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Dynamical sampling ☆

A. Aldroubi [a,*], C. Cabrelli [b,c], U. Molter [b,c], S. Tang [a]

[a] *Department of Mathematics, Vanderbilt University, Nashville, TN 37240-0001, USA*
[b] *Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina*
[c] *CONICET, Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina*

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Let $Y = \{f(i), Af(i), \dots, A^{l_i}f(i) : i \in \Omega\}$, where A is a bounded operator on $\ell^2(I)$. The problem under consideration is to find necessary and sufficient conditions on $A, \Omega, \{l_i : i \in \Omega\}$ in order to recover any $f \in \ell^2(I)$ from the measurements Y. This is the so-called dynamical sampling problem in which we seek to recover a function f by combining coarse samples of f and its futures states $A^l f$. We completely solve this problem in finite dimensional spaces, and for a large class of self adjoint operators in infinite dimensional spaces. In the latter case, although Y can be complete, using the Müntz–Szász Theorem we show it can never be a basis. We can also show that, when Ω is finite, Y is not a frame except for some very special cases. The existence of these special cases is derived from Carleson's Theorem for interpolating sequences in the Hardy space $H^2(\mathbb{D})$. Finally, using the recently proved Kadison–Singer/Feichtinger theorem we show that the set obtained by normalizing the vectors of Y can never be a frame when Ω is finite.

1. Introduction

Dynamical sampling refers to the process that results from sampling an evolving signal f at various times and asks the question: when do coarse samplings taken at varying times contain the same information as a finer sampling taken at the earliest time? In other words, under what conditions on an evolving system, can time samples be traded for spatial samples? Because dynamical sampling uses samples from varying

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* Corresponding author.
E-mail addresses: aldroubi@math.vanderbilt.edu (A. Aldroubi), cabrelli@dm.uba.ar (C. Cabrelli), umolter@dm.uba.ar (U. Molter), stang@math.vanderbilt.edu (S. Tang).

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time levels for a single reconstruction, it departs from classical sampling theory in which a signal f does not evolve in time and is to be reconstructed from its samples at a single time $t = 0$, see [1,2,5,7,8,11,19,20,28,23,30,33,37,43,44], and references therein.

The general dynamical sampling problem can be stated as follows: Let f be a function in a separable Hilbert space $\mathcal{H}$, e.g., $\mathbb{C}^d$ or $\ell^2(\mathbb{N})$, and assume that f evolves through an evolution operator $A : \mathcal{H} \to \mathcal{H}$ so that the function at time n has evolved to become $f^{(n)} = A^n f$. We identify $\mathcal{H}$ with $\ell^2(I)$ where $I = \{1, \dots, d\}$ in the finite dimensional case, $I = \mathbb{N}$ in the infinite dimensional case. We denote by $\{e_i\}_{i\in I}$ the standard basis of $\ell^2(I)$.

The time-space sample at time $t \in \mathbb{N}$ and location $p \in I$, is the value $A^t f(p)$. In this way we associate with each pair $(p, t) \in I \times \mathbb{N}$ a sample value.

The general dynamical sampling problem can then be described as: Under what conditions on the operator A, and a set $S \subset I \times \mathbb{N}$, can every f in the Hilbert space H be recovered in a stable way from the samples in S.

At time $t = n$, we sample f at the locations $\Omega_n \subset I$ resulting in the measurements $\{f^{(n)}(i) : i \in \Omega_n\}$. Here $f^{(n)}(i) =< A^n f, e_i >$.

The measurements $\{f^{(0)}(i) : i \in \Omega_0\}$ that we have from our original signal $f = f^{(0)}$ will contain in general insufficient information to recover f. In other words, f is undersampled. So we will need some extra information from the iterations of f by the operator A: $\{f^{(n)}(i) = A^n f(i) : i \in \Omega_n\}$. Again, for each n, the measurements $\{f^{(n)}(i) : i \in \Omega_n\}$ that we have by sampling our signals $A^n f$ at Ω_n are insufficient to recover $A^n f$ in general.

Several questions arise. Will the combined measurements $\{f^{(n)}(i) : i \in \Omega_n\}$ contain in general all the information needed to recover f (and hence $A^n f$)? How many iterations L will we need (i.e., $n = 1, \dots, L$) to recover the original signal? What are the right “spatial” sampling sets Ω_n we need to choose in order to recover f? In what way all these questions depend on the operator A?

The goal of this paper is to answer these questions and understand completely this problem that we can formulate as:

Let A be the evolution operator acting in $\ell^2(I)$, $\Omega \subset I$ be a fixed set of locations, and $\{l_i : i \in \Omega\}$ where l_i is a positive integer or $+\infty$.

Problem 1.1. Find conditions on A, Ω and $\{l_i : i \in \Omega\}$ such that any vector $f \in \ell^2(I)$ can be recovered from the samples $Y = \{f(i), Af(i), \dots, A^{l_i} f(i) : i \in \Omega\}$ in a stable way.

Note that, in Problem 1.1, we allow l_i to be finite or infinite. Note also that, Problem 1.1 is not the most general problem since the way it is stated implies that $\Omega = \Omega_0$ and $\Omega_n = \{i \in \Omega_0 : l_i \geq n\}$. Thus, an underlying assumption is that $\Omega_{n+1} \subset \Omega_n$ for all $n \geq 0$. For each $i \in \Omega$, let S_i be the operator from $\mathcal{H} = \ell^2(I)$ to $\mathcal{H}_i = \ell^2(\{0, \dots, l_i\})$, defined by $S_i f = (A^j f(i))_{j=0,\dots,l_i}$ and define S to be the operator $S = S_0 \oplus S_1 \oplus \dots$

Then f can be recovered from $Y = \{f(i), Af(i), \dots, A^{l_i} f(i) : i \in \Omega\}$ in a stable way if and only if there exist constants $c_1, c_2 > 0$ such that

$$c_1\|f\|_2^2 \leq \|\mathcal{S}f\|_2^2 = \sum_{i\in\Omega} \|S_i f\|_2^2 \leq c_2\|f\|_2^2. \tag{1}$$

Using the standard basis $\{e_i\}$ for $\ell^2(I)$, we obtain from (1) that

$$c_1\|f\|_2^2 \leq \sum_{i\in\Omega} \sum_{j=0}^{l_i} |\langle f, A^{*j} e_i\rangle|^2 \leq c_2\|f\|_2^2.$$

Thus we get

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Lemma 1.2. *Every $f \in \ell^2(I)$ can be recovered from the measurements set $Y = \{f(i), Af(i), \dots, A^{l_i} f(i) : i \in \Omega\}$ in a stable way if and only if the set of vectors $\{A^{*j} e_i : i \in \Omega, j = 0, \dots, l_i\}$ is a frame for $\ell^2(I)$.*

1.1. Connections to other fields

The dynamical sampling problem has similarities to other areas of mathematics. For example, in wavelet theory [9,16,17,25,34,38,42], a high-pass convolution operator H and a low-pass convolution operator L are applied to the function f. The goal is to design operators H and L so that reconstruction of f from samples of Hf and Lf is feasible. In dynamical sampling there is only one operator A, and it is applied iteratively to the function f. Furthermore, the operator A may be high-pass, low-pass, or neither and is given in the problem formulation, not designed.

In inverse problems (see [36] and the references therein), a single operator B, that often represents a physical process, is to be inverted. The goal is to recover a function f from the observation Bf. If B is not bounded below, the problem is considered an ill-posed inverse problem. Dynamical sampling is different because $A^n f$ is not necessarily known for any n; instead f is to be recovered from partial knowledge of $A^n f$ for many values of n. In fact, the dynamical sampling problem can be phrased as an inverse problem when the operator B is the operation of applying the operators $A, A^2, \dots, A^L$ and then subsampling each of these signals accordingly on some sets Ω_n for times $t = n$.

The methods that we develop for studying the dynamical sampling problem are related to methods in spectral theory, operator algebras, and frame theory [2,10,13,15,18,20–22,45]. For example, the proof of Theorem 3.15, below, use the newly proved [35] Kadison–Singer/Feichtinger conjecture [14,12]. Another example is the existence of cyclic vectors that form frames, which is related to Carleson's Theorem for interpolating sequences in the Hardy space $H^2(\mathbb{D})$ (c.f., Theorem 3.16).

Application to Wireless Sensor Networks (WSN) is a natural setting for dynamical sampling. In WSN, large amounts of physical sensors are distributed to gather information about a field to be monitored, such as temperature, pressure, or pollution. WSN are used in many industries, including the health, military, and environmental industries (c.f., [29,31,39,32,41,40] and the reference therein). The goal is to exploit the evolutionary structure and the placement of sensors to reconstruct an unknown field. The idea is simple. If it is not possible to place sampling devices at the desired locations, then we may be able to recover the desired information by placing the sensors elsewhere and use the evolution process to recover the signals at the relevant locations. In addition, if the cost of a sensor is expensive relative to the cost of activating the sensor, then, we may be able to recover the same information with fewer sensors, each being activated more frequently. In this way, reconstruction of a signal becomes cheaper. In other words we perform a time-space trade-off.

1.2. Contribution and organization

In Section 2 we present the results for the finite dimensional case. Specifically, Subsection 2.1 concerns the special case of diagonalizable operators acting on vectors in $\mathbb{C}^d$. This case is treated first in order to give some intuition about the general theory. For example, Theorem 2.2 explains the reconstruction properties for the examples below: Consider the following two matrices acting on $\mathbb{C}^5$.

$$P = \begin{pmatrix} 9/2 & 1/2 & -7 & 5 & -3 \\ 15/2 & 3/2 & -11 & 5 & -7 \\ 5 & 0 & -7 & 5 & -5 \\ 4 & 0 & -4 & 3 & -4 \\ 1/2 & 1/2 & -1 & 0 & 1 \end{pmatrix} \quad Q = \begin{pmatrix} 3/2 & -1/2 & 2 & 0 & 1 \\ 1/2 & 5/2 & 0 & 0 & -1 \\ 0 & 0 & 3 & 0 & 0 \\ 1 & 0 & -1 & 3 & -1 \\ -1/2 & -1/2 & 1 & 0 & 3 \end{pmatrix}.$$

For the matrix P, Theorem 2.2 shows that any $f \in \mathbb{C}^5$ can be recovered from the data sampled at the single "spacial" point $i = 2$, i.e., from

$$Y = \{f(2), Pf(2), P^2 f(2), P^3 f(2), P^4 f(2)\}.$$

However, if $i = 3$, i.e., $Y = \{f(3), Pf(3), P^2 f(3), P^3 f(3), P^4 f(3)\}$ the information is not sufficient to determine f. In fact if we do not sample at $i = 1$, or $i = 2$, the only way to recover any $f \in \mathbb{C}^5$ is to sample at all the remaining "spacial" points $i = 3, 4, 5$. For example, $Y = \{f(i), Pf(i) : i = 3, 4, 5\}$ is enough data to recover f, but $Y = \{f(i), Pf(i), ..., P^{Lf}(i) : i = 3, 4\}$, is not enough information no matter how large L is.

