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Implicitization of rational hypersurfaces via linear syzygies: A practical overview ☆

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We unveil in concrete terms the general machinery of the syzygy-based algorithms for the implicitization of rational surfaces in terms of the monomials in the polynomials defining the parametrization, following and expanding our joint article with M. Dohm. These algebraic techniques, based on the theory of approximation complexes due to J. Herzog, A. Simis and W. Vasconcelos, were introduced for the implicitization problem by J.-P. Jouanolou, L. Busé, and M. Chardin. Their work was inspired by the practical method of moving curves, proposed by T. Sederberg and F. Chen, translated into the language of syzygies by D. Cox. Our aim is to express the theoretical results and resulting algorithms into very concrete terms, avoiding the use of the advanced homological commutative algebraic tools which are needed for their proofs.

1. Introduction

Let $\mathbb{K}$ be a field. We can assume $\mathbb{K} = \mathbb{Q}$ (or any computable field) when dealing with implementations. All the varieties, rings and vector spaces we will consider are understood to be taken over $\mathbb{K}$. Consider a rational parametrization

$$\begin{aligned} \mathbb{K}^2 &\overset{f}{\dashrightarrow} \mathbb{K}^3 \\ s=(s_1,s_2) &\mapsto \left(\frac{f_1(s)}{f_0(s)}, \frac{f_2(s)}{f_0(s)}, \frac{f_3(s)}{f_0(s)}\right) \end{aligned} \tag{1.1}$$

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of a (hyper)surface $\mathscr{S} := (F = 0) \subset \mathbb{A}^3$, where $F \in \mathbb{K}[T_1, T_2, T_3]$ is a non-constant polynomial and $f_i \in \mathbb{K}[s_1, s_2]$. (As usual, the dashed arrow means that f is defined on a dense open set of $\mathbb{K}^2$.) An important problem in computer aided geometric design is to switch from parametric to implicit representations of rational surfaces (Hoffmann, 1989), that is the *parametrization* f is assumed to be known and one seeks for the *implicit equation* F (which is defined only up to multiplicative constant). In fact, we will assume that f is given and our aim will not be to get the implicit equation F of $\mathscr{S}$ written in terms of its monomials, but a *matrix representation* of the surface.

Definition 1.1. A matrix representation M of $\mathscr{S}$ is a matrix with entries in $\mathbb{K}[T_1, T_2, T_3]$, generically of full rank, which verifies the following condition: for any point $p \in \mathbb{K}^3$, the rank of $M(p)$ drops if and only if p lies on $\mathscr{S}$.

The use of matrix representations goes back to Manocha and Canny (1991), and to Chionh and Goldman (1992). Having the matrix M is sufficiently good for many purposes and it is cheaper to compute. The well-developed theory and tools of linear algebra can be applied to solve geometric problems. We can certainly use the (numerical) rank dropping condition in Definition 1.1 to check membership in $\mathscr{S}$, and, moreover, the whole structure of minors of M is related to the singularities of the parametrization (Botbol et al., 2014) and gives a way to invert it when the fiber has a single point (Busé, 2014; Botbol et al., 2014). Matrix representations are also useful for solving intersection problems as is shown in Aruliah et al. (2007), Thang et al. (2009), Diaz-Toca et al. (2013), Busé (2014). Much of the computational difficulty in these problems lies on computing ranks for polynomial matrices (cf. Henrion and Sebek, 1999 as well as Section 5 in the nice and interesting paper Busé, 2014).

The motivation for this paper is to present in the simplest possible terms procedures for the implicitization of rational surfaces via matrix representations, based on the syzygies $(h_0, \ldots, h_3)$ of the input polynomials, that is, 4-tuples of polynomials in the s variables verifying the linear relation $\sum_{i=0}^{3} h_i f_i = 0$. The theoretical justification is not naive and requires a good command of techniques of (homological) commutative algebra. However, the algorithms do not require a heavy background and are easy to explain. We will show that they perform very well, and moreover, they work even better in the presence of base points.

Call T_1, T_2, T_3 the coordinates in the target of f. Our question is an instance of elimination of variables, where we want to find the algebraic relations among the variables T_1, T_2, T_3 under the assumption that $f_0(s)T_i - f_i(s) = 0, i = 1, 2, 3$, for some s in the domain of f. The eliminant polynomial by excellence is the determinant $\det(A)$, a polynomial with integer coordinates on the coefficients of a square matrix A, which vanishes on those coefficients for which there exists a nonzero solution x to the equations $A \cdot x = 0$. Elimination of variables is done in the literature through different incarnations of the following general strategy:

(1) Reduce the problem to a linear algebra problem.
(2) Hide the variables one wants to eliminate in the (typically monomial) bases.
(3) Use determinants.

This strategy is also the core in our syzygy-based algorithms.

The following short account of the approach of the use of syzygies in our context is reconstructed from David Cox's lecture at the Conference PASI on Commutative Algebra and its connections to Geometry honoring Wolmer Vasconcelos, held in Brazil in 2009 (Corso and Polini, 2011, Mini Course 1). The use of syzygies for the implicitization of (conic) surfaces goes back to Steiner in 1832 (Steiner, 1832). In 1887, Meyer describes in Meyer (1887) syzygies of three polynomials and makes a general conjecture proved by Hilbert in 1890 (Hilbert, 1890). Surface implicitization by eliminating parameters was studied by Salmon in 1862 (Salmon, 1958) and Dixon in 1908 using resultants (Dixon, 1908). In 1995, Sederberg and Chen reintroduced the use of syzygies, by a method termed as *Moving curves and surfaces* (Sederberg and Chen, 1995). Cox realized they were using syzygies (Cox, 2001), and produced several papers with other coauthors (Cox et al., 1998b; Busé et al., 2003; Cox, 2003a; Zheng et al., 2003; Chen et al., 2005). In 2002, Busé and Jouanolou (2003) abstracted and generalized on a sound basis the method of Sederberg–Chen via approximation complexes, a tool in homological commutative algebra that had been developed by Herzog et al. (1982, 1983a, 1983b). Busé, Chardin, Jouanolou and Simis produced further advances in the homogeneous case (Busé and Jouanolou, 2003; Busé and Chardin, 2005; Chardin, 2006; Busé et al., 2009, 2010). Goldman et al. studied the cases of planar and space curves (Jia and Goldman, 2009; Hoffman et al., 2010; Jia et al., 2010). A generalization of the linear syzygy method when the support of the input polynomials is a square (that is, bihomogeneous of degree (d,d)) was proposed by Busé and Dohm (2007), and for any polygon by Botbol et al. (2009), and Botbol (2009, 2011a, 2011b). This method is particularly adapted when the polynomials defining the parametrization are *sparse*, which is often the case. This will be our point of view in this article. So, we want to solve the following problem.

Problem. Given a rational parametrization f as in (1.1), find a matrix representation M of the surface $\mathscr{S}$ by means of syzygies and the monomial structure of $f_0, \dots, f_3$.

The main general algorithmic answer to this problem is given in Algorithm 3.1 (see Theorem 3.3). Our assumption that the dimension of $\mathscr{S}$ is 2 is equivalent to the fact, when we extend the map to the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$, that for almost all $p = f(s)$ in the image of f, the number of preimages by f is finite. This number is called the degree of f and noted $\deg(f)$. The matrix representations M of $\mathscr{S}$ provided by Algorithms 3.1 and 3.6 moreover satisfy that the greatest common divisor of all minors of M of maximal size equals $F^{\deg(f)}$.

We present in Section 2 the first naive linear algebra algorithm to compute the implicit equation F, which requires to solve a huge linear algebra system. Moreover, this naive method "forgets" the parametrization and thus in general it is not useful in Computer Aided Geometric Design. In Section 3 we recall previous results on the implicitization of curves and surfaces using syzygies and present our general methods of implicitization via linear syzygies, which requires to solve considerably smaller systems. We highlight in Section 3.1 the main elimination step, which was termed *instant elimination* in Eisenbud (2004) (see also the references therein).

In Section 3.3 we present in Theorem 3.8 a refinement of Theorem 3.3 for bihomogeneous parametrizations, in the same spirit. Technicalities are avoided in our presentation in these sections, and in particular in the statement of our main results Theorems 3.3 and 3.8.

Detailed hypotheses and proofs are deferred to Section 4, where we introduce the necessary background on toric geometry. We collect in Appendix A a general overview of the rationale of the tools and results from homological commutative algebra required for the proofs. A reader only interested in the application of our results, can skip these two sections, except for the explanation in Remark 4.7 which addresses the question of running our main algorithm without checking hypotheses on the base points.

Section 5 illustrates the practicality and advantages of our approach. For our computations, we use implementations in Macaulay 2, which need different type of homogenizations to use current routines (via a toric embedding or a multihomogenization via an abstract toric Cox ring) (Botbol and Dohm, 2010; Botbol, 2010).[1] For the best performance of our algorithms, it would be important to design ad-hoc structured linear algebra strategies to compute syzygies in the sparse case.

