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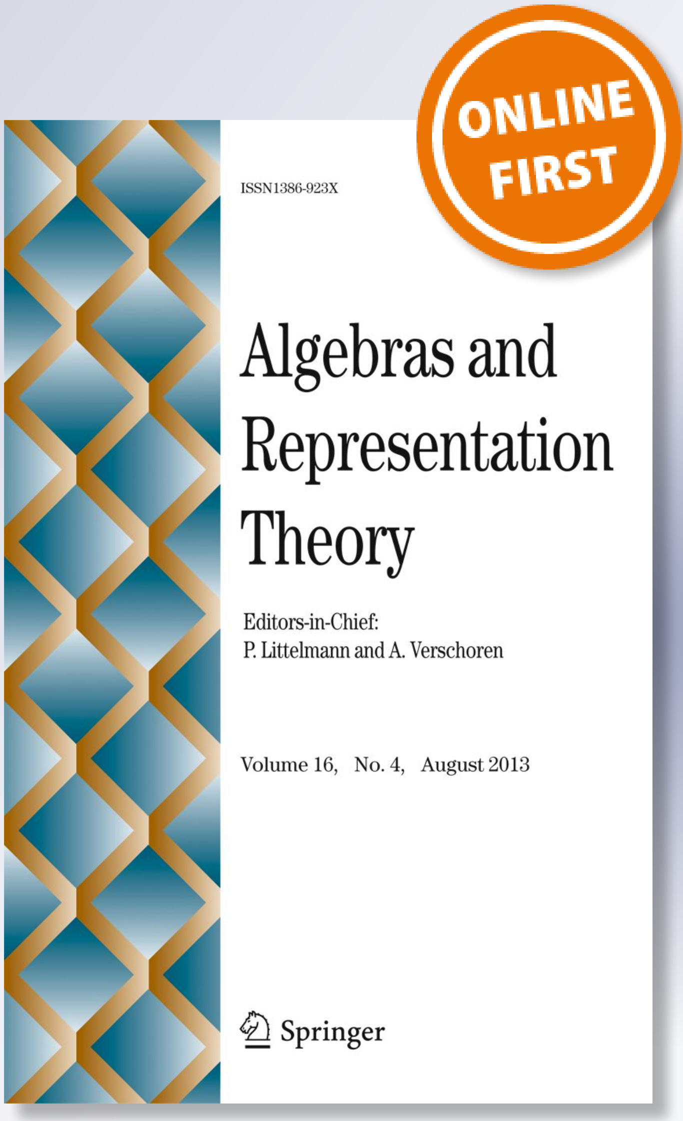

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τ-Tilting Modules Over One-Point Extensions by a Projective Module

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Abstract Let A be the one point extension of an algebra B by a projective B-module. We prove that the extension of a given support τ-tilting B-module is a support τ-tilting A-module; and, conversely, the restriction of a given support τ-tilting A-module is a support τ-tilting B-module. Moreover, we prove that there exists a full embedding of quivers between the corresponding poset of support τ-tilting modules.

Keywords One-point extension · Tilting modules · Poset · τ-tilting modules

Mathematics Subject Classification (2010) 16G20 · 16E10 · 16E30

1 Introduction

Tilting theory plays an important role in representation theory of finite dimensional algebras. In particular, the concept of tilting modules were introduced in the early eighties, see for example [5–7]. The mutation process is an essential concept in tilting theory. The basic idea of a mutation is to replace an indecomposable direct summand of a tilting module by another indecomposable module in order to obtain a new tilting module. In that sense, any almost complete tilting module is a direct summand of at most two tilting modules, but it is not always exactly two. The mutation process is possible only when we have two complements. This suggests to consider a larger class of objects. In [1], T. Adachi, O. Iyama and I. Reiten introduced a class of modules called support τ-tilting modules, which contains the classical tilting modules, see Definition 2.8. Furthermore, the almost complete support

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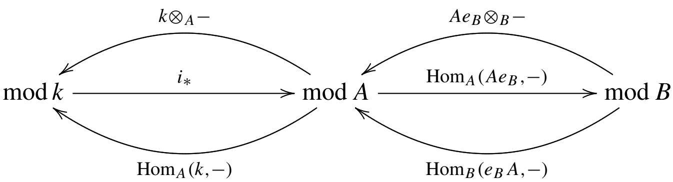
τ-tilting modules have the desired property concerning complements, that is, they have exactly two complements. A motivation to define support τ-tilting modules come from cluster tilting theory, since the mutation there is always possible to do. Moreover, in [1, Theorem 4.1] the authors showed that there is a deep connection between τ-tilting theory and cluster-tilting theory. They also showed that the notion of support τ-tilting modules is connected with silting theory, see [1, Theorem 3.2].

Since τ-tilting theory is a generalization of tilting theory, many properties of tilting modules are preserved by support τ-tilting modules. In [2], for one point extension algebras I. Assem, D. Happel and S. Trepode studied how to extend and restrict tilting modules. More precisely, if $A = B[P_0]$ is the one-point extension of an algebra B by a projective B-module P_0, they showed how to construct in a natural way a tilting A-module from a tilting B-module and conversely, given a tilting B-module they constructed a tilting A-module. Motivated by this fact, in this article we shall study the behavior of support τ-tilting modules for one-point extension. Let e_B be the identity in B. Since $e_B A e_B \cong B$ and $A/Ae_B A \cong k$, we have a recollement of mod A by mod B and mod k as follows (see Definition 2.1)



We denote $\mathcal{R} = \mathrm{Hom}_A(Ae_B, -)$ and $\mathcal{E} = \mathrm{Hom}_A(e_B A, -)$. We prove the following result:

Theorem A *Let B be a finite dimensional k-algebra over an algebraically closed field k. Let $A = B[P_0]$ be the one-point extension of B by a projective B-module P_0 and $S = i^* k$. Then,*

(a) *If M is a basic support τ-tilting B-module then $\mathcal{E}M \oplus S$ is a support τ-tilting A-module.*
(b) *If T is a basic support τ-tilting A-module then $\mathcal{R}T$ is a support τ-tilting B-module.*

As a direct consequence, we obtain that the functors $\mathcal{R}$ and $\mathcal{E}$ induce morphisms r from $\mathrm{s}\tau - \mathrm{tilt}\, A$ to $\mathrm{s}\tau - \mathrm{tilt}\, B$ and e from $\mathrm{s}\tau - \mathrm{tilt}\, B$ to $\mathrm{s}\tau - \mathrm{tilt}\, A$ such that $re = \mathrm{id}_{\mathrm{s}\tau - \mathrm{tilt}\, B}$, where $\mathrm{s}\tau - \mathrm{tilt}\, B$ ($\mathrm{s}\tau - \mathrm{tilt}\, A$, respectively) is the set of isomorphism classes of basic support τ-tilting modules over B (A, respectively). Moreover, as a corollary of Theorem A we obtain a particular case of [8, Theorem 3.15].

Corollary *There is a bijection between*

$$\mathrm{s}\tau - \mathit{tilt}\, B \leftrightarrow \mathrm{s}\tau - \mathit{tilt}_S A := \{M \in \mathrm{s}\tau - \mathit{tilt}\, A \;/\; S \in \mathit{add}\, M\}$$

In [2, Proposition 6.1] *the authors proved that if B is a hereditary algebra, $A = B[P_0]$ and T a tilting B-module then $End_A eT$ is a one-point extension of $End_B T$. In this work,* *we generalize the same result for any algebra B, $A = B[P_0]$ and T a τ-tilting B-module. On the other hand, in* [2, Theorem 5.2]*, the authors also showed that there exists a full embedding of quivers between the poset of tilting modules. We prove that the above mentioned result still holds true for support τ-tilting modules, as we state in the next theorem. We denote by $Q(\mathrm{s}\tau - tilt\, B)$ the support τ-tilting quiver, see Definition 2.19.*

Theorem B *Let B be a finite dimensional k-algebra over an algebraically closed field k and $A = B[P_0]$ be the one-point extension of B by a projective B-module P_0. Then the map $e : \mathrm{s}\tau - tilt\, B \to \mathrm{s}\tau - tilt\, A$ induces a full embedding of quivers $e : Q(\mathrm{s}\tau - tilt\, B) \to Q(\mathrm{s}\tau - tilt\, A)$.*

Finally, we point out some technical properties concerning the successors and the predecessors of a support τ-tilting module which belong to the image of e.

We observe that most of the statements fail if we drop the assumption that the module P_0 is projective.

This paper is organized as follows. In the first section, we present some notations and preliminaries results. Section 2 is dedicated to prove Theorem A and the results concerning the relationship between the support τ-tilting B-modules and the support τ-tilting A-modules. We study their torsion pairs and their endomorphism algebras. In Section 3, we prove Theorem B and state some technical consequences.

2 Preliminaries

Throughout this paper, all algebras are basic connected finite dimensional algebras over an algebraically closed field k.

