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On second iterated clique graphs that are also third iterated clique graphs

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Abstract

Iterated clique graphs arise when the clique operator is applied to a graph more than once. Determining whether a graph is a clique graph or an iterated clique graph is usually a difficult task. The fact that being a clique graph and being an iterated clique graph are not equivalent things has been proved recently. However, it is still unknown whether the classes of second iterated clique graphs and third iterated clique graphs are the same. In this work we find classes of graphs, defined by means of conditions on the clique size and the structure of the clique intersections, whose second iterated clique graphs are also third iterated clique graphs.

Keywords: Clique graph, clique operator, iterated clique graph, line graph

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1 Introduction

Let G be a simple graph. A *clique* of G is a maximal set of pairwise adjacent vertices of G. The set of cliques of G is denoted by $\mathcal{C}(G)$.

The *clique graph* of G, or $K(G)$, is the graph whose vertices are the cliques of G, in a way that two different cliques C and C' of G are adjacent in $K(G)$ if and only if $C \cap C' \neq \emptyset$. Denote the class of all graphs by $\mathcal{G}$. The function $K : \mathcal{G} \rightarrow \mathcal{G}$ that assigns to each graph its clique graph is called the *clique operator*. G is said to be a clique graph if $G = K(H)$, for some graph H. Thus, $K(\mathcal{G})$ is the class of clique graphs.

Given $n > 1$, the function K^n is the composition of the clique operator with itself n times. The *n-th iterated clique graph* of G is the graph $K^n(G)$. G is an *n-th iterated clique graph* if $G = K^n(H)$, for some graph H. Thus, $K^n(\mathcal{G})$ is the class of n-th iterated clique graphs.

Determining whether a given graph is a clique graph is usually a complicated task. Actually, the clique graph recognition problem is NP-complete [1]. The complexity of recognizing iterated clique graphs has not been established yet. Furthermore, very little is known about the relationship between the classes $K(\mathcal{G})$, $K^2(\mathcal{G})$, $K^3(\mathcal{G})$ ... and their differences.

It is now known that $K(\mathcal{G})$ and $K^2(\mathcal{G})$ are different because the clique graph of the octahedron is not a second iterated clique graph [2]. However, no other example of a considerably different graph in $K(\mathcal{G}) \setminus K^2(\mathcal{G})$ is known and it is unknown whether $K^n(\mathcal{G}) = K^{n+1}(\mathcal{G})$ for $n > 2$.

This work is focused on comparing $K^2(\mathcal{G})$ and $K^3(\mathcal{G})$. We prove that, for every graph G with cliques of size at most 3, $K^2(G)$ is a graph in $K^3(\mathcal{G})$. The proof of this result involves line graphs. Afterwards, we identify some patterns in the proof to attain a more general result that allows cliques of size larger than 3 in the graphs that we consider. As a result of this work, we get a wide variety of graphs in $K^2(\mathcal{G}) \cap K^3(\mathcal{G})$, in a good starting point for proving that $K^2(\mathcal{G}) = K^3(\mathcal{G})$, in case that this is really true. If false, it could help in the refinement of the search for a graph in $K^2(\mathcal{G}) \setminus K^3(\mathcal{G})$, since many possible counterexamples are discarded.

2 Results

As it was said in the Introduction, it is known that $K(\mathcal{G}) \neq K^2(\mathcal{G})$ because the clique graph of the octahedron is a graph in $K(\mathcal{G}) \setminus K^2(\mathcal{G})$ [2]. The octahedron and its clique graph are members of a special class of graphs, called *octahedral graphs*.

For $n \geq 3$, define the *n-th octahedral graph*, or O_n, as the graph with vertex set $\{1, 2, ..., n, 1', 2', ..., n'\}$, where the only missing edges are those of the form ii', with $1 \leq i \leq n$. Thus, the octahedron is equivalent to the graph O_3, and $K(O_3) = O_4$. More generally, it holds that $K(O_n) = O_{2^{n-1}}$ for every $n \geq 3$ [3].