2. A naive linear algebra answer

The convex hull in $\mathbb{R}^n$ of the exponents of the monomials occurring in a nonzero (Laurent) polynomial h in n variables is called the Newton polytope $\mathcal{N}(h)$ of h. When h is a polynomial in (s_1, s_2) of degree (at most) d, its Newton polygon $\mathcal{N}(h)$ is contained in the triangle Δ_d with vertices $(0,0), (d,0), (0,d)$. The Euclidean area $\mathrm{vol}(\Delta_d)$ of this triangle is $d^2/2$ and its lattice area $\mathrm{vol}_{\mathbb{Z}}(\Delta_d)$ is equal to $2 \cdot d^2/2 = d^2$, which is always an integer.

We have the following classical result (cf. for instance Busé and Jouanolou, 2003):

[1] Routine updates at: http://mate.dm.uba.ar/~nbotbol/Macaulay2/BigradedImplicit.m2, http://mate.dm.uba.ar/~nbotbol/Macaulay2/MatrixRepToric.m2.

Theorem 2.1. *For generic polynomials $f_0, \ldots, f_3$ of degree d, the degree of the implicit equation F is d^2 and its Newton polytope is the tetrahedron with vertices* $(0,0,0)$, $(d^2,0,0)$, $(0,d^2,0)$, $(0,0,d^2)$.

In the sparse case, the following generalization holds (Sturmfels and Yu, 1994).

Theorem 2.2. *For generic polynomials $f_0, \ldots, f_3$ with the same Newton polygon P, the degree of F is the lattice area $v = \mathrm{vol}_{\mathbb{Z}}(P)$ and its Newton polytope is the tetrahedron with vertices* $(0,0,0)$, $(v,0,0)$, $(0,v,0)$, $(0,0,v)$.

A first naive algorithm would then be the following. Assume the Newton polytope $\mathcal{N}(F)$ of F is known (as in the previous theorems) and number $m_1, \ldots, m_N \in \mathbb{N}^3$ the integer points (also called lattice points) in $\mathcal{N}(F)$. Consider indeterminates $c = (c_1, \ldots, c_N)$ and write $F = \sum_{i=1}^N c_i T^{m_i}$. Substitute $T = f(s)$ and equate to zero the coefficient of each power of (s_1, s_2) that occurs (clearing the denominator). This sets a system $\mathcal{L}$ of linear equations in c, with solution space of dimension 1. Any nonzero solution c of $\mathcal{L}$ will give a choice of implicit equation F.

This solves the problem, but, which is the size of this linear system $\mathcal{L}$?

The number of lattice points in Δ_d equals $\binom{d^2+3}{3}$. In the sparse case, the number of lattice points of a given lattice polygon P can be computed via a theorem of Ehrhart valid in any dimension (Ehrhart, 1967), which amounts to Pick's formula in the case dimension 2. Given a positive integer t, we denote by tP the Minkowski sum of P with itself t times, i.e. $tP = \{p_1 + \cdots + p_t,\ p_i \in P \text{ for } i = 1, \ldots, t\}$. The number of lattice points in tP equals

$$\#(tP \cap \mathbb{Z}^2) = \mathrm{vol}(P)t^2 + \frac{1}{2}\mathrm{vol}_{\mathbb{Z}}(\partial P)t + 1, \tag{2.1}$$

where $\mathrm{vol}_{\mathbb{Z}}(\partial P)$ denotes the number of lattice points in the boundary of P. In particular, $\#(P \cap \mathbb{Z}^2) = \mathrm{vol}(P) + \frac{1}{2}\mathrm{vol}_{\mathbb{Z}}(\partial P) + 1$.

The proof of the following result is straightforward:

Lemma 2.3. *In case $f_0, \ldots, f_3$ are generic polynomials of degree d in (s_1, s_2), the number of unknowns in the linear system $\mathcal{L}$ in the coefficients of the implicit equation F is $\binom{d^2+3}{3}$ (approximately $d^6/6$) and the number of equations is $\binom{d^3+2}{2}$ (approximately $d^6/2$).*

For any lattice polygon P and generic polynomials f_i with Newton polytope P, the linear system $\mathcal{L}$ has $\binom{\mathrm{vol}_{\mathbb{Z}}(P)+3}{3}$ (approximately $\mathrm{vol}_{\mathbb{Z}}(P)^3/6$) variables and $\frac{\mathrm{vol}_{\mathbb{Z}}(P)^3}{2} + \frac{\mathrm{vol}_{\mathbb{Z}}(P)^2}{2}\mathrm{vol}_{\mathbb{Z}}(P) + 1$ equations (approximately $\mathrm{vol}_{\mathbb{Z}}(P)^3/2$).

We will see in Remark 3.5 of Section 3 that the size of the involved linear systems in the syzygy based methods is drastically smaller.

3. The main algorithm based on linear syzygies

Our main result is Theorem 3.3, which has a wide applicability. We distill and state it in naive terms, which do not call upon the more sophisticated tools recalled in Section 4 and Appendix A required for its proof. This is why we postpone the detail of Hypotheses 4.4 and 4.8 until Section 4. Our approach is an inhomogeneous translation of the basic general algorithm for the sparse case in Botbol et al. (2009), which were inspired by the methods (Busé and Jouanolou, 2003) for classical homogeneous polynomials.

Before moving to the implicitization of rational surfaces, we recall the practical approach of moving lines proposed by Sederberg and Chen (1995) for the implicitization of planar curves.

3.1. Curves

A planar rational curve $\mathscr{C}$ over a field $\mathbb{K}$ is given as the image of a map

$$\begin{aligned} \mathbb{K}^1 &\overset{f}{\dashrightarrow} \mathbb{K}^2 \\ s &\mapsto \left(\frac{f_1(s)}{f_0(s)}, \frac{f_2(s)}{f_0(s)}\right), \end{aligned}$$

with $f_i \in \mathbb{K}[s]$ polynomials of degree d in s. We can assume without loss of generality that $\gcd(f_0, f_1, f_2) = 1$. Remark that a linear syzygy can be represented as a linear form $L = h_0T_0 + h_1T_1 + h_2T_2$ in the new variables $T = (T_0, T_1, T_2)$ with $h_i \in \mathbb{K}[s]$ such that

$$\sum_{i=0,1,2} h_i f_i = 0.$$

With this incarnation, a linear syzygy was termed a *moving line*. For any $\nu \in \mathbb{N}$, consider the finite-dimensional $\mathbb{K}$-vector space $\mathrm{Syz}(f)_\nu$ of linear syzygies satisfying $\deg(h_i) \leq \nu$, and call $N(\nu)$ its dimension.

Pick a $\mathbb{K}$-basis $h^i = (h^i_0, h^i_1, h^i_2), i = 1, \dots, N(\nu)$ of $\mathrm{Syz}(f)_\nu$. Consider the standard monomial basis $\{1, s, \dots, s^\nu\}$ of polynomials in s of degree at most ν and write for each syzygy h^i:

$$\begin{aligned} L_i = L_i(s, T) &= \sum_{j=0,1,2} h^i_j(s)T_j = \sum_{j=0,1,2} \left(\sum_{k=0}^{\nu} c^i_{jk} s^k\right) T_j \\ &= \sum_{k=0}^{\nu} \left(\sum_{j=0,1,2} c^i_{jk} T_j\right) s^k. \end{aligned}$$

Let M_ν be the $N(\nu) \times (\nu + 1)$ matrix of coefficients of the L_i's with respect to the basis $\{1, s, \dots, s^\nu\}$:

$$M_\nu = \left(\sum_{j=0,1,2} c^i_{jk} T_j\right)_{i=1,\dots,N(\nu), k=0,\dots,\nu}.$$

Observe that the variable s has disappeared. This is the **main elimination step**!

It is known that for $\nu = d - 1$, the matrix M_ν is a square matrix with $\det(M_\nu) = F^{\deg(f)}$, where F is an implicit equation of $\mathscr{C}$. In case $\nu \geq d$, then M_ν is a non-square matrix with more columns than rows, but still the greatest common divisor of its minors of maximal size equals $F^{\deg(f)}$. In both cases, for $\nu \geq d - 1$, a point $P \in \mathbb{P}^2$ lies on $\mathscr{C}$ iff the rank of $M_\nu(P)$ drops.

In other words, one can always represent the curve as a square matrix of linear syzygies, which gives a *matrix representation* of the implicit equation. In principle, one could now actually calculate the implicit equation, but the matrix M_ν is easier to get and well suited for numerical methods (Aruliah et al., 2007). As we remarked in the surface case, testing whether a point p lies on the curve only requires computing the rank of M_ν evaluated in p. Also, the singularities of $\mathcal{C}$ can be read off from M_ν (Jia and Goldman, 2009; Cox et al., 2013; Busé and D'Andrea, 2012).

In the absence of common zeros of f_0, f_1, f_2, it is possible to find the implicit equation via a resultant computation. Note that for a parametrization with polynomials of degree d, the Sylvester resultant matrix uses a matrix of size $2d$, while the syzygy method uses a matrix of size d, as the Bézout resultant (indeed, one first needs to solve a linear system of size $d \times 2d$ to find a basis of syzygies of degree $d - 1$).

3.2. The general method of implicitization via linear syzygies for surfaces

Assume we are given a rational parametrization of a surface $\mathscr{S}$ as in (1.1). We aim at finding a matrix representation for $\mathscr{S}$. Note that we can in principle assume that $(f_0, \dots, f_3)$ are Laurent polynomials admitting negative exponents, but after multiplying them by a common monomial, we get a new rational parametrization of $\mathscr{S}$ defined by polynomials $f_i \in \mathbb{K}[s_1, s_2]$. We will then assume, without loss of generality, that we have a lattice polygon P which lies in the first orthant of $\mathbb{R}^2$ and contains the Newton polytopes of $f_0, \dots, f_3$.

