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THE BOHNENBLUST-HILLE INEQUALITY COMBINED WITH AN INEQUALITY OF HELSON

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Abstract. We give a variant of the Bohenblust-Hille inequality which, for certain families of polynomials, leads to constants with polynomial growth in the degree.

1. Introduction

Hardy and Littlewood showed in [7] that there exists a constant $K > 0$ such that for every $f \in H^1$ we have

$$\left(\int_{\mathbb{D}} |f(z)|^2 dm(z) \right)^{1/2} \leq K \int_{\mathbb{T}} |f(w)| d\sigma(w) \,,$$

where dm and $d\sigma$ denote respectively the normalised Lebesgue measures on the complex unit disc $\mathbb{D}$ and the torus (or unit circle) $\mathbb{T}$. Equivalently, this means that the Hardy space $H_1(\mathbb{T})$ is contained in the Bergman space $B_2(\mathbb{D})$. Shapiro [12, pp. 117-118] showed that the inequality holds with $K = \pi$, and Mateljević [10] (see also [11, 13]) showed that actually the constant could be taken to be $K = 1$. A simple reformulation of the Bergman norm then gives that if $\sum_{n=0}^{\infty} a_n z^n$ is the Fourier series expansion of $f \in H^1(\mathbb{D})$, we have

$$\left(\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} \right)^{1/2} \leq \int_{\mathbb{T}} |f(w)| d\sigma(w).$$

A few years later Helson in [9] generalised this inequality to functions in N variables. For $n \in \mathbb{N}$ denote by $d(n)$ the number of divisors and by $p^\alpha = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ the prime decomposition of n. Then we have that for every $f \in H^1(\mathbb{T}^N)$ with Fourier series expansion $\sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha$

$$\left(\sum_{\alpha \in \mathbb{N}_0^N} \frac{|c_\alpha|^2}{d(p^\alpha)} \right)^{1/2} \leq \int_{\mathbb{T}^N} |f(w)| d\sigma(w). \tag{1.1}$$

Given a multiindex α, we write $\alpha + 1 = (\alpha_1 + 1) \cdots (\alpha_k + 1)$. Note that, with this notation, we have $d(p^\alpha) = \alpha + 1$.

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On the other hand, by the Bohnenblust-Hille inequality [4] as presented in [5] there is a constant $C > 0$ such that for every m-homogeneous polynomial in N variables $P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha$ with $z \in \mathbb{C}^N$ we have

$$\Big(\sum_{|\alpha|=m} |c_\alpha|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \leq C^m \sup_{z \in \mathbb{D}^N} |P(z)|. \tag{1.2}$$

The proof of this inequality given in [5] consists basically of two steps: first to decompose the sum in (1.2) as the product of certain mixed sums and second to bound each one of these sums by a term including $\|P\|$, the supremum of $|P|$ on $\mathbb{D}^N$. For this second step usually the following result of Bayart [1] is used: for every m-homogeneous polynomial in N variables we have

$$\Big(\sum_{|\alpha|=m} |c_\alpha|^2 \Big)^{1/2} \leq 2^{m/2} \int_{\mathbb{T}^N} \Big| \sum_{|\alpha|=m} c_\alpha w^\alpha \Big| d\sigma(w). \tag{1.3}$$

Very recently, it was proved in [2, Corollary 5.3] that for every $\varepsilon > 0$ there exists $\kappa > 0$ such that we can take $\kappa(1+\varepsilon)^m$ as the constant in (1.2). Our aim in this note is to get a variant of (1.2) by using (1.1) instead of (1.3). With this variant, we see that for polynomials P, each of whose monomials involve a uniformly bounded number of variables, the obtained constants have polynomial growth in m.

2. Main result and some remarks

The following is our main result.