2.1 Subcategories

For an algebra A we denote by $\mathrm{mod}\, A$ the category of finitely generated left A-modules. An algebra B is called a *full subcategory* of A if there exists an idempotent $e \in A$ such that $B = eAe$. An algebra B is called *convex* in A if, whenever there exists a sequence $e_i = e_{i_0}, e_{i_1}, \cdots e_{i_t} = e_j$ of primitive orthogonal idempotents such that $e_{i_{l+1}} A e_{e_{i_l}} \neq 0$ for $0 \leq l < t$, $ee_i = e_i$ and $ee_j = e_j$, then $ee_{i_l} = e_{i_l}$, for each l.

For a subcategory C of $\mathrm{mod}\, A$, we define full subcategories

$$\mathcal{C}^{\perp} = \{X \in \mathrm{mod}\, A \mid \mathrm{Hom}_A(\mathcal{C}, X) = 0\}$$

and,

$$\mathcal{C}^{\perp_1} = \{X \in \mathrm{mod}\, A \mid \mathrm{Ext}^1_A(\mathcal{C}, X) = 0\}.$$

Dually, the categories ${}^{\perp}\mathcal{C}$ and ${}^{\perp_1}\mathcal{C}$ are defined. In particular, if X is an A-module, we can define the full subcategories $X^{\perp}$ y ${}^{\perp}X$ of $\mathrm{mod}\, A$ as follows:

$$X^{\perp} = (\mathrm{add}\, X)^{\perp}$$

$${}^{\perp}X = {}^{\perp}(\mathrm{add}\, X)$$

where $\mathrm{add}\, X$ means the full subcategory of $\mathrm{mod}\, A$ whose objects are the direct sums of direct summands of X.

Recall that a subcategory $\mathcal{X}$ of an additive category $\mathcal{C}$ is said to be *contravariantly finite* in $\mathcal{C}$ if for every object M in $\mathcal{C}$ there exist some $X \in \mathcal{X}$ and a morphism $f : X \to M$ such that for every $X' \in \mathcal{X}$ the sequence $\mathrm{Hom}_{\mathcal{C}}(X', X) \xrightarrow{f} \mathrm{Hom}_{\mathcal{C}}(X', M) \to 0$ is exact. Dually we define *covariantly finite subcategories* in $\mathcal{C}$. Furthermore, a subcategory of $\mathcal{C}$ is said to be *functorially finite* in $\mathcal{C}$ if it is both contravariantly and covariantly finite in $\mathcal{C}$.

A full subcategory $\mathcal{T}$ of $\mathrm{mod}\, A$ is a *torsion class* (*torsion free class*, respectively) if it is closed under factor modules (submodules, respectively) and extensions. A pair $(\mathcal{T}, \mathcal{F})$ is called a torsion pair if $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$. We say that $X \in \mathcal{T}$ is Ext-*projective* if $\mathrm{Ext}^1_A(X, \mathcal{T}) = 0$. If $\mathcal{T}$ is functorially finite in $\mathrm{mod}\, A$, then there are only finitely many indecomposable Ext-projective modules in $\mathcal{T}$ up to isomorphism, and we denote by $P(\mathcal{T})$ the direct sum of the Ext-projective modules in $\mathcal{T}$.

We denote by D the usual standard duality $\mathrm{Hom}_k(-, k) : \mathrm{mod}\, A \to \mathrm{mod}\, A^{op}$, see [3, I, 2.9].

For an A-module X, we denote by $\mathrm{Fac}\, X$ the full subcategory of $\mathrm{mod}\, A$ whose objects are the factor modules of finite direct sums of copies of X.

Finally, we say that an A-module X is *basic* if the indecomposable direct summands of X are pairwise non-isomorphic.

2.2 One-point extension algebras

Let B be an algebra and P_0 be a fixed projective B-module. We denote by $A = B[P_0]$ the one-point extension of B by P_0, which is, the matrix algebra

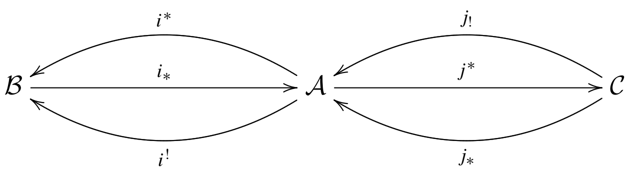
$$A = \begin{pmatrix} B & P_0 \\ 0 & k \end{pmatrix}$$

with the ordinary matrix addition and the multiplication induced by the module structure of P_0.

It is well-known that B is a full convex subcategory of A, and that there is a unique indecomposable projective A-module $\widetilde{P}$ which is not a projective B-module. Moreover, the simple top S of $\widetilde{P}$ is an injective A-module and $\mathrm{pd}_A S \leq 1$, where by $\mathrm{pd}_A S$ we mean the projective dimension of the simple S.

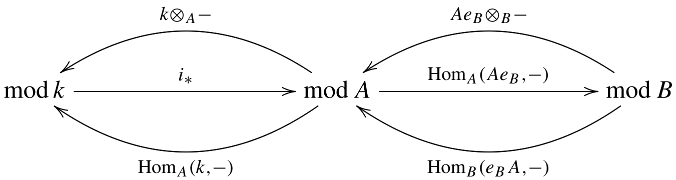
On the other hand, it is known that $\mathrm{mod}\, A$ has a decomposition by $\mathrm{mod}\, B$ and $\mathrm{mod}\, k$, which is a recollement. We recall the definition of *recollement between abelian categories*.

Definition 2.1 A recollement of an abelian category $\mathcal{A}$ by abelian categories $\mathcal{B}$ and $\mathcal{C}$, denoted by $\mathrm{R}(\mathcal{B}, \mathcal{A}, \mathcal{C})$, is a diagram of additive functors as follows, satisfying the conditions below.



(1) $(j_!, j^*, j_*)$ and $(i^*, i_*, i^!)$ are adjoint triples.
(2) The functors i_*, $j^!$ and j_* are fully faithful.
(3) $\mathrm{Im} i_* = \ker j^*$.

Let e_B be the identity of B. Then, $e_B A e_B \cong B$ and $A/Ae_B A \cong k$. We have the following recollement



We called the functor $\mathrm{Hom}_A(Ae_B, -)$ the *restriction functor* and we denote it by $\mathcal{R}$. Similarly, we called the functor $\mathrm{Hom}_A(e_B A, -)$ the *extension functor* and we denote it by $\mathcal{E}$.

The next proposition lists some properties of $\mathrm{R}(\mathcal{B}, \mathcal{A}, \mathcal{C})$ that can be obtained from the definition of recollement (see for instance [9]).

Proposition 2.2 *The following properties hold for a recollement* $\mathrm{R}(\mathcal{B}, \mathcal{A}, \mathcal{C})$.

a) *The functors i_* and j^* are exact.*
b) *The compositions $i^* j_!$ and $i^! j_*$ are identically zero.*
c) *The units $Id_{\mathcal{B}} \to i^! i_*$ and $Id_{\mathcal{C}} \to j^* j_!$ and the counits $i^* i_* \to Id_{\mathcal{B}}$ and $j^* j_* \to Id_{\mathcal{C}}$ are natural isomorphisms.*
d) *If $\mathcal{C}$ has enough projective and injective objects, then $j_!$ preserves projective objects and j_* preserves injective objects.*

It follows from the definition of recollement that the restriction functor is exact and $\mathcal{R}\mathcal{E} \cong \mathrm{Id}_{\mathrm{mod}\, B}$. Moreover, since $e_B A$ is a projective B-module, $\mathcal{E}$ is also exact. If we consider mod B embedded in mod A under the usual embedding functor, then $\mathcal{R}X$ is a submodule of X.

In [9], C. Psaroudakis studied homological aspects of recollements of abelian categories. In particular, the author studied when the exact functor j^* induces , restricted to suitable subcategories, natural isomorphisms $(j^*)^m : \mathrm{Ext}^n_{\mathcal{A}}(Z, W) \to \mathrm{Ext}^n_{\mathcal{C}}(j^*(Z), j^*(W))$. For the convenience of the reader, we recall here some of these results.

Definition 2.3 [9, Definition 3.1] For $0 \le k \le \infty$, the *right k-perpendicular subcategory* $i_*(\mathcal{B})^{0\perp_k}$ of $\mathcal{B}$ in $\mathcal{A}$ is defined by

$$i_*(\mathcal{B})^{0\perp_k} = \{A \in \mathcal{A} \mid \mathrm{Ext}^n_{\mathcal{A}}(i_*(B), A) = 0, \forall B \in \mathcal{B} \text{ and } 0 \le n \le k\}$$

and dually the *left k-perpendicular subcategory* ${}^{0\perp_k} i_*(\mathcal{B})$ of $\mathcal{B}$ in $\mathcal{A}$ is defined by

$${}^{0\perp_k} i_*(\mathcal{B}) = \{A \in \mathcal{A} \mid \mathrm{Ext}^n_{\mathcal{A}}(A, i_*(B)) = 0, \forall B \in \mathcal{B} \text{ and } 0 \le n \le k\}$$

Since $i_*(k) \cong S$, the right 1-perpendicular category $i_*(\mathrm{mod}\, k)^{0\perp_1}$ is

$$i_*(\mathrm{mod}\, k)^{0\perp_1} = \{M \in \mathrm{mod}\, A \mid \mathrm{Hom}_A(S, M) = 0 \text{ and } \mathrm{Ext}^1_A(S, M) = 0\} = S^{\perp} \cap S^{\perp_1}$$

which coincides with the usual right perpendicular category of add S. We denote this subcategory by S^{perp}. It follows from [9, Proposition 3.2], that if $M \in \mathrm{mod}\, B$ then $\mathcal{E}M \in S^{perp}$.