We saw that in the curve case, it is always possible to find a square matrix representation. In the surface case, however, linear syzygies provide in general rectangular matrix representations and the implicit equation (raised to the degree of the map f) equals the great common divisor of the maximal minors (or the determinant of a complex). A recent paper by Busé (2014) presents a very interesting square matrix representation out of a matrix representation M when we work over the real numbers, by considering the square matrix MM^t. This approach is natural because of the properties of the rank of a real matrix with respect to its singular value decomposition. The determinant of MM^t gives an implicit equation for $\mathscr{S}$ (in general, it gives F with multiplicity), which is moreover a sum of squares. As Busé observes, for complex matrices it would be enough to replace the transpose M^t by the conjugate transpose.

The paper (Chen et al., 2005) translated to the general case of (parametric) surfaces the method of moving lines developed for curves, by considering a so-called μ-basis. They showed that μ-bases always exist, they are a basis of the moving plane module of the rational surface, and that they are a basis of the moving surface ideal of the rational surface if the base points of the parametrization are local complete intersections. As no bound on the degrees of the basis of syzygies is given, the associated implementation requires the use of Gröbner basis computations (except for special cases, as in Wang and Chen, 2012). By introducing a fixed support $2P$ for the syzygies in Algorithm 3.1, we are able to find the implicit equation under the general strategy explained in the introduction.

The use of quadratic relations (i.e. linear syzygies among the products $f_i f_j$ of any of two of the polynomials f_i defining the parametrization) was proposed to construct square matrices (Sederberg and Chen, 1995; Cox et al., 1998b; Cox, 2001; D'Andrea, 2001; Adkins et al., 2005). Khetan and D'Andrea generalized in 2006 (Khetan and D'Andrea, 2006) the method of moving quadrics to the toric case. These syzygies among the $f_i f_j$ are termed as "quadratic syzygies", even if they are linear relations on these products. This is why in the context of this paper, one uses the redundant name "linear syzygies" for the standard linear relations among the given polynomials f_i.

The choice of the quadratic syzygies is in general not canonical and the cost of computing syzygies is increased. Note that syzygies in $(f_0, \dots, f_3)$ and the implicit equation F have a common shape. Indeed, linear syzygies $h = (h_0, \dots, h_3)$ of degree ν correspond to polynomials $H(s, T) = \sum_{i=0}^{3} h_i(s) T_i$ such that $\sum_{i=0}^{3} h_i(s) f_i(s) = 0$, with $\deg(H)$ in the s variables equal to ν, and $\deg(H)$ in the T variables equal to 1. Also, quadratic syzygies of degree ν' correspond to polynomials $H(s, T) = \sum_{i \le j=0}^{3} h_{i,j}(s) T_i T_j$ such that $\sum_{i,j=0}^{3} h_{i,j}(s) f_i f_j(s) = 0$, with $\deg(H)$ in the s variables equal to ν', and $\deg(H)$ in the T variables equal to 2. The implicit equation (of degree D) is a polynomial $H(s, T) = \sum_{|\alpha| \le D} h_\alpha T^\alpha$ such that $\sum_\alpha h_\alpha f_1^{\alpha_1}(s) = 0$. Thus, $\deg(H)$ in the s variables equals 0, and $\deg(H)$ in T variables equals D. So to go from linear syzygies to the implicit equation, in some sense one has to play the game of lowering the degree in the s variables to 0 (which increases the degree in the T variables up to D).

We now present our main general algorithm to construct matrix representations of parametrized surfaces. Clearly, given any lattice polygon $P \subset \mathbb{R}^2$, $2P = \{p_1 + p_2, p_i \in P\}$ is again a lattice polygon. Moreover, in dimension two, any lattice polygon is normal, which is implicitly used in the algorithm. This means that $2P \cap \mathbb{Z}^2 = \{p_1 + p_2, p_i \in P \cap \mathbb{Z}^2\}$.

Algorithm 3.1. The following algorithm produces a matrix of polynomials in (T_1, T_2, T_3) out of the input polynomials $f_0, \dots, f_3$ in variables $s = (s_1, s_2)$:

- **INPUT:** A lattice polytope P and polynomials $(f_0(s), f_1(s), f_2(s), f_3(s))$ with no common factor and Newton polytopes $\mathcal{N}(f_i)$ contained in P.

- **STEP 1:** Let $(h_0^{(j)},\dots,h_3^{(j)})$, $j=1,\dots,N$, be a $\mathbb{K}$-basis of the syzygies $(h_0,\dots,h_3)$ with $\mathcal{N}(h_i)\subset 2P$.
- **STEP 2:** Represent the syzygies as linear forms $L_j = h_0^{(j)}T_0+\cdots+h_3^{(j)}T_3$. Write $h_i^{(j)} = \sum_{\beta\in 2P\cap\mathbb{Z}^2} h_{i,\beta}^{(j)}s^\beta$ and switch:

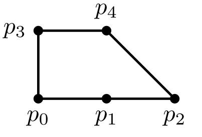
$$L_j=\sum_i h_i^{(j)}T_i=\sum_\beta\left(\sum_i h_{i,\beta}^{(j)}T_i\right)s^\beta.$$

- **OUTPUT:** The matrix M of linear forms $\ell_{j,\beta}:=\sum_i h_{i,\beta}^{(j)}T_i$.

We illustrate the steps in Algorithm 3.1 in the following example.

Example 3.2. Let P be the lattice polygon with vertices $(0,0)$, $(0,1)$, $(2,0)$ and $(1,1)$, with lattice points $p_0,\dots,p_4$, as in the figure. We consider the following four polynomials with support in P, where we denote $s:=(s_1,s_2)$, and given $p:=(i,j)$ we write $s^p:=s_1^is_2^j$:

$$\begin{aligned}
f_0&=1+3s_1+s_1^2+2s_2+s_1s_2=s^{p_0}+3s^{p_1}+s^{p_2}+2s^{p_3}+s^{p_4},\\
f_1&=5s^{p_0}-s^{p_1}-s^{p_2}+2s^{p_3}-s^{p_4},\\
f_2&=7s^{p_0}+3s^{p_1}+2s^{p_2}+6s^{p_3}+3s^{p_4},\\
f_3&=11s^{p_0}+0s^{p_1}+4s^{p_2}+3s^{p_3}+5s^{p_4}.
\end{aligned}$$



To compute the syzygies in Step 1, we consider the morphism $(a_0,a_1,a_2,a_3)\mapsto\sum_i a_if_i$, where a_i are polynomials with support in $2P$. Let B be the matrix of this map in the monomials bases. Since $2P$ has 12 lattice points and $\sum_i a_if_i$ has support in $3P$, which has 22 lattice points, then B is a *Sylvester* matrix of size 22×48. It can be easily checked that B is full ranked (same as for generic polynomials). Thus, the kernel of B has dimension $N=48-22=26$, which is the number of linearly independent syzygies.

To construct the matrix M, assume that we choose as our first syzygy the following 4-tuple of polynomials $(h_0^{(1)},\dots,h_3^{(1)})$ with $\mathcal{N}(h_i^{(1)})\subset 2P$:

$$\begin{aligned}
h_0^{(1)}&=-196s^{2p_0}+504s^{p_0+p_1}-257s^{p_0+p_2}+672s^{p_0+p_3}+234s^{p_0+p_4},\\
h_1^{(1)}&=-237s^{p_0+p_2}+420s^{p_0+p_3}-168s^{p_0+p_4},\\
h_2^{(1)}&=28s^{2p_0}+10s^{p_0+p_2}-364s^{p_0+p_3}+226s^{p_0+p_4},\quad\text{and}\\
h_3^{(1)}&=-216s^{p_0+p_4}.
\end{aligned}$$

We consider $L_1=h_0^{(1)}T_0+\cdots+h_3^{(1)}T_3$ and we write

$$\begin{aligned}
L_1={}&(-196T_0+28T_2)s^{2p_0}+(504T_0)s^{p_0+p_1}+(-257T_0-237T_1+10T_2)s^{p_0+p_2}\\
&+(672T_0+420T_1-364T_2)s^{p_0+p_3}+(234T_0-168T_1+226T_2-216T_3)s^{p_0+p_4},
\end{aligned}$$

which gives the first column of the 26×12-matrix M (computed with the computer-algebra system *Macaulay2*, Grayson and Stillman)

```
| -196T_0+28T_2                 0                             0                             ...
| 504T_0                        -196T_0+28T_2                 0                             ...
| -257T_0-237T_1+10T_2          504T_0                        -196T_0+28T_2                 ...
| 672T_0+420T_1-364T_2          0                             0                             ...
| 234T_0-168T_1+226T_2-216T_3   672T_0+420T_1-364T_2          0                             ...
| 0                             -257T_0-237T_1+10T_2          504T_0                        ...
| 0                             234T_0-168T_1+226T_2-216T_3   672T_0+420T_1-364T_2          ...
| 0                             0                             -257T_0-237T_1+10T_2          ...
| 0                             0                             234T_0-168T_1+226T_2-216T_3...
```

```
| 0                          0                          0                          ...
| 0                          0                          0                          ...
| 0                          0                          0                          ...

                         12                         26
Matrix (QQ[T , T , T , T ])   <--- (QQ[T , T , T , T ])
            0   1   2   3              0   1   2   3
```

The columns of this matrix M are given by a choice of a basis of syzygies with support in $2P$. The corresponding linear forms $L_1, \dots, L_N$ are known as the *moving planes* defining the surface parametrized by $f_1, \dots, f_3$. The associated rational map f has $\deg(f) = 1$. It can be checked that the common factor of any maximal minor of M is the degree 3 implicit equation of the closed image of f:

$$F = 2643T_0^3 + 2905T_0^2T_1 + 1345T_0T_1^2 + 91T_1^3 - 8T_0^2T_2 - 444T_0T_1T_2 + 284T_1^2T_2 + \cdots,$$

as asserted by Theorem 3.3 below.

