equals the $\alpha(k_i)$-th row of W, which is $F(v_{\alpha(k_i)})$. Then, we have that

$$\det W_{j-1} = \prod_{\ell=1}^{d_j} \frac{1}{v_j - v_{\alpha(k_\ell)}} \det W_j$$

or, equivalently,

$$\det W_j = \det W_{j-1} \prod_{\ell=1}^{d_j} (v_j - v_{\alpha(k_\ell)}).$$

In this way, we can prove by reverse induction in j that for $j = r, \ldots, 2$,

$$\det W = \det W_{j-1} \prod_{\substack{e\in E\\ \beta(e)\ge j}} (v_{\beta(e)} - v_{\alpha(e)}),$$

and at the end we obtain

$$\det W = \det W_1 \prod_{e\in E} (v_{\beta(e)} - v_{\alpha(e)}). \tag{3}$$

The next step is to bound $|\det W_1|$ using Hadamard inequality. For $1 \le j \le r$, keeping the notation of the above paragraphs, the j-th row of W_1 is $F[v_{\alpha(k_1)}, \ldots, v_{\alpha(k_{d_j})}, v_j]$ and by Lemma 9 its norm equals

$$\left(\sum_{i=d_j}^{r-1} \Big| \sum_{\substack{(t_1,\ldots,t_{d_j},t_{d_j+1})\in\mathbb{N}_0^{d_j+1}\\ t_1+\cdots+t_{d_j}+t_{d_j+1}=i-d_j}} \left(\prod_{\ell=1}^{d_j} v_{\alpha(k_\ell)}^{t_\ell} \right) v_j^{t_{d_j+1}} \Big|^2 \right)^{1/2}.$$

Note that for each $d_j \le i \le r-1$ there are $\binom{i}{d_j}$ terms. Since for $1 \le \ell \le d_j$ we have that $|v_{\alpha(k_\ell)}| \le |v_j|$, we have

$$\left(\sum_{i=d_j}^{r-1}\Big|\sum_{\substack{(t_1,\dots,t_{d_j},t_{d_j+1})\in\mathbb{N}_0^{d_j+1}\\ t_1+\cdots+t_{d_j}+t_{d_j+1}=i-d_j}}\left(\prod_{\ell=1}^{d_j}v_{\alpha(k_\ell)}^{t_\ell}\right)v_j^{t_{d_j+1}}\Big|^2\right)^{1/2} \leq$$

$$\left(\sum_{i=d_j}^{r-1}\binom{i}{d_j}^2|v_j|^{2(i-d_j)}\right)^{1/2} \leq$$

$$\left(\sum_{i=d_j}^{r-1}\binom{i}{d_j}^2\right)^{1/2}\max\{1,|v_j|\}^{r-1-d_j} \leq$$

$$\left(\frac{r}{\sqrt{3}}\right)^{d_j}r^{1/2}\max\{1,|v_j|\}^{r-1-d_j}$$

by Lemma 10. By Hadamard inequality,

$$|\det W_1| \leq$$

$$\prod_{j=1}^{r}\left(\frac{r}{\sqrt{3}}\right)^{d_j}r^{1/2}\max\{1,|v_j|\}^{r-1-d_j} = \tag{4}$$

$$\left(\frac{r}{\sqrt{3}}\right)^{\#E}r^{r/2}\prod_{j=1}^{r}\max\{1,|v_j|\}^{r-1-d_j}.$$

Finally, using equations (1), (2), (3), (4) and Lemma 11,

$$\prod_{(v_i,v_j)\in E}|v_i-v_j| =$$

$$\prod_{e\in E}|v_{\beta(e)}-v_{\alpha(e)}| =$$

$$|\det W||\det(W_1)|^{-1} \geq$$

$$|\mathrm{sDisc}_{d-r}(P)|^{1/2}\,|a_d|^{-(r-1)}\left(\prod_{j=1}^{r}\max\{1,|v_j|\}^{-(r-1-d_j)}\right)\left(\frac{r}{\sqrt{3}}\right)^{-\#E}r^{-r/2}\left(\prod_{j=1}^{r}m_j\right)^{-1/2} \geq$$

$$|\mathrm{sDisc}_{d-r}(P)|^{1/2}\,\mathrm{M}(P)^{-(r-1)}\left(\frac{r}{\sqrt{3}}\right)^{-\#E}r^{-r/2}\left(\frac{1}{3}\right)^{\min\{d,2d-2r\}/6}$$

as we wanted to prove. $\square$

We include below some remarks considering cases in which the bound in Theorem 6 can be slightly improved.