Once it was proved that O_4 is a graph in $K(\mathcal{G}) \setminus K^2(\mathcal{G})$, it was thought that $K^2(O_3)$, i.e., O_8, was a natural candidate for a graph in $K^2(\mathcal{G}) \setminus K^3(\mathcal{G})$. What is more, it was conjectured by one of the authors [2] that $K^n(O_3)$ is in $K^n(\mathcal{G}) \setminus K^{n+1}(\mathcal{G})$ for every $n \geq 1$. However, this conjecture turned out to be short-lived.

Recall that the *line graph* of a graph G, or $L(G)$, is the graph whose vertices are the edges of G, two of them being adjacent if and only if they have a common endpoint (see Figure 1 for an example). It turns out that $K^2(O_3) = K^3(L(O_3))$. Therefore, the statement $K^n(O_3) \in K^n(\mathcal{G}) \setminus K^{n+1}(\mathcal{G})$ is false for every $n \geq 2$.

A review of the operations that allow us to verify that $K^2(O_3) = K^3(L(O_3))$ reveals that the fact that all the cliques of O_3 have size 3 is useful. Actually, line graphs allow to prove that every graph G with all its cliques of size 3 has $K^2(G)$ in $K^3(\mathcal{G})$.

Theorem 2.1 *Let G be a graph such that every clique of G has size 3. Then $K^2(G) = K^3(L(G))$.*

Outline of the proof. Set $G' = K(L(G))$. Let V' be the set of nonsimplicial vertices of G. It is not difficult to prove that G' is equal (isomorphic) to the graph with vertex set $V' \cup \mathcal{C}(G)$, where two vertices in V' are adjacent in G' if and only they are adjacent in G; two cliques of G are adjacent in G' if and only if they share two vertices; and $v \in V'$ and $T \in \mathcal{C}(G)$ are adjacent in G' if and only if $v \in T$.

Consider the graph G'' with vertex set $V(G) \cup \mathcal{C}(G)$, such that the rules of adjacency in G'' are the same as in G'. It can be proved that $K^2(G') = K^2(G'')$.

G'' has two types of cliques, namely, those of the form $C \cup \{C\}$, where C is a clique of G and those of the form $\{u, v\} \cup \{C \in \mathcal{C}(G) : \{u, v\} \subseteq C\}$, where uv is an edge of G contained in at least two of its cliques.

It also holds that two cliques of G'' have nonempty intersection if and only if the intersection has a vertex of G. Let G''' be the graph obtained from G by adding, for each clique C of G'' containing exactly two vertices of G, the vertex C and making it adjacent to the vertices of $V(G) \cap C$. By the first sentence of this paragraph and the characterization of the cliques of G'', we have that $K^2(G'') = K^2(G''')$.

G is a subgraph of G''' and $\mathcal{C}(G) \subseteq \mathcal{C}(G''')$. Let $f : V(K^2(G)) \longrightarrow V(K^2(G'''))$ be a function that assigns to each $H \in V(K^2(G))$ an element $I \in V(K^2(G'''))$ such that $H \subseteq I$. It can be proved that f is a graph isomorphism between $K^2(G)$ and $K^2(G''')$. Therefore, $K^2(G) = K^2(G')$.

□

As a corollary of Theorem 2.1, we have that $K^2(G)$ is in $K^3(\mathcal{G})$ for every graph G whose cliques have size at most 3.

Corollary 2.2 *Let G be a graph such that every clique of G has size at most 3. Then there exists a graph G' such that $G \in K(\mathcal{G})$ and $K^2(G') = K^2(G)$.*

Proof. If every clique of G has size 3, then we can directly apply the previous theorem. Otherwise, for every clique C of G with size less than 3, add simplicial vertices to C until its size becomes 3, to obtain a graph $G^\star$. It is clear that $K(G) = K(G^\star)$ and that every clique of $G^\star$ has size 3. Thus we can apply the previous theorem to $G^\star$ to obtain the desired graph G'. □

An equality like that of Theorem 2.1 becomes false when the graph G has a clique of size larger than 3. What is more, $K^2(G)$ and $K^3(L(G))$ might be very different graphs. To illustrate this, set G equal to the complete graph K_4. It is easy to check that $L(K_4) = O_3$ (see Figure 1). Thus, $K^2(G)$ is the graph consisting of a single vertex, whereas $K^3(L(G)) = O_{128}$.

