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A nodal inverse problem for a quasi-linear ordinary differential equation in the half-line

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Abstract

In this paper we study an inverse problem for a quasi-linear ordinary differential equation with a monotonic weight in the half-line. First, we find the asymptotic behavior of the singular eigenvalues, and we obtain a Weyl-type asymptotics imposing an appropriate integrability condition on the weight. Then, we investigate the inverse problem of recovering the coefficients from nodal data. We show that any dense subset of nodes of the eigenfunctions is enough to recover the weight.

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Inverse problems, eigenvalues, nodal points, singular problem, p-Laplacian
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1. Introduction

In this work we deal with the inverse nodal problem for the following weighted singular quasi-linear ordinary differential equation

$$-(|u'|^{p-2}u')' = \lambda\sigma(t)|u|^{p-2}u \qquad t \geq 0, \tag{1}$$

with boundary conditions

$$u(0) = 0 \text{ and } \lim_{t\to+\infty} \frac{u(t)}{\sqrt{t}} = 0, \tag{2}$$

where $1 < p < \infty$, λ is a real parameter, and the weight σ satisfies:

(H1) σ is a continuous, positive and monotonic function.

(H2) $t^p\sigma \in L^1([0,\infty))$.

(H3) $\int_0^\infty \sigma(t)^{1/p} dt = 1$.

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The eigenvalue problem (1)-(2) was studied first in the linear case $p = 2$ by Einar Hille in [21]. Later, the problem appeared related to the number of negative eigenvalues of Schrodinger equations, and was studied by Bargmann, Calogero, and Cohn among several other physicists and mathematicians, see [31] for a survey. Also, Birman and Solomyak were interested in this problem, see [2], which appears in the theory of infinite waveguides.

For general p, the existence of a sequence of eigenvalues $\{\lambda_n\}_{n\geq 1}$ was proved by Kusano and Naito in [24]. As in the linear case, each eigenvalue is simple, and any eigenfunction u_n associated to λ_n has exactly n zeros in $[0, \infty)$. We call X the nodal set of problem (1)-(2), which can be indexed as a double sequence

$$X = \Big\{ \{x_j^n\}_{n\geq 1, 1\leq j\leq n} : 0 = x_1^n < \cdots < x_j^n < \cdots < x_n^n,\ u_n(x_j^n) = 0. \Big\}.$$

The monotonicity of σ is not necessary to obtain these results, and other weaker conditions can be imposed instead of $t^p\sigma \in L^1$. However, this is the key assumption in order to prove a Weyl-type asymptotic formula for the eigenvalues.

For $p = 2$, the following asymptotic formula holds:

$$\lambda_n = \left(\frac{\pi n}{\int_0^\infty \sqrt{\sigma} dt} \right)^2 + o(n^2). \tag{3}$$

Three different proofs were given of this behavior. The first one is due to Hille, who used in [21] a shooting argument. A different proof was given in [3] and [29], and a third proof can be found in Chapter 4 of [31]. Moreover, assuming only $t\sigma \in L^1([0, \infty))$, the eigenvalues satisfy a non Weyl-type asymptotic, namely $cn^\alpha < \lambda_n < Cn^\alpha$, for some positive constants c, C, and $1 \leq \alpha < 2$. In all of these proofs, the monotonicity of σ was needed, as well in the fourth proof that we provide here.

In this paper we are interested in the characterization of σ in terms of the set of nodes X. This kind of inverse problem in finite intervals was introduced first by McLaughlin in [28], and followed quickly by Hald and McLaughlin in [18] and -with different methods- by Shen in [33]. Today, a large body of literature was developed on these problems, see for example [16, 19, 20, 27, 34].

Let us observe that the eigenfunctions of problem (1) with the weight $\hat{\sigma} = c\sigma$ does not change, and the set X remains the same (the constant c will appear in the eigenvalue $\hat{\lambda}_n = \lambda_n/c$). Therefore, some kind of normalization must be imposed on σ, being a local one like $\sigma(t_0) = 1$, or a global one like $\|\sigma^a\|_1 = 1$ for some $a > 0$. We assume $(H3)$ since this integral will appear in the eigenvalue estimates, and sometimes we can simplify the notation.

Let us note that we are dealing here with an inverse problem for the *weight* in equation (1). A different problem is to determine a *potential* q in the following equation:

$$-(|u'|^{p-2}u')' + q(t)|u|^{p-2}u = \lambda|u|^{p-2}u \qquad t \geq 0.$$

This problem has received a lot of attention, in both the linear (see for instance [4, 22, 25, 28, 30, 36, 38]) and quasilinear cases (see [8, 23, 26, 35]), always for bounded intervals.