The following result describes the quotient category $\mathcal{C}$ of a recollement.

Lemma 2.4 [9, Proposition 3.2] *Let* $\mathrm{R}(\mathcal{B}, \mathcal{A}, \mathcal{C})$ *be a recollement of abelian categories and assume that* $\mathcal{C}$ *has enough projective and injective objects. Then we have the following equivalences:*

$$j^*|_{{}^{0\perp_1} i_*(\mathcal{B})} : \quad {}^{0\perp_1} i_*(\mathcal{B}) \xrightarrow{\simeq} \mathcal{C} \xleftarrow{\simeq} i_*(\mathcal{B})^{0\perp_1} \quad : j^*|i_*(\mathcal{B})^{0\perp_k}$$

By Lemma 2.4, we have that mod B and S^{perp} are equivalent categories. Namely, if $X \in S^{perp}$ then $X \to \mathcal{E}\mathcal{R}X$ is a funtorial isomorphism.

Proposition 2.5 [9, Theorem 3.10] *Let* $\mathrm{R}(\mathcal{B}, \mathcal{A}, \mathcal{C})$ *be a recollement of abelian categories and assume that* $\mathcal{A}$ *and* $\mathcal{C}$ *have enough projective and injective objects. Then the following statements are equivalent.*

i) *The map* $j^*_{Z,W} : Ext^n_{\mathcal{A}}(Z, W) \to Ext^n_{\mathcal{C}}(j^*(Z), j^*(W))$ *is invertible,* $\forall W \in \mathcal{A}$ *(resp.* $\forall Z \in \mathcal{A}$*), and* $0 \leq n \leq k$.
ii) $Z \in i_*(\mathcal{B})^{0\perp_k}$ *(resp.* $W \in {}^{0\perp_k} i_*(\mathcal{B})$*).*

Remark 2.6 We state here some particular cases of Proposition 2.5 that are going to be useful in this work.

1. $\mathrm{Ext}^1_A(X, \mathcal{E}M) \cong \mathrm{Ext}^1_B(\mathcal{R}X, M)$.
2. If $X \in S^{perp}$, then $\mathrm{Ext}^1_A(\mathcal{E}M, X) \cong \mathrm{Ext}^1_B(M, \mathcal{R}X)$.

Lemma 2.7 [2, Proposition 2.5] *Let* X *be an* A*-module.* $Hom_A(S, X) = 0$ *if and only if* S *is not a direct summand of* X.

2.3 τ-tilting Theory

We recall some results on τ-tilting modules. For a detail account on τ-tilting theory we refer the reader to [1].

Definition 2.8 [1, Definition 0.1] Let A be a finite dimensional algebra.

(a) An A-module M is τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.
(b) An A-module M is τ-tilting (almost complete τ-tilting, respectively) if M is τ-rigid and $|M| = |A|$ ($|M| = |A| - 1$, respectively).
(c) An A-module M is support τ-tilting if there exists an idempotent e of A such that M is a τ-tilting $A/\langle e\rangle$-module.

For the convenience of the reader we state [4, Proposition 5.8] and [1, Proposition 2.4] which will be useful for our further purposes.

Proposition 2.9 [4, Proposition 5.8] *Let* $X, Y \in mod\, A$*. The following conditions hold.*

1. $Hom_A(X, \tau Y) = 0$ *if and only if* $Ext^1_A(M, Fac\, N) = 0$.
2. M *is* τ*-rigid if and only if* M *is Ext-projective in* $Fac\, M$.

Lemma 2.10 [1, Proposition 2.4] *Let* A *be a finite dimensional algebra. Let* X *be in* $mod\, A$ *with a projective presentation* $P_1 \xrightarrow{p} P_0 \to X \to 0$*. For* $Y \in mod\, A$*, we have that if the map* $Hom_A(p, Y)$ *is surjective, then* $Hom_A(Y, \tau X) = 0$*. Moreover, the converse holds if the projective presentation is minimal.*

The next result gives a relationship between the torsion classes and the support τ -tilting modules. We denote by $\mathrm{s}\tau - \mathrm{tilt}\, A$ the set of isomorphism classes of basic support τ-tilting A-modules and by $f - \mathrm{tors}\, A$ the set of functorially finite torsion classes in mod A.

Theorem 2.11 [1, Theorem 2.7] *There is a bijection between f − tors A and* sτ − *tilt A given by $\mathcal{T} \to P(\mathcal{T})$ with inverse $M \to Fac\, M$.*

Remark 2.12 Note that the inclusion in $f - \mathrm{tors}\, A$ gives rise to a partial order on $\mathrm{s}\tau - \mathrm{tilt}\, A$, as follows: "$U \leq T$ if and only if $\mathrm{Fac}\, U \subset \mathrm{Fac}\, T$". Then, $\mathrm{s}\tau - \mathrm{tilt}\, A$ is a partially ordered set.

For τ-tilting modules, we have a result which is an analog to Bongartz's Lemma for tilting modules. For the convenience of the reader we state it below.

Theorem 2.13 [1, Theorem 2.10] *Let U be a τ-rigid A-module. Then, $\mathcal{T} = {}^{\perp}(\tau U)$ is a sincere functorially finite torsion class and $T = P(\mathcal{T})$ is a τ-tilting A-module satisfying $U \in add\, T$ and ${}^{\perp}(\tau U) = Fac\, T$.*

The support τ-tilting module $P({}^{\perp}(\tau U))$ is said to be the *Bongartz completion* of U.

We have the following characterizations for a τ-rigid module to be a τ-tilting module.

Theorem 2.14 [1, Theorem 2.12] *The following conditions are equivalent for a τ-rigid module T.*

(a) *T is τ-tilting.*
(b) *T is maximal τ-rigid, i.e., if $T \oplus X$ is τ-rigid for some A-module X, then $X \in add\, T$.*
(c) *${}^{\perp}(\tau T) = Fac\, T$.*

In [8], G. Jasso proved another criterion to decide when a τ-rigid module is a support τ-tilting module, as we state below.

Lemma 2.15 *Let A be a finite dimensional algebra. Let M be a τ-rigid A-module. Then the following are equivalent:*

(1) *M is a support τ-tilting A-module.*
(2) *There exists an exact sequence*

$$A \xrightarrow{f} M_0 \to M_1 \to 0$$

where $M_0, M_1 \in add\, M$ and f is a left $addM$-approximation of A.

Sometimes, it is convenient to see the support τ-tilting A-modules and the τ-rigid A-modules, as certain pair of A-modules. More precisely,

Definition 2.16 [1, Definition 0.3] Let (M, P) be a pair with $M \in \mathrm{mod}\, A$ and P a projective A-module.

(a) If M is τ-rigid and $\mathrm{Hom}_A(P, M) = 0$ then (M, P) is a τ-rigid pair.
(b) If (M, P) is τ-rigid and $|M| + |P| = |A|$ ($|M| + |P| = |A| - 1$, respectively) then (M, P) is a support τ-tilting (almost complete support τ-tilting, respectively) pair.

It follows from [1, Proposition 2.3], that the notions of support τ -tilting modules and of support τ -tilting pairs are essentially the same.

We say that $(X, 0)$ $((0, X)$, respectively) with X an indecomposable module is a complement of an almost complete support τ-tilting pair (U, Q) if $(U \oplus X, Q)$ $((U, Q \oplus X)$, respectively) is a support τ-tilting pair.

Theorem 2.17 [1, Theorem 2.18] *Any basic almost complete support τ-tilting pair for mod A has exactly two complements.*

Two completions (T, P) and (T', P') of an almost complete support τ-tilting pair (U, Q) are called mutations one of each other. We write $(T', P') = \mu_{(X,0)}(T, P)$ $((T', P') = \mu_{(0,X)}(T, P)$, respectively) if $(X, 0)$ $((0, X)$, respectively) is a complement of (U, Q) giving rise to (T, P).

Definition 2.18 [1, Definition 2.28] Let $T = X \oplus U$ and T' be support τ-tilting A-modules such that $T' = \mu_X T$ for some indecomposable A-module X. We say that T' is a left mutation (right mutation, respectively) of T and we write $T' = \mu_X^- T$ $(T = \mu_X^+ T$, respectively) if the following equivalent conditions are satisfied.

(a) $T > T'$ $(T < T'$, respectively).
(b) $X \notin \operatorname{Fac} U$ $(X \in \operatorname{Fac} U$, respectively).
(c) ${}^{\perp}(\tau U) \subseteq {}^{\perp}(\tau X)$ $({}^{\perp}(\tau U) \nsubseteq {}^{\perp}(\tau X)$, respectively).