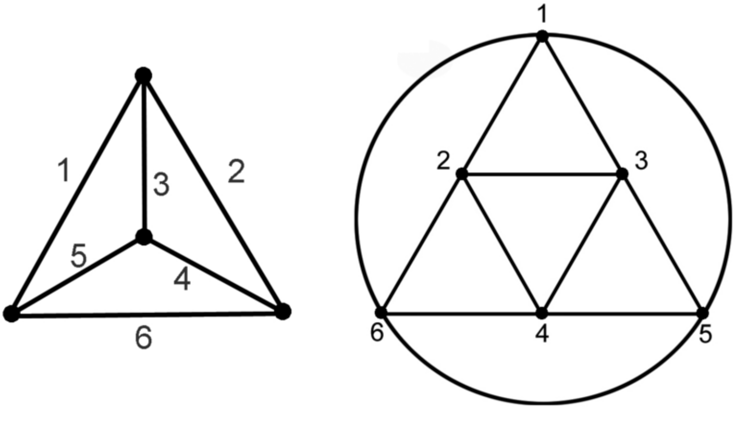

Fig. 1. The line graph of K_4 is equal to the octahedron.

However, we can try to identify more graphs whose second iterated clique graphs are also third iterated clique graphs through a generalization of the constructions in the proof of Theorem 2.1, without the need of referring to line graphs.

Recall that the graph G'' in the proof of Theorem 2.1 can be viewed as the graph whose vertex set is $G \cup \mathcal{C}(G)$, such that two vertices of G are adjacent in G'' if and only if they are adjacent in G; $v \in V(G)$ and $C \in \mathcal{C}(G)$ are adjacent in G'' if and only if $v \in C$ and $C_1, C_2 \in \mathcal{C}(G)$ are adjacent in G'' if and only if $|C_1 \cap C_2| > 1$.

In the general case, we suggest using that type of construction to obtain a new graph H such that H is a clique graph and $K^2(G) = K^2(H)$. These conditions would imply that $K^2(G)$ is in $K^3(\mathcal{G})$.

However, this type of construction also fails for small graphs. Consider for example the graph G_1 in Figure 2. It has cliques $C_1 = \{1,2,3\}$, $C_2 = \{2,3,5,6\}$ and $C_3 = \{5,6,7\}$. Thus, the vertices of $K^2(G_1)$ are $\{C_1, C_2\}$ and $\{C_2, C_3\}$. If we apply the construction recently described to G_1, we obtain the lower graph of Figure 2. This graph, unlike G_1, has a second iterated clique graph with three vertices, namely, $\{\{1,2,3,C_1\}, \{2,3,5,6,C_2\}, \{2,3,C_1,C_2\}\}$, $\{\{5,6,7,C_3\}, \{2,3,5,6,C_2\}, \{5,6,C_2,C_3\}\}$ and $\{\{2,3,5,6,C_2\}, \{2,3,C_1,C_2\}, \{5,6,C_2,C_3\}\}$. Similarly, the construction fails when applied to G_2. Now we show that some restrictions can be added to make the construction work.