We wish to observe that the determination of the nodes of eigenfunctions is technically possible in several cases of interest. Historically, it goes back to experiments performed in 1680 by Robert Hooke, who obtained the nodal lines of vibrating plates by covering them with sand and observing where the sand accumulate. Later, in 1787 Chladni repeated and published this kind of experiments in [9], and in 1831 Faraday described the Faraday ripples or waves which appears in the surfaces of a fluid contained in a vibrating recipient, see [13]. The vibrations of beams, strings, cable tensors and many other structures are nowadays monitored, and the damage of the material is inferred from the behavior of the zeros of eigenfunctions (see for instance [11]). To this end, several procedures are used: the zeros can be determined by scanning the vibrating body with a laser and measuring the Doppler shift in the backscatter, see [20]; another method due to Cha, Dym and Wong consists on attach a lumped mass, and whenever the mass is located in a node, it will not be affected by the original vibrations and the mass will remain stationary, see [6].

To our knowledge, the nodal inverse problem in the half-line was not studied before, see [15] for a survey of other spectral inverse problems in the half-line, and [7] for the classical Gelfand-Levitan-Marchenko techniques in quantum scattering. We believe that the reason is that nodal inverse problems were solved by using very precise estimates on the nodal lengths and the eigenvalues. For weighted problems, the eigenvalue estimates involve the total variation of $\log(\sigma)$ and hold only for weights σ which are bounded away from zero; in problem (1)-(2) this condition implies that every solution oscillate and posses infinitely many zeros. In the determination of a potential, the length of the nodal domains is needed, for intervals of length L these lengths are known with high precision, and they are about $jL/n + O(1/n^2)$, a kind of estimate which seems difficult to generalize to infinite intervals.

Here we overcame this problem following the ideas recently introduced in [32]. First, we define a family of probability measures $\{\mu_n\}_{n\geq 1}$, such that μ_n is uniformly spread on the zeros of the $n-$th. eigenfunction, and we show that there exists a weak limit μ. Then, we characterize the limit measure in terms of the weight σ.

However, we need to solve two problems in order to carry on this idea. First, we need an estimate of the eigenvalues like the one in equation (3), which is known only for $p = 2$. Second, the measures μ_n are defined over the half line, so we need to prove the tightness of the sequence $\{\mu_n\}_{n\geq 1}$ in order to recover some compactness and get a probability measure as a limit. Let us introduce this concept here, since it plays a key role in the rest of the work.

Definition 1.1. *A sequence $\{\mu_n\}_{n\geq 1}$ of Borel probability measures on $\mathbb{R}$ is called tight if for every $\varepsilon > 0$ there exists a compact set K_ε and n_0 such that*

$$\mu_n(K_\varepsilon) \geq 1 - \varepsilon$$

for all $n \geq n_0$.

In Section §2 we introduce some definitions, a short review of results for eigenvalue problems, and some auxiliary results about convergence of measures and its distribution functions.

In Section §3 we solve one of our problems, showing that $\{\mu_n\}_{n\geq 1}$ is tigh:

Theorem 1.2. *Let X be the nodal set of problem (1)-(2), with σ satisfying $(H1)$-$(H3)$, and let $\{\mu_n\}_{n\geq 1}$ be the sequence of measures defined as*

$$\mu_n = n^{-1} \sum_{j=1}^{n} \delta_{x_j^n}$$

where $x_j^n \in X$, and $\delta_y = \delta(x - y)$ is the Dirac's delta function centered at y. Then $\{\mu_n\}_{n\geq 1}$ is tight.

In Section §4 we study the asymptotic behavior of eigenvalues. We need here the constant π_p defined in Section §2, and we have the following result:

Theorem 1.3. *Let $\{\lambda_n\}_{n\geq 1}$ be the sequence of eigenvalues of problem (1)-(2), with σ satisfying $(H1)$-$(H3)$. Then*

$$\lambda_n = \pi_p^p n^p + o(n^p) \tag{4}$$

as $n \to \infty$.

The proof of Theorem 1.3 is simpler than the ones in [3, 21, 29, 31], and it is based only on the Sturm's comparison theorem. The proof in [3, 29] relies on Hilbert space techniques which are not available here, the original one of Hille using Prufer's transformation needs $\sigma \in C^1([0, \infty))$, and the one in [31] recover the result as a corollary of a precise estimate of the spectral counting function.

Let us remark that hypotheses $(H3)$ is not necessary here, and without assuming it we get

$$\lambda_n = \left(\frac{\pi_p n}{\int_0^\infty \sigma^{1/p} dt} \right)^p + o(n^p).$$

The proof of this general case follows easily by normalizing σ in the equation, and rescaling the eigenvalues.

From Theorem 1.2 and Helly's selection theorem (see Section §2), we obtain a convergent subsequence of $\{\mu_n\}_{n\geq 1}$. In Section §5 we show that the full sequence converges and we characterize the limit measure:

Theorem 1.4. *Let $\{\mu_n\}_{n\geq 1}$ be the sequence of measures defined in Theorem 1.2. Then, μ_n converge weakly to μ, where*

$$\mu([a,b]) = \int_a^b \sigma^{1/p} dt$$

for any $[a,b] \subset [0,\infty)$.