Definition 2.19 [1, Definition 2.29] The support τ-tilting quiver $Q(\mathrm{s}\tau - \operatorname{tilt} A)$ of A is defined as follows:

- The set of vertices consists of the isomorphisms classes of basic support τ-tilting A-modules.
- There is an arrow from T to U if U is a left mutation of T.

Remark 2.20 Note that this exchange graph is n-regular, where $n = |A|$ is the number of non-isomorphic simple A-modules.

It follows from [1, Corollary 2.34] that the exchange quiver $Q(\mathrm{s}\tau - \operatorname{tilt} A)$ coincides with the Hasse quiver of the partially ordered set $\mathrm{s}\tau - \operatorname{tilt} A$.

3 Extension and Restriction Maps

Throughout this section, we assume that A is the one-point extension of B by a projective B-module P_0. We study the relationship between the support τ-tilting B-modules and the support τ-tilting A-modules.

We start with a remark which shall be very useful for our purposes.

Remark 3.1 Let Y be an A-module such that $\operatorname{Ext}_A^1(S, Y) = 0$. Then $Y = Y' \oplus S^r$ with $Y' \in S^{\mathrm{perp}}$ and $r \geq 0$. In fact, first assume that $\operatorname{Hom}_A(S, Y) = 0$. Then, by Lemma (2.7) we have that $Y = Y'$ and $r = 0$. Now if $\operatorname{Hom}_A(S, Y) \neq 0$, then again, by Lemma (2.7) we have that S is a direct summand of Y, namely, $Y = S \oplus Z$. Note that $\operatorname{Ext}_A^1(S, Z) = 0$. If $\operatorname{Hom}_A(S, Z) = 0$ we are done. Otherwise, S is a direct summand of Z and $Z = Z_1 \oplus S$. Moreover, $Y = S^2 \oplus Z'$. Iterating this argument over Z_i, for $i = 1, \ldots, r - 1$, we get $Y = Y' \oplus S^r$.

Proposition 3.2 *Let B be an algebra and $A = B[P_0]$. Then,*

(a) *If (M, Q) is a basic τ-rigid (support τ-tilting, respectively) pair for mod B, then $(\mathcal{E}M \oplus S, Q)$ is a τ-rigid (support τ-tilting, respectively) pair for mod A.*
(b) *If (T, P) is a basic τ-rigid (support τ-tilting, respectively) pair for mod A, then $(\mathcal{R}T, P^*)$ is a τ-rigid (support τ-tilting, respectively) pair for mod B, where P^* is the projective B-module which is obtained by P removing the projective A-module $\widetilde{P}$.*

Proof (a). Consider (M, Q) a τ-rigid pair for mod B. By Proposition 2.9, we have that $\mathrm{Ext}^1_B(M, \mathrm{Fac}\, M) = 0$. Let us show that $\mathrm{Ext}^1_A(\mathcal{E}M \oplus S, \mathrm{Fac}\,(\mathcal{E}M \oplus S)) = 0$.

Note that, $\mathrm{Fac}\,(\mathcal{E}M \oplus S) = \mathrm{Fac}\,(\mathcal{E}M) \oplus \mathrm{Fac}\, S$. That is, if $N \in \mathrm{Fac}\,(\mathcal{E}M \oplus S)$, then $N = N' \oplus S^r$ with $N' \in \mathrm{Fac}\, M$ and $r \geq 0$. Indeed, if $\mathrm{Hom}_A(S, N) = 0$, then according to Lemma (2.7) S is not a direct summand of N and therefore $N \in \mathrm{Fac}\,(\mathcal{E}M)$. Otherwise, S is a direct summand of N. Then, $N = N' \oplus S^k$ with $\mathrm{Hom}_A(S, N') = 0$. Since $N \in \mathrm{Fac}\,(\mathcal{E}M \oplus S)$, we have $N' \in \mathrm{Fac}\,(\mathcal{E}M \oplus S)$. Therefore, since $\mathrm{Hom}_A(S, N') = 0$, $N' \in \mathrm{Fac}\,(\mathcal{E}S)$ and the assertion is shown. Conversely, it is clear that if $N \in \mathrm{Fac}\,(\mathcal{E}M) \oplus \mathrm{Fac}\, S$, then $N \in \mathrm{Fac}\,(\mathcal{E}M \oplus S)$. Then, $\mathrm{Ext}^1_A(\mathcal{E}M \oplus S, \mathrm{Fac}\,(\mathcal{E}M \oplus S)) = \mathrm{Ext}^1_A(\mathcal{E}M \oplus S, \mathrm{Fac}\,(\mathcal{E}M) \oplus \mathrm{Fac}\, S)$ and, moreover, both equal to $\mathrm{Ext}^1_A(\mathcal{E}M, \mathrm{Fac}\,(\mathcal{E}M)) \oplus \mathrm{Ext}^1_A(S, \mathrm{Fac}\,(\mathcal{E}M)) \oplus \mathrm{Ext}^1_A(\mathcal{E}M \oplus S, \mathrm{Fac}\, S)$. If $X \in \mathrm{Fac}\, S$, then $X \cong S^k$, with $k \geq 0$. Since S is an injective module, we have that $\mathrm{Ext}^1_A(\mathcal{E}M \oplus S, \mathrm{Fac}\, S) = 0$.

Now, we show that $\mathrm{Ext}^1_A(S, \mathrm{Fac}\,(\mathcal{E}M)) = 0$. Consider $Y \in \mathrm{Fac}\,(\mathcal{E}M)$. By definition, there exists an epimorphism $f : N \to Y$, with $N \in \mathrm{add}\,(\mathcal{E}M)$. Applying $\mathrm{Hom}_A(S, -)$ we have

$$\mathrm{Ext}^1_A(S, N) \to \mathrm{Ext}^1_A(S, Y) \to \mathrm{Ext}^2_A(S, \mathrm{Ker} f)$$

since $N \in \mathrm{add}\,(\mathcal{E}M)$ and $\mathrm{pd}_A S \leq 1$ then $\mathrm{Ext}^1_A(S, N) = 0$ and $\mathrm{Ext}^2_A(S, \mathrm{Ker} f) = 0$, respectively . Thus, $\mathrm{Ext}^1_A(S, Y) = 0$. Then, $\mathrm{Ext}^1_A(S, \mathrm{Fac}\,(\mathcal{E}M)) = 0$.

Finally, we prove that $\mathrm{Ext}^1_A(\mathcal{E}M, \mathrm{Fac}\,(\mathcal{E}M)) = 0$. Let $W \in \mathrm{Fac}\,(\mathcal{E}M)$. By definition, there exists an epimorphism $g : Z \to W$, with $Z \in \mathrm{add}\,(\mathcal{E}M)$. Applying the functor $\mathcal{R}$ to g, we get that $\mathcal{R}W \in \mathrm{Fac}\, M$, because $\mathcal{R}Z \in \mathrm{add}\,(M)$. Since M is a τ-rigid B-module, then $\mathrm{Ext}^1_B(M, \mathcal{R}W) = 0$.

On the other hand, since $W \in \mathrm{Fac}\,(\mathcal{E}M)$ and $\mathcal{E}M \in S^{\mathrm{perp}}$, then $\mathrm{Ext}^1_A(S, W) = 0$. By Remark 3.1, we have that $W = S^j \oplus W'$, with $W' \in S^{\mathrm{perp}}$ and $j \geq 0$. Thus, by Proposition (2.5),

$$\begin{aligned} \mathrm{Ext}^1_A(\mathcal{E}M, W) &= \mathrm{Ext}^1_A(\mathcal{E}M, W') \oplus \mathrm{Ext}^1_A(\mathcal{E}M, S^j) \\ &= \mathrm{Ext}^1_B(M, \mathcal{R}W') \\ &= 0. \end{aligned}$$

Therefore, $\mathrm{Ext}^1_A(\mathcal{E}M \oplus S, \mathrm{Fac}\,(\mathcal{E}M \oplus S)) = 0$. Moreover, by Proposition 2.9, $\mathcal{E}M \oplus S$ is a τ-rigid A-module. It is left to show that $\mathrm{Hom}_A(Q, \mathcal{E}M \oplus S) = 0$. We have that

$$\begin{aligned} \mathrm{Hom}_A(Q, \mathcal{E}M \oplus S) &\cong \mathrm{Hom}_A(Q, \mathcal{E}M) \oplus \mathrm{Hom}_A(Q, S) \\ &\cong \mathrm{Hom}_B(\mathcal{R}Q, M) \\ &\cong \mathrm{Hom}_B(Q, M) \\ &\cong 0 \end{aligned}$$

where $\mathrm{Hom}_A(Q, S) = 0$ because Q is a B-module. Hence $(\mathcal{E}M \oplus S, Q)$ is a τ-rigid pair for mod A.