Theorem 2.1. *Let $\Lambda \subseteq \{\alpha \in \mathbb{N}_0^N : |\alpha| = m\}$ be an indexing set. Then for every family $\big(c_\alpha\big)_{\alpha \in \Lambda}$ we have*

$$\Big(\sum_{\alpha \in \Lambda} \Big(\frac{|c_\alpha|}{\sqrt{\alpha+1}} \Big)^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \leq m^{\frac{m-1}{2m}} \Big(1 - \frac{1}{m-1}\Big)^{m-1} \sup_{z \in \mathbb{D}^N} \Big| \sum_{\alpha \in \Lambda} c_\alpha z^\alpha \Big|.$$

We give several remarks before we present the proof.

Remarks 2.2.

(1) It is easy to see that $\sqrt{\alpha+1} \leq \sqrt{2}^m$. Hence the preceding inequality includes the hypercontractive version of the Bohnenblust-Hille inequality from (1.2) as a special case.

(2) Thanks to the term $\sqrt{\alpha+1}$, the constants in the previous inequality grow much more slowly than the constants in (1.2). Actually, we have

$$m^{\frac{m-1}{2m}} \big(1 - \frac{1}{m-1}\big)^{m-1} = \frac{\sqrt{m}}{e} \, \big(1 + o(m)\big).$$

(3) Let $\mathrm{vars}(\alpha)$ denote the number of different variables involved in the monomial z^α. In other words, $\mathrm{vars}(\alpha) = \mathrm{card}\big\{j : \alpha_j \neq 0\big\}$. Given M we consider the set

$$\Lambda_{N,M} = \big\{\alpha \in \mathbb{N}_0^N : |\alpha| = m \text{ and } \mathrm{vars}(\alpha) \leq M\big\}$$

(note that if $M \geq N$, then $\Lambda_{N,M} = \Lambda_{N,N}$). An application of Lagrange multipliers gives that for any $\alpha \in \Lambda_{N,M}$ we have for every N and M,

$$\alpha + 1 = (\alpha_1 + 1) \cdots (\alpha_k + 1) \cdots \leq \Big(\frac{m}{M} + 1\Big)^M.$$

Combining this with Theorem 2.1 we obtain for every m, N, M,

$$\Big(\sum_{\alpha\in\Lambda_{N,M}} |c_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \leq \Big(\frac{m}{M}+1\Big)^{M/2}\Big(\sum_{\alpha\in\Lambda_{N,M}}\Big(\frac{|c_\alpha|}{\sqrt{\alpha+1}}\Big)^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}}$$
$$\leq \Big(\frac{m}{M}+1\Big)^{M/2} m^{\frac{m-1}{2m}}\Big(1-\frac{1}{m-1}\Big)^{m-1}\sup_{z\in\mathbb{D}^N}\Big|\sum_{\alpha\in\Lambda_{N,M}} c_\alpha z^\alpha\Big|;$$

hence

$$\Big(\sum_{\alpha\in\Lambda_{N,M}} |c_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \leq 2^{\frac{M}{2}} m^{\frac{M+1}{2}} \sup_{z\in\mathbb{D}^N}\Big|\sum_{\alpha\in\Lambda_{N,M}} c_\alpha z^\alpha\Big|. \tag{2.1}$$

This means that for polynomials whose monomials have a uniformly bounded number M of different variables, we get a Bohnenblust-Hille type inequality with a constant of polynomial growth in m. We remark that the dimension N plays no role in this inequality; the only important point here is the number of different variables in each monomial. As a consequence, an analogue of (2.1) holds for m-homogeneous polynomials on c_0: let $P : c_0 \to \mathbb{C}$ be an m-homogeneous polynomial and

$$\Lambda_M = \{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \colon |\alpha| = m \text{ and } \operatorname{vars}(\alpha) \leq M\}.$$

Then for every M and m

$$\Big(\sum_{\alpha\in\Lambda_M} |c_\alpha(P)|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \leq 2^{\frac{M}{2}} m^{\frac{M+1}{2}} \, \|P\|,$$

where the $c_\alpha(P)$ are the coefficients of P and $\|P\|$ is the supremum of $|P|$ on the unit ball of c_0.

(4) In [6, Theorem 5.3] a very general version of the Bohnenblust-Hille inequality is given, involving operators with values on a Banach lattice. A straightforward combination of the proof of Theorem 2.1 (see the final section) and the arguments presented in [6, Theorem 5.3] easily gives a version of Theorem 2.1 in that setting.