At first sight, it seems that all the zeros of any subsequence of eigenfunctions are needed to determine σ. However, there are a lot of redundant information in the full set X, and in Section §6 we show that any dense subset of nodes is enough to characterize the weight σ. Previously, we will show in a short Lemma that X is dense in the half-line.

Theorem 1.5 (Uniqueness given a dense subset of X). *Let $S = \{x^n_{j(n)}\} \subset X$ be any dense subset of zeros of eigenfunctions of problem (1)-(2). Then, there exists a unique weight σ satisfying $(H1)$-$(H3)$ such that $x^n_{j(n)}$ is the $j(n)-$th. zero of the $n-$th. eigenfunction.*

Our proof is partially based on the ideas of Hald and McLaughlin in [19], although we avoided the use of the delicate estimates of the lengths of the nodal domains which require high regularity of the weight. We wish to stress that this result is new even for $p = 2$ and finite intervals $[a,b]$, since Hald and McLaughlin proved the uniqueness of the weight given a dense subset only for smooth weights (namely $\sigma'' \in L^1([a,b])$).

For weights $\sigma \in BV([a,b])$, a *twin* dense set was required in [20, 32], that is, if a nodal point x^n_j belongs to the subset S, then also x^n_{j-1} or x^n_{j+1} belongs to S. The proof of Theorem 1.5 can be extended to include nodal inverse problems for Sturm-Liouville operators in finite intervals with weights $\sigma^{1/p} \in BV$, without the monotonicity condition.

2. Preparatory results

Let us collect some results from measure theory and quasilinear eigenvalue problems.

2.1. Probability measures and Helly's theorem

We need the following definitions and results concerning probability measures.

Definition 2.1. *A sequence $\{\mu_n\}_{n\geq 1}$ of Borel probability measures on $\mathbb{R}$ converges weakly to a measure μ if*

$$\int_{\mathbb{R}} f d\mu_n \to \int_{\mathbb{R}} f d\mu$$

for every $f \in C_b(\mathbb{R})$.

Prokhorov's theorem states that a sequence $\{\mu_n\}_{n\geq 1}$ of Borel probability measures on $\mathbb{R}$ is tight if and only if its closure is weakly compact. A stronger result is due to Helly, and we state it in terms of the distribution functions

$$F_n(x) = \mu_n(-\infty, x]$$

Theorem 2.2 (Helly's selection theorem). *Let $\{F_n\}_{n\geq 1}$ be a sequence of non-decreasing real functions on $\mathbb{R}$ satisfying $0 \leq F_n(x) \leq 1$ for all $x \in \mathbb{R}$ and $n \geq 1$. Then, there exists a subsequence $\{F_{n_j}\}$ converging pointwise to a real function F. If the limit function F is continuous, then this convergence is uniform on compact sets of $\mathbb{R}$.*

For a proof, see Billingsley [1]. In particular, when F_n are associated to a tight sequence of measures which converges weakly to μ and the distribution function of μ is continuous, the convergence is uniform in $\mathbb{R}$. The key point is that there are no loss of mass at infinity:

Lemma 2.3. *Let $\{F_n\}_{n\geq 1}$ be a sequence of distribution functions associated to a tight sequence of probability measures $\{\mu_n\}_{n\geq 1}$ supported on $\mathbb{R}$, and suppose that $\{F_n\}_{n\geq 1}$ converges pointwise to a continuous function F which is the distribution function of some probability measure μ. Then, $\{F_n\}_{n\geq 1}$ converges uniformly to F.*

Proof. Let us fix some $\varepsilon > 0$. Since $\{\mu_n\}_{n\geq 1}$ is tight, there exists $T > 0$ and n_0 such that $\mu_n([-T,T]) = F_n(T) - F_n(-T) > 1-\varepsilon$ for $n > n_0$. Hence, we have

$$F(T) - F(-T) \geq 1 - \varepsilon.$$

We choose $\delta > 0$ such that $|F(x) - F(y)| \leq \varepsilon$ whenever $x, y \in [-T, T]$ and $|x-y| \leq \delta$. We choose a family of points $-T = y_1 < y_2 < \cdots < y_k = T$, satisfying $|y_j - y_{j+1}| = \delta$. Hence,

$$|F(y_j) - F_n(y_j)| < \varepsilon$$

for any $n \geq n_1$ due to the pointwise convergence to F. Now, for $x \in (y_j, y_{j+1})$, and using the monotonicity of F_n,

$$\begin{aligned}
F(x) - F_n(x) &\leq F(x) - F_n(y_j) \\
&\leq |F(x) - F(y_j)| + |F(y_j) - F_n(y_j)| \\
&< 2\varepsilon. \\
F(x) - F_n(x) &\geq F(x) - F_n(y_{j+1}) \\
&= F(x) - F(y_{j+1}) + F(y_{j+1}) - F_n(y_{j+1}) \\
&> -2\varepsilon,
\end{aligned}$$

and thus $|F(x) - F_n(x)| < 2\varepsilon$ for $n \geq \max\{n_0, n_1\}$ and $x \in [-T, T]$.