For the matrix Q, Theorem 2.2 implies that it is not possible to reconstruct $f \in \mathbb{C}^5$ if the number of sampling points is less than 3. However, we can reconstruct any $f \in \mathbb{C}^5$ from the data

$$\begin{aligned} Y = \{&f(1), Qf(1), Q^2 f(1), Q^3 f(1), Q^4 f(1),\\ &f(2), Qf(2), Q^2 f(2), Q^3 f(2), Q^4 f(2),\\ &f(4), Qf(4)\}. \end{aligned}$$

Yet, it is not possible to recover f from the set $Y = \{Q^l f(i) : i = 1, 2, 3, l = 0, \dots, L\}$ for any L. Theorem 2.2 gives all the sets Ω such that any $f \in \mathbb{C}^5$ can be recovered from $Y = \{A^l f(i) : i \in \Omega, l = 0, ..., l_i\}$.

In Subsection 2.2 Problem 1.1 is solved for the general case in $\mathbb{C}^d$, and Corollary 2.7 elucidates the example below: Consider

$$R = \begin{pmatrix} 0 & -1 & 4 & -1 & 2 \\ 2 & 1 & -2 & 1 & -2 \\ -1/2 & -1/2 & 3 & 0 & 1 \\ 1/2 & -1/2 & 0 & 2 & 0 \\ -1/2 & -1/2 & 2 & -1 & 2 \end{pmatrix}.$$

Then, Corollary 2.7 shows that Ω must contain at least two "spacial" sampling points for the recovery of functions from their time-space samples to be feasible. For example, if $\Omega = \{1, 3\}$, then $Y = \{R^l f(i) : i \in \Omega, l = 0, \dots, L\}$ is enough recover $f \in \mathbb{C}^5$. However, if Ω is changed to $\Omega = \{1, 2\}$, then $Y = \{R^l f(i) : i \in \Omega, l = 0, \dots, L\}$ does not provide enough information.

The dynamical sampling problem in infinite dimensional separable Hilbert spaces is studied in Section 3. For this case, we restrict ourselves to certain classes of self adjoint operators in $\ell^2(\mathbb{N})$. In light of Lemma 1.2, in Subsection 3.1, we characterize the sets $\Omega \subset \mathbb{N}$ such that $\mathcal{F}_\Omega = \{A^j e_i : i \in \Omega, j = 0, \dots, l_i\}$ is complete in $\ell^2(\mathbb{N})$ (Theorem 3.3). However, we also show that if Ω is a finite set, then $\{A^j e_i : i \in \Omega, j = 0, \dots, l_i\}$ is never a basis (see Theorem 3.8). It turns out that the obstruction to being a basis is redundancy. This fact is proved using the beautiful Müntz–Szász Theorem 3.5 below.

Although $\mathcal{F}_\Omega = \{A^j e_i : i \in \Omega, j = 0, \dots, l_i\}$ cannot be a basis, it should be possible that $\mathcal{F}_\Omega$ is a frame for sets $\Omega \subset \mathbb{N}$ with finite cardinality. It turns out however, that except for special cases, if Ω is a finite set, then $\mathcal{F}_\Omega$ is not a frame for $\ell^2(\mathbb{N})$.

If Ω consists of a single vector, we are able to characterize completely when $\mathcal{F}_\Omega$ is a frame for $\ell^2(\mathbb{N})$ (Theorem 3.16), by relating our problem to a theorem by Carleson on interpolating sequences in the Hardy spaces $H^2(\mathbb{D})$.

2. Finite dimensional case

In this section we will address the finite dimensional case. That is, our evolution operator is a matrix A acting on the space $\mathbb{C}^d$ and $I = \{1, \dots, d\}$. Thus, given A, our goal is to find necessary and sufficient

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conditions on the set of indices $\Omega \subset I$ and the numbers $\{l_i\}_{i\in\Omega}$ such that every vector $f \in \mathbb{C}^d$ can be recovered from the samples $\{A^j f(i) : i \in \Omega,\ j = 0, \dots, l_i\}$ or equivalently (using Lemma 1.2), the set of vectors

$$\{A^{*j} e_i : i \in \Omega,\ j = 0, \dots, l_i\} \text{ is a frame of } \mathbb{C}^d. \tag{2}$$

(Note that this implies that we need at least d space-time samples to be able to recover the vector f.)

The problem can be further reduced as follows: Let B be any invertible matrix with complex coefficients, and let Q be the matrix $Q = BA^*B^{-1}$, so that $A^* = B^{-1}QB$. Let b_i denote the ith column of B. Since a frame is transformed to a frame by invertible linear operators, condition (2) is equivalent to $\{Q^j b_i : i \in \Omega,\ j = 0, \dots, l_i\}$ being a frame of $\mathbb{C}^d$.

This allows us to replace the general matrix A^* by a possibly simpler matrix and we have:

Lemma 2.1. *Every $f \in \mathbb{C}^d$ can be recovered from the measurement set $Y = \{A^j f(i) :\ i \in \Omega, j = 0, \dots, l_i\}$ if and only if the set of vectors $\{Q^j b_i : i \in \Omega,\ j = 0, \dots, l_i\}$ is a frame for $\mathbb{C}^d$.*

We begin with the simpler case when A^* is a diagonalizable matrix.

2.1. Diagonalizable transformations

Let $A \in \mathbb{C}^{d\times d}$ be a matrix that can be written as $A^* = B^{-1}DB$ where D is a diagonal matrix of the form

$$D = \begin{pmatrix} \lambda_1 I_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 I_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n I_n \end{pmatrix}. \tag{3}$$

In (3), I_k is an $h_k \times h_k$ identity matrix, and $B \in \mathbb{C}^{d\times d}$ is an invertible matrix. Thus A^* is a diagonalizable matrix with distinct eigenvalues $\{\lambda_1, \dots, \lambda_n\}$.

Using Lemma 2.1 and $Q = D$, Problem 1.1 becomes the problem of finding necessary and sufficient conditions on vectors b_i and numbers l_i, and the set $\Omega \subset \{1, \dots, m\}$ such that the set of vectors $\{D^j b_i : i \in \Omega,\ j = 0, \dots, l_i\}$ is a frame for $\mathbb{C}^d$. Recall that the Q-*annihilator* q_b^Q *of a vector* b is the monic polynomial of smallest degree, such that $q_b^Q(Q)b \equiv 0$. Let P_j denote the orthogonal projection in $\mathbb{C}^d$ onto the eigenspace of D associated to the eigenvalue λ_j. Then we have:

Theorem 2.2. *Let $\Omega \subset \{1, \dots, d\}$ and $\{b_i : i \in \Omega\}$ vectors in $\mathbb{C}^d$. Let D be a diagonal matrix and r_i the degree of the D-annihilator of b_i. Set $l_i = r_i - 1$. Then $\{D^j b_i :\ i \in \Omega, j = 0, \dots, l_i\}$ is a frame of $\mathbb{C}^d$ if and only if $\{P_j(b_i) : i \in \Omega\}$ form a frame of $P_j(\mathbb{C}^d)$, $j = 1, \dots, n$.*

As a corollary, using Lemma 2.1 we get

Theorem 2.3. *Let $A^* = B^{-1}DB$, and let $\{b_i : i \in \Omega\}$ be the column vectors of B whose indices belong to Ω. Let r_i be the degree of the D-annihilator of b_i and let $l_i = r_i - 1$. Then $\{A^{*j} e_i :\ i \in \Omega, j = 0, \dots, l_i\}$ is a frame of $\mathbb{C}^d$ if and only if $\{P_j(b_i) : i \in \Omega\}$ form a frame of $P_j(\mathbb{C}^d)$, $j = 1, \dots, n$.*

Equivalently, any vector $f \in \mathbb{C}^d$ can be recovered from the samples

$$Y = \{f(i), Af(i), \dots, A^{l_i} f(i) : i \in \Omega\}$$

if and only if $\{P_j(b_i) : i \in \Omega\}$ form a frame of $P_j(\mathbb{C}^d)$, $j = 1, \dots, n$.

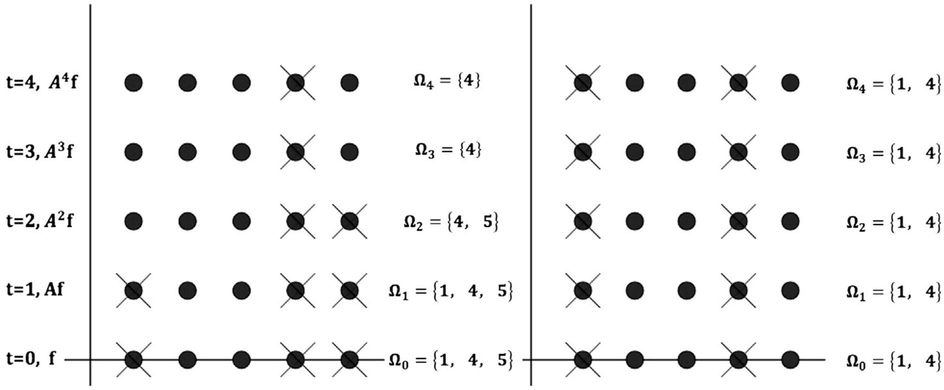

Fig. 1. Illustration of a time-space sampling pattern. Crosses correspond to time-space sampling points. Left panel: $\Omega = \Omega_0 = \{1,4,5\}$. $l_1 = 1, l_4 = 4, l_5 = 3$. Right panel: $\Omega = \Omega_0 = \{1,4\}$. $L = 4$.

Example 2.2 in [3] can be derived from Theorem 2.3 when all the eigenvalues have multiplicity 1, and when there is a single sampling point at location i.

Note that, in the previous Theorem, the number of time-samples l_i depends on the sampling point i. If instead the number of time-samples L is the same for all $i \in \Omega$, (note that $L \geq max\{l_i : i \in \Omega\}$ is an obvious choice, but depending on the vectors b_i it may be possible to choose $L \leq min\{l_i : i \in \Omega\}$), then we have the following Theorems (see Fig. 1).

Theorem 2.4. *Let D be a diagonal matrix, $\Omega \subset \{1,\dots,d\}$ and $\{b_i : i \in \Omega\}$ be a set of vectors in $\mathbb{C}^d$ such that $\{P_j(b_i) : i \in \Omega\}$ form a frame of $P_j(\mathbb{C}^d)$, $j = 1,\dots,n$. Let L be any fixed integer, then $E = \bigcup_{\{i\in\Omega: b_i\neq 0\}} \{b_i, Db_i, \dots, D^L b_i\}$ is a frame of $\mathbb{C}^d$ if and only if $\{D^{L+1}b_i, : i \in \Omega\} \subset \operatorname{span}(E)$.*

Proof. Note that if $\{D^{L+1}b_i : i \in \Omega\} \subset \operatorname{span}(E)$ then $D(\operatorname{span}(E)) \subset \operatorname{span}(E)$. Therefore by Theorem 2.3, E is a frame of $\mathbb{C}^d$. □

As a corollary, for our original Problem 1.1 we obtain

Theorem 2.5. *Let $A^* = B^{-1}DB$, L be any fixed integer, and let $\{b_i : i \in \Omega\}$ be a set of vectors in $\mathbb{C}^d$ such that $\{P_j(b_i) : i \in \Omega\}$ form a frame of $P_j(\mathbb{C}^d)$, $j = 1,\dots,n$. Then $\{A^{*j}e_i : \ i \in \Omega,\ j = 0,\dots,L\}$ is a frame of $\mathbb{C}^d$ if and only if $\{D^{L+1}b_i : i \in \Omega\} \subset \operatorname{span}\left(\{D^j b_i : i \in \Omega\ , j = 0,\dots,L\}\right)$.*

Equivalently any $f \in \mathbb{C}^d$ can be recovered from the samples

$$Y = \{f(i), Af(i), A^2f(i), \dots, A^L f(i) : i \in \Omega\},$$

if and only if $\{D^{L+1}b_i : i \in \Omega\} \subset \operatorname{span}\left(\{D^j b_i : i \in \Omega\ , j = 0,\dots,L\}\right)$.