Note that we have to write the lattice points in $2P$ as a sum of two points in P, but in general there is not a unique way of doing this. In our example, for instance, $p_0 + p_2 = p_1 + p_1$, so a choice was made. In fact, it is possible to make a coherent choice in general with the use of weight vectors, but any choice will work since in the quotient ring A defined in (4.7) below, it holds that $X_0.X_2$ and X_1^2 are identified.

We now state our main result. The proof will be given in Section 4.

Theorem 3.3. *Given $(f_0(s), f_1(s), f_2(s), f_3(s))$ with no common factor, with Newton polytopes contained in P and satisfying Hypotheses 4.4 below, Algorithm 3.1 computes a presentation matrix of the implicit equation of the rational map f. That is, the rank of the matrix M drops precisely when evaluated at the points in the closure of the image of f.*

Moreover, the implicit equation F can be computed (up to multiplication by a nonzero constant) as

$$F^{\deg(f)} = \gcd(\textit{maximal minors of } M). \tag{3.1}$$

The main ingredient for the validity of Algorithm 3.1 to give a matrix representation is the choice $\mathbf{2}P \cap \mathbb{Z}^2$ of the support of the linear syzygies. Again, the "instant" elimination is done in STEP 2, where the s variables give the monomial basis which is used to compute the matrix M (and thus they disappear from the output!).

In fact, Algorithm 3.3 can be run without checking Hypotheses 4.4. We point out in Remark 4.7 the possible outcomes. The general algorithm can be refined using Theorem 11 in Botbol et al. (2009).

Theorem 3.4. *Assume f satisfies the hypotheses of Theorem 3.3. If the lattice polygon P can be written as a multiple $P = dP'$ of another lattice polygon P'* without interior lattice points, *then we can consider in* **STEP 1** *of Algorithm 3.1 syzygies $(h_0, \dots, h_3)$ with smaller support $N(h_i)$ contained in $(2d-1)P'$ (which is strictly contained in $2P$), and the* **OUTPUT** *will still be a matrix representation for f. Moreover, in case P' is the unit simplex, it is enough to consider syzygies with support inside $(2d-2)P'$.*

We then have the following comparison between the general syzygy method and the naive linear algebra method described in Section 2.

Remark 3.5. Assume that P is the triangle of size d. Then, as it is enough to consider syzygies of degree $2d-2$, they can be found by solving a linear system on $4\binom{2d}{2}$ variables with $\binom{3d}{2}$ equations. That is, both sizes, as well as the vector space dimension of the space of syzygies in this degree, are quadratic in d. The matrix M has then a number of rows quadratic in d. The number of its columns equals $\binom{2d}{2}$, again quadratic in d. Comparing with the sizes in Lemma 2.3, which are of degree 6 in d, we observe that the syzygy method is a great improvement on the naive linear algebra method!

The same improvement occurs for any lattice polygon P. Using (2.1), we see that syzygies with support in $2P$ can be obtained by solving a system with approximately $9\,\mathrm{vol}(P)$ equations in $16\,\mathrm{vol}(P)$ variables and both row and column sizes of the matrix representation M are of the order of $\mathrm{vol}(P)$ and not of its cube, as in Lemma 2.3.

3.3. The bihomogeneous case and beyond

As we have mentioned, the main motivation for the implicitization problem comes from Computer Aided Geometric Design and geometric modeling. In this area, bihomogeneous surfaces (corresponding to rectangular support P) are known as tensor product surfaces, and they play a central role, in particular the Bézier surfaces. Quoting Dietz (1998): "*In current CAD systems tensor product surface representations with their rectangular structure are a de facto standard*". These surfaces (called *NURBS*) are given by pieces of parametrized surfaces cut by curves. So, it is necessary to control the location of the parameter, which can be achieved by computing the kernel of the matrix representation we give, as explained in Busé (2014).

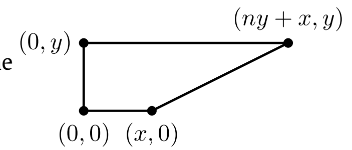
Due to the nature of the base locus of the parametrization, many of the current geometric modeling systems do not satisfy the hypotheses to be detailed in Hypotheses 4.4 needed for Theorem 3.3 to hold, if considered as homogeneous polynomials (with P an equilateral triangle). But if we use a rectangle P as the input in Algorithm 3.1, it is possible to get a full-ranked matrix representation by Theorem 3.3. In this bihomogeneous case, the detailed study of regularity in Botbol and Chardin (in press) allows to get the following improvement in the support of the proposed linear syzygies in STEP 1 of Algorithm 3.1: it is enough that the support of these syzygies is contained in a polygon obtained by only enlarging the rectangle support P of the input polynomials (approximately) to its double in *only* the horizontal *or* the vertical direction, instead of considering syzygies with support in (the lattice points of) $2P$.

A more general result can be obtained for bigraded toric surfaces, and in particular for lattice polygons defining a Hirzebruch surface, that is, for Hirzebruch quadrilaterals $H_{a,b,n}$ with vertices $(0,0), (a,0), (0,b)$ and $(a+nb, b)$, for any $a, b, n \in \mathbb{N}$. We state this extension in Theorem 3.8 below.

Algorithm 3.6. Take as **INPUT** a Hirzebruch lattice polygon $P = H_{a,b,n}$ and bivariate polynomials (f_0, f_1, f_2, f_3) with Newton polygons contained in $H_{a,b,n}$, and which satisfy the Hypotheses 4.8. Run Algorithm 3.1 with the following modification: in **STEP 1** consider a basis of syzygies with support in the smaller lattice quadrilaterals $H_{2a-1,b-1,n}$ (or $H_{a-1,2b-1,n}$ instead). The **OUTPUT** is the corresponding matrix M of linear forms.

In most cases, it is convenient to consider syzygies with support in $H_{2a-1,b-1,n}$ rather than in $H_{a-1,2b-1,n}$ since the first one has less lattice points.

In general, a Hirzebruch lattice polygon $H_{x,y,n}$ has the shape in the diagram on the right.



Remark 3.7. Note that for $n = 0$, $H_{a,b,0}$ is the standard lattice rectangle with vertices in $(0,0), (a,0), (0,b), (a,b)$, and thus Algorithm 3.6 works in particular in a standard bihomogeneous setting.

Theorem 3.8. *Given $(f_0(s), f_1(s), f_2(s), f_3(s))$ with no common factor, with Newton polytopes contained in $H_{a,b,n}$ and satisfying Hypotheses 4.8 below, Algorithm 3.6 computes a presentation matrix of the implicit equation of the rational map f. That is, the rank of the output matrix M drops precisely when evaluated at the points in the closure of the image of f.*

Moreover, the implicit equation F can be computed (up to multiplication by a nonzero constant) as

$$F^{\deg(f)} = \gcd(\textit{maximal minors of } M). \tag{3.2}$$

The proof of Theorem 3.8 will be also given in Section 4.

4. The hypotheses via toric geometry and the proofs of our main results

In this section we will recall a minimum of theoretical tools from toric geometry in order to state the hypotheses needed for Theorems 3.3 and 3.8 and to give their proofs. The main homological commutative algebra tools that are the core of the proofs are recalled in Appendix A.

We refer to Cox (2003b), Fulton (1993), Cox et al. (2011) and Gel'fand et al. (1994, Chs. 5&6) for the general notions, and to Khetan and D'Andrea (2006, §2), Botbol et al. (2009), Botbol (2011a) for applications to the implicitization problem. Any reader only interested in the application of Algorithm 3.1 or its bihomogeneous (toric) refinement given in Algorithm 3.6 can skip this section.

As usual, we denote by $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$ the multiplicative group of units of $\mathbb{K}$. The first observation is that we can equivalently consider our parametrization (1.1) as a map $\tilde{f} : \mathbb{K}^2 \dashrightarrow \mathbb{P}^3(\mathbb{K})$ or $\tilde{f} : (\mathbb{K}^*)^2 \dashrightarrow \mathbb{P}^3(\mathbb{K})$ with image inside 3-dimensional projective space, and domain a dense open set U in affine space $\mathbb{K}^2$ or the torus $(\mathbb{K}^*)^2$ over $\mathbb{K}$, given by

$$s \mapsto (f_0(s) : f_1(s) : f_2(s) : f_3(s)), \tag{4.1}$$

for any $s \in U$, and we have the commutative diagram

$$\begin{tikzcd} (\mathbb{K}^*)^2 \arrow[r, dashed, "f"] \arrow[dr, dashed, "\tilde{f}"'] & \mathbb{K}^3 \arrow[d, hook, "\iota"] \\ & \mathbb{P}^3. \end{tikzcd} \tag{4.2}$$

In fact, if F is the implicit equation of the (closure of the) image of f, the (closure of the) image of $\tilde{f}$ is the closure of $\mathscr{S}$ under the standard embedding $\mathbb{K}^3 \hookrightarrow \mathbb{P}^3(\mathbb{K})$ and its equation is the homogenization of F.