In addition, if (M, Q) is a support τ-tilting pair, then $|M| + |Q| = |B|$. Since $\mathcal{E}$ is a faithful functor, then $|M| = |\mathcal{E}M|$. Moreover, since $\mathcal{E}M \in S^{\text{perp}}$ then S is not a direct summand of $\mathcal{E}M$. Hence, $|\mathcal{E}M \oplus S| = |\mathcal{E}M| + 1$ and

$$\begin{aligned} |\mathcal{E}M \oplus S| + |Q| &= 1 + |\mathcal{E}M| + |Q| \\ &= 1 + |B| \\ &= |A|. \end{aligned}$$

(b). Let (T, P) be a τ-rigid pair for mod A. Consider

$$P_1 \xrightarrow{p} P_0 \to T \to 0 \tag{1}$$

a minimal projective presentation of T. Then, since $\mathcal{R}$ preserves projective modules we have that

$$\mathcal{R}P_1 \xrightarrow{\mathcal{R}p} \mathcal{R}P_0 \to \mathcal{R}T \to 0$$

is a projective presentation of $\mathcal{R}T$. According to Lemma 2.10, we have to show that $\text{Hom}(\mathcal{R}p, \mathcal{R}T)$ is a surjective map. Let $f \in \text{Hom}_B(\mathcal{R}P_1, \mathcal{R}T)$. Since S is a injective simple A-module, then $\mathcal{R}P_1 \cong P_1$. The morphism f induces a morphism $\widetilde{f} \in \text{Hom}_A(P_1, T)$ given by $\widetilde{f} = if$, where $i : \mathcal{R}T \to T$ is the natural inclusion. Since T is a τ-rigid A-module and Eq. 1 is a minimal projective presentation it follows from Lemma (2.10) that there exists a morphism $g : P_0 \to T$ such that $\widetilde{f} = gp$. Then, we have that $\mathcal{R}\widetilde{f} = \mathcal{R}g\mathcal{R}p$. Therefore, $f = \widetilde{g}\mathcal{R}p$ with $\widetilde{g} \in \text{Hom}_B(\mathcal{R}P_0, \mathcal{R}T)$. Hence, $\mathcal{R}T$ is a τ-rigid B-module.

Since $\mathcal{R}T$ is a submodule of T, it follows that $\text{Hom}_A(P^*, \mathcal{R}T) = 0$. Therefore, $(\mathcal{R}T, P^*)$ is a τ-rigid pair for mod B.

In addition, if (T, P) is a support τ-tilting pair for mod A, we shall show that $(\mathcal{R}T, P^*)$ is a support τ-tilting pair for mod B. It follows from Lemma 2.15, that there exists an exact sequence

$$A \xrightarrow{f} T_0 \to T_1 \to 0 \tag{2}$$

where $T_0, T_1 \in \text{add}\, T$ and f is a left add T-approximation of A. Since B is a direct summand of A, we have morphisms $B \xrightarrow{i} A$ and $A \xrightarrow{\pi} B$ where i is the natural inclusion, π the canonical projection and $\pi i = \text{Id}_B$. Thus, we obtain the following exact sequence

$$B \xrightarrow{\mathcal{R}i\,\mathcal{R}f} \mathcal{R}T_0 \to \mathcal{R}T_1 \to 0 \tag{3}$$

It is left to prove that $\mathcal{R}i\,\mathcal{R}f$ is a left add $\mathcal{R}T$-approximation of B. Let $h : B \to U$, with $U \in \text{add}\,\mathcal{R}T$. Then, there exists $U' \in \text{add}\, T$ such that U is a direct summand of $\mathcal{R}U'$. Then we have a morphism $\widetilde{h} = i_2 i_1 h\pi : A \to U'$, where $i_1 : U \to \mathcal{R}U'$ and $i_2 : \mathcal{R}U' \to U'$ are the natural inclusions. Since f is a left add T-approximation of A, there exists $g : T_0 \to U'$ such that

$$gf = \widetilde{h}. \tag{4}$$

Applying the functor $\mathcal{R}$ to Eq. 4 we obtain a morphism $\widetilde{g} : \mathcal{R}T_0 \to U$ such that $\widetilde{g}\mathcal{R}f\mathcal{R}i = h$. Hence, $(\mathcal{R}T, P^*)$ is a support τ-tilting pair for mod B. □

It follows from Proposition 3.2 that we get morphisms between the corresponding posets of support τ-tilting modules, as we state in the following theorem.

Theorem 3.3 *The functors $\mathcal{E}$ and $\mathcal{R}$ induce two maps:*

$$\begin{aligned} e : s\tau - tilt\, B &\to s\tau - tilt\, A \\ (M, Q) &\to (\mathcal{E}M \oplus S, Q) \end{aligned}$$

and,

$$r : s\tau - tilt\, A \rightarrow s\tau - tilt\, B$$
$$(T, P) \rightarrow (\widehat{T}, P^*)$$

where $\widehat{T}$ *is a (unique up to isomorphism) basic* τ*-rigid* B*-module such that add* $\widehat{T}$ = *add* $\mathcal{R}T$*. Moreover, the composition* $re = id_{s\tau - tilt\, B}$.

Proof By Theorem 3.2, r and e are maps. Moreover, since $\mathcal{R}\mathcal{E} \cong \mathrm{id}_{\mathrm{mod}\, B}$ we have that $re = \mathrm{id}_{s\tau-\mathrm{tilt}\, B}$. □

In [8], G. Jasso studied which are all the basic support τ-tilting modules that have as direct summand a given basic τ-rigid A-module. More precisely, let U be a τ-rigid A-module and denote by T_U the Bongartz completion of U in mod A. Consider $C = \mathrm{End}_A T_U / < e_U >$, where e_U is the idempotent corresponding to the projective $\mathrm{End}_A T_U$-module $\mathrm{Hom}_A(T_U, U)$. Then, the author proved that there exists a bijection between $s\tau - \mathrm{tilt}\, C$ and

$$s\tau - \mathrm{tilt}_U\, A := \{M \in s\tau - \mathrm{tilt}\, A \;/\; U \in \mathrm{add}\, M\}.$$

In particular, if we consider $U = S$ then, C is isomorphic to B. As a corollary of Theorem 3.2, we obtain a special case of [8, Theorem 3.15].

Corollary 3.4 *There is a bijection between*

$$s\tau - tilt\, B \leftrightarrow s\tau - tilt\,_S A = \{M \in s\tau - tilt\, A \;/\; S \in add\, M\}$$

Proof Let $T \in s\tau - \mathrm{tilt}_S\, A$. Then $T = T' \oplus S$. We have to show that there exists a B-module M such that $T = \mathcal{E}M \oplus S$. Since T is basic, then $\mathrm{Hom}_A(S, T') = 0$. Since also $\mathrm{Ext}^1_A(S, T') = 0$, we have that $T' \in S^{\mathrm{perp}}$. The B-module $M = \mathcal{R}T'$ satisfies $T = S \oplus T' \cong S \oplus \mathcal{E}M$. Moreover, since T is basic so is $\mathcal{E}M$, hence so is $M \cong \mathcal{R}\mathcal{E}M \cong \mathcal{R}T$. □

Now, we discuss the torsion pairs corresponding to a τ-tilting module T. We recall that if T is a τ-tilting module over an algebra C, then T determines a torsion pair $({}^{\perp}\tau T, T^{\perp})$ in mod C. We start with the following lemma.

Lemma 3.5 *Let* T *be a* τ*-rigid* A*-module and* X *be a* B*-module. If* $X \in {}^{\perp}(\tau_B \mathcal{R}T)$ *then* $\mathcal{E}X \in {}^{\perp}(\tau_A T)$.

Proof Let $X \in {}^{\perp}(\tau_B \mathcal{R}T)$. Then, $\mathrm{Hom}_B(X, \tau_B \mathcal{R}T) = 0$. By Proposition 2.9, we have that $\mathrm{Ext}^1_B(\mathcal{R}T, \mathrm{Fac}\, X) = 0$. We shall prove that $\mathrm{Ext}^1_A(T, \mathrm{Fac}\,(\mathcal{E}X)) = 0$.

Let $Y \in \mathrm{Fac}\,(\mathcal{E}X)$, then there exists an epimorphism $f : M \rightarrow Y$, with $M \in \mathrm{add}\,(\mathcal{E}X)$. Since $\mathcal{E}X \in S^{\mathrm{perp}}$, then $\mathrm{Ext}^1_A(S, Y) = 0$. Thus, by Remark (3.1), we have that $Y = Y' \oplus S^r$, with $Y' \in S^{\mathrm{perp}}$ and $r \geq 0$.

Applying the functor $\mathcal{R}$ to the morphism $f : M \rightarrow Y' \oplus S^r$, we obtain that $\mathcal{R}Y' \in \mathrm{Fac}\, X$, and thus $\mathrm{Ext}^1_B(\mathcal{R}T, \mathcal{R}Y') = 0$. Then, by Proposition 2.5 $\mathrm{Ext}^1_A(T, \mathcal{E}\mathcal{R}Y') = 0$. Since $Y' \in S^{\mathrm{perp}}$, then $\mathrm{Ext}^1_A(T, Y') = 0$. Therefore,

$$\begin{aligned} \mathrm{Ext}^1_A(T, Y) &\cong \mathrm{Ext}^1_A(T, Y' \oplus S^r) \\ &\cong \mathrm{Ext}^1_A(T, Y') \oplus \mathrm{Ext}^1_A(T, S^r) \\ &\cong 0 \end{aligned}$$

because S is an injective module.