For $x > T$, since $|F_n(T) - F(T)| < \varepsilon$, we have $F(T) > 1 - 2\varepsilon$, and the monotonicity of F, which is a distribution function of some measure, implies that $F(x) - F(T) < 2\varepsilon$. Hence,

$$|F_n(x) - F(x)| \leq |F_n(x) - F_n(T)| + |F_n(T) - F(T)| + |F(T) - F(x)| < 4\varepsilon.$$

A similar inequality holds for $x < -T$, and the result is proved. □

2.2. *Quasilinear eigenvalue problems*

The p-Laplacian eigenvalue problem in bounded intervals was thoroughly studied in recent years. We will state without proofs several results, see the book [12] for details.

Two key result which will be needed are the following ones:

Theorem 2.4. *[Sturm's comparison and oscillation theorem] Let $\sigma(t) \leq \rho(t)$ be two positive continuous functions, and let u, v be solutions of the following problems:*

$$\begin{aligned}
-(|u'|^{p-2}u')' &= \sigma(t)|u|^{p-2}u, \\
-(|v'|^{p-2}v')' &= \rho(t)|v|^{p-2}v.
\end{aligned}$$

Then, between two zeros of a solution u, any solution v has at least one zero. Moreover, given two solutions u_1, u_2 of the first equation, their zeros alternate.

Also, the eigenvalues of the following problems in $[a, b]$

$$\begin{aligned}
-(|u'|^{p-2}u')' = \lambda\sigma(t)|u|^{p-2}u, \qquad u(a) = u(b) = 0 \\
-(|v'|^{p-2}v')' = \mu\rho(t)|v|^{p-2}v, \qquad v(a) = v(b) = 0
\end{aligned}$$

satisfy $\mu_k(\rho) \leq \lambda_k(\sigma)$ for any $k \geq 1$,with strict inequalities if $\sigma \not\equiv \rho$.

Theorem 2.5. *[Domain monotonicity of the eigenvalues] Let $a < b < c$, and let ρ be a positive continuous function. Then the eigenvalues of the following problems*

$$\begin{aligned}
-(|u'|^{p-2}u')' = \lambda\rho(t)|u|^{p-2}u, \qquad u(a) = u(b) = 0 \\
-(|v'|^{p-2}v')' = \mu\rho(t)|v|^{p-2}v, \qquad v(a) = v(c) = 0
\end{aligned}$$

satisfy $\mu_k \leq \lambda_k$ for any $k \geq 1$.

When σ is a constant, the eigenvalues and eigenfunctions can be computed explicitly in terms of generalized trigonometric functions. We call $\sin_p(x)$ the solution of the initial value problem

$$-(|u'|^{p-2}u')' = (p-1)|u|^{p-2}u, \qquad u(0) = 0, \quad u'(0) = 1,$$

and let $\hat{\pi}_p$ the first zero of $\sin_p(t)$, given by

$$\hat{\pi}_p = 2\int_0^1 \frac{dt}{\sqrt[p]{1-t^p}}.$$

Let us note that there are alternative definitions of $\sin_p$ and π_p, depending on the presence or not of the factor $p-1$ in the equation. It is convenient to introduce

$$\pi_p = \sqrt[p]{p-1}\hat{\pi}_p,$$

and now we have following characterization of the spectrum:

Theorem 2.6. *Let $\{\lambda_n\}_{n\geq 1}$, $\{v_n\}_{n\geq 1}$ be the eigenvalues and eigenfunctions of*

$$-(|v'|^{p-2}v')' = \lambda|v|^{p-2}v, \qquad v(a) = v(b) = 0.$$

Then

$$\lambda_n = \left(\frac{\pi_p n}{b-a}\right)^p, \qquad v_n(t) = \sin_p\left(\frac{\hat{\pi}_p nt}{b-a}\right).$$

Moreover, the $n-$th. eigenvalue is simple, and the associated eigenfunction v_n has n nodal domains, that is, v_n has $n+1$ simple zeros in $[a,b]$.