Similarly, we can consider our rational parametrization from any algebraic variety which contains the domain of f as a *dense* subset. We will choose embedded or abstract compact toric varieties to get a degree or multidegree notion that will allow us to get homological arguments to "bound" the support of the syzygies in Theorems 3.3 and 3.8.

4.1. Toric embeddings

Let $\tilde{f}$ be a rational map as in (4.1). The base points of the parametrization are the common zeros of $f_0, \ldots, f_3$, that is, the points where the map is not defined. We assume that $\tilde{f}$ is a generically finite map onto its image and hence it parametrizes an irreducible surface $\mathscr{S} \subset \mathbb{P}^3$. We also assume without loss of generality that $\gcd(f_0, f_1, f_2, f_3) = 1$, which means that there are only finitely many base points.

Let $P \subset \mathbb{R}^2$ be a lattice polygon with $m+1$ lattice points, which contains the Newton polygons $\mathcal{N}(f_0), \ldots, \mathcal{N}(f_3)$. Write $P \cap \mathbb{Z}^2 = \{p_0, \ldots, p_m\}$. The polygon P determines a projective toric surface $\mathscr{T}_P \subseteq \mathbb{P}^m$ as the closed image of the embedding

$$(\mathbb{K}^*)^2 \xrightarrow{\rho} \mathbb{P}^m$$
$$(s_1, s_2) \mapsto (\ldots : s^{p_i} : \ldots)$$

where $i = 0, \ldots, m$. For example, the unit triangle with vertices $(0, 1)$, $(1, 0)$ and $(0, 0)$ (or any lattice translate of it) corresponds to $\mathbb{P}^2$, and any lattice rectangle gives a Segre–Veronese projective embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, which are special cases of toric embeddings.

Example 4.1. Assume P is the unit square, with $m+1=4$ integer points:

$$p_0=(0,0),\, p_1=(1,0),\, p_2=(0,1),\, p_3=(1,1).$$

A polynomial f_i with Newton polytope contained in P looks like

$$f_i(s)=a_{(0,0)}+a_{(1,0)}s_1+a_{(0,1)}s_2+a_{(1,1)}s_1s_2. \tag{4.3}$$

We take 4 new variables $(X_0:X_1:X_2:X_3)$ as the homogeneous coordinates in $\mathbb{P}^3$. The toric variety $\mathscr{T}_P$ is the projective variety in $\mathbb{P}^3$ cut out by the relation $X_0X_3-X_1X_2=0$. This binomial equation comes from the primitive affine relation $p_0+p_3=p_1+p_2$, which implies the multiplicative relation $s^{p_0}s^{p_3}=s^{p_1}s^{p_2}$ between the monomials with these exponents. The coordinate ring of $\mathscr{T}_P$ is the quotient ring $\mathbb{K}[X_0,\dots,X_3]/\langle X_0X_3-X_1X_2\rangle$.

In general, we will call $(X_0:\cdots:X_m)$ the homogeneous coordinate variables in $\mathbb{P}^m$. Write $P\cap\mathbb{Z}^2=\{p_0,\dots,p_m\}$. We set one variable X_i for each integer point p_i in P and we record multiplicatively (by binomial equations) the affine relations among these points. These binomials generate the toric ideal $J_P=J(\mathscr{T}_P)$, which defines the variety $\mathscr{T}_P\subset\mathbb{P}^m$. To each

$$f_i(s)=\sum_{i=0}^{m}a_{p_i}s^{p_i}, \tag{4.4}$$

we associate the homogeneous linear form

$$g_i(s)=\sum_{i=0}^{m}a_{p_i}X_i. \tag{4.5}$$

For instance, in Example 4.1, the polynomial f_i gets translated to

$$g_i(X_0,\dots,X_3)=a_{(0,0)}X_0+a_{(1,0)}X_1+a_{(0,1)}X_2+a_{(1,1)}X_3,$$

and over $\mathscr{T}_P$, we have the relation $X_0X_3-X_1X_2=0$.

The rational map $\tilde f$ factorizes through $\mathscr{T}_P$ in the following way

$$\begin{tikzcd} (\mathbb{K}^*)^2 \arrow[r, dashed, "\tilde f"] \arrow[d, hook, "\rho"] & \mathbb{P}^3 \\ \mathscr{T}_P \arrow[ur, dashed, "g"'] & \end{tikzcd} \tag{4.6}$$

where $g=(g_0:g_1:g_2:g_3)$ is given by four homogeneous linear polynomials g_0,g_1,g_2,g_3 in $m+1$ variables. Thus, we have a new homogeneous parametrization g of the closed image of $\tilde f$ from $\mathscr{T}_P$. The polynomials g_i generate an ideal I in the coordinate ring

$$A=\mathbb{K}[X_0,\dots,X_m]/J_P \tag{4.7}$$

of $\mathscr{T}_P$. This ideal I defines the structure of the base locus in $\mathscr{T}_P$.

The embedding $\rho:(\mathbb{K}^*)^2\to\mathbb{P}^3$ provides a $\mathbb{Z}$-grading in the coordinate ring A of $\mathscr{T}_P$, which is used to study the map g with the tools recalled in Appendix A.

4.2. Abstract toric varieties and Cox rings

Given a lattice polygon P, one can also associate to it an abstract compact toric variety X_P that naturally contains the torus $(\mathbb{K}^*)^2$ as a dense open set (via the map we call j below), adjoining a torus invariant divisor to each edge of P. We refer the reader to Cox et al. (1998a, 2011) for the theory and details.

The map $\tilde f$ also defines a rational map $\overline{f}$ that makes the following diagram commutative:

$$\begin{tikzcd}
X_P \arrow[dr, dashed, "\overline{f}"] & \\
(\mathbb{K}^*)^2 \arrow[u, hook, "j"'] \arrow[r, dashed, "\tilde{f}"'] \arrow[d, hook, "\rho"'] & \mathbb{P}^3 \\
\mathscr{T}_P \arrow[ur, dashed, "g"'] &
\end{tikzcd} \tag{4.8}$$

Example 4.2. Assume P is the unit square, with $N = 4$ edges: the segments $E_1 = [(0,0),(1,0)]$, $E_2 = [(0,0),(0,1)]$, $E_3 = [(0,1),(1,1)]$, and $E_4 = [(1,0),(1,1)]$. The respective inner normal vectors $\eta_1 = (0,1)$, $\eta_2 = (1,0)$, $\eta_3 = (0,-1)$, $\eta_4 = (-1,0)$ satisfy the linear relations $\eta_1 + \eta_3 = 0$, $\eta_2 + \eta_4 = 0$, which give rise to two homogeneities. We introduce four associated variables $Y = (Y_1, \dots, Y_4)$. A polynomial f_i with Newton polytope P as in (4.3) defines a bihomogeneous polynomial (in (Y_1, Y_3) and (Y_2, Y_4)):

$$\overline{f}_i(Y) = a_{(0,0)} Y_3 Y_4 + a_{(1,0)} Y_1 Y_4 + a_{(0,1)} Y_2 Y_3 + a_{(1,1)} Y_1 Y_2.$$

These polynomials $\overline{f}_i$ define the map $\overline{f} = (\overline{f}_0 : \dots, \overline{f}_3)$.

The main motivation for this change of perspective comes again from the commutative algebra results needed for the proof of Theorems 3.3 and 3.6. The Cox ring of X_P is endowed with a more natural multigrading, which is finer than the grading obtained via the embedded projective variety $\mathscr{T}_P$. Also, this point of view has an impact in the computations, as the number of variables to eliminate is smaller (one for each edge of P, instead of one for each lattice point in P). In our small Example 4.2, there are four edges and four lattice points, but the number of edges can remain constant while the number of lattice points goes to infinity.

4.3. Precise hypotheses and proof of Theorem 3.3

In this subsection we detail the precise hypotheses that ensure the validity of Theorem 3.3 and we prove it, based on results in Botbol et al. (2009). We first need to recall a few standard definitions from commutative algebra.

Definition 4.3. Given (nonzero) homogeneous polynomials $(g_0, \dots, g_3)$, defining a rational map $g : \mathscr{T}_P \dashrightarrow \mathbb{P}^3$ as in (4.6), a point $p \in \mathscr{T}_P$ is a base point of g if it is a common zero set of $g_0, \dots, g_3$, that is, if p is a zero of the ideal $I \subset A$ in $\mathscr{T}_P$.

Let $p \in \mathscr{T}_P$ be a base point of g. The local ring of p is the ring $A_p = \{h_1/h_2,\ h_i \in A,\ h_2(p) \neq 0\}$, with the natural operations induced from A (in turn, naturally induced from the polynomial ring). Let I_p be the ideal generated by (the classes of) $g_0, \dots g_3$ in A_p. We say that p is a local complete intersection base point if I_p can be generated by only 2 elements. We say that p is an almost complete intersection base point if I_p can be generated with 3 elements.