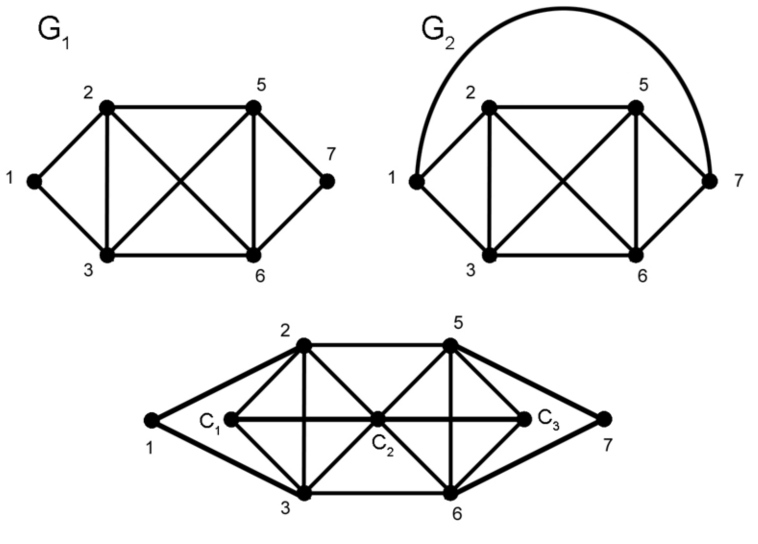

Fig. 2.

Theorem 2.3 *Let G be a G_1 and G_2 free graph (see Figure 2) such that $|C_1 \cap C_2| \leq 2$ for every $C_1, C_2 \in \mathcal{C}(G)$ with $C_1 \neq C_2$. Let G' be the graph such that $V(G') = V(G) \cup \mathcal{C}(G)$ and $E(G') = E(G) \cup \{vC : \ v \in V(G),\ C \in C(G),\ v \in C\} \cup \{CC' : \ C, C' \in \mathcal{C}(G),\ |C \cap C'| = 2\}$. Then $G' \in K(\mathcal{G})$ and*

$K^2(G) = K^2(G')$.

Outline of the proof. The proof shares several ideas with the proof of Theorem 2.1. It is first necessary to prove that every clique of G' is of the form $C \cup \{C\}$, for some clique C of G or of the form $\{u, v\} \cup \{C \in \mathcal{C}(G) : \{u, v\} \subseteq C\}$, where uv is an edge of G contained in more than one clique.

Then it can be proved that two cliques of G' have nonempty intersection if and only if the intersection contains a vertex of G. Let G'' be the graph obtained from G by adding, for each edge e contained in at least two cliques of G, a vertex v_e that is made adjacent to the endpoints of e. As a consequence of the first paragraph and the description of the cliques of G', we have that $K^2(G') = K^2(G'')$.

G is clearly a subgraph of G'' and $\mathcal{C}(G) \subseteq \mathcal{C}(G'')$. Let $f : \mathcal{C}(K(G)) \longrightarrow \mathcal{C}(K(G''))$ be a function such that, for every $A \in \mathcal{C}(K(G))$, $f(A)$ is a clique of $K(G'')$ such that $A \subseteq f(A)$. It can be proved that f is an isomorphism between $K^2(G)$ and $K^2(G'')$. This implies that $K^2(G) = K^2(G')$.

Finally, it is necessary to prove that G' is indeed a clique graph. Assume that G' is connected. Then G' can be seen as the two section graph of the family $\{\{u, v\} \cup \{C \in \mathcal{C}(G) : \{u, v\} \subseteq C\}\}_{uv \in E(G)}$. By the characterization of clique graphs by Roberts and Spencer [4], it suffices to show that this family is Helly. A simple way to prove the Helly property in this case is considering any set T of three pairwise adjacent vertices of G' and proving that the subfamily $\mathcal{F}$ of $\{\{u, v\} \cup \{C \in \mathcal{C}(G) : \{u, v\} \subseteq C\}\}_{uv \in E(G)}$ with all the members that have at least two elements of T is such that the intersection of all its elements is nonempty.

$\square$

References

[1] L. Alcón, L. Faria, C.M.H. de Figueiredo and M. Gutierrez, *The complexity of clique graph recognition*, Theor. Comput. Sci., **410** (2009), 2072-2083.

[2] Pablo De Caria, The images of the clique operator and its square are different, Journal of Graph Theory, **77** (2014), 39-57.

[3] V. Neumann-Lara, On clique divergent graphs, in Problèmes Combinatoires et Théorie des Graphes (Colloques internationaux CNRS, 260) Paris, Editions du CNRS (1978), 313-315.

[4] F. Roberts and J. Spencer, A characterization of clique graphs, J. Combin. Theory B, **10** (1971), 102-108.