Proof. For the proof we just apply Lemma 2.1 and Theorem 2.4. □

A special case of Theorem 2.5 is [3, Theorem 3.2]. There, since the operator A is a convolution operator in $\ell^2(\mathbb{Z}_d) \approx \mathbb{C}^d$, the matrix B is the Fourier matrix whose columns consist of the discrete, complex exponentials. The set Ω consists of the union of a uniform grid $m\mathbb{Z}_d$ and an extra sampling set Ω_0. In [3, Theorem 3.2] L can be chosen to be any number larger than m.

Theorems 2.3 and 2.5 will be consequences of our general results but we state them here to help the comprehension of the general results below.

2.2. General linear transformations

For a general matrix we will need to use the reduction to its Jordan form. To state our results in this case, we need to introduce some notations and describe the general Jordan form of a matrix with complex entries. (For these and other results about matrix or linear transformation decompositions see for example [27].)

A matrix J is in Jordan form if

$$J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_n \end{pmatrix}. \tag{4}$$

In (4), for $s = 1, \dots, n$, $J_s = \lambda_s I_s + N_s$ where I_s is an $h_s \times h_s$ identity matrix, and N_s is a $h_s \times h_s$ nilpotent block-matrix of the form:

$$N_s = \begin{pmatrix} N_{s1} & 0 & \cdots & 0 \\ 0 & N_{s2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_{s\gamma_s} \end{pmatrix} \tag{5}$$

where each N_{si} is a $t_i^s \times t_i^s$ cyclic nilpotent matrix,

$$N_{si} \in \mathbb{C}^{t_i^s \times t_i^s}, \quad N_{si} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{6}$$

with $t_1^s \geq t_2^s \geq \dots$, and $t_1^s + t_2^s + \dots + t_{\gamma_s}^s = h_s$. Also $h_1 + \dots + h_n = d$. The matrix J has d rows and distinct eigenvalues λ_j, $j = 1, \dots, n$.

Let k_j^s denote the index corresponding to the first row of the block N_{sj} from the matrix J, and let $e_{k_j^s}$ be the corresponding element of the standard basis of $\mathbb{C}^d$. (That is a cyclic vector associated to that block.) We also define $W_s := \operatorname{span}\{e_{k_j^s} : j = 1, \dots, \gamma_s\}$, for $s = 1, \dots, n$, and P_s will again denote the orthogonal projection onto W_s. Finally, recall that the J *annihilator* q_b^J *of a vector* b is the monic polynomial of smallest degree, such that $q_b^J(J)b \equiv 0$. Using the notations and definitions above we can state the following theorem:

Theorem 2.6. *Let J be a matrix in Jordan form, as in (4). Let $\Omega \subset \{1, \dots, d\}$ and $\{b_i : i \in \Omega\}$ be a subset of vectors of $\mathbb{C}^d$, r_i be the degree of the J-annihilator of the vector b_i and let $l_i = r_i - 1$.*

Then the following propositions are equivalent.

i) *The set of vectors $\{J^j b_i : \ i \in \Omega, j = 0, \dots, l_i\}$ is a frame for $\mathbb{C}^d$.*
ii) *For every $s = 1, \dots, n$, $\{P_s(b_i), i \in \Omega\}$ form a frame of W_s.*

Now, for a general matrix A, using Lemma 2.1 we can state:

Corollary 2.7. *Let A be a matrix, such that $A^* = B^{-1}JB$, where $J \in \mathbb{C}^{d\times d}$ is the Jordan matrix for A^*. Let $\{b_i : i \in \Omega\}$ be a subset of the column vectors of B, r_i be the degree of the J-annihilator of the vector b_i, and let $l_i = r_i - 1$.*

Then, every $f \in \mathbb{C}^d$ can be recovered from the measurement set $Y = \{(A^j f)(i) : \ i \in \Omega, j = 0, \dots, l_i\}$ of $\mathbb{C}^d$ if and only if $\{P_s(b_i), i \in \Omega\}$ form a frame of W_s.

In other words, we will be able to recover f from the measurements Y, if and only if the *Jordan*-vectors of A^* (i.e. the columns of the matrix B that reduces A^* to its Jordan form) corresponding to Ω satisfy that their projections on the spaces W_s form a frame.

Remark 2.8. We want to emphasize at this point, that given a matrix in Jordan form there is an obvious choice of vectors in order that their iterations give a frame of the space (namely, the cyclic vectors $e_{k_j^s}$ corresponding to each block). However, we are dealing here with a much more difficult problem. The vectors b_i are given beforehand, and we need to find conditions in order to decide if their iterations form a frame.

The following theorem is just a statement about replacing the *optimal* iteration of each vector b_i by any fixed number of iterations. The idea is, that we iterate a fixed number of times L but we do not need to know the degree r_i of the J-annihilator for each b_i. Clearly, if $L \geq \max\{r_i - 1 : i \in \Omega\}$ then we can always recover any f from Y. But the number of time iterations L may be smaller than any $r_i - 1$, $i \in \Omega$. In fact, for practical purposes it might be better to iterate, than to try to figure out which is the degree of the annihilator for b_i.

Theorem 2.9. *Let $J \in \mathbb{C}^{d\times d}$ be a matrix in Jordan form (see (4)). Let $\Omega \subset \{1, \dots, d\}$, and let $\{b_i : i \in \Omega\}$ be a set of vectors in $\mathbb{C}^d$, such that for each $s = 1, \dots, n$ the projections $\{P_s(b_i) : i \in \Omega\}$ onto W_s form a frame of W_s. Let L be any fixed integer, then $E = \bigcup_{\{i\in\Omega : b_i \neq 0\}} \{b_i, Jb_i, \dots, J^L b_i\}$ is a frame of $\mathbb{C}^d$ if and only if $\{J^{L+1}b_i : i \in \Omega\} \subset \operatorname{span}(E)$.*

As a corollary we immediately get the solution to Problem 1.1 in finite dimensions.

Corollary 2.10. *Let $\Omega \subset I$, $A^* = B^{-1}JB$, and L be any fixed integer. Assume that $\{P_s(b_i) : i \in \Omega\}$ form a frame of W_s and set $E = \{J^s b_i : i \in \Omega, s = 0, \dots, L,\}$. Then any $f \in \mathbb{C}^d$ can be recovered from the samples $Y = \{f(i), Af(i), A^2f(i), \dots, A^L f(i) : i \in \Omega\}$, if and only if $\{J^{L+1}b_i : i \in \Omega\} \subset \operatorname{span}(E\})$.*

2.3. Proofs

In order to introduce some needed notations, we first recall the standard decomposition of a linear transformation acting on a finite dimensional vector space that produces a basis for the Jordan form.

Let V be a finite dimensional vector space of dimension d over $\mathbb{C}$ and let $T : V \longrightarrow V$ be a linear transformation. The characteristic polynomial of T factorizes as $\chi_T(x) = (x - \lambda_1)^{h_1} \dots (x - \lambda_n)^{h_n}$ where $h_i \geq 1$ and $\lambda_1, \dots, \lambda_n$ are distinct elements of $\mathbb{C}$. The minimal polynomial of T will be then $m_T(x) = (x - \lambda_1)^{r_1} \dots (x - \lambda_n)^{r_n}$ with $1 \leq r_i \leq h_i$ for $i = 1, \dots, n$. By the primary decomposition theorem, the subspaces $V_s = \operatorname{Ker}(T - \lambda_s I)^{r_s}$, $s = 1, \dots, n$ are invariant under T (i.e. $T(V_s) \subset V_s$) and we have also that $V = V_1 \oplus \dots \oplus V_n$.

Let T_s be the restriction of T to V_s. Then, the minimal polynomial of T_s is $(x-\lambda_s)^{r_s}$, and $T_s = N_s + \lambda_s I_s$, where N_s is nilpotent of order r_s and I_s is the identity operator on V_s. Now for each s we apply the cyclic decomposition to N_s and the space V_s to obtain:

$$V_s = V_{s1} \oplus \dots \oplus V_{s\gamma_s}$$

where each V_{sj} is invariant under N_s, and the restriction operator N_{sj} of N_s to V_{sj} is a *cyclic nilpotent* operator on V_{sj}.

Finally, let us fix for each j a cyclic vector $w_{sj} \in V_{sj}$ and define the subspace $W_s = \operatorname{span}\{w_{s1} \dots w_{s\gamma_s}\}$, $W = W_1 \oplus \dots \oplus W_n$ and let P_{W_s} be the projection onto W_s, with $I_W = P_{W_1} + \dots + P_{W_n}$.

With this notation we can state the main theorem of this section:

Theorem 2.11. *Let $\{b_i : i \in \Omega\}$ be a set of vectors in V. If the set $\{P_{W_s} b_i : i \in \Omega\}$ is complete in W_s for each $s = 1, \dots, n$, then the set $\{b_i, Tb_i, \dots, T^{l_i} b_i : i \in \Omega\}$ is a frame of V, where r_i is the degree of the T-annihilator of b_i and $l_i = r_i - 1$.*

To prove Theorem 2.11, we will first concentrate on the case where the transformation T has minimal polynomial consisting of a unique factor, i.e. $m_T(x) = (x - \lambda)^r$, so that $T = \lambda I_d + N$, and $N^r = 0$ but $N^{r-1} \neq 0$.

2.4. Case $T = \lambda I_d + N$

Remark 2.12. It is not difficult to see that, in this case, given some $L \in \mathbb{N}$, $\{T^j b_i : \ i \in \Omega, j = 0, \dots, L\}$ is a frame for V if and only if $\{N^j b_i : \ i \in \Omega, j = 0, \dots, L\}$ is a frame for V. In addition, since $N^r b_i = 0$ we need only to iterate to $r - 1$. In fact, we only need to iterate each b_i to $l_i = r_i - 1$ where r_i is the degree of the N annihilator of b_i.

Definition 2.13. A matrix $A \in \mathbb{C}^{d \times d}$ is *perfect* if $a_{ii} \neq 0, i = 1, \dots, d$ and $\det(A_i) \neq 0, i = 1, \dots, d$ where $A_s \in \mathbb{C}^{s \times s}$ is the submatrix of A, $A_s = \{a_{i,j}\}_{i,j=1,\dots,s}$.

We need the following lemma that is straightforward to prove.

Lemma 2.14. *Let $A \in \mathbb{C}^{d \times d}$ be an invertible matrix. Then there exists a perfect matrix $B \in \mathbb{C}^{d \times d}$ that consists of row (or column) permutations of A.*

Proof. The proof is by induction on d, which is the number of rows (or columns) of the matrix. The case of $d = 1$ is obvious, so let A be an invertible $d \times d$ matrix with entries $a_{i,j}$ and assume that the lemma is true for dimension $d - 1$. Let us expand the determinant of A using the last column, i.e.:

$$\det(A) = \sum_{i=1}^{d} (-1)^{i+d} \, a_{i,d} \, \det(A^{(i,d)}),$$

where $A^{(i,j)}$ denotes the $(d-1) \times (d-1)$ submatrix of A that is obtained by removing the row i and the column j from A.