We have similar definitions for the map $\tilde{f} : (\mathbb{K}^*)^2 \dashrightarrow \mathbb{P}^3$.

For a given lattice polygon P, here are the hypotheses we need in terms of g:

Hypotheses 4.4. There are only finitely many base points of g on $\mathscr{T}_P$ which are local complete intersections.

We cannot easily find hypotheses on f equivalent to Hypotheses 4.4. Given a lattice polygon P, an edge E of P, and a polynomial f_i with $\mathcal{N}(f_i)$ contained in P as in (4.4), the restriction $f_{i|_E}$ of f_i to E is defined as the sub-sum of the monomials with exponents p_i in E. We have the following partial translation.

Proposition 4.5. *Let f, $\mathscr{T}$ and g be as in (4.6). Then*

(1) *There are only finitely many base points of g on $\mathscr{T}_P$ if and only if there are only finitely many isolated base points of f in the torus and for each edge E of P, at least one of the restrictions $f_{i|E}$ is nonzero.*
(2) *If g has finitely many isolated base points on $\mathscr{T}_P$ which are local complete intersections, then the base points of f in the torus are local complete intersections.*

Proof. The map ρ defines an isomorphism between $(\mathbb{K}^*)^2$ and its image (which is an open dense subset of $\mathscr{T}_P$), sending a base point q of $\tilde{f}$ (that is, a point where $f_0(q) = \cdots = f_3(q) = 0$) to a base point $p = \rho(q)$ of g, and reciprocally, any base point p of g in $\rho(\mathbb{K})^2$ is the image of a base point q of f. Moreover, we have an isomorphism between the ideal generated by $f_0, \ldots, f_3$ at q and I_p. Any base point of g outside the image of ρ cannot be seen in the torus. But these points are either the fixed torus points corresponding to the finitely many vertices of P, or they lie in the torus of the toric divisor D_E in $\mathscr{T}_P$ associated to an edge E of P. As D_E has dimension 1, there are finitely many solutions as long as at least one of the $f_{i|E}$ is nonzero. $\square$

The following example shows that the converse of item (2) in Proposition 4.5 does not hold.

Example 4.6. Consider the sparse parametrization with 6 monomials: $(f_0, f_1, f_2, f_3) = (st^6 + 2, st^5 - 3st^3, st^4 + 5s^2t^6, 2 + s^2t^6)$. Then, f has no base points in the torus. But if we consider their standard homogenizations to degree 8 polynomials (that is, we take P equal to 8 times the standard unit simplex in the plane), the corresponding homogeneous polynomials $g_0, \ldots, g_3$ have one base point "at infinity" which is not even an almost locally complete intersection.

We now give the proof of Theorem 3.3.

Proof of Theorem 3.3. Given $(f_0(s), f_1(s), f_2(s), f_3(s))$ with no common factor and Newton polytopes contained in P, the corresponding polynomials g_i associated to f_i are homogeneous of degree $d = 1$ and satisfy Hypotheses 4.4.

From Botbol et al. (2009, Cor. 14) one has that for $d = 1$, the matrix of coefficients of a K-basis of the module of Syzygies of g in any degree $\nu \geq 2$ with respect to a K-basis of the graded piece A_ν of A, is a matrix representation for the closure of the image of f, which equals the closure of the image of g.

In particular, we can take $\nu = 2$. In STEP 1 of Algorithm 3.1, the syzygies $(h_0^{(j)}, \ldots, h_3^{(j)})$ for $j = 1, \ldots, N$ with $\mathcal{N}(h_i^{(j)}) \subset 2P$ for all i, j, provide a $\mathbb{K}$-basis of the module of syzygies of g in degree 2, since classes of monomials of degree 2 in A correspond to monomials in the s variables with exponents in $2P$.

Equality (3.1) follows from Theorem 13 in Botbol et al. (2009). $\square$

In principle, given a rational map $\tilde{f}$, we could take any lattice polygon P containing the union $\mathcal{N}(f)$ of Newton polytopes of $f_0, \ldots, f_3$. Note that the hypothesis that f is generically finite implies that $\mathcal{N}(f)$ is two-dimensional. Taking P strictly containing $\mathcal{N}(f)$ will increase the number of exponents and will in general produce bad behavior of g at the fixed points in $\mathscr{T}_P$ corresponding to the vertices of P which do not lie in $\mathcal{N}(f)$.

Remark 4.7. We can check algorithmically if $f_0, \ldots, f_3$ have finitely many solutions over $(\overline{\mathbb{K}}^*)^2$ and if for any edge E at least one of the restrictions $f_{i|E}$ is nonzero. So, by Proposition 4.5, we can check whether g has finitely many base points in $\mathscr{T}_P$.

Assume the dimension of the base locus of g is zero. As we remarked in Example 4.6, even if we could check the local behavior of the base points of f in the torus, this would not imply the satisfiability of Hypotheses 4.4. But what if we **don't check** this and run Algorithm 3.1? ... **nothing bad**!

We then check whether the output matrix M has full rank:

- If the rank of M is not maximal, then there is at least one base point p of g which is not an almost local complete intersection. In this case, we cannot get the implicit equation, but we get a certificate of the bad behavior of the base locus (without computing it).
- If the rank of M is maximal, it may happen that the its rank drops when evaluated at points outside $\mathscr{S}$ due to the existence of an almost complete intersection but non-complete intersection base point. In this case, the greatest common divisor of the maximal minors of M would have irreducible factors other than the implicit equation F. In fact, the existence of other irreducible factors is equivalent to the fact that there exists a base point which is an almost local complete intersection but not a local complete intersection.

4.4. *The hypotheses and proof of Theorem 3.8*

In this subsection we detail the precise hypotheses that ensure the validity of Theorem 3.8 and we prove it, based on results in Botbol (2011b).

Given P and f, here are the hypotheses we need in terms of the map $\overline{f}$ in (4.8):

Hypotheses 4.8. There are only finitely many base points of $\overline{f}$ on X_P which are local complete intersections.

Again, we cannot easily find hypotheses on f equivalent to Hypotheses 4.4, since good algebraic behavior of the base points in the torus does not imply the same behavior for the possible base points of $\overline{f}$ at the invariant divisors in X_P associated with the edges of P.

Proposition 4.9. *Let f, X_P and $\overline{f}$ as in Hypotheses 4.8. Then*

(1) *There are only finitely many isolated base points of $\overline{f}$ on X_P if and only if there are only finitely many isolated base points of f in the torus and for each edge of P, at least one of the restrictions of the f_i is nonzero.*
(2) *If $\overline{f}$ has finitely many base points on X_P which are local complete intersections, then the base points of f in the torus are local complete intersections.*

We next give the proof of Theorem 3.8.

Proof of Theorem 3.8. By hypothesis, there are only finitely many isolated base points of $\overline{f}$ on the toric variety X_P associated with $P := H_{a,b,n}$, which are local complete intersections. There are four primitive inner normal vectors of P: $\eta_1 = (0,1)$, $\eta_2 = (0,1)$, $\eta_3 = (-1,0)$, $\eta_4 = (-1,n)$, which satisfy the linear relations $\eta_3 = -\eta_1$, $\eta_4 = n\eta_2 - \eta_1$. So any multidegree ν can be described by a "bidegree" (ν_1,ν_2) given by the degrees with respect to the first normals and which fixes (up to translation) the associated polytope P_ν with the same normals as P. Thus, by Botbol (2011b, Thm. 5.5) the matrix of coefficients of a K-basis of the module of Syzygies of $\overline{f}$ in any bidegree (ν_1,ν_2) with $\nu_1 \geq 2a-1$ and $\nu_2 \geq b-1$ (or $\nu_1 \geq a-1$, $\nu_2 \geq 2b-1$),[2] with respect to a K-basis of the bigraded piece (ν_1,ν_2) of the Cox ring of X_P, is a matrix representation for the closure of the image $\mathscr{S}$ of f (which equals the closure of the image of $\overline{f}$).

Taking $(\nu_1,\nu_2) = (2a-1, b-1)$ one has that in STEP 1 a basis of syzygies $(h_0^{(j)},\dots,h_3^{(j)})$, for $j = 1,\dots,N$ with all $\mathcal{N}(h_i^{(j)})$ with support in the quadrilateral $H_{2a-1,b-1,n}$, provides a $\mathbb{K}$-basis of the module of syzygies of $\overline{f}$ in bidegree $(2a-1,b-1)$. Hence, the matrix M of coefficients of such syzygies obtained in STEP 2 gives a representation matrix for $\mathscr{S}$.

Equality (3.2) also follows from Theorem 5.5 in Botbol (2011b). $\square$

---

[2] The choice of the bidegree is less obvious than in the graded case. For further details, see definition of $\mathfrak{R}_B(\gamma)$ in Botbol (2011b, Thm. 5.5), or the analysis of the bidegree in the standard bigraded case in Botbol (2011b, Sec. 7.1).