See, for instance, Del Pino, Drabek and Manasevich [10]. There exist similar formula for eigenvalues $\{\nu_n\}_{n\geq 0}$ corresponding to the Neumann boundary condition $u'(a) = u'(b) = 0$, and eigenvalues $\{\eta_n\}_{n\geq 1}$ corresponding to the mixed boundary condition $u(a) = u'(b) = 0$ or $u'(a) = u(b) = 0$, namely

$$\nu_n = \left(\frac{\pi_p(n-1)}{b-a}\right)^p, \qquad \eta_n = \left(\frac{\pi_p n}{2(b-a)}\right)^p.$$

On the other hand, there are no explicit expressions for weighted problems, and there are several bounds available, see [31]. We state the main result for the weighted problem that we will need later:

Theorem 2.7. *[Section §5 in [39], and Theorem 1.6 in [14]] Let $\sigma \in L^1([a,b])$ be a positive function, and let $\{\lambda_n\}_{n\geq 1}$, $\{v_n\}_{n\geq 1}$ be the eigenvalues and eigenfunctions of*

$$-(|v'|^{p-2}v')' = \lambda\sigma(t)|v|^{p-2}u \qquad v(a) = v(b) = 0. \tag{5}$$

Then the $n-$th. eigenvalue is simple, and the associated eigenfunction v_n has n nodal domains, that is, v_n has $n+1$ simple zeros in $[a,b]$. Moreover, the asymptotic behavior of the eigenvalues is given by

$$\lambda_n = \frac{\pi_p^p n^p}{\left(\int_a^b \sigma^{1/p}(t)dt\right)^p} + o(n^p) \tag{6}$$

as n goes to infinity.

Finally, we will need the following bound which goes back to Nehari, Calogero and Cohn in the linear case $p = 2$:

Theorem 2.8. *[Theorem 1.1 in [5]] Let $\sigma \in L^1([a,b])$ be a non negative monotonic function, and let λ_1 be the first eigenvalue of*

$$-(|u'|^{p-2}u')' = \lambda\sigma(t)|u|^{p-2}u \qquad u(a) = u(b) = 0. \tag{7}$$

Then

$$\frac{\pi_p}{2} \leq \lambda_1^{1/p}\int_a^b \sigma^{1/p}(t)dt. \tag{8}$$

3. Tightness of the sequence of measures

The following upper bound of λ_n will be used in the proofs of Theorems 1.2 and 1.3.

Lemma 3.1. *Let $\{\lambda_n\}_{n\geq 1}$ be the sequence of eigenvalues of problem (1)-(2), with σ satisfying $(H1)$-$(H3)$. Suppose that u_n has a zero greater than $t_0 \in (0,\infty)$. Then*

$$\lambda_n \leq \frac{\pi_p^p n^p}{t_0^p \sigma(t_0)} \tag{9}$$

Proof. Let us consider the following eigenvalue problem,

$$\begin{array}{rcll} -(|v'|^{p-2}v')' &=& \hat{\lambda}\sigma(t_0)|v|^{p-2}v & t \in (0,t_0) \\ v(0) &=& 0 & \\ v(t_0) &=& 0, & \end{array} \tag{10}$$

the eigenvalues are given by $\hat{\lambda}_n = \pi_p^p n^p / \sigma(t_0)$.

By the Sturm's comparison theorem 2.4, since $\sigma(t_0) \leq \sigma$ in $[0,t_0]$, and u_n has less than n zeros in $[0,t_0]$, we have

$$\lambda_n \leq \hat{\lambda}_n = \frac{\pi_p^p n^p}{t_0^p \sigma(t_0)},$$

and the proof is finished. □

Proof of Theorem 1.2. Let $\varepsilon > 0$ be fixed. From (H3), we can choose $T > 1$ such that

$$\int_T^\infty \sigma^{1/p}(t)dt < \varepsilon.$$

Let n_0 such that u_n has at least two zeros greater than T if $n \geq n_0$. Let $y_1^n < \cdots < y_{k(n)}^n$ be the zeros of u_n in $[T,\infty)$, and let us find an upper bound for $k(n)$.

Applying inequality (8) between two consecutive zeros, and using that the first eigenvalue between two zeros coincides with λ_n, we get

$$\begin{aligned} k(n) - 1 &\leq \frac{2}{\pi_p} \sum_{j=1}^{k(n)-1} \lambda_n^{1/p} \int_{y_j^n}^{y_{j+1}^n} \sigma^{1/p}(t)dt \\ &\leq \frac{2\lambda_n^{1/p}}{\pi_p} \int_T^\infty \sigma^{1/p}(t)dt \\ &\leq \frac{\varepsilon 2\lambda_n^{1/p}}{\pi_p}. \end{aligned}$$

Using Theorem 2.5 and Lemma 3.1, we can bound $\lambda_n \leq \hat{\lambda}_n$, the n-th. eigenvalue of problem (10) with $x_0 = 1$, and from the explicit formula for the eigenvalues in Theorem 2.6, we obtain

$$k(n) \leq \frac{\varepsilon 2\hat{\lambda}_n^{1/p}}{\pi_p} + 1 \leq \frac{\varepsilon 2n}{\sigma(1)^{1/p}} + 1.$$