3. The proof

Let us fix some notation before we prove our main result. We are going to use the following indexing sets:

$$\mathcal{M}(m,N) = \{\boldsymbol{i} = (i_1,\dots,i_m)\colon 1 \leq i_j \leq N,\ j = 1,\dots,m\},$$
$$\mathcal{J}(m,N) = \{\boldsymbol{i} \in \mathcal{M}(m,N) \in\colon 1 \leq i_1 \leq \cdots \leq i_m \leq N\}.$$

In $\mathcal{M}(m,N)$ we define an equivalence relation by $\boldsymbol{i} \sim \boldsymbol{j}$ if there is a permutation σ of $\{1,\dots,N\}$ such that $j_k = i_{\sigma(k)}$ for every k. With this, if matrix $\big(a_{i_1,\dots,i_m}\big)$ is symmetric, then we have

$$\sum_{\boldsymbol{i}\in\mathcal{M}(m,N)} a_{\boldsymbol{i}} = \sum_{\boldsymbol{i}\in\mathcal{J}(m,N)}\sum_{\boldsymbol{j}\in[\boldsymbol{i}]} a_{\boldsymbol{j}} = \sum_{\boldsymbol{i}\in\mathcal{J}(m,N)} \operatorname{card}[\boldsymbol{i}] a_{\boldsymbol{i}}.$$

Also, given $\boldsymbol{i} \in \mathcal{M}(m-1,N)$ and $j \in \{1,\dots,N\}$, for $1 \leq k \leq m-1$ we define $(\boldsymbol{i},_k j) = (i_1,\dots,i_{k-1},j,i_k,\dots,i_{m-1}) \in \mathcal{M}(m,N)$ (that is, we put j in the k-th position, shifting the rest to the right).

There is a one-to-one correspondence between $\mathcal{J}(m,N)$ and $\{\alpha \in \mathbb{N}_0^N \colon |\alpha| = m\}$ defined as follows. For each $\boldsymbol{i}$ we define $\alpha = (\alpha_1, \ldots, \alpha_N)$ by $\alpha_r = \operatorname{card}\{j \colon i_j = r\}$ (i.e. α_r counts how many times r appears in $\boldsymbol{i}$); on the other hand, given α we define $\boldsymbol{i} = (1, \overset{\alpha_1}{\ldots}, 1, \ldots, N, \overset{\alpha_N}{\ldots}, N) \in \mathcal{J}(m,N)$.

Each m-homogeneous polynomial on N variables has a unique symmetric m-linear form $L : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \to \mathbb{C}$ such that $P(z) = L(z, \ldots, z)$ for every z. If (c_α) are the coefficients of the polynomial and $a_{i_1,\ldots,i_m} = L(e_{i_1}, \ldots, e_{i_m})$ is the matrix of L, we have $c_\alpha = \operatorname{card}[\boldsymbol{i}] a_{\boldsymbol{i}}$, where α and $\boldsymbol{i}$ are related to each other.

Finally, if α and $\boldsymbol{i}$ are related and $p_1 < p_2 < \cdots$ denotes the sequence of prime numbers, we will write $p^\alpha = p_1^{\alpha_1} \cdots p_N^{\alpha_N} = p_{i_1} \cdots p_{i_m} = p_{\boldsymbol{i}}$.

Proof of Theorem 2.1. We follow essentially the guidelines of the proof of the Bohnenblust-Hille inequality as presented in [5]. First of all let us assume that $c_\alpha = 0$ for every $\alpha \notin \Lambda$; then we have