Since $\det(A)$ is different from zero, there exists $i \in \{1, \dots, d\}$ such that $a_{i,d}$ and $det(A^{(i,d)})$ are both different from zero. Let B be the matrix obtained from A by interchanging row i with row d. So the $(d-1) \times (d-1)$ submatrix B_{d-1} of B obtained by removing row d and column d from B, is invertible and the element of B, $b_{d,d} = a_{i,d}$ is not zero.

We now apply the inductive hypothesis to the matrix B_{d-1}. So there exits some permutation of the rows of B_{d-1} such that the matrix is perfect. If we apply the same permutation to the firs $d - 1$ rows of B, we obtain a matrix $\tilde{B}$ such that $\tilde{B}_{d-1}$ is perfect and its (d, d)th entry is non zero. Therefore $\tilde{B}$ is perfect and has been obtained from A by permutation of the rows. □

If N is nilpotent of order r, then there exist $\gamma \in \mathbb{N}$ and invariant subspaces $V_i \subset V$, $i = 1, \dots, \gamma$ such that

$$V = V_1 \oplus \cdots \oplus V_\gamma, \quad \dim(V_j) = t_j, \ t_j \geq t_{j+1}, \ j = 1, \dots, \gamma - 1,$$

and $N = N_1 + \cdots + N_\gamma$, where $N_j = P_j N P_j$ is a cyclic nilpotent operator in V_j, $j = 1, \dots, \gamma$. Here P_j is the projection onto V_j. Note that $t_1 + \cdots + t_\gamma = d$.

For each $j = 1, \dots, \gamma$, let $w_j \in V_j$ be a cyclic vector for N_j. Note that the set $\{w_1, \dots, w_\gamma\}$ is a linearly independent set.

Let $W = \operatorname{span}\{w_1, \dots, w_\gamma\}$. Then, we can write $V = W \oplus NW \oplus \dots \oplus N^{r-1}W$. Furthermore, the projections P_{N^jW} satisfy $P^2_{N^jW} = P_{N^jW}$, and $I = \sum_{j=0}^{r-1} P_{N^jW}$.

Finally, note that

$$N^s P_W = P_{N^sW} N^s. \tag{7}$$

With the notation above, we have the following theorem:

Theorem 2.15. *Let N be a nilpotent operator on V. Let $B \subset V$ be a finite set of vectors such that $\{P_W(b) : b \in B\}$ is complete in W. Then*

$$\bigcup_{b \in B} \{b, Nb, \dots, N^{l_b} b\} \quad \textit{is a frame for } V,$$

where $l_b = r_b - 1$ and r_b is the degree of the N-annihilator of b.

Proof. In order to prove Theorem 2.15, we will show that there exist vectors $\{b_1, \dots, b_\gamma\}$ in B, where $\gamma = \dim(W)$, such that

$$\bigcup_{i=1}^{\gamma} \{b_i, Nb_i, \dots, N^{t_i-1} b_i\} \quad \text{is a basis of } V.$$

Recall that t_i are the dimensions of V_i defined above. Since $\{P_W(b) : b \in B\}$ is complete in W and $\dim(W) = \gamma$ it is clear that we can choose $\{b_1, \dots, b_\gamma\} \subset B$ such that $\{P_W(b_i) : i = 1, \dots, \gamma\}$ is a basis of W. Since $\{w_1, \dots, w_\gamma\}$ is also a basis of W, there exist unique scalars $\{\theta_{i,j} : i, j = 1, \dots, \gamma\}$ such that,

$$P_W(b_i) = \sum_{j=1}^{\gamma} \theta_{ij} w_j, \tag{8}$$

with the matrix $\Theta = \{\theta_{i,j}\}_{i,j=1,\dots,\gamma}$ invertible. Thus, using Lemma 2.14 we can relabel the indices of $\{b_i\}$ in such a way that Θ is *perfect*. Therefore, without loss of generality, we can assume that $\{b_1, \dots, b_\gamma\}$ are already in the right order, so that Θ is perfect.

We will now prove that the d vectors $\{b_i, Nb_i, \dots, N^{t_i-1} b_i\}_{i=1,\dots,\gamma}$ are linearly independent. For this, assume that there exist scalars α_j^s such that

$$0 = \sum_{j=1}^{\gamma} \alpha_j^0 b_j + \sum_{j=1}^{p_1} \alpha_j^1 N b_j + \dots + \sum_{j=1}^{p_{r-1}} \alpha_j^{r-1} N^{r-1} b_j, \tag{9}$$

where $p_s = \max\{j : t_j > s\} = \dim N^s W, s = 1, \dots, r-1$ (note that $p_s \geq 1$, since $N^{r-1} b_1 \neq 0$).

Note that since $V = W \oplus NW \oplus \dots \oplus N^{r-1}W$, for any vector $x \in V$, $P_W(Nx) = 0$. Therefore, if we apply P_W on both sides of (9), we obtain

$$\sum_{j=1}^{\gamma} \alpha_j^0 P_W b_j = 0.$$

Since $\{P_W b_i : i = 1, \dots, \gamma\}$ are linearly independent, we have $\alpha_j^0 = 0,\ j = 1, \dots, \gamma$. Hence, if we now apply P_{NW} to (9), we have as before that

$$\sum_{j=1}^{p_1} \alpha_j^1 P_{NW} N b_j = 0.$$

Using the commutation property of the projection, (7), we have

$$\sum_{j=1}^{p_1} \alpha_j^1 N P_W b_j = 0.$$

In matrix notation, this is

$$[\alpha_1^1 \dots \alpha_{p_1}^1] \Theta_{p_1} \begin{bmatrix} Nw_1 \\ \vdots \\ Nw_{p_1} \end{bmatrix} = 0.$$

Note that by definition of p_1, $Nw_1, \dots, Nw_{p_1}$ span NW, and since the dimension of NW is exactly p_1, $Nw_1, \dots, Nw_{p_1}$ are linearly independent vectors. Therefore $[\alpha_1^1 \dots \alpha_{p_1}^1]\Theta_{p_1} = 0$. Since Θ is perfect, $[\alpha_1^1 \dots \alpha_{p_1}^1] = [0 \dots 0]$. Iterating the above argument, the Theorem follows. $\square$

Proof of Theorem 2.11. We will prove the case when the minimal polynomial has only two factors. The general case follows by induction.

That is, let $T : V \to V$ be a linear transformation with characteristic polynomial of the form $\chi_T(x) = (x - \lambda_1)^{h_1}(x - \lambda_2)^{h_2}$. Thus, $V = V_1 \oplus V_2$ where V_1, V_2 are the subspaces associated to each factor, and $T = T_1 \oplus T_2$. In addition, $W = W_1 \oplus W_2$ where W_1, W_2 are the subspaces of the cyclic vectors from the cyclic decomposition of N_1 with respect of V_1 and of N_2 with respect to V_2.

Let $\{b_i : i \in \Omega\}$ be vectors in V that satisfy the hypothesis of the Theorem. For each b_i we write $b_i = c_i + d_i$ with $c_i \in V_1$ and $d_i \in V_2$, $i \in \Omega$. Let r_i, m_i and n_i be the degrees of the annihilators $q_{b_i}^T$, $q_{c_i}^{T_1}$ and $q_{d_i}^{T_2}$, respectively. By hypothesis $\{P_{W_1} c_i : i \in \Omega\}$ and $\{P_{W_2} d_i : i \in \Omega\}$ are complete in W_1 and W_2, respectively. Hence, applying Theorem 2.15 to N_1 and N_2 we conclude that $\bigcup_{i \in \Omega}\{T_1^j c_i, j = 0, 1, \dots m_i - 1\}$ is complete in V_1, and that $\bigcup_{i \in \Omega}\{T_2^j d_i, j = 0, 1, \dots n_i - 1\}$ is complete in V_2.

We will now need a Lemma (recall that q_b^T is the T-annihilator of the vector b):

Lemma 2.16. *Let T be as above, and $V = V_1 \oplus V_2$. Given $b \in V$, $b = c + d$ then $q_b^T = q_c^{T_1} q_d^{T_2}$ where $q_c^{T_1}$ and $q_d^{T_2}$ are coprime. Further let $u \in V_2$, $u = q_c^{T_1}(T_2)d$. Then $q_u^{T_2}$ coincides with $q_d^{T_2}$.*

Proof. The fact that $q_b^T = q_c^{T_1} q_d^{T_2}$ with coprime $q_c^{T_1}$ and $q_d^{T_2}$ is a consequence of the decomposition of T.

Now, by definition of $q_u^{T_2}$ we have that

$$0 = q_u^{T_2}(T_2)(u) = q_u^{T_2}(T_2)(q_c^{T_1}(T_2)d) = (q_u^{T_2} q_c^{T_1})(T_2)d.$$

Thus, $q_d^{T_2}$ has to divide $q_u^{T_2} \cdot q_c^{T_1}$, but since $q_d^{T_2}$ is coprime with $q_c^{T_1}$, we conclude that

$$q_d^{T_2} \quad \text{divides} \quad q_u^{T_2}. \tag{10}$$

On the other hand

$$\begin{aligned} 0 = q_d^{T_2}(T_2)(d) &= q_c^{T_1}(T_2)(q_d^{T_2}(T_2)d) = (q_c^{T_1} q_d^{T_2})(T_2)d \\ &= (q_d^{T_2} q_c^{T_1})(T_2)d = q_d^{T_2}(T_2)(q_c^{T_1}(T_2)d) = q_d^{T_2}(T_2)(u), \end{aligned}$$

and therefore

$$q_u^{T_2} \quad \text{divides} \quad q_d^{T_2}. \tag{11}$$

From (10) and (11) we obtain $q_d^{T_2} = q_u^{T_2}$. □

Now, we continue with the proof of the Theorem. Recall r_i, m_i and n_i be the degrees of $q_{b_i}^T$, $q_{c_i}^{T_1}$ and $q_{d_i}^{T_2}$, respectively, and let $l_i = r_i - 1$. Also note that by Lemma 2.16 $r_i = m_i + n_i$. In order to prove that the set $\{b_i, Tb_i, \dots, T^{l_i}b_i : i \in \Omega\}$ is complete in V, we will replace this set with a new one in such a way that the dimension of the span does not change.

For each $i \in \Omega$, let $u_i = q_{c_i}^{T_1}(T_2)d_i$. Now, for a fixed i we leave the vectors $b_i, Tb_i, \dots, T^{m_i-1}b_i$ unchanged, but for $s = 0, \dots, n_i - 1$ we replace the vectors $T^{m_i+s}b_i$ by the vectors $T^{m_i+s}b_i + \beta_s(T)b_i$ where β_s is the polynomial $\beta_s(x) = x^s q_{c_i}^{T_1}(x) - x^{m_i+s}$.

Note that $\operatorname{span}\{b_i, Tb_i, \dots, T^{m_i+s}b_i\}$ remains unchanged, since $\beta_s(T)b_i$ is a linear combination of the vectors $\{T^s b_i, \dots, T^{m_i+s-1}b_i\}$.