Remark 12 *Following the notation in Theorem 6, for $1 \le j \le r$ let $\tilde{d}_j$ be the total degree of vertex v_j and let $\tilde{d} = \min\{\tilde{d}_j \,|\, 1 \le j \le r\}$. If P is a monic polynomial then*

$$\prod_{(v_i,v_j)\in E} |v_i - v_j| \ge |\mathrm{sDisc}_{d-r}(P)|^{1/2}\, \mathrm{M}(P)^{-(r-1-\frac{1}{2}\tilde{d})} \left(\frac{r}{\sqrt{3}}\right)^{-\#E} r^{-r/2} \left(\frac{1}{3}\right)^{\min\{d,2d-2r\}/6}.$$

Indeed, taking into account that $|v_{\alpha(e)}| \le |v_{\beta(e)}|$ for every $e \in E$, we change the last part of the proof of Theorem 6 as follows:

$$\begin{aligned}
\prod_{(v_i,v_j)\in E} |v_i - v_j| &= \\
\prod_{e\in E} |v_{\beta(e)} - v_{\alpha(e)}| &= \\
|\det W|\,|\det(W_1)|^{-1} &\ge \\
|\mathrm{sDisc}_{d-r}(P)|^{1/2} \left(\prod_{j=1}^{r} \max\{1,|v_j|\}^{-(r-1-d_j)}\right) \left(\prod_{e\in E} \frac{\max\{1,|v_{\alpha(e)}|\}^{1/2}}{\max\{1,|v_{\beta(e)}|\}^{1/2}}\right) \left(\frac{r}{\sqrt{3}}\right)^{-\#E} r^{-r/2} \left(\prod_{j=1}^{r} m_j\right)^{-1/2} &\ge \\
|\mathrm{sDisc}_{d-r}(P)|^{1/2} \left(\prod_{j=1}^{r} \max\{1,|v_j|\}^{-(r-1-\frac{1}{2}\tilde{d}_j)}\right) \left(\frac{r}{\sqrt{3}}\right)^{-\#E} r^{-r/2} \left(\frac{1}{3}\right)^{\min\{d,2d-2r\}/6} &\ge \\
|\mathrm{sDisc}_{d-r}(P)|^{1/2}\, \mathrm{M}(P)^{-(r-1-\frac{1}{2}\tilde{d})} \left(\frac{r}{\sqrt{3}}\right)^{-\#E} r^{-r/2} \left(\frac{1}{3}\right)^{\min\{d,2d-2r\}/6}&.
\end{aligned}$$

The next remark considers the case where a number of small distances is guaranteed by some extra information. More explicitly, suppose that a particular polynomial P is given, and after some computations several pairs of "very close" roots $(v_{\alpha(1)}, v_{\beta(1)}), \ldots, (v_{\alpha(\ell)}, v_{\beta(\ell)})$ are obtained, where $\ell \ge \#E$ and for $1 \le i \le \ell$,

$$|v_{\alpha(i)} - v_{\beta(i)}| \le \left(\frac{\sqrt{3}}{r}\right)^{1+\Delta_i}$$

with $\Delta_1 \ge \cdots \ge \Delta_\ell \ge 0$ (note that these pairs could possible have common vertices). The idea is to use this information to improve the general bound from Theorem 6 by a factor of

$$\left(\frac{r}{\sqrt{3}}\right)^{\Delta_{\#E+1}+\cdots+\Delta_\ell}.$$

In this way, the closer the roots that have been discovered are, the more the bound is improved.

It could be particularly useful to bound the minimal distance between different roots when at least two pairs of "very close" roots have been discovered, taking E as the set with only one edge joining a pair of closest roots.

Remark 13 *Following the notation in Theorem 6, suppose that $r > 2$ and that there exist at least ℓ distinct pairs of roots $(v_{\alpha(1)}, v_{\beta(1)}), \ldots, (v_{\alpha(\ell)}, v_{\beta(\ell)})$ whose distance is less than $\frac{\sqrt{3}}{r}$ (not necessarily these pairs of roots should be connected by edges in E). For $1 \le i \le \ell$, let Δ_i such that*

$$|v_{\alpha(i)} - v_{\beta(i)}| \le \left(\frac{\sqrt{3}}{r}\right)^{1+\Delta_i}$$

and renumber these pairs such that

$$\Delta_1 \geq \cdots \geq \Delta_\ell \geq 0.$$

Then, if $\#E < \ell$*,*

$$\prod_{(v_i,v_j)\in E} |v_i - v_j| \ \geq \ |\mathrm{sDisc}_{d-r}(P)|^{1/2}\mathrm{M}(P)^{-(r-1)}\Big(\frac{r}{\sqrt{3}}\Big)^{-\#E+\Delta_{\#E+1}+\cdots+\Delta_\ell} r^{-r/2}\Big(\frac{1}{3}\Big)^{\min\{d,2d-2r\}/6}.$$