Now,

$$\mu_n([T,\infty)) = \frac{\#\{j : x_j^n \in [T,\infty)\}}{n} = \frac{k(n)}{n} \leq \frac{2\varepsilon}{\sigma(1)^{1/p}} + \frac{1}{n},$$

and therefore the sequence $\{\mu_n\}_{n\geq 1}$ is tight. The theorem is proved. □

Remark 3.2. *Observe that the monotonicity of σ is needed since the proof depends on Theorem 2.8. With little extra effort, the proof can be extended for functions σ which are decreasing in $[x_0,\infty)]$ for some x_0. We only need to take $T > x_0$, and in the last step of the proof we can consider any interval $[x_0, x_0+\delta]$ instead of $[0,1]$.*

4. Asymptotic behavior of Eigenvalues

Proof of Theorem 1.3. In order to prove that $\lambda_n = \pi_p^p n^p + o(n^p)$, it is enough to show that

$$\lim_{n\to\infty} \frac{\lambda_n}{\pi_p^p n^p} = 1. \tag{11}$$

Let $\varepsilon > 0$ be fixed. As before, there exists $T > 0$ such that

$$\int_T^\infty \sigma^{1/p}(t)dt < \varepsilon. \tag{12}$$

Let us compare the eigenvalues of problem (1)-(2) with $\{\hat\lambda_n\}_{n\geq 1}$, the eigenvalues of

$$-(|v'|^{p-2}v')' = \hat\lambda\sigma(t)|v|^{p-2}v \qquad v(0) = v(T) = 0, \tag{13}$$

Let us recall that the eigenvalues satisfy

$$\hat\lambda_n = \frac{\pi_p^p n^p}{\left(\int_0^T \sigma^{1/p}(t)dt\right)^p} + o(n^p),$$

and we fix some n_0 satisfying two conditions. We need first that the absolute value of the error term $o(n)$ in the previous formula is bounded above by εn^p. The second condition will be imposed below.

We claim that $\lambda_n \leq \hat\lambda_n$ for any $n \geq 1$. If not, from Sturm's comparison theorem 2.4, u_n has at least n zeros in $(0, T)$, but also $u_n(0) = 0$, and u_n has only n zeros, a contradiction. Hence,

$$\lambda_n \leq \hat\lambda_n \leq \frac{\pi_p^p n^p}{(1-\varepsilon)^p} + \varepsilon n^p = \pi_p^p n^p + \left(\frac{1-(1-\varepsilon)^p}{(1-\varepsilon)^p} + \varepsilon\right) n^p \tag{14}$$

for $n \geq n_0$ due to $(H3)$ and the bound (12). Let us call

$$C_\varepsilon := \frac{1-(1-\varepsilon)^p}{(1-\varepsilon)^p} + \varepsilon,$$

and observe that $C_\varepsilon = O(\varepsilon)$ as $\varepsilon \to 0^+$.

In order to obtain a lower bound for λ_n, we estimate the number of zeros of u_n in $[0, T]$ and $[T, \infty)$. As in the proof of Theorem 1.2 in Section §3, we call $k(n)$ the number of zeros of u_n in (T, ∞), and we have

$$k(n) \leq \frac{\varepsilon 2\lambda_n^{1/p}}{\pi_p} + 1 \leq \varepsilon n \frac{2(\pi_p^p + C_\varepsilon)^{1/p}}{\pi_p(1-\varepsilon)} + 1,$$

the last inequality due to the previous bound obtained in (14). For brevity, let us call

$$D_\varepsilon := \frac{2(\pi_p^p + C_\varepsilon)^{1/p}}{\pi_p(1-\varepsilon)}$$

and let us note that $D_\varepsilon \to 2$ as $\varepsilon \to 0^+$.

Thus, u_n has at least $n - k(n) \geq n(1 - \varepsilon D_\varepsilon) - 1$ zeros in $[0, T]$, and let us define

$$m = \left\lfloor n(1-\varepsilon D_\varepsilon) - 1 \right\rfloor - 1,$$

where $\lfloor x \rfloor$ denotes the integer part of x. Here we impose the second condition on n_0: we need $m \geq n_0$ in order to use again the bound on the error term in the asymptotic formula of the eigenvalues.

Let v_m the eigenfunction corresponding to $\hat\lambda_m$ in problem (13). Comparing the number of zeros in $[0, T]$ of u_n and v_m, Sturm's comparison theorem 2.4 implies

$$\hat\lambda_m \leq \lambda_n.$$

Using the asymptotic formula for $\hat{\lambda}_m$, and recalling $(H3)$, we get

$$\lambda_n \geq \hat{\lambda}_m \geq \pi_p^p m^p - \varepsilon m^p \geq (\pi_p^p - \varepsilon)\Big(n(1-\varepsilon D_\varepsilon) - 3\Big)^p,$$

namely

$$\lambda_n \geq \pi_p^p n^p \Big(\frac{\pi_p^p - \varepsilon}{\pi_p^p}\Big)\Big(1 - \varepsilon D_\varepsilon - \frac{3}{n}\Big)^p. \tag{15}$$

Finally, from bounds (14) and (15), we get

$$\Big(\frac{\pi_p^p - \varepsilon}{\pi_p^p}\Big)\Big(1 - \varepsilon D_\varepsilon - \frac{3}{n}\Big)^p \leq \frac{\lambda_n}{\pi_p^p n^p} \leq 1 + C_\varepsilon,$$

and the desired limit (11) follows. The proof is finished □

5. The nodal inverse problem

We prove now that $\mu_n \to \mu$, and we characterize the limit measure in terms of the weight σ.