Then, $\mathrm{Ext}^1_A(T, \mathrm{Fac}\,(\mathcal{E}X)) = 0$ and, by Proposition 2.9, we get the result. □

Definition 3.6 Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair for mod A.

1. If each indecomposable A-module lies either in $\mathcal{T}$ or in $\mathcal{F}$, then $(\mathcal{T}, \mathcal{F})$ is called *splitting*.
2. If $\mathcal{T}$ is closed under submodules then $(\mathcal{T}, \mathcal{F})$ is called *hereditary*.

Theorem 3.7 (i) *Let T be a τ-tilting B-module and X be a B-module. Then the following conditions hold.*

(a) *$X \in^{\perp} \tau_B T$ if and only if $\mathcal{E}X \in^{\perp} (\tau_A \mathcal{E}T)$.*
(b) *$X \in T^{\perp}$ if and only if $\mathcal{E}X \in \mathcal{E}T^{\perp}$.*

(ii) *Let T be a τ-tilting A-module. Then the following conditions hold.*

(a) *If $(^{\perp}\tau_A T, T^{\perp})$ is a hereditary torsion pair for mod A then $(^{\perp}(\tau_B \mathcal{R}T), (\mathcal{R}T)^{\perp})$ is an hereditary torsion pair for mod B.*
(b) *If $(^{\perp}\tau_A T, T^{\perp})$ is a splitting torsion pair for mod A then $(^{\perp}(\tau_B \mathcal{R}T), (\mathcal{R}T)^{\perp})$ is a splitting torsion pair for mod B.*

Proof $(i).(a)$. Since T is a τ-tilting A-module, we know that $^{\perp}\tau_A T = \mathrm{Fac}\,T$. Then the result follows from the fact that $X \in \mathrm{Fac}\,T$ if and only if $\mathcal{E}X \in \mathrm{Fac}\,\mathcal{E}T$.

$(i).(b)$. Follows from the fact that

$$\begin{aligned}\mathrm{Hom}_A(\mathcal{E}T, \mathcal{E}X) &\cong \mathrm{Hom}_B(\mathcal{R}\mathcal{E}T, X)\\ &\cong \mathrm{Hom}_B(T, X).\end{aligned}$$

$(ii).(a)$. Consider $(^{\perp}\tau_A T, T^{\perp})$ a hereditary torsion pair for mod A. Let $X \in^{\perp} (\tau_B \mathcal{R}T)$ and Y be a submodule of X. Then, we shall show that $Y \in^{\perp} (\tau_B \mathcal{R}T)$.

Since $X \in^{\perp} (\tau_B \mathcal{R}T)$, by Lemma 3.5, we have that $\mathcal{E}X \in^{\perp} \tau_A T$. Then $\mathcal{E}N \in^{\perp} \tau_A T$, because $\mathcal{E}N$ is a submodule of $\mathcal{E}M$. Since $^{\perp}\tau_A T = \mathrm{Fac}\,T$, then $\mathcal{E}N \in \mathrm{Fac}\,T$. Thus, $N \in \mathrm{Fac}\,\mathcal{R}T =^{\perp} (\tau_B \mathcal{R}T)$. Therefore $(^{\perp}(\tau_B \mathcal{R}T), (\mathcal{R}T)^{\perp})$ is a hereditary torsion pair for mod B.

$(ii).(b)$. Suppose $(^{\perp}\tau_A T, T^{\perp})$ is a splitting torsion pair for mod A and consider $X \in \mathrm{mod}\,B$. Since $\mathcal{E}X \in \mathrm{mod}\,A$, we have that either $\mathcal{E}X \in^{\perp} \tau_A T = \mathrm{Fac}\,T$ or $\mathcal{E}X \in T^{\perp}$. Therefore, $X \in^{\perp} (\tau_B \mathcal{R}T)$ or $X \in (\mathcal{R}T)^{\perp}$ and the assertion is shown. □

We end this section computing the endomorphism algebra of eT, when T is a τ-tilting B-module. Recall that $\nu_C = DC \otimes_C \_$ is the *Nakayama functor* for an algebra C.

Theorem 3.8 *Let T be a τ-tilting B-module. Then, $End_A eT$ is the one-point extension of $End_B T$ by the module $Hom_B(T, \nu_B P_0)$.*

Proof Note that

$$\mathrm{End}_A eT = \mathrm{End}_A(\mathcal{E}T \oplus S) \cong \begin{pmatrix} \mathrm{End}_A(\mathcal{E}T) & \mathrm{Hom}_A(\mathcal{E}T, S) \\ \mathrm{Hom}_A(S, \mathcal{E}T) & \mathrm{End}_A S \end{pmatrix}.$$

Since $\mathrm{End}_A S \cong k$ and $\mathcal{E}T \in S^{\mathrm{perp}}$, it is left to prove that $\mathrm{Hom}_A(\mathcal{E}T, S) \cong \mathrm{Hom}_B(T, \nu_B P_0)$.

Consider the Auslander-Reiten sequence

$$0 \to \tau_A S \to E \to S \to 0 \tag{5}$$

in mod A. By [3, IV, 3.9], E is an injective module. We claim that $\mathcal{R}E \cong \nu_B P_0$. Indeed, applying $\mathcal{R}$ to the sequence (5), we obtain $\mathcal{R}E \cong \mathcal{R}(\tau_A S)$.

On the other hand, consider the projective resolution of S,

$$0 \to P_0 \to P \to S.$$

By [3, IV, 2.4], there exists an exact sequence

$$0 \to \tau_A S \nu_A P_0 \to \nu_A P \to \nu_A S \to 0 \quad (6)$$

where $\nu_A P \cong S$ and $\nu_A P_0 = \bigoplus_x I_x^A$, if $P_0 = \bigoplus_x P_x^A$ where P_x^A is the indecomposable projective A-module at the vertex x. By [2, Lemma 4.5], $I_x^A = \mathcal{E} I_x^B$. Then, applying the functor $\mathcal{R}$ to Eq. 6 we obtain that $\mathcal{R}(\tau_A S) \cong \mathcal{R}(\nu_A P_0) \cong \nu_B P_0$. Therefore,

$$\begin{aligned} \mathcal{R}E &\cong \mathcal{R}(\tau_A S) \\ &\cong \nu_B P_0. \end{aligned}$$

Applying $\mathrm{Hom}_A(\mathcal{E}T, -)$ to the sequence (5) yields an exact sequence as follows

$$0 \to \mathrm{Hom}_A(\mathcal{E}T, \tau_A S) \to \mathrm{Hom}_A(\mathcal{E}T, E) \to \mathrm{Hom}_A(\mathcal{E}T, S) \to \mathrm{Ext}_A^1(\mathcal{E}T, \tau_A S).$$

Since $\mathrm{pd}_A S \leq 1$, the Auslander-Reiten formula yields $\mathrm{Hom}_A(\mathcal{E}T, \tau_A S) = 0$. On the other hand, since $\mathrm{Ext}_A^1(\mathcal{E}T, \tau_A S) \cong D\overline{\mathrm{Hom}}_A(S, \mathcal{E}T)$ and $\mathrm{Hom}_A(S, \mathcal{E}T) = 0$, we obtain that $\mathrm{Ext}_A^1(\mathcal{E}T, \tau_A S) = 0$. Thus, $\mathrm{Hom}_A(\mathcal{E}T, E) \cong \mathrm{Hom}_A(\mathcal{E}T, S)$.

Finally, since $E \in S^{\mathrm{perp}}$, then

$$\begin{aligned} \mathrm{Hom}_A(\mathcal{E}T, S) &\cong \mathrm{Hom}_A(\mathcal{E}T, E) \\ &\cong \mathrm{Hom}_A(T, \mathcal{R}E) \\ &\cong \mathrm{Hom}_B(T, \nu_B P_0) \end{aligned}$$

proving the result. □

4 The Quiver of Support τ-Tilting Modules

Now we focus our attention on the quivers of the support τ-tilting modules. We shall compare $Q(\mathrm{s}\tau - \mathrm{tilt}\, B)$ and $Q(\mathrm{s}\tau - \mathrm{tilt}\, A)$. Our aim is to show that the morphism e states in Corollary 3.3 is a full embedding between the posets of support τ-tilting modules. We start with the following theorem.

Theorem 4.1 (a) *The maps* $e : \mathrm{s}\tau - \mathit{tilt}\, B \to \mathrm{s}\tau - \mathit{tilt}\, A$ *and* $r : \mathrm{s}\tau - \mathit{tilt}\, A \to \mathrm{s}\tau - \mathit{tilt}\, B$ *are morphisms of posets.*

(b) *An arrow* $\alpha : (M_1, Q_1) \to (M_2, Q_2)$ *in* $Q(\mathrm{s}\tau - \mathit{tilt}\, B)$ *induces an arrow* $e\alpha : e(M_1, Q_1) \to e(M_2, Q_2)$ *in* $Q(\mathrm{s}\tau - \mathit{tilt}\, A)$.