Indeed, suppose that

$$0 < \omega_1 \leq \cdots \leq \omega_{\binom{r}{2}}$$

are the ordered distances between pairs of roots of P*. By the assumptions, for* $1 \leq i \leq \ell$ *there are at least* i *distances less than or equal to* $\left(\frac{\sqrt{3}}{r}\right)^{1+\Delta_i}$ *and then we have that* $\omega_i \leq \left(\frac{\sqrt{3}}{r}\right)^{1+\Delta_i}$*. Consider* $\tilde{E}$ *the set of* ℓ *edges whose lengths are* $\omega_1, \ldots, \omega_\ell$*, this is to say, the set formed by the* ℓ *edges with smallest lengths. Then, applying the bound in Theorem 6 to* $\tilde{G} = (\{v_1, \ldots, v_r\}, \tilde{E})$ *we obtain the following bound for the product of the lengths of the edges in* E*:*

$$\begin{aligned}
\prod_{(v_i,v_j)\in E} |v_i - v_j| \qquad & \geq \\
\prod_{i=1}^{\#E} \omega_i \qquad & = \\
\Big(\prod_{(v_i,v_j)\in \tilde{E}} |v_i - v_j|\Big)\Big(\prod_{i=\#E+1}^{\ell} \omega_i^{-1}\Big) \qquad & \geq \\
|\mathrm{sDisc}_{d-r}(P)|^{1/2}\,\mathrm{M}(P)^{-(r-1)}\Big(\frac{r}{\sqrt{3}}\Big)^{-\ell} r^{-r/2}\Big(\frac{1}{3}\Big)^{\min\{d,2d-2r\}/6}\Big(\prod_{i=\#E+1}^{\ell}\Big(\frac{r}{\sqrt{3}}\Big)^{1+\Delta_i}\Big) \qquad & = \\
|\mathrm{sDisc}_{d-r}(P)|^{1/2}\mathrm{M}(P)^{-(r-1)}\Big(\frac{r}{\sqrt{3}}\Big)^{-\#E+\Delta_{\#E+1}+\cdots+\Delta_\ell} r^{-r/2}\Big(\frac{1}{3}\Big)^{\min\{d,2d-2r\}/6}.
\end{aligned}$$

Finally, as an application of Theorem 6, we give a simplified proof of [8, Theorem 9] with smaller constants.

Theorem 14 *Let* $P \in \mathbb{C}[X]$ *be a polynomial of degree* d *with exactly* $r \geq 2$ *distinct complex roots and let* $V = \{v_1, \ldots, v_r\} \subset \mathbb{C}$ *be the set of roots. For any root* v *of* P*, we denote by* $\mathrm{sep}(P, v)$ *the distance from* v *to (one of) its closest different root of* P*. Then, for any* $V' \subset V$*,*

$$\prod_{v\in V'} \mathrm{sep}(P, v) \geq |\mathrm{sDisc}_{d-r}(P)|\,\mathrm{M}(P)^{-2(r-1)}\Big(\frac{r}{\sqrt{3}}\Big)^{-\#V'} r^{-r}\,\Big(\frac{1}{3}\Big)^{\min\{d,2d-2r\}/3}.$$

Proof: For each $v \in V$, we take $\tilde{v}$ as (one of) its closest different root of P. We consider the multigraph $G = (V, E)$ where E is the multiset of edges of type $(v, \tilde{v})$ with $v \in V'$. Note that each edge in E can occur at most 2 times (one for each of its vertex). We divide E in two sets E_0 and E_1, with E_0 having all the elements in E and E_1 having the elements that occur twice in E. Applying Theorem 6 to (V, E_0) and

(V, E_1) and taking into account that $\#E_0 + \#E_1 = \#V'$, we obtain

$$\prod_{v \in V'} \mathrm{sep}(P, v) =$$

$$\Big(\prod_{(v_i,v_j) \in E_0} |v_i - v_j| \Big) \Big(\prod_{(v_i,v_j) \in E_1} |v_i - v_j| \Big) \geq$$

$$|\mathrm{sDisc}_{d-r}(P)| \mathrm{M}(P)^{-2(r-1)} \left(\frac{r}{\sqrt{3}} \right)^{-\#V'} r^{-r} \left(\frac{1}{3} \right)^{\min\{d, 2d-2r\}/3}$$

as we wanted to prove. □

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