$$\begin{aligned}
\Big(\sum_{\alpha \in \Lambda} \Big(\frac{|c_\alpha|}{\sqrt{\alpha+1}} \Big)^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} &= \Big(\sum_{\boldsymbol{i} \in \mathcal{J}(m,N)} \Big| \operatorname{card}[\boldsymbol{i}] \frac{a_{\boldsymbol{i}}}{\sqrt{d(p_{\boldsymbol{i}})}} \Big|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \\
&= \Big(\sum_{\boldsymbol{i} \in \mathcal{M}(m,N)} \frac{1}{\operatorname{card}[\boldsymbol{i}]} \Big| \operatorname{card}[\boldsymbol{i}] \frac{a_{\boldsymbol{i}}}{\sqrt{d(p_{\boldsymbol{i}})}} \Big|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \\
&= \Big(\sum_{\boldsymbol{i} \in \mathcal{M}(m,N)} \Big| \operatorname{card}[\boldsymbol{i}]^{1-\frac{m+1}{2m}} \frac{a_{\boldsymbol{i}}}{\sqrt{d(p_{\boldsymbol{i}})}} \Big|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} .
\end{aligned}$$

We now use an inequality due to Blei [3, Lemma 5.3] (see also [5, Lemma 1]): for any family of complex numbers $\big(b_{\boldsymbol{i}}\big)_{\boldsymbol{i} \in \mathcal{M}(m,N)}$ we have

$$\sum_{\boldsymbol{i} \in \mathcal{M}(m,N)} |b_{\boldsymbol{i}}|^{\frac{2m}{m+1}} \leq \prod_{k=1}^m \Big(\sum_{j=1}^N \Big(\sum_{\boldsymbol{i} \in \mathcal{M}(m-1,N)} |b_{(\boldsymbol{i},_k j)}|^2 \Big)^{1/2} \Big)^{\frac{2}{m-1}} . \tag{3.1}$$

Using this and the fact that $\operatorname{card}[(\boldsymbol{i},_k j)] \leq m \operatorname{card}[\boldsymbol{i}]$ we get

$$\begin{aligned}
&\Big(\sum_{\alpha \in \Lambda} \Big(\frac{|c_\alpha|}{\sqrt{\alpha+1}} \Big)^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \\
&\qquad \leq \prod_{k=1}^m \Big(\sum_{j=1}^N \Big(\sum_{\boldsymbol{i} \in \mathcal{M}(m-1,N)} \Big| \operatorname{card}[(\boldsymbol{i},_k j)]^{\frac{m-1}{2m}} \frac{a_{(\boldsymbol{i},_k j)}}{\sqrt{d(p_{(\boldsymbol{i},_k j)})}} \Big|^2 \Big)^{1/2} \Big)^{\frac{1}{m}} \\
&\qquad \leq \prod_{k=1}^m \Big(\sum_{j=1}^N \Big(\sum_{\boldsymbol{i} \in \mathcal{M}(m-1,N)} \Big| \operatorname{card}[\boldsymbol{i}]^{\frac{m-1}{2m}} m^{\frac{m-1}{2m}} \frac{a_{(\boldsymbol{i},_k j)}}{\sqrt{d(p_{(\boldsymbol{i},_k j)})}} \Big|^2 \Big)^{1/2} \Big)^{\frac{1}{m}} \\
&\qquad = m^{\frac{m-1}{2m}} \prod_{k=1}^m \Big(\sum_{j=1}^N \Big(\sum_{\boldsymbol{i} \in \mathcal{M}(m-1,N)} \operatorname{card}[\boldsymbol{i}] \Big| \frac{a_{(\boldsymbol{i},_k j)}}{\sqrt{d(p_{(\boldsymbol{i},_k j)})}} \Big|^2 \Big)^{1/2} \Big)^{\frac{1}{m}} .
\end{aligned}$$

We now bound each one of the sums in the product. We use the fact that the coefficients $a_{\boldsymbol{j}}$ are symmetric. Also, if q divides $p_{i_1} \cdots p_{i_m} = p_{\boldsymbol{i}}$, then it also divides $p_{i_1} \cdots p_{i_m} p_j = p_{(\boldsymbol{i},_k j)}$; hence $d(p_{\boldsymbol{i}}) \leq d(p_{(\boldsymbol{i},_k j)})$ for every $\boldsymbol{i}$ and every j. This altogether gives