Proof (*a*). Let (M_1, Q_1) and (M_2, Q_2) be support τ-tilting pairs for mod B such that $(M_1, Q_1) < (M_2, Q_2)$. We have to prove that $(\mathcal{E}M_1 \oplus S, Q_1) < (\mathcal{E}M_2 \oplus S, Q_2)$, or equivalently, $\mathrm{Fac}\,(\mathcal{E}M_1 \oplus S) \subseteq \mathrm{Fac}\,(\mathcal{E}M_2 \oplus S)$. Since $\mathrm{Fac}\,(\mathcal{E}M_1 \oplus S) = \mathrm{Fac}\,(\mathcal{E}M_1) \oplus \mathrm{Fac}\, S$, we only have to show that $\mathrm{Fac}\,(\mathcal{E}M_1) \subseteq \mathrm{Fac}\,(\mathcal{E}M_2)$.

Since $\mathrm{Fac}\, M_1 \subseteq \mathrm{Fac}\, M_2$, there exists an epimorphism $f : Z \to M_1$, with $Z \in \mathrm{add}\, M_2$. Applying the exact functor $\mathcal{E}$ to f, we obtain an epimorphism $\mathcal{E}f : \mathcal{E}Z \to \mathcal{E}M_1$, where $\mathcal{E}Z \in \mathrm{add}\, \mathcal{E}M_2$. Then, $\mathcal{E}M_1 \in \mathrm{Fac}\,(\mathcal{E}M_2)$. Therefore, $\mathrm{Fac}\,(\mathcal{E}M_1) \subseteq \mathrm{Fac}\,(\mathcal{E}M_2)$.

Conversely. Let (T_1, P_1) and (T_2, P_2) be support τ-tilting pairs for mod A, such that $(T_1, P_1) < (T_2, P_2)$. We claim that $\mathcal{R}T_1 \in \operatorname{Fac} \mathcal{R}T_2$. In fact, since $\operatorname{Fac} T_1 \subseteq \operatorname{Fac} T_2$, there exists an epimorphism $g : W \to T_1$, with $W \in \operatorname{add} T_2$. Applying the exact functor $\mathcal{R}$ to g, we obtain an epimorphism $\mathcal{R}g : \mathcal{R}W \to \mathcal{R}T_2$, where $\mathcal{R}W \in \operatorname{add} \mathcal{R}T_2$. Therefore, $\mathcal{R}T_1 \in \operatorname{Fac} (\mathcal{R}T_2)$.

(*b*). Let $\alpha : (M_1, Q_1) \to (M_2, Q_2)$ be an arrow in $Q(\mathrm{s}\tau - \operatorname{tilt} B)$. Then, there exists an almost complete support τ-tilting pair for mod B, let denote it (U, P), which is a direct summand of (M_1, Q_1) and (M_2, Q_2). Since e is a morphism of posets, we have $e(M_1, Q_1) < e(M_2, Q_2)$. Observe that $e(U, P) = (\mathcal{E}U \oplus S, P)$ is an almost complete support τ-tilting pair for mod A, since

$$\begin{aligned} |\mathcal{E}U \oplus S| + |Q| &= |\mathcal{E}U| + 1 + |Q| \\ &= |U| + |Q| + 1 \\ &= n - 1. \end{aligned}$$

Moreover, $e(U, P) = (\mathcal{E}U \oplus S, P)$ is a direct summand of $e(M_1, Q_1)$ and $e(M_2, Q_2)$. Thus, by definition, we have that $e(M_2, Q_2) = \mu^-_{\mathcal{E}X} e(M_1, Q_1)$. Hence, there exists an arrow $e\alpha : e(M_1, Q_1) \to e(M_2, Q_2)$ in $Q(\mathrm{s}\tau - \operatorname{tilt} A)$. □

Remark 4.2 The above theorem shows that the extension functor behaves well respect to the mutation of support τ-tilting modules. In some way, the extension functor commutes with the mutation.

Proof of Theorem B By Theorem 4.1 and since $re = \mathrm{Id}_{\mathrm{s}\tau - \operatorname{tilt} B}$, the map e is an embedding of quivers. Hence, we only have to show that if there exists an arrow $e(M, P) \to e(N, Q)$ in $Q(\mathrm{s}\tau - \operatorname{tilt} A)$, then there exist an arrow $(M, P) \to (N, Q)$ in $Q(\mathrm{s}\tau - \operatorname{tilt} B)$.

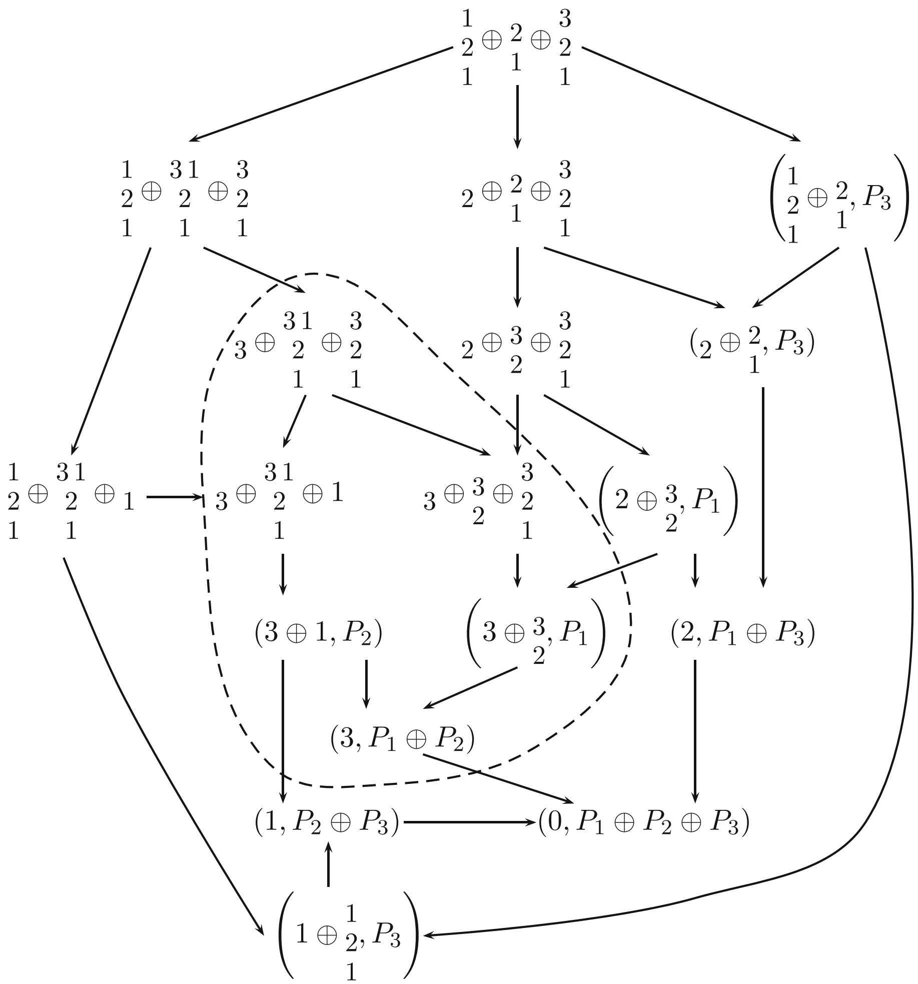
We know that $e(M, P) = (\mathcal{E}M \oplus S, P)$ and $e(N, Q) = (\mathcal{E}N \oplus S, Q)$. Since there exists an arrow from $e(M, P)$ to $e(N, Q)$, then there is an almost complete support τ-tilting module, (U, L), which is a direct summand of $e(M, P)$ and $e(N, Q)$. Since S is a direct summand of $e(M, P)$ and $e(N, Q)$, then S is a direct summand of U. Thus $U = U' \oplus S$, with $U' \in S^{\mathrm{perp}}$. Then, $|\mathcal{R}U| + |L| = |\mathcal{R}U'| + |L| = |U'| + |L| = n - 2$. Note that L is a projective B-module, since $\operatorname{Hom}_A(L, S) = 0$. Therefore, we have that (U', L) is an almost complete support τ-tilting pair for mod B which is a direct summand of (M, P) and (N, Q). Since r is a morphism of posets, there exists an arrow $(M, P) \to (N, Q)$ in $Q(\mathrm{s}\tau - \operatorname{tilt} B)$. □

We illustrate the above theorem with the following example.

Example 4.3 Let B be the algebra given by the quiver $1 \underset{\beta}{\overset{\alpha}{\leftrightarrows}} 2$ with the relation $\alpha\beta = 0$.

We denote all the modules by their composition factors. Consider $A = B[P_2]$, the one-point of B by the projective $P_2 = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$. Then A is given by the quiver $1 \underset{\beta}{\overset{\alpha}{\leftrightarrows}} 2 \overset{\gamma}{\longleftarrow} 3$ with relation the $\alpha\beta = 0$.

The quiver $Q(\mathrm{s}\tau - \mathrm{tilt}\, A)$ is the following



Then, the image of the quiver $Q(\mathrm{s}\tau - \mathrm{tilt}\, B)$ under e is the subquiver indicated by dotted lines.

For the remainder of this section, we state some technical results about the local behavior of $Q(\mathrm{s}\tau - \mathrm{tilt}\, A)$. We are interested to know when the image of e is closed under successors. The next theorem gives us an answer for a particular case.