Now we observe that:

$$T^{m_i+s}b_i + \beta_s(T)b_i = \left[T_1^{m_i+s}c_i + \beta_s(T_1)c_i\right] + \left[T_2^{m_i+s}d_i + \beta_s(T_2)d_i\right].$$

The first term of the sum on the right hand side of the equation above is in V_1 and the second in V_2. By definition of β_s we have:

$$T_1^{m_i+s}c_i + \beta_s(T_1)c_i = T_1^{m_i+s}c_i + T_1^s q_{c_i}^{T_1}(T_1)c_i - T_1^{m_i+s}c_i = T_1^s q_{c_i}^{T_1}(T_1)c_i = 0,$$

and

$$T_2^{m_i+s}d_i + \beta_s(T_2)d_i = T_2^s q_{c_i}^{T_1}(T_2)(d_i) = T_2^s u_i.$$

Thus, for each $i \in \Omega$, the vectors $\{b_i, \dots, T^{l_i}b_i\}$ have been replaced by the vectors $\{b_i, \dots, T^{m_i-1}b_i, u_i, \dots, T^{n_i-1}u_i\}$ and both sets have the same span.

To finish the proof we only need to show that the new system is complete in V.

Using Lemma 2.16, we have that for each $i \in \Omega$,

$$\dim(\operatorname{span}\{u_i, \dots, T_2^{n_i-1}u_i\}) = \dim(\operatorname{span}\{d_i, \dots, T_2^{n_i-1}d_i\}) = n_i,$$

and since each $T_2^s u_i \in \operatorname{span}\{d_i, \dots, T_2^{n_i-1}d_i\}$ we conclude that

$$\operatorname{span}\{u_i, \dots, T_2^{n_i-1}u_i : i \in \Omega\} = \operatorname{span}\{d_i, \dots, T_2^{n_i-1}d_i : i \in \Omega\}. \tag{12}$$

Now assume that $x \in V$ with $x = x_1 + x_2$, $x_i \in V_i$. Since by hypothesis $\operatorname{span}\{c_i, \dots, T_1^{m_i-1}c_i : i \in \Omega\}$ is complete in V_1, we can write

$$x_1 = \sum_{i\in\Omega} \sum_{j=0}^{m_i-1} \alpha_j^i T_1^j c_i, \tag{13}$$

for same scalars α_j^i, and therefore,

$$\sum_{i\in\Omega} \sum_{j=0}^{m_i-1} \alpha_j^i T^j b_i = x_1 + \sum_{i\in\Omega} \sum_{j=0}^{m_i-1} \alpha_j^i T_2^j d_i = x_1 + \tilde{x}_2, \tag{14}$$

since $\sum_{i\in\Omega}\sum_{j=0}^{m_i-1}\alpha_j^i T_2^j d_i = \tilde{x}_2$ is in V_2 by the invariance of V_2 by T. Since by hypothesis $\{T_2^j d_i : i \in \Omega,$ $j = 1, \dots, n_i - 1\}$ is complete in V_2, by equation (12), $\{T_2^j u_i : i \in \Omega, \quad j = 1, \dots, n_i - 1\}$ is also complete in V_2, and therefore there exist scalars β_j^i,

$$x_2 - \tilde{x}_2 = \sum_{i\in\Omega}\sum_{j=0}^{n_i-1}\beta_j^i T_2^j u_i,$$

and so

$$x = \sum_{i\in\Omega}\sum_{j=0}^{m_i-1}\alpha_j^i T^j b_i + \sum_{i\in\Omega}\sum_{j=0}^{n_i-1}\beta_j^i T_2^j u_i,$$

which completes the proof of Theorem 2.11 for the case of two coprime factors in the minimal polynomial of J. The general case of more factors follows by induction adapting the previous argument. $\square$

Theorem 2.6 and Theorem 2.9 and its corollaries are easy consequences of Theorem 2.11.

Proof of Theorem 2.9. Note that if $\{J^{L+1}b_i : i \in \Omega\} \subset \mathrm{span}(E)$, then $\{J^{L+2}b_i : i \in \Omega\} \subset \mathrm{span}(E)$ as well. Continuing in this way, it follows that for each $i \in \Omega$, $\mathrm{span}(E)$ contains all the powers $J^j b_i$ for any j. Therefore, using Theorem 2.6, it follows that $\mathrm{span}(E)$ contains a frame of $\mathbb{C}^d$, so that, $\mathrm{span}(E) = \mathbb{C}^d$ and E is a frame of $\mathbb{C}^d$. The converse is obvious. $\square$

The proof of Theorem 2.5 uses a similar argument.

Although Theorem 2.2 is a direct consequence of Theorem 2.6, we will give a simpler proof for this case.

Proof of Theorem 2.2. Let $\{P_j(b_i) : i \in \Omega\}$ form a frame of $P_j(\mathbb{C}^d)$, for each $j = 1, \dots, n$. Since we are working with finite dimensional spaces, to show that $\{D^j b_i : \ i \in \Omega, j = 0, \dots, l_i\}$ is a frame of $\mathbb{C}^d$, all we need to show is that it is complete in $\mathbb{C}^d$. Let x be any vector in $\mathbb{C}^d$, then $x = \sum_{j=1}^{n} P_j x$. Assume that $\langle D^l b_i, x\rangle = 0$ for all $i \in \Omega$ and $l = 0, \dots, l_i$. Since $l_i = r_i - 1$, where r_i is the degree of the D-annihilator of b_i, we have that $\langle D^l b_i, x\rangle = 0$ for all $i \in \Omega$ and $l = 0, \dots, d$. In particular, since $n \le d$, $\langle D^l b_i, x\rangle = 0$ for all $i \in \Omega$ and $l = 0, \dots, n$. Then

$$\langle D^l b_i, x\rangle = \sum_{j=1}^{n}\langle D^l b_i, P_j x\rangle = \sum_{j=1}^{n}\lambda_j^l\langle P_j b_i, P_j x\rangle = 0, \tag{15}$$

for all $i \in \Omega$ and $l = 0, \dots, n$. Let z_i be the vector $\big(\langle P_j b_i, P_j x\rangle\big) \in \mathbb{C}^n$. Then for each i, (15) can be written in matrix form as $Vz_i = 0$ where V is the $n \times n$ Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}, \tag{16}$$

which is invertible since, by assumption, the λ_js are distinct. Thus, $z_i = 0$. Hence, for each j, we have that $\langle P_j b_i, P_j x\rangle = 0$ for all $i \in \Omega$. Since $\{P_j(b_i) : i \in \Omega\}$ form a frame of $P_j(\mathbb{C}^d)$, $P_j x = 0$. Hence, $P_j x = 0$ for $j = 1, \dots, n$ and therefore $x = 0$. $\square$

Remark 2.17. Given a general linear transformation $T : V \longrightarrow V$, the cyclic decomposition theorem gives the rational form for the matrix of T in some special basis. A natural question is then if we can obtain a similar result to Theorem 2.11 for this decomposition. (Rational form instead of Jordan form.) The answer is no. That is, if a set of vectors b_i with $i \in \Omega$ where Ω is a finite subset of $\{1, \dots, d\}$ when projected onto the subspace generated by the cyclic vectors, is complete in this subspace, this does not necessarily imply that its iterations $T^j b_i$ are complete in V. The following example illustrates this fact for a single cyclic operator.

- Let T be the linear transformation in $\mathbb{R}^3$ given as multiplication by the following matrix M.

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The matrix M is in rational form with just one cyclic block. The vector $e_1 = (1, 0, 0)$ is cyclic for M. However it is easy to see that there exists a vector $b = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$ such that $P_W(b) = x_1 \neq 0$ (here W is $\text{span}\{e_1\}$), but $\{b, Mb, M^2 b\}$ are linearly dependent, and hence do not span $\mathbb{R}^3$. So our proof for the Jordan form uses the fact that the cyclic components in the Jordan decomposition are nilpotent!

3. Dynamical sampling in infinite dimensions

In this section we consider the dynamical sampling problem in a separable Hilbert space $\mathcal{H}$, that without any loss of generality can be considered to be $\ell^2(\mathbb{N})$. The evolution operators that we will consider belong to the following class $\mathcal{A}$ of bounded self adjoint operators:

$$\mathcal{A} = \{A \in \mathcal{B}(\ell^2(\mathbb{N})) : A = A^*, \text{and there exists a basis of } \ell^2(\mathbb{N}) \text{ of eigenvectors of } A\}.$$

The notation $\mathcal{B}(\mathcal{H})$ stands for the bounded linear operators on the Hilbert space $\mathcal{H}$. So, if $A \in \mathcal{A}$ there exists an unitary operator B such that $A = B^* D B$ with $D = \sum_j \lambda_j P_j$ with pure spectrum $\sigma_p(A) = \{\lambda_j : j \in \mathbb{N}\} \subset \mathbb{R}$, with $\sup_j |\lambda_j| < +\infty$ and orthogonal projections $\{P_j\}$ such that $\sum_j P_j = I$ and $P_j P_k = 0$ for $j \neq k$. Note that the class $\mathcal{A}$ includes all the bounded self-adjoint compact operators.

Recall that a set $\{v_k\}$ in a Hilbert space $\mathcal{H}$ is

- **complete**, if $\langle f, v_k \rangle = 0 \ \forall k \implies f = 0$,
- **minimal** if $\forall j, v_j \notin \overline{\text{span}}\{v_k\}_{k \neq j}$,
- a **frame** if there exist constants $C_1, C_2 > 0$ such that for all $f \in \mathcal{H}$, $A\|f\|_{\mathcal{H}}^2 \le \sum_k |\langle f, v_k \rangle|^2 \le B\|f\|_{\mathcal{H}}^2$, and
- a **Riesz basis**, if it is a basis which is also a frame.

Remark 3.1. Note that by the definition of $\mathcal{A}$, we have that for any $f \in \ell^2(\mathbb{N})$ and $l = 0, 1, \dots$

$$\langle f, A^l e_j \rangle = \langle f, B^* D^l B e_j \rangle = \langle Bf, D^l b_j \rangle \quad \text{and} \quad \|A^l\| = \|D^l\|.$$

It follows that $\mathcal{F}_\Omega = \{A^l e_i : \ i \in \Omega, l = 0, \dots, l_i\}$ is complete, (minimal, frame) if and only if $\{D^l b_i : i \in \Omega, l = 0, \dots, l_i\}$ is complete (minimal, frame).

3.1. Completeness

In this section, we characterize the sampling sets $\Omega \subset \mathbb{N}$ such that a function $f \in \ell^2(\mathbb{N})$ can be recovered from the data

$$Y = \{f(i), Af(i), A^2f(i), \dots, A^{l_i}f(i) : i \in \Omega\}$$

where $A \in \mathcal{A}$, and $0 \le l_i \le \infty$.

Definition 3.2. Given $A \in \mathcal{A}$, for each set Ω we consider the set of vectors $O_\Omega := \{b_j = Be_j : j \in \Omega\}$, where e_j is the jth canonical vector of $\ell^2(\mathbb{N})$. For each $b_i \in O_\Omega$ we define r_i to be the degree of the D-annihilator of b_i if such an annihilator exists, or we set $r_i = \infty$. Since B is unitary, this number r_i is also the degree of the A-annihilator of e_i, for the remainder of this paper we let $l_i = r_i - 1$. Also, for convenience of notation, let $\Omega_\infty := \{i \in \Omega : l_i = \infty\}$.