Proof of Theorem 1.4. Let $\{F_n\}_{n\geq 1}$ be the sequence of distribution functions associated to the measures $\{\mu_n\}_{n\geq 1}$. From Helly's selection theorem 2.2, there exists a converging subsequence $\{F_{n_j}\}_{j\geq 1}$ and also a limit function F. Observe that Prokhorov's theorem implies that there exists a probability measure $\mu = dF$, and

$$\int_0^\infty f(x) d\mu_{n_j} \to \int_0^\infty f(x) d\mu,$$

for any $f \in C_b(\mathbb{R})$, which in turns implies that $\mu_n([a, b]) \to \mu([a, b])$.

Let us show that, for any $x > 0$, we have

$$\lim_{n\to\infty} F_n(x) = \int_0^x \sigma^{1/p}(t) dt. \tag{16}$$

Let us fix $\varepsilon > 0$. Now, being $\{\mu_n\}_{n\geq 1}$ a tight sequence, there exists $T > 0$ such that $\mu_n([0, T]) > 1 - \varepsilon/2$, and then we have $F_n(T) > 1 - \varepsilon/2$. Moreover, we can take T big enough such that, once again,

$$\int_T^\infty \sigma^{1/p}(t) dt < \frac{\varepsilon}{2}. \tag{17}$$

First, we consider $x \leq T$. We subdivide the interval $[0, x]$ in M subintervals of length $h = x/M$. The length h is small enough in order to have

$$0 \leq S_i^{1/p} - s_i^{1/p} < \frac{\varepsilon}{4T}$$

for $1 \leq i \leq M$, where

$$\begin{aligned} s_i &= \inf\{\sigma(t) : t \in [(i-1)h, ih)\}, \\ S_i &= \sup\{\sigma(t) : t \in [(i-1)h, ih)\}. \end{aligned}$$

Since σ is continuous, is Riemann integrable in $[0, T]$ and

$$\sum_{i=1}^M h s_i^{1/p} \leq \int_0^x \sigma^{1/p}(t) dt \leq \sum_{i=1}^M h S_i^{1/p}. \tag{18}$$

Moreover, we have chosen h in order to have

$$\sum_{i=1}^M h S_i^{1/p} - \sum_{i=1}^M h s_i^{1/p} < \sum_{i=1}^M \frac{\varepsilon h}{4T} < \frac{\varepsilon}{4}. \tag{19}$$

Let us approximate the measures μ_n in each interval. Let k_i^n be the number of zeros of u_n in the intervals $[(i-1)h, ih)$, with $1 \leq i \leq M$ (we take the last one as closed, i.e., $[(M-1)h, Mh]$). Hence,

$$\frac{1}{n}\sum_{i=1}^{M} k_i^n = F_n(x).$$

We can estimate each k_i^n by using the Sturm's comparison theorem 2.4. Since $s_i \leq \sigma \leq S_i$ in $[(i-1)h, ih)$, comparing with the explicit solution $\sin_p(t)$ for constant coefficient problems we obtain

$$\frac{\lambda_n^{1/p} s_i^{1/p} h}{\pi_p} - 1 \leq k_i^n \leq \frac{\lambda_n^{1/p} S_i^{1/p} h}{\pi_p} + 1.$$

Thus,

$$\frac{\lambda_n^{1/p}}{n\pi_p}\sum_{i=1}^{M} s_i^{1/p} h - \frac{M}{n} \leq F_n(x) \leq \frac{\lambda_n^{1/p}}{n\pi_p}\sum_{i=1}^{M} S_i^{1/p} h + \frac{M}{n}. \tag{20}$$

Therefore, from inequalities (18), (19) and (20), we get

$$\begin{aligned} F_n(x) - \int_0^x \sigma^{1/p}(t)dt \leq\ & \frac{\lambda_n^{1/p}}{n\pi_p}\sum_{i=1}^{M} S_i^{1/p} h + \frac{M}{n} - \sum_{i=1}^{M} h s_i^{1/p} \\ \leq\ & \frac{\lambda_n^{1/p}}{n\pi_p}\sum_{i=1}^{M}\left(S_i^{1/p} h - h s_i^{1/p}\right) + \frac{M}{n} + \left(\frac{\lambda_n^{1/p}}{n\pi_p} - 1\right)\sum_{i=1}^{M} s_i^{1/p} h \\ \leq\ & \frac{\varepsilon\lambda_n^{1/p}}{4n\pi_p} + \frac{M}{n} + \left|\frac{\lambda_n^{1/p}}{n\pi_p} - 1\right| \\ =\ & O(\varepsilon), \end{aligned}$$

for n big enough, from Theorem 1.3, and using that M is fixed.