$$\begin{aligned}\sum_{j=1}^{N}\Big(\sum_{\boldsymbol{i}\in\mathcal{M}(m-1,N)}\operatorname{card}[\boldsymbol{i}]\Big|\frac{a_{(\boldsymbol{i},_k j)}}{\sqrt{d(p_{(\boldsymbol{i},_k j)})}}\Big|^2\Big)^{1/2}\\ =\sum_{j=1}^{N}\Big(\sum_{\boldsymbol{i}\in\mathcal{J}(m-1,N)}\operatorname{card}[\boldsymbol{i}]^2\frac{|a_{(\boldsymbol{i},_k j)}|^2}{d(p_{(\boldsymbol{i},_k j)})}\Big)^{1/2}\\ \le\sum_{j=1}^{N}\Big(\sum_{\boldsymbol{i}\in\mathcal{J}(m-1,N)}\frac{|\operatorname{card}[\boldsymbol{i}]a_{(\boldsymbol{i},_k j)}|^2}{d(p_{\boldsymbol{i}})}\Big)^{1/2}.\end{aligned}$$

Let us note that what we have here are the coefficients of an $(m-1)$-homogeneous polynomial in N variables. We now use (1.1) to conclude our argument:

$$\begin{aligned}&\sum_{j=1}^{N}\Big(\sum_{\boldsymbol{i}\in\mathcal{J}(m-1,N)}\frac{|\operatorname{card}[\boldsymbol{i}]a_{(\boldsymbol{i},_k j)}|^2}{d(p_{\boldsymbol{i}})}\Big)^{1/2}\\ &\le\sum_{j=1}^{N}\int_{\mathbb{T}^N}\Big|\sum_{\boldsymbol{i}\in\mathcal{J}(m-1,N)}\operatorname{card}[\boldsymbol{i}]a_{(\boldsymbol{i},_k j)}w_{i_1}\cdots w_{i_{m-1}}\Big|d\sigma(w)\\ &\le\int_{\mathbb{T}^N}\sum_{j=1}^{N}\Big|\sum_{\boldsymbol{i}\in\mathcal{M}(m-1,N)}a_{(\boldsymbol{i},_k j)}w_{i_1}\cdots w_{i_{m-1}}\Big|d\sigma(w)\\ &\le\sup_{z\in\mathbb{D}^N}\sum_{j=1}^{N}\Big|\sum_{\boldsymbol{i}\in\mathcal{M}(m-1,N)}a_{(\boldsymbol{i},_k j)}z_{i_1}\cdots z_{i_{m-1}}\Big|\\ &=\sup_{z\in\mathbb{D}^N}\sup_{y\in\mathbb{D}^N}\Big|\sum_{j=1}^{N}\sum_{\boldsymbol{i}\in\mathcal{M}(m-1,N)}a_{(\boldsymbol{i},_k j)}z_{i_1}\cdots z_{i_{m-1}}y_j\Big|\\ &\le\Big(1-\frac{1}{m-1}\Big)^{m-1}\sup_{z\in\mathbb{D}^N}\Big|\sum_{\alpha\in\Lambda}c_\alpha z^\alpha\Big|,\end{aligned}$$

where the last inequality follows from a result of Harris [8, Theorem 1] (see also [5, (13)]). This completes the proof. □

As we have already mentioned, very recently [2, Corollary 5.3] has shown that for every $\varepsilon>0$ there exists $\kappa>0$ such that (1.2) holds with $\kappa(1+\varepsilon)^m$. The main idea for the proof is to replace (3.1) by a similar inequality in which we have mixed sums with k and $m-k$ indices (instead of 1 and $m-1$, as we have here). This allows us to use instead of (1.3) the following inequality:

$$\Big(\sum_{|\alpha|=m}|c_\alpha|^2\Big)^{1/2}\le c_p^m\Big(\int_{\mathbb{T}^N}\Big|\sum_{|\alpha|=m}c_\alpha w^\alpha\Big|^p d\sigma(w)\Big)^{\frac{1}{p}}\quad\text{for } 1\le p\le 2.$$

A good control on the constants c_p (that tend to 1 as p goes to 2) gives an improvement on the constant in (1.2) presented in [2]. In our setting, by dividing by $\alpha+1$, we are using (1.1), which already has constant 1. Hence this new approach does not improve the constants in our setting.

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