Theorem 3.3. *Let $A \in \mathcal{A}$ and $\Omega \subset \mathbb{N}$. Then the set $\mathcal{F}_\Omega = \left\{A^le_i : i \in \Omega, l = 0, \dots, l_i\right\}$ is complete in $\ell^2(\mathbb{N})$ if and only if for each j, the set $\left\{P_j(b_i) : i \in \Omega\right\}$ is complete on the range E_j of P_j.*

Remarks 3.4.

i) Note that Theorem 3.3 implies that $|\Omega| \ge \sup_j \dim(E_j)$. Thus, if some eigenspace has infinite dimension or if $\sup_j \dim(E_j) = +\infty$, then it is necessary to have infinitely many “spacial” sampling points in order to recover f. In particular if Ω is finite, a necessary condition on A in order for $\mathcal{F}_\Omega$ to be complete is that for all j, $\dim(E_j) < M < +\infty$ for some positive constant M.

ii) Theorem 3.3 can be extended to a larger class of operators. For example, for the class of operators $\widetilde{\mathcal{A}}$ in $\mathcal{B}(\ell^2(\mathbb{N}))$ in which $A \in \widetilde{\mathcal{A}}$ if $A = B^{-1}DB$ where with $D = \sum_j \lambda_j P_j$ with pure spectrum $\sigma_p(A) = \{\lambda_j : j \in \mathbb{N}\} \subset \mathbb{C}$ and orthogonal projections $\{P_j\}$ such that $\sum_j P_j = I$ and $P_jP_k = 0$ for $j \ne k$.

Proof of Theorem 3.3. By Remark 3.1, to prove the theorem we only need to show that $\left\{D^lb_i : i \in \Omega, l = 0, \dots, l_i\right\}$ is complete if and only if for each j, the set $\left\{P_j(b_i) : i \in \Omega\right\}$ is complete in the range E_j of P_j.

Assume that $\left\{D^lb_i : i \in \Omega, l = 0, \dots, l_i\right\}$ is complete. For a fixed j, let $g \in E_j$ and assume that $< g, P_jb_i >= 0$ for all $i \in \Omega$. Then for any $l = 0, 1, \dots, l_i$, we have

$$\lambda_j^l < g, P_jb_i >=< g, \lambda_j^l P_jb_i >=< g, P_jD^lb_i >=< g, D^lb_i >= 0.$$

Since $\left\{D^lb_i : i \in \Omega, l = 0, \dots, l_i\right\}$ is complete in $\ell^2(\mathbb{N})$, $g = 0$. It follows that $\left\{P_j(b_i) : i \in \Omega\right\}$ is complete on the range E_j of P_j.

Now assume that $\left\{P_j(b_i) : i \in \Omega\right\}$ is complete in the range E_j of P_j. Let $S = \overline{\operatorname{span}}\left\{D^lb_i : i \in \Omega, l = 0, \dots, l_i\right\}$. Clearly $DS \subset S$. Thus S is invariant for D. Since D is self-adjoint, $S^\perp$ is also invariant for D. It follows that the orthogonal projection $P_{S^\perp}$ commutes with D. Thus $P_{S^\perp} = \sum_j P_jP_{S^\perp}P_j$. Multiplying this last expression by P_k from the right, we get that $P_{S^\perp}P_k = P_kP_{S^\perp}P_k$. Multiplying to the left, we get that $P_kP_{S^\perp} = P_kP_{S^\perp}P_k$. Hence, $P_{S^\perp}$ commutes with P_k for all k. Therefore, for each $i \in \Omega$, $0 = P_jP_{S^\perp}(b_i) = P_{S^\perp}P_j(b_i)$.

So $P_{S^\perp}$ is zero in E_j for all j (since $\{P_j(b_i) : i \in \Omega\}$ is complete in E_j). Hence $P_{S^\perp}$ is zero everywhere which implies that $S^\perp$ is the zero subspace. That is $S = \ell^2(\mathbb{N})$, and $\mathcal{F}_\Omega$ is complete which finishes the proof of the theorem. □

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3.2. Minimality and bases for the dynamical sampling in infinite dimensional Hilbert spaces

In this section we will show, that if $\Omega \subset \mathbb{N}$ is finite, and the set $\mathcal{F}_\Omega = \{A^l e_i : i \in \Omega, l = 0, \dots, l_i\}$ is complete, then it can *never* be minimal, and hence the set $\mathcal{F}_\Omega$ is never a basis. In some sense, the set $\mathcal{F}_\Omega$ contains many "redundant vectors" which prevents it from being a basis. However, since $\mathcal{F}_\Omega$ is complete, this redundancy may help $\mathcal{F}_\Omega$ to be a frame. We will discuss this issue in the next section. For this section, we need the celebrated Müntz–Szász Theorem characterizing the sequences of monomials that are complete in $C[0, 1]$ or $C[a, b]$ [24]:

Theorem 3.5 *(Müntz–Szász Theorem). Let $0 \le n_1 \le n_2 \le \dots$ be an increasing sequence of nonnegative integers that goes to $+\infty$. Then*

(1) *$\{x^{n_k}\}$ is complete in $C[0, 1]$ if and only if $n_1 = 0$ and $\sum_{k=2}^{\infty} 1/n_k = \infty$.*
(2) *If $0 < a < b < \infty$, then $\{x^{n_k}\}$ is complete in $C[a, b]$ if and only if $\sum_{k=2}^{\infty} 1/n_k = \infty$.*

We are now ready to state the main results of this section.

Theorem 3.6. *Let $A \in \mathcal{A}$ and let Ω be a non-empty subset of $\mathbb{N}$. If there exists $b_i \in O_\Omega$ such that $r_i = \infty$, then the set $\mathcal{F}_\Omega$ is not minimal.*

As an immediate corollary we get

Theorem 3.7. *Let $A \in \mathcal{A}$ and let Ω be a finite subset of $\mathbb{N}$. If $\mathcal{F}_\Omega = \{A^l e_i : i \in \Omega, l = 0, \dots, l_i\}$ is complete in $\ell^2(\mathbb{N})$, then $\mathcal{F}_\Omega$ is not minimal in $\ell^2(\mathbb{N})$.*

Proof. Since $\mathcal{F}_\Omega \in \ell^2(\mathbb{N})$, there exists some b_i with $r_i = \infty$ and then Theorem 3.6 applies. □

Another immediate corollary is

Theorem 3.8. *Let $A \in \mathcal{A}$ and let Ω be a finite subset of $\mathbb{N}$. Then the set $\mathcal{F}_\Omega = \{A^l e_i : i \in \Omega, l = 0, \dots, l_i\}$ is not a basis for $\ell^2(\mathbb{N})$.*

Proof. A basis is a complete set, so the result is a consequence of Theorem 3.7. □

Remarks 3.9.

(1) Theorem 3.8 remains true for the class of operators $A \in \widetilde{\mathcal{A}}$ described in Remark 3.4.
(2) Theorems 3.7 and 3.8 do not hold in the case of Ω being an infinite set. A trivial example is when $A = I$ is the identity matrix and $\Omega = \mathbb{N}$. A less trivial example is when $B \in \ell^2(\mathbb{Z})$ is the symmetric bi-infinite matrix with entries $B_{ii} = 1$, $B_{i(i+1)} = 1/4$ and $B_{i(i+k)} = 0$ for $k \ge 2$. Let $\Omega = 3\mathbb{Z}$ and $D_{kk} = 2$ if $k = 3\mathbb{Z}$, $D_{kk} = 1$ if $k = 3\mathbb{Z} + 1$, and $D_{kk} = -1$ if $k = 3\mathbb{Z} + 2$. Then $\mathcal{F}_\Omega = \{A^l e_i : i \in \Omega, l = 0, \dots, 2\}$ is a basis for $\ell^2(\mathbb{Z})$. In fact $\mathcal{F}_\Omega$ is a Riesz basis of $\ell^2(\mathbb{Z})$. Examples in which the Ω is nonuniform can be found in [4].

Proof of Theorem 3.6. Again, using Remark 3.1, we will show that $\{D^l b : l = 0, 1, \dots\}$ is not minimal. We first assume that $D = \sum_j \lambda_j P_j$ is non-negative, i.e., $\lambda_j \ge 0$ for all $j \in \mathbb{N}$. Since $A \in B(\ell^2(\mathbb{N}))$, we also have that $0 \le \lambda_j \le \|D\| < \infty$. Let $b \in O_\Omega$ be such that its D-annihilator has degree $r = \infty$ and let n_k be any increasing sequence of nonnegative integers such that $\sum_{k=2}^{\infty} 1/n_k = \infty$.

Fix $f \in \overline{\operatorname{span}}\{D^l b : l = 0, 1, \dots\}$. Then for any $\epsilon > 0$, there exists a polynomial p such that $\|f - p(D)b\|_2 \leq \epsilon/2$. Since the polynomial p is a continuous function on $C[0, \|D\|]$, (by the Müntz–Szász Theorem) there exists a polynomial $g \in \operatorname{span}\{1, x^{n_k} : k \in \mathbb{N}\}$ such that $\sup \left\{|p(x) - g(x)| : x \in [0, \|D\|]\right\} \leq \frac{\epsilon}{2\|b\|_2}$.

Now we note that

$$\|p(D)b - g(D)b\|^2_{\ell^2(\mathbb{N})} = \sum_j |p(\lambda_j) - g(\lambda_j)|^2 |b_j|^2 \leq (\epsilon/2)^2.$$

Hence

$$\|f - g(D)b\|_2 \leq \|f - p(D)b\|_2 + \|p(D)b - g(D)b\|_2 \leq \epsilon$$

Therefore $\overline{\operatorname{span}}\{b, D^{n_k} b : k \in \mathbb{N}\} = \overline{\operatorname{span}}\{D^l b : l = 0, 1, \dots\}$ and we conclude that $\{D^l b : l = 0, 1, \dots\}$ is not minimal.

If the assumption about the non-negativity of $D = \sum_j \lambda_j P_j$ is removed, then by the previous argument $\{D^{2l} b : l = 0, 1, \dots\}$ is not minimal hence $\{D^l b : l = 0, 1, \dots\}$ is not minimal either, and the proof is complete. $\square$

The following corollary of Theorem 3.6 will be needed later.

Corollary 3.10. *Let b be such that its D-annihilator has degree $r = \infty$. If there exists an increasing sequence $\{n_k : k \in \mathbb{N}\}$ of positive integers such that $\sum_{k=2}^{\infty} \frac{1}{n_k} = +\infty$, then the collection $\{D^{n_k} b : k \in \mathbb{N}\}$ is not minimal.*

Proof. Pick a subsequence $\{n_{k_j}\}$ of $\{n_k\}$ such that $\sum_{j=2}^{\infty} \frac{1}{n_{k_j}} = +\infty$ and apply the same argument as in the proof of the theorem. $\square$

3.3. Frames in infinite dimensional Hilbert spaces

In the previous sections, we have seen that although the set $\mathcal{F}_\Omega = \left\{A^l e_i : \ i \in \Omega, l = 0, \dots, l_i\right\}$ can be complete for appropriate sets Ω, it cannot form a basis for $\ell^2(\mathbb{N})$ if Ω is a finite set, in general. The main reason is that $\mathcal{F}_\Omega$ cannot be minimal, which is necessary to be a basis. On the other hand, the non-minimality is a statement about redundancy. Thus, although $\mathcal{F}_\Omega$ cannot be a basis, it is possible that $\mathcal{F}_\Omega$ is a frame for sets $\Omega \subset \mathbb{N}$ with finite cardinality. Being a frame is in fact desirable since in this case we can reconstruct any $f \in \ell^2(\mathbb{N})$ in stable way from the data $Y = \{f(i), Af(i), A^2 f(i), \dots, A^{l_i} f(i) : i \in \Omega\}$.