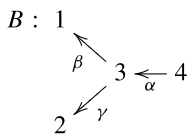
Theorem 4.4 *Let* (T, P) *and* (T', P') *be basic support* τ*-tilting pairs for mod* A *such that there exists an arrow* $(T, P) \to (T', P')$ *in* $Q(\mathrm{s}\tau - \mathit{tilt}\, A)$*. If* $(T, P) = e(M, Q)$ *and* $\mathit{Hom}_A(\mathcal{E}M, S) \neq 0$ *then there exists a support* τ*-tilting pair* (N, R) *in* $\mathrm{s}\tau - \mathit{tilt}\, B$ *such that* $(T', P') = e(N, R)$.

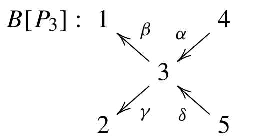
Proof Let $(T, P) = e(M, Q)$ be a support τ-tilting pair for $\operatorname{mod} A$ such that $\operatorname{Hom}_A(\mathcal{E}M, S) \neq 0$. Then, by Shur's Lemma $S \in \operatorname{Fac}(\mathcal{E}M)$. We claim that S is a direct summand of T' where (T', P') is a support τ-tilting pair such that there exists an arrow from (T, P) to (T', P') in $Q(\mathrm{s}\tau{-}\mathrm{tilt}\, A)$. In fact, otherwise $(T', P') = \mu_S(T, P)$. Moreover, since there exists an arrow from (T, P) to (T', P') in $Q(\mathrm{s}\tau - \mathrm{tilt}\, A)$ then $(T', P') = \mu_S^-(T, P)$. Therefore, it follows by Definition 2.18 that $S \notin \operatorname{Fac}(\mathcal{E}M)$, which is a contradiction. Hence, $T' = S \oplus Y$.

Since $S \oplus Y$ is a basic τ-rigid module, then $\operatorname{Ext}^1_A(S, Y) = 0$ and $\operatorname{Hom}_A(S, Y) = 0$. Then $Y \in S^{perp}$ and therefore $Y \cong \mathcal{E}\mathcal{R}Y$. Furthermore, since $\operatorname{Hom}_A(P', S \oplus Y) = 0$ we have that P' is a projective B-module. Considering the support τ-tilting pair $(\mathcal{R}Y, P')$ we obtain the result. $\square$

The following example shows that the condition $\operatorname{Hom}_A(\mathcal{E}M, S) \neq 0$ in Theorem 4.4 can not be removed.

Example 4.5 Consider the following algebras:





$\alpha\beta = 0$ $\qquad\qquad$ $\alpha\beta = 0$

It is not hard to see that $(1 \oplus 4, P_2 \oplus P_3)$ is an almost complete support τ-tilting pair for $\operatorname{mod} A$ and their complements are $(5, 0)$ and $(0, P_5)$. Moreover, there exists an arrow $(1 \oplus 5 \oplus 4, P_2 \oplus P_3) \to (1 \oplus 4, P_2 \oplus P_3 \oplus P_5)$ in $Q(\mathrm{s}\tau - \mathrm{tilt}\, A)$.

Note that a support τ-tilting pair, (U, P), belongs to the image of e if and only if S is a direct summand of U. Then, $(1 \oplus 5 \oplus 4, P_2 \oplus P_3)$ belongs to the image of e, but $(1 \oplus 4, P_2 \oplus P_3 \oplus P_5)$ does not belong to the image of e.

Suppose that we have a pair (M, Q) in $Q(\mathrm{s}\tau - \mathrm{tilt}\, A)$ which belongs to the image of e. Then, the following result gives information about the predecessors of (M, Q).

Theorem 4.6 *Let* (T, P) *be a support* τ*-tilting pair such that there exists a support* τ*-tilting pair* (M, Q) *in* $\mathrm{s}\tau - \mathrm{tilt}\, B$ *with* $(T, P) = e(M, Q) = (\mathcal{E}M \oplus S, P)$. *Then there is exactly one immediate predecessor of* (T, P) *in* $Q(\mathrm{s}\tau - \mathrm{tilt}\, A)$ *which does not belong to the image of e if and only if* $\operatorname{Hom}_A(\mathcal{E}M, S) \neq 0$.

Proof Suppose that there is exactly one immediate predecessor of (T, P) in $Q(\mathrm{s}\tau - \mathrm{tilt}\, A)$ which does not belong to the image of e and assume that $\operatorname{Hom}(\mathcal{E}M, S) = 0$. Then, $S \notin \operatorname{Fac}(\mathcal{E}M)$. By definition $\mu_S(T, P)$ is a left mutation of (T, P) and there exists an arrow from (T, P) to $\mu_S(T, P)$ in $Q(\mathrm{s}\tau - \mathrm{tilt}\, A)$. Therefore, all the predecessors (T', P') of (T, P) satisfy that $T' = S \oplus M$ with $M \in S^{perp}$. Then, all the predecessors belong to the image of e, which is a contradiction.

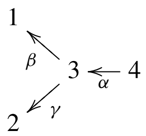
Conversely. Let $(T, P) \in \mathrm{s}\tau - \mathrm{tilt}\, A$ such that $(T, P) = (\mathcal{E}M \oplus S, P)$ and $\operatorname{Hom}_A(\mathcal{E}M, S) \neq 0$. We show that there is only one immediate predecessor of (T, P) which does not have S as a direct summand.

By definition of $Q(\mathrm{s}\tau - \mathrm{tilt}\, A)$, there is at most one immediate predecessor of (T, P) such that S is not a direct summand. Assume that all immediate predecessors of (T, P)

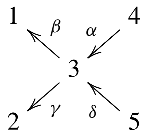
have the simple S as a direct summand. Then there exists an immediate successor of (T, P), let say (T', P') in $Q(s\tau - \text{tilt}\, A)$, such that S is not a direct summand of (T', P'). Thus, by construction, we have $(T', P') = \mu_S^+(T, P)$. It follows by Definition 2.18 that $S \notin \text{Fac}\,(\mathcal{E}M)$ and thus $\text{Hom}_A(\mathcal{E}M, S) = 0$, which is a contradiction. Therefore, we prove that there is exactly one immediate predecessor of (T, P) such that S is not a direct summand. □

We end up this section showing an example that if we extend by a non-projective module, then neither the restriction nor the extension define maps between the corresponding posets of support τ-tilting modules.

Example 4.7 Let B be the following algebra



and let $A = B[X]$, where $X = \begin{smallmatrix}3\\2\end{smallmatrix}$. Then A is given by the quiver



with the relation $\delta\beta = 0$.

1. Extending the τ-tilting B-module $M = \begin{smallmatrix}4\\3\\2\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\1\end{smallmatrix}$ we get the A-module $eM = \begin{smallmatrix}45\\3\\2\end{smallmatrix} \oplus \begin{smallmatrix}5\\3\end{smallmatrix} \oplus \begin{smallmatrix}45\\3\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\1\end{smallmatrix} \oplus 5$ which is not τ-tilting because $\text{Hom}_A(\begin{smallmatrix}4\\3\\1\end{smallmatrix}, \tau_A 5) \neq 0$.
2. Restricting the τ-tilting A-module $T = 4 \oplus 5 \oplus \begin{smallmatrix}45\\3\end{smallmatrix} \oplus \begin{smallmatrix}45\\3\\2\end{smallmatrix} \oplus 1$ yields the B-module $\mathcal{R}T = 4 \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\2\end{smallmatrix} \oplus 1$ which is not τ-tilting because $\text{Hom}_B(1, \tau_B \begin{smallmatrix}4\\3\\2\end{smallmatrix}) \neq 0$.

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References

1. Adachi, T., Iyama, O., Reiten, I.: τ-tilting theory. Compos. Math. **150**(03), 415–452 (2014)
2. Assem, I., Happel, D., Trepode, S.: Extending tilting modules to one-point extensions by projectives. Comm. Algebra **35**(10), 2983–3006 (2007)
3. Assem, I., Simson, D., Skowronski, A.: Elements of the Representation Theory Of Associative Algebras. London Math. Soc. Student Texts 65. Cambridge University Press, Cambridge (2006)
4. Auslander, M., Smalø, S.O.: Almost split sequences in subcategories. J. Algebra **69**(2), 426–454 (1981)
5. Bongartz, K.: Tilted Algebras. Proc. ICRA III (Puebla 1980), Lecture Notes in Math. No. 903, Springer-Verlar, 2936

6. Brenner, S., Butler, M.C.R.: Generalizations of Bernstein-Gelfand-Ponomarev Reflection Functors. Lecture Notes in Math, vol. 839, pp. 103–169. Springer, Berlin (1980)
7. Happel, D., Ringel, C.M.: Tilted algebras. Trans. Amer. Math. Soc. **274**, 399–443 (1982)
8. Jasso, G.: Reduction of τ-tilting modules and torsion pairs. International Mathematics Research Notices, rnu163 (2014)
9. Psaroudakis, C.: Homological theory of recollements of abelian categories. J. Algebra **398**, 63–110 (2014)