A lower bound can be found in much the same way, and the limit (16) is proved for $x \leq T$ since ε is arbitrary. Finally, for $x > T$ and n big enough, we have

$$\begin{aligned} \left|F_n(x) - \int_0^x \sigma^{1/p}(t)dt\right| \leq\ & |F_n(x) - F_n(T)| + \\ & + \left|F_n(T) - \int_0^T \sigma^{1/p}(t)dt\right| \\ & + \left|\int_0^T \sigma^{1/p}(t)dt - \int_0^x \sigma^{1/p}(t)dt\right| \\ <\ & \frac{\varepsilon}{2} + \varepsilon + \frac{\varepsilon}{2}, \end{aligned}$$

and the theorem is proved. □

6. Uniqueness given any dense subset of nodes

The following Lemma shows that X is dense in the half-line.

Lemma 6.1. *Let X be the nodal set of problem (1)-(2), with σ satisfying $(H1)$-$(H3)$. Then X is dense in* $[0, \infty)$.

Proof. It is enough to show that $(a,b)\cap X\neq\emptyset$ for any subinterval $(a,b)\subset[0,\infty)$.

Let us consider the Dirichlet eigenvalue problem

$$(|v'|^{p-2}v')' = \mu\sigma(t)|v|^{p-2}v, \qquad v(a)=v(b)=0,$$

and let μ_1 be the first eigenvalue.

Let λ_k be an eigenvalue of problem (1)-(2) greater that μ_1. Thus, Theorem 2.4 implies that the associated eigenfunction u_k has at least a zero in (a,b). The proof is finished.

We are ready to prove that any dense set of nodes characterize the weight.

Proof of Theorem 1.5. Let $\{F_n\}_{n\geq1}$, $\{G_n\}_{n\geq1}$ be the distribution functions of the measures $\{\mu_n\}_{n\geq1}$, $\{\hat{\mu}_n\}_{n\geq1}$ corresponding to the zeros of problem (1)-(2) with weights σ and ρ respectively, both satisfying $(H1)$-$(H3)$.

Let us assume that there are some dense subset of zeros $S=\{x^n_{j(n)}\}\subset X$ such that

$$F_n(x^n_{j(n)}) = G_n(x^n_{j(n)}).$$

.

We know from Theorem 1.4 and Lemma 2.3 that $\{F_n\}_{n\geq1}$ and $\{G_n\}_{n\geq1}$ converge uniformly to F and G, where

$$F(x)=\int_0^x \sigma^{1/p}(t)dt, \qquad G(x)=\int_0^x \rho^{1/p}(t)dt.$$

Let us fix some $\varepsilon>0$. Let us recall that the uniform convergence implies that there exists some n_0 such that, for $n\geq n_0$, we have

$$\|F_n-F\|_\infty+\|G_n-G\|_\infty<\varepsilon/2.$$

Moreover, from the proof of Lemma 2.3, we know that there exists some $\delta>0$ such that

$$|F(x)-F(y)|+|G(x)-G(y)|<\varepsilon/2$$

whenever $|x-y|<\delta$.

Let $x\in[0,\infty)$, and we want to show that $|F(x)-G(x)|<\varepsilon$. Since S is dense, we can choose some zero $y^n_{j(n)}\in S$ satisfying $|x-y^n_{j(n)}|<\delta$ and $n\geq n_0$. Now,

$$\begin{aligned}
|F(x)-G(x)| &\leq |F(x)-F(y^n_{j(n)})|+|F(y^n_{j(n)})-F_n(y^n_{j(n)})| \\
&\quad +|F_n(y^n_{j(n)})-G_n(y^n_{j(n)})| \\
&\quad +|G_n(y^n_{j(n)})-G(y^n_{j(n)})|+|G(y^n_{j(n)})-G(x)| \\
&< \tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2},
\end{aligned}$$

and we obtain the desired bound.

Since ε is arbitrary, we have $F(x)=G(x)$ for any x, and the continuity of the weights together with the Fundamental Theorem of Calculus implies $\sigma=\rho$. The proof is finished. □

Remark 6.2. *The results obtained in [18, 19] for $p=2$ in finite intervals require a dense set of twin nodes. The previous proof can be easily extended to these problems assuming only that the weight σ is continuous at right, and $\sqrt{\sigma}\in BV([0,1])$, combined with the results in [32].*

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