In this section we will show that, except for some special case of the eigenvalues of A, if Ω is a finite set, i.e., $|\Omega| < \infty$, then $\mathcal{F}_\Omega$ can never be a frame for $\ell^2(\mathbb{N})$. Thus essentially, either the eigenvalues of A are *nice*, as we will make precise below, in which case we can choose Ω to consist of just one element whose iterations may be a frame, or, the only hope for $\mathcal{F}_\Omega$ to be a frame for $\ell^2(\mathbb{N})$ is that Ω is infinite in which case it needs to be well-spread over $\mathbb{N}$.

Theorem 3.11. *Let $A \in \mathcal{A}$ and let $\Omega \subset \mathbb{N}$ be a finite subset of $\mathbb{N}$. If $\mathcal{F}_\Omega = \left\{A^l e_i : \ i \in \Omega, l = 0, \dots, l_i\right\}$ is a frame, with constants C_1 and C_2, then*

$$\inf\{\|A^l e_i\|_2 : \ i \in \Omega, l = 0, \dots, l_i\} = 0.$$

Proof. If $\mathcal{F}_\Omega$ is a frame, then it is complete. Therefore, since Ω is finite, there exists $i_0 \in \Omega$ with $l_{i_0} = +\infty$. We have:

$$\sum_{l=0}^{+\infty} \|A^l e_{i_0}\|^4 = \sum_{l=0}^{+\infty} | < A^l e_{i_0}, A^l e_{i_0} > |^2 = \sum_{l=0}^{+\infty} | < e_{i_0}, A^{2l} e_{i_0} > |^2 \leq C_2.$$

As a consequence $\|A^l e_{i_0}\|$ goes to zero with l. □

Therefore, when $|\Omega| < \infty$, the only possibility for $\mathcal{F}_\Omega$ to be a frame, is that

$$\inf\{\|A^l e_i\|_2 : \ i \in \Omega, l = 0, \dots, l_i\} = 0$$

and

$$\sup\{\|A^l e_i\|_2 : \ i \in \Omega, l = 0, \dots, l_i\} \leq C < \infty.$$

We have the following theorem to establish for which finite sets Ω, $\mathcal{F}_\Omega$ is not a frame for $\ell^2(\mathbb{N})$.

Theorem 3.12. *Let $A \in \mathcal{A}$ and let Ω be a finite subset of $\mathbb{N}$. For $\mathcal{F}_\Omega = \{A^l e_i : \ i \in \Omega, l = 0, \dots, l_i\}$ to be a frame, it is necessary that 1 or -1 are cluster points of $\sigma(A)$.*

Since a compact self-adjoint operator on a Hilbert space either has finitely many eigenvalues or the eigenvalues form a sequence that goes to zero, we have the following corollary:

Corollary 3.13. *Let A be a compact self-adjoint operator, and $\Omega \subset \mathbb{N}$ be a finite set. Then $\mathcal{F}_\Omega = \{A^l e_i : \ i \in \Omega, l = 0, \dots, l_i\}$ is not a frame.*

Remark 3.14. Theorems 3.11 and 3.12 can be generalized to the class $\mathcal{A}$ defined in (ii) of Remark 3.4.

Proof of Theorem 3.12. If $\mathcal{F}_\Omega$ is a frame then it is complete in $\ell^2(\mathbb{N})$, then the set $\Omega_\infty := \{i \in \Omega : l_i = \infty\}$ is nonempty.

Using again the completeness of $\mathcal{F}_\Omega$ we see that the set

$$J = \{j \in \mathbb{N} : P_j b_i = 0, \ \forall i \in \Omega_\infty\},$$

must be finite. (For this note that if J is infinite then $\bigoplus_{j \in J} E_j$ is infinite dimensional and cannot be generated by the finite set of vectors $\{D^l b_i : \ i \in \Omega \setminus \Omega_{+\infty}, l = 0, \dots, l_i\}$.)

If there exist $j \in \mathbb{N}$ and $i \in \Omega_\infty$ such that $|\lambda_j| \geq 1$ and $P_j b_i \neq 0$ then for $x = P_j b_i$ we have

$$\sum_l |\langle x, D^l b_i \rangle|^2 = \sum_l |\lambda_j|^{2l} \|P_j b_i\|_2^4 = \infty.$$

Thus, $\mathcal{F}_\Omega$ is not a frame.

Otherwise, let $r := \sup_{j \in \mathbb{N}} \{|\lambda_j| : P_j b_i \neq 0 \text{ for some } i \in \Omega_\infty\}$.

Since -1 or 1 are not cluster points of $\sigma(A)$, $r < 1$. But

$$\|D b_i\|_2 \leq \sup_{j \in \mathbb{N}} \{|\lambda_j| : P_j b_i \neq 0\} \|b_i\|_2 \quad \forall i \in \Omega_\infty,$$

and therefore we have that $\|D^l b_i\|_2 \leq r^l \|b_i\|_2$. Now given $\epsilon > 0$, there exists N such that

$$\sum_{i \in \Omega_\infty} \sum_{l > N} \|D^l b_i\|_2^2 \leq \epsilon.$$

Choose $f \in \ell^2(\mathbb{N})$ such that $\|f\|_2 = 1$, $< f, D^l b_i >= 0$ for all $i \in \Omega - \Omega_\infty$ and $l = 0, \dots, l_i$ and such that $< f, D^l b_i >= 0$ for all $i \in \Omega_\infty$ and $l = 0, \dots, N$. Then

$$\sum_{i\in\Omega}\sum_{l=0}^{l_i} | < f, D^l b_i > |^2 \leq \epsilon = \epsilon \|f\|_2.$$

Since ϵ is arbitrary, the last inequality implies that $\mathcal{F}_\Omega$ is not a frame since it cannot have a positive lower frame bound. □

Although Theorem 3.12 states that $\mathcal{F}_\Omega$ is not a frame for $\ell^2(\mathbb{N})$, it could be that after normalization of the vectors in $\mathcal{F}_\Omega$, the new set $\mathcal{Z}_\Omega$ is a frame for $\ell^2(\mathbb{N})$. It turns out that the obstruction is intrinsic. In fact, this case is even worse, since $\mathcal{Z}_\Omega$ is not a frame even if 1 or -1 is (are) a cluster point(s) of $\sigma(A)$.

Theorem 3.15. *Let $A \in \mathcal{A}$ and let $\Omega \subset \mathbb{N}$ be a finite set. Then the unit norm sequence $\left\{\frac{A^l e_i}{\|A^l e_i\|_2} : i \in \Omega, l = 0, \dots, l_i\right\}$ is not a frame.*

Proof. Note that by Remark 3.1, $\left\{\frac{A^l e_i}{\|A^l e_i\|_2} : i \in \Omega, l = 0, \dots, l_i\right\}$ is a frame if and only if $\mathcal{Z}_\Omega = \left\{\frac{D^l b_i}{\|D^l b_i\|_2} : i \in \Omega, l = 0, \dots, l_i\right\}$ is a frame.

Assume that $\mathcal{Z}_\Omega$ is a frame. Since it is norm-bounded (actually unit norm), the Kadison–Singer/Feichtinger conjectures proved recently [35] applies, and $\mathcal{Z}_\Omega$ is the finite union of Riesz sequences $\bigcup_{j=1}^{N} R_j$.

Because $\mathcal{Z}_\Omega$ is complete, there exists some b such that its D-annihilator has degree $r = \infty$, $j \in \{1, \dots, N\}$ and an increasing sequence of positive integers $\{n_k\}$ with $\sum_{k\geq 2} \frac{1}{n_k} = +\infty$ such that

$$S = \left\{\frac{D^{n_k} b}{\|D^{n_k} b\|_2} : k \in \mathbb{N}\right\} \subset R_j.$$

The set S is a Riesz sequence, because it is a subset of a Riesz sequence. On the other hand, S is not minimal by Corollary 3.10, which is a contradiction since a Riesz sequence is always a minimal set. □

We will now concentrate on the case when there is a cluster point of $\sigma(A)$ at 1 or -1, and we start with the case where Ω consists of a single sampling point, i.e., $O_\Omega = \{b\}$. Let us denote by r_b, the degree of the D-annihilator of b and $l_b = r_b - 1$ if r_b is finite or $l_b = +\infty$ otherwise.

Since $A \in \mathcal{A}$, $A = B^*DB$, by Remark 3.1 $\mathcal{F}_\Omega$ is a frame of $\ell^2(\mathbb{N})$ if and only if there exists a vector $b = Be_j$ for some $j \in \mathbb{N}$ that corresponds to the sampling point, and $\{D^l b : l = 0, 1, \dots\}$ is a frame for $\ell^2(\mathbb{N})$.

For this case, Theorem 3.3 implies that if $\mathcal{F}_\Omega$ is a frame of $\ell^2(\mathbb{N})$, then the projection operators P_j used in the description of the operator $A \in \mathcal{A}$ must be of rank 1. Moreover, the vector b corresponding to the sampling point must have infinite support, otherwise l_b will be finite and $\mathcal{F}_\Omega$ cannot be complete in $\ell^2(\mathbb{N})$. Moreover, for this case in order for $\mathcal{F}_\Omega$ to be a frame, it is necessary that $|\lambda_k| < 1$ for all k, otherwise, if there exists $\lambda_{j_0} \geq 1$ then for $x = P_{j_0} b$ (note that by Theorem 3.3 $P_{j_0} b \neq 0$) we would have

$$\sum_n |\langle x, D^n b\rangle|^2 = \sum_n |\lambda_{j_0}|^{2n} \|P_{j_0} b\|_2^4 = \infty,$$

which is a contradiction.

In addition, if $\mathcal{F}_\Omega$ is a frame, then the sequence $\{\lambda_k\}$ cannot have a cluster point a with $|a| < 1$. To see this, suppose there is a subsequence $\lambda_{k_s} \to a$ for some a with $|a| < 1$, and let W be the orthogonal sum of the eigenspaces associated to the eigenvalues λ_{k_s}. Then W is invariant for D. Set $D_1 = D|_W$, and $\tilde{b} = P_W b$

where P_W is the orthogonal projection on W. Then, by Theorem 3.12, $\{D_1^j \tilde{b} : j = 0, 1, \dots\}$ cannot be a frame for W. It follows that $\mathcal{F}_\Omega$ cannot be a frame for $\ell^2(\mathbb{N})$, since the orthogonal projection of a frame onto a closed subspace is a frame of the subspace.

Thus the only possibility for $\mathcal{F}_\Omega$ to be a frame of $\ell^2(\mathbb{N})$ is that $|\lambda_k| \to 1$. These remarks allow us to characterize when $\mathcal{F}_\Omega$ is a frame for the situation when $|\Omega| = 1$.

Theorem 3.16. *Let $D = \sum_j \lambda_j P_j$ be such that P_j have rank 1 for all $j \in \mathbb{N}$, and let $b := \{b(k)\}_{k \in \mathbb{N}} \in \ell^2(\mathbb{N})$. Then $\{D^l b : l = 0, 1, \dots\}$ is a frame if and only if*

i) $|\lambda_k| < 1$ *for all* k.
ii) $|\lambda_k| \to 1$.
iii) $\{\lambda_k\}$ *satisfies Carleson's condition*

$$\inf_n \prod_{k \neq n} \frac{|\lambda_n - \lambda_k|}{|1 - \bar{\lambda}_n \lambda_k|} \geq \delta. \tag{17}$$

for some $\delta > 0$.
iv) $b(k) = m_k \sqrt{1 - |\lambda_k|^2}$ *for some sequence* $\{m_k\}$ *satisfying* $0 < C_1 \leq |m_k| \leq C_2 < \infty$.