5. Examples

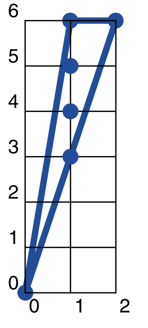
This section consists of four examples which highlight the usefulness of our approach. Example 5.1 is taken from a case studied in Botbol et al. (2009, Ex. 18) of a sparse parametrization where projective implicitization does not work due to the nature of the base locus of the map, but Algorithm 3.1 is applicable with a right choice of polygon P read from the monomials of the input polynomials. In Example 5.2 we show how the method in Algorithm 3.1 works for a parametrization given by fewnomials of high degree, where classical resultant tools fail due to the computational complexity. In Example 5.3, classical resultant tools fail because of the existence of a base point in the torus. Finally, in Example 5.4 we compare the methods in Algorithms 3.1 and 3.6.

5.1. A very sparse parametrization

Consider the sparse parametrization with 6 monomials given in Botbol et al. (2009, Ex. 18): $(f_0, f_1, f_2, f_3) = (st^6 + 2, st^5 - 3st^3, st^4 + 5s^2t^6, 2 + s^2t^6)$. The matrix representation can be computed using the package *MatrixRepToric.m2* (Botbol and Dohm, 2010) in the computer algebra software *Macaulay2* (Grayson and Stillman).

One first defines the map f given by polynomials in the ring $S = \mathbb{Q}[s,t]$ (note that for easiness of typing, we call the variables (s,t) instead of (s_1, s_2)):

```
S = QQ[s,t];
f = {s*t^6+2, s*t^5-3*s*t^3, s*t^4+5*s^2*t^6, 2+s^2*t^6};
```



Consider P the lattice triangle with vertices $(0,0)$, $(1,6)$ and $(2,6)$.
One can compute P by the command:

```
P = polynomialsToPolytope L
```

The lattice-points of P can be computed using the auxiliary Macaulay2 package `Polyhedra` as:

```
latticePoints P
```

By taking syzygies with support in $2P$, on gets a matrix representation of size 17×34. The greatest common divisor of the 17-minors of this matrix is the homogeneous implicit equation of the surface:

$$\begin{aligned}
&2809T_0^2T_1^4 + 124\,002T_1^6 - 5618T_0^3T_1^2T_2 + 66\,816T_0T_1^4T_2 + 2809T_0^4T_2^2 \\
&\quad - 50\,580T_0^2T_1^2T_2^2 + 86\,976T_1^4T_2^2 + 212T_0^3T_2^3 - 14\,210T_0T_1^2T_2^3 + 3078T_0^2T_2^4 \\
&\quad + 13\,632T_1^2T_2^4 + 116T_0T_2^5 + 841T_2^6 + 14\,045T_0^3T_1^2T_3 - 169\,849T_0T_1^4T_3 \\
&\quad - 14\,045T_0^4T_2T_3 + 261\,327T_0^2T_1^2T_2T_3 - 468\,288T_1^4T_2T_3 - 7208T_0^3T_2^2T_3 \\
&\quad + 157\,155T_0T_1^2T_2^3T_3 - 31\,098T_0^2T_2^3T_3 - 129\,215T_1^2T_2^3T_3 - 4528T_0T_2^4T_3 \\
&\quad - 12\,673T_2^5T_3 - 16\,695T_0^2T_1^2T_3^2 + 169\,600T_1^4T_3^2 + 30\,740T_0^3T_2T_3^2 \\
&\quad - 433\,384T_0T_1^2T_2T_3^2 + 82\,434T_0^2T_2^2T_3^2 + 269\,745T_1^2T_2^2T_3^2 + 36\,696T_0T_2^3T_3^2 \\
&\quad + 63\,946T_2^4T_3^2 + 2775T_0T_1^2T_3^3 - 19\,470T_0^2T_2T_3^3 + 177\,675T_1^2T_2T_3^3 \\
&\quad - 85\,360T_0T_2^2T_3^3 - 109\,490T_2^3T_3^3 - 125T_1^2T_3^4 + 2900T_0T_2T_3^4 \\
&\quad + 7325T_2^2T_3^4 - 125T_2T_3^5
\end{aligned}$$

We can set $T_0 = 1$ to get the affine equation.

The map g is computed with the following command:

```
g = teToricRationalMap f;
```

The matrix representation and the implicit equation are computed as follows:

```
M = representationMatrix (teToricRationalMap f,2);
implicitEq (L,2)
```

Notice that the 2 as second parameter in the computation of M is precisely the 2 in the support $2P$ of the syzygies. For a deeper understanding of the choice of the parameter 2, see Appendix A.

In the language of Section 4, the coordinate ring associated to $\mathscr{T}_P$ is $A = \mathbb{K}[X_0, \ldots, X_5]/J_P$, where $J_P = (X_3^2 - X_2X_4, X_2X_3 - X_1X_4, X_2^2 - X_1X_3, X_1^2 - X_0X_5)$. The parametrization g over $\mathscr{T}_P$ is given by $(g_0, g_1, g_2, g_3) = (2X_0 + X_5, 2X_0 + X_4, -3X_1 + X_3, X_2 + 5X_5)$. This matrix can be computed as the right-most map of the $\nu_0 = 2d = 2$ strand of a graded complex as explained in Appendix A. The method fails over $\mathbb{P}^2$ (i.e. $P =$ the triangle with vertices $(0, 0), (8, 0), (0, 8)$) due to the nature of the base locus. One can see this just by computing a matrix representation and verifying that it is not full-ranked.

5.2. Fewnomials with high degree

This example contributes to show how the method works fine for high degree fewnomials involved in the parametrization.

Consider the polynomials $(f_0, f_1, f_2, f_3) = (1 + st + s^{37}, s^7 + s^{47}, s^{37} + s^{59}, s^{61})$. Let $f : \mathbb{C}^2 \dashrightarrow \mathbb{C}^3$ be the parametrization that maps $(s, t) \mapsto (f_1/f_0, f_2/f_0, f_3/f_0)(s, t)$. The implicit equation of the closure of the image of f could be computed by eliminating the variables (s, t) as follows (using general elimination procedures based on Gröbner bases in Macaulay2):

```
R = QQ[s,t, x, y, z, w]
f0 = 1 + s*t + s^37; f1 = s^7 + s^47; f2 = s^37 + s^59; f3 = s^61
L1 = x*f1 - y*f0; L2 = x*f2 - z*f0; L3 = x*f3 - w*f0
eliminate ({s,t}, ideal(L1,L2,L3))
```

In a 2014 standard desktop computer this routine does not end after one hour of computation. We also tried the well implemented *eliminate* command in Singular (Decker et al., 2012), but with the same lack of answer after a couple of hours of computation.

By homogenizing with an auxiliary variable u we could try eliminate (s, t, u) using Macaulay resultant methods, but we easily figure out that the homogenized forms $\overline{L}_1, \overline{L}_2, \overline{L}_3$ vanish identically over the point $(s, t, u) = (0 : 1 : 0)$. This implies in particular that the Macaulay resultant $\mathrm{Res}_{(s,t,u)}(\overline{L}_1, \overline{L}_2, \overline{L}_3)$ is identically zero.

Finally one can compute the implicit equation (and matrix representation) by implementing the implicitization techniques described in this article. A not very efficient (but efficient enough) routine in Botbol (2010) gives the toric map g in less than 2 minutes and the desired matrix representation M in less than one more minute.

5.3. Fewnomials with base points in the torus

This example shows a case where classical resultants cannot be applied to compute the implicit equation, but the techniques in the paper can. Anyway, we recall that the aim of the matrix representations is to provide a better and more complete tool for representing a surface, and hence, the point presented with this example is just one extra advantage of the method.

Consider the following parametrization $(f_0, f_1, f_2, f_3) = (1 - ts, -ts^{36} + 1, -t(-s^{38} + t), s^{37} - t)$ given by four polynomials that define a parametrization $f : \mathbb{C}^2 \dashrightarrow \mathbb{C}^3$ that maps $(s, t) \mapsto (f_1/f_0, f_2/f_0, f_3/f_0)(s, t)$. The implicit equation cannot be computed by eliminating the variables (s, t) with classical resultants, because the point $(1, 1)$ is in the base locus. However, this fact does not imply any problem in Algorithm 3.1.

With Algorithm 3.6 with a rectangular $P = H_{38,2,0}$, it takes 0.058 seconds in a standard 2014 desktop computer to get a matrix representation M of the closure of the image of f. The size of M is 152×194 and the gcd of its maximal minors has degree 110.

5.4. Comparison with and without embedding

While Algorithm 3.1 holds with great generality, when dealing with polynomials with rectangular support (which can be interpreted as bihomogeneous polynomials), Algorithm 3.6 provides a smaller matrix representation.

Consider the following four polynomials $f_0, \ldots, f_3$:

$$
\begin{aligned}
f_0 &= 3s_1^2s_2 - 2s_1s_2^2 - s_1^2 + s_1s_2 - 3s_1 - s_2 + 4 - s_2^2,\\
f_1 &= 3s_1^2s_2 - s_1^2 - 3s_1s_2 - s_1 + s_2 + s_2^2 + s_2^2 + s_1^2s_2^2,\\
f_2 &= 2s_1^2s_2^2 - 3s_1^2s_2 - s_1^2 + s_1s_2 + 3s_1 - 3s_2 + 2 - s_2^2, \quad \text{and}\\
f_3 &= 2s_1^2s_2^2 - 3s_1^2s_2 - 2s_1s_2^2 + s_1^2 + 5s_1s_2 - 3s_1 - 3s_2 + 4 - s_2^2.
\end{aligned}
$$

The Newton polytope $P = \mathcal{N}(f)$ is the rectangle $\{(x, y) : 0 \le x, y \le 2)\}$. If we apply Algorithm 3.1 (as we illustrated in Example 3.2), we obtain a matrix representation M of size 25×51.