This theorem implies the following Corollary:

Corollary 3.17. *Let $A = B^* D B \in \mathcal{A}$, and $D = \sum_j \lambda_j P_j$ be such that P_j have rank 1 for all $j \in \mathbb{N}$. Then, there exists $i_0 \in \mathbb{N}$ such that $\mathcal{F}_\Omega = \{A^l e_{i_0} : l = 0, \dots\}$ is a frame for $\ell^2(\mathbb{N})$, if and only if $\{\lambda_j\}$ satisfy the conditions of Theorem 3.16 and there exists $i_0 \in \mathbb{N}$, such that $b = B e_{i_0}$ satisfies the condition* iv *of Theorem 3.16.*

Theorem 3.16 follows from the discussion above and the following two Lemmas

Lemma 3.18. *Let D be as in Theorem 3.16 and assume that $|\lambda_k| < 1$ for all k. Let $b^0(k) = \sqrt{1 - |\lambda_k|^2}$, and assume that $b^0 \in \ell^2(\mathbb{N})$. Let $b \in \ell^2(\mathbb{N})$.*

Then, $\{D^l b : l \in \mathbb{N}\}$ is a frame for $\ell^2(N)$ if and only if $\{D^l b^0 : l \in \mathbb{N}\}$ is a frame and there exist C_1 and C_2 such that $b(k)/b^0(k) = m_k$ satisfies $0 < C_1 \leq |m_k| \leq C_2 < \infty$.

Note that by assumption $\sum_{k=1}^{\infty} (1 - |\lambda_k|^2) < +\infty$ since $b^0 \in \ell^2(\mathbb{N})$. In particular $|\lambda_k| \to 1$.

Lemma 3.19. *Let $D = \sum_j \lambda_j P_j$ be such that $|\lambda_k| < 1$, $\lambda_k \longrightarrow 1$ and let $b^0(k) = \sqrt{1 - |\lambda_k|^2}$. Then the following are equivalent:*

i)

$$\{b^0, Db^0, D^2 b^0, \dots\} \text{ is a frame for } \ell^2(N)$$

ii)

$$\inf_n \prod_{k \neq n} \frac{|\lambda_n - \lambda_k|}{|1 - \bar{\lambda}_n \lambda_k|} \geq \delta.$$

for some $\delta > 0$.

In Lemma 3.19, the assumption $\lambda_k \longrightarrow 1$ can be replaced by $\lambda_k \longrightarrow -1$ and the lemma remains true. Its proof, below, is due to J. Antezana [6] and is a consequence of a theorem by Carleson [26] about interpolating sequences in the Hardy space $H^2(\mathbb{D})$ of the unit disk in $\mathbb{C}$.

Proof of Lemma 3.18. Let us first prove the sufficiency. Assume that $\{D^l b^0 : l \in \mathbb{N}\}$ is a frame for $\ell^2(\mathbb{N})$ with positive frame bounds A, B, and let $b \in \ell^2(\mathbb{N})$ such that $b(k) = m_k b^0(k)$ with $0 < C_1 \leq |m_k| \leq C_2 < \infty$. Let $x \in \ell^2(\mathbb{N})$ be an arbitrary vector and define $y_k = \overline{m_k} x_k$. Then $y \in \ell^2(\mathbb{N})$ and $C_1\|x\|_2 \leq \|y\| \leq C_2\|x\|_2$. Hence

$$C_1^2 A\|x\|_2^2 \leq \sum_l |\langle y, D^l b^0\rangle|^2 = \sum_l \langle x, D^l b\rangle|^2 \leq C_2^2 B\|x\|_2^2,$$

and therefore $\{D^l b : l \in \mathbb{N}\}$ is a frame for $\ell^2(\mathbb{N})$.

Conversely, let $b \in \ell^2(\mathbb{N})$ and assume that $\{D^l b : l \in \mathbb{N}\}$ is a frame for $\ell^2(N)$ with frame bounds A' and B'. Then for any vector e_k of the standard orthonormal basis of $\ell^2(\mathbb{N})$, we have

$$A' \leq \sum_{l=0}^{\infty} | < e_k, D^l b > |^2 = \frac{|b(k)|^2}{1 - |\lambda_k|^2} \leq B'.$$

Thus $\sqrt{A'} b^0(k) \leq |b(k)| \leq \sqrt{B'} b^0(k)$ for all k. Thus, the sequence $\{m_k\} \subset \mathbb{C}$ defined by $b(k) = m_k b^0(k)$ satisfies $\sqrt{A'} \leq |m_k| \leq \sqrt{B'}$.

Let $x \in \ell^2(\mathbb{N})$ be an arbitrary vector and define now $y_k = \frac{1}{m_k} x_k$. Then $y \in \ell^2(\mathbb{N})$ and

$$\frac{A'}{B'}\|x\|_2^2 \leq \sum_l |\langle x, D^l b^0\rangle|^2 = \sum_l |\langle y, D^l b\rangle|^2 \leq \frac{B'}{A'}\|x\|_2^2,$$

and so $\{D^l b^0 : l \in \mathbb{N}\}$ is a frame for $\ell^2(\mathbb{N})$. $\square$

The proof of Lemma 3.19 relies on a Theorem by Carleson on interpolating sequences in the Hardy space $H^2(\mathbb{D})$ on the open unit disk $\mathbb{D}$ in the complex plane. If $H(\mathbb{D})$ is the vector space of holomorphic functions on $\mathbb{D}$, $H^2(\mathbb{D})$ is defined as

$$H^2(\mathbb{D}) = \Big\{ f \in H(\mathbb{D}) : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ for some sequence } \{a_n\} \in \ell^2(\mathbb{N}) \Big\}.$$

Endowed with the inner product between $f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} a'_n z^n$ defined by $\langle f, g\rangle = \sum a_n \overline{a'_n}$, $H^2(\mathbb{D})$ becomes a Hilbert space isometrically isomorphic to $\ell^2(\mathbb{N})$ via the isomorphism $\Phi(f) = \{a_n\}$.

Definition 3.20. A sequence $\{\lambda_k\}$ in $\mathbb{D}$ is an interpolating sequence for $H^2(\mathbb{D})$ if for any sequence $\{c_k\}$ such that $\sum_k |c_k|^2(1 - |\lambda_k|^2) < +\infty$, there exists a function $f \in H^2(\mathbb{D})$ such that $f(\lambda_k) = c_k$.

Proof of Lemma 3.19. Let $\mathcal{T}_k$, denote the vector in $\ell^2(\mathbb{N})$ defined by $\mathcal{T}_k = (1, \lambda_k, \lambda_k^2, \dots)$, and $x \in \ell^2(\mathbb{N})$. Then

$$\sum_{l=0}^{\infty} | < x, D^l b^0 > |^2 = \sum_{l=0}^{\infty} \Big| \sum_{k=1}^{\infty} x_k \lambda_k^l \sqrt{1 - \lambda_k^2} \Big|^2 = \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} \frac{< \mathcal{T}_s, \mathcal{T}_t >}{\|\mathcal{T}_s\|_2 \|\mathcal{T}_t\|_2} x_s \overline{x_t}.$$

Thus, for $\{D^l b^0 : l = 0, 1, \dots\}$ to be a frame of $\ell^2(\mathbb{N})$, it is necessary and sufficient that the Gramian $G_\Lambda = \{G_\Lambda(s, t)\} = \left\{ \frac{<\mathcal{T}_s, \mathcal{T}_t>}{\|\mathcal{T}_s\|_2 \|\mathcal{T}_t\|_2} \right\}$ be a bounded invertible operator on $\ell^2(N)$ (Note that G_Λ is then the frame operator for $\{D^l b^0 : l = 0, 1, \dots\}$).

Equivalently, $\{D^l b^0 : l = 0, 1, \dots\}$ is a frame of $\ell^2(\mathbb{N})$ if and only if the sequence $\{\widetilde{\mathcal{T}}_j = \frac{\mathcal{T}_j}{\|\mathcal{T}_j\|_2}\}$ is a Riesz basic sequence in $\ell^2(\mathbb{N})$, i.e., there exist constants $0 < C_1 \le C_2 < \infty$ such that

$$C_1\|c\|_2^2 \le \|\sum_j c_j \widetilde{\mathcal{T}}_j\|_2^2 \le C_2\|c\|_2^2 \quad \text{for all } c \in \ell^2(\mathbb{N}).$$

By the isometric map Φ from $\ell^2(\mathbb{N})$ to $H^2(\mathbb{D})$ defined above, $\{D^l b^0 : l = 0, 1, \dots\}$ is a frame of $\ell^2(\mathbb{N})$ is a frame if and only if the sequence $\{\tilde{k}_{\lambda_j} = \Phi(\widetilde{\mathcal{T}}_j)\}$ is a Riesz basic sequence in $H^2(\mathbb{D})$.

Let $k_{\lambda_j} = \Phi(\mathcal{T}_j)$. It is not difficult to check that for any $f \in H^2(\mathbb{D})$, $\langle f, k_{\lambda_j}\rangle = f(\lambda_j)$ and that $\{\lambda_j\}$ is an interpolating sequence in $H^2(\mathbb{D})$ if and only if $G_\Lambda = \big(\langle \tilde{k}_{\lambda_j}, \tilde{k}_{\lambda_j}\rangle\big)$ is a bounded invertible operator on $\ell^2(\mathbb{N})$. By Carleson' s Theorem [26], this happens if and only if (17) is satisfied. □

Frames of the form $\{D^l b_i : i \in \Omega, l = 0 \dots, l_i\}$ for the case when $|\Omega| \ge 1$ or when the projections P_j have finite rank but possibly greater than or equal to 1 can be easily found by using Theorem 3.16. For example, if $|\Omega| = 2$, $P_j(\ell^2(\mathbb{N}))$ has dimension 1 for $j \in \mathbb{N}$, b_1, $\{\lambda_k\}$ satisfies the conditions of Theorem 3.16 and b_2 is such that $b_2(k) = m_k\sqrt{1 - |\lambda_k|^2}$ for some sequence $\{m_k\}$ satisfying $|m_k| \le C < \infty$. To construct frames for the case when the projections P_j have finite rank but possibly greater than or equal to 1, we note that there exist orthogonal subspaces $W_1, \dots, W_N$ of $\ell^2(\mathbb{N})$ such that operator D_i on each W_i either has finite dimensional range, or satisfies the condition of Theorem 3.16.

4. Concluding remarks

In this paper we have studied the sets of spatial sampling locations Ω that allow us to reconstruct a function f from the samples of $\{f(i), Af(i), \dots, A^{l_i}f(i) : i \in \Omega\}$. The finite dimensional case is completely resolved and we find necessary and sufficient conditions on Ω, l_i, and A for the stable recovery of f.

For the case where $\mathcal{H} \approx \ell^2(\mathbb{N})$, we restricted ourselves to the subclass $\mathcal{A}$ of self-adjoint diagonalizable operators. Without stability requirements, the sets Ω for which a reconstruction of f is possible are completely characterized. For the case where Ω is an infinite set, there are examples for which the stable reconstruction of f is possible as in [4], and it is not difficult to construct other examples of infinite sets Ω for which stable reconstruction is possible as well. However, the problem of finding necessary and sufficient conditions for stable reconstruction is still open.

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