The associated toric variety X_P can be identified with the $(2, 2)$ Segre–Veronese embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^8$ (see Busé and Dohm, 2007; Botbol et al., 2009; Botbol, 2011a).

By means of Algorithm 3.6 we get a matrix representation M from a basis of linear syzygies of bidegree $(2.2 - 1, 2 - 1) = (3, 1)$. This matrix representation can be computed using the algorithm developed in Botbol (2010) and implemented in M2, as the matrix M_ν for bidegree $\nu = (3, 1)$, and one obtains a square 8×8-matrix. Its determinant equals the implicit equation F:

```
                  8             7                6  2           5  3          4  4
F = 63569053X  - 159051916X X  + 175350068X X  - 82733240X X  + 2363584X X  +  ...
             0              0 1              0 1            0 1           0 1
```

Notice that the matrix $M_{(3,1)}$ is considerably smaller than the 25×51-matrix M because instead of considering syzygies with support in the rectangle $2P = \{(x, y) : 0 \le x, y \le 4\}$, the syzygies are taken with support in the smaller rectangle $\{(x, y) : 0 \le x \le 3, 0 \le y \le 1\}$.

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Appendix A. Commutative algebra tools

This appendix is devoted to highlight the tools of homological commutative algebra and algebraic geometry that are needed to justify the validity of Algorithms 3.1 and 3.6, and to explain the choice of the support of the syzygies in STEP 2 which define the matrix representation of the parametrized surface.

Given P, the toric embedding $\rho : (\mathbb{K}^*)^2 \to \mathbb{P}^m$ in Section 4.1 provides a $\mathbb{Z}$-grading in the coordinate ring $A = \mathbb{K}[X_0, \ldots, X_m]/J_P$ in (4.7) of $\mathscr{T}_P$, which can be used to study the map g in Diagram (4.6) and its associated Rees and symmetric algebras, denoted by $\mathrm{Rees}_A(I)$ and $\mathrm{Sym}_A(I)$ respectively. Notice also that the graded ring A coincides with the affine semigroup ring of the lattice polytope P, which is *Cohen–Macaulay* and normal because P has dimension 2.

The grading in A plays a key role in the elimination process. The matrix representation M of Section 3.2 depends on a choice of degree ν, as was shown in Section 3.1 for the case of curves. The reason why ν needs to be considered is rather technical, and a complete explanation involves sheaf cohomology. From a more naive point of view, the implicit equation of the surface $\mathscr{S} := \overline{\mathrm{im}(g)}$ is written in the variables $T = (T_0, \dots, T_3)$ but depends on the algebraic relations among the polynomials g_i, which lie in A. Fixing a degree ν in A can be thought as *eliminating the variables of A, by hiding them in the monomial basis of the graded piece* A_ν. In turn, recall that the variables $X = (X_0, \dots, X_m)$ in A code monomials in the original s variables, with exponents in the lattice points in P.

More geometrically, consider the graph variety Γ of g where both group of variables X and T are involved. The elimination process can be understood geometrically as projecting Γ via π_2 in the following diagram

$$\begin{tikzcd}
\Gamma \subset \mathscr{T}_P \times \mathbb{P}^3 \arrow[d, "\pi_1"] \arrow[dr, "\pi_2"] & \\
\mathscr{T}_P \arrow[r, dashed, "g"'] & \mathbb{P}^3 \supset \overline{\mathrm{im}(g)} = \overline{\mathrm{im}(f)}.
\end{tikzcd}$$

In the correspondence between subvarieties of $\mathscr{T}_P \times \mathbb{P}^3$ and bigraded algebras, the inclusion of the graph $\Gamma \subset \mathscr{T}_P \times \mathbb{P}^3$ corresponds to the surjection $A[T_0, T_1, T_2, T_3] \twoheadrightarrow \mathrm{Rees}_A(I)$, the Rees algebra of the ideal I generated by $g_0, \dots, g_3$ over the coordinate ring A. The projection $\pi_2(\Gamma)$ corresponds to eliminating the variables X_i of $\mathrm{Rees}_A(I)$. We denote by $I(\pi_2(\Gamma))$ the defining ideal of $\pi_2(\Gamma) \subset \mathbb{P}^3$.

How to eliminate the X variables from $\mathrm{Rees}_A(I)$ algebraically? A standard procedure is to find a free graded presentation $F_1 \xrightarrow{M} F_2 \to \mathrm{Rees}_A(I) \to 0$ and a degree ν (in the X variables) such that the Fitting ideal $\mathfrak{F}(M_\nu)$ generated by the maximal minors of M_ν (in the graded strand $(F_1)_\nu \xrightarrow{M_\nu} (F_2)_\nu \to \mathrm{Rees}_A(I)_\nu \to 0$) computes $I(\pi_2(\Gamma))$. It happens that no universal way to compute such a free presentation is available, so the idea is to "approximate" $\mathrm{Rees}_A(I)$ by the (hopefully) similar algebra $\mathrm{Sym}_A(I)$ that admits such a universal resolution. These resolutions of the symmetric algebras are known as approximation complexes, they were introduced in Herzog et al. (1982, 1983a) and their application on elimination theory was done in Busé and Jouanolou (2003), Busé (2006). The last map of the approximation complex is the following in our case:

$$Z_1[T_0, T_1, T_2, T_3] \xrightarrow{M'} A[T_0, T_1, T_2, T_3] \to \mathrm{Sym}_A(I) \to 0,$$

where $Z_1 = \left\{(a_0, a_1, a_2, a_3) \in A^4 : \sum a_i g_i = 0\right\}$ is the first module of syzygies of g_0, g_1, g_2, g_3 and $M' = [T_0\ T_1\ T_2\ T_3]^t$, that is,

$$M' \cdot (a_0, a_1, a_2, a_3) := \sum a_i T_i.$$

The cokernel of M' is $\mathrm{Sym}_A(I) = A[T_0, T_1, T_2, T_3]/J$, where

$$J := \{\sum a_i T_i : a_i \in A[T_0, T_1, T_2, T_3] \text{ and } \sum a_i g_i = 0\}.$$

We can recognize the origin of the linear forms L_i in STEP 2 of our algorithms!

But there is a remaining question: which is the relation between $\mathrm{Rees}_A(I)$ and $\mathrm{Sym}_A(I)$? Which variety does $\mathrm{Sym}_A(I)$ define? Can we use $\mathfrak{F}(M'_\nu)$ to compute $\mathcal{I}_{\pi_2(\Gamma)}$ for some ν? The answer is that in case there are finitely many base points and for each base point p, the local I_p is a local complete intersection, then $\mathrm{Rees}_A(I)$ and $\mathrm{Sym}_A(I)$ define the same scheme in $\mathscr{T} \times \mathbb{A}^4$ (thus, in $\mathscr{T} \times \mathbb{P}^3$). As $\mathrm{Rees}_A(I)$ is $\mathfrak{m}$-torsion free, both algebras coincide module the $\mathfrak{m}$-torsion of $\mathrm{Sym}_A(I)$, $\mathrm{Rees}_A(I) \cong \mathrm{Sym}_A(I)/H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))$, where $\mathfrak{m}$ is the maximal ideal generated by $X_0, \dots, X_m$. Thus if I is a local complete intersection and ν is such that $H^0_{\mathfrak{m}}(\mathrm{Sym}_A(I))_\nu = 0$, then $\mathrm{Rees}_A(I)_\nu \cong \mathrm{Sym}_A(I)_\nu$. This happens for $\nu \geq \nu_0 := 2$ by Theorem 11 in Botbol et al. (2009) (in fact, that result also proves Theorem 3.4 as remarked before). In particular, in this case, $\mathfrak{F}(M'_\nu)$ computes $\mathcal{I}_{\pi_2(\Gamma)}$ for any $\nu \geq \nu_0$.

In fact, the condition of I being a local complete intersection can be relaxed to the condition of being locally an almost complete intersection. (i.e. I_p can be generated by 3 elements, for any p in the finite set $V(I)$). In this case, $\dim(\mathrm{Sym}_A(I)) = \dim(\mathrm{Rees}_A(I))$. Since there is always a surjective map $\mathrm{Sym}_A(I) \twoheadrightarrow \mathrm{Rees}_A(I)$ then $V(\mathrm{Sym}_A(I)) = \Gamma \cup U$, where U has the same dimension. In particular, $\pi_2(\Gamma) \cup \pi_2(U) = \pi_2(V(\mathrm{Sym}_A(I)))$. For $\nu \geq \nu_0$, $\mathrm{Sym}_A(I)$ is $\mathfrak{m}$-torsion free, and $\mathfrak{F}(M'_\nu)$ computes $I(\pi_2(V(\mathrm{Sym}_A(I)))$ for any $\nu \geq \nu_0$. So, the gcd H of the maximal minors of M'_ν contains the homogenization of the implicit equation F as a factor.

In the bigraded case of Hirzebruch surfaces, in particular in the standard bigraded case, the basic ideas are similar, but new technical details have to be managed in order to determine the bidegrees for which the torsion of the symmetric algebra vanishes. We refer the reader to Botbol (2011b) for the details and proofs.

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