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INDUCED CURRENT IN QUANTUM AND CLASSICAL RATCHETS

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In a previous work, we described transport in a classical, externally driven, overdamped ratchet. A transport current arises under two possible conditions: either by increasing the external driving or by adding an optimal amount of noise when the system operates below threshold. In this work, we study the underdamped case. In order to obtain transport it is necessary for the presence of both — a damping mechanism and the lack of symmetries in the potential. Some interesting properties were found: under particular conditions the system could be considered as a mass separation device, and for a specific range of the control parameter, the maximum Lyapunov exponent is reduced when noise is added to the system. We also study analytically and numerically the quantum analog of the same system and explore the conditions to find transport.

Keywords: Brownian motors; noise; ratchets.

1. Introduction

Transport phenomena play a crucial role in many problems of physics, biology and social sciences. In particular, it has been of increasing interest in the transport properties of the Brownian ratchets (BR). These are devices out of thermal equilibrium in which a nonzero net drift velocity may be obtained when fluctuations interact with the broken symmetry of the potential [Astumian *et al.*, 1994; Bier, 1997; Parrondo *et al.*, 2003; Harmer *et al.*, 1999; Denisov *et al.*, 2007; Flach *et al.*, 2000]. There are many areas in which BR's play an important role. However, in the past years, this subject has deserved a great deal of attention because of their technological applications in nanoscales and microscales devices [Rousselet *et al.*, 1994; Faucheux *et al.*, 1995].

Several works have been devoted to the study of transport and directional current in classical ratchets under the influence of external perturbations, with and without noise, and as a function of the intensity of an external driving. In these models, a net transport can be induced with fluctuations associated with additive forces and noise. Transport is also found without noise, in overdamped [Popescu *et al.*, 2000; Fendrik *et al.*, 2006], or underdamped systems [Mateos, 1995; Jung *et al.*, 1995; Parrondo *et al.*, 2002; Marchesoni *et al.*, 2006]. One purpose of this paper is to investigate the phenomenon of current inversion as a function of the specific frictional coefficient whose relevance is that it provides the conceptual basis of a mass separator device. We show that these inertial ratchets may exhibit a very complex and rich dynamics, that involves regular and chaotic motion. Moreover, they can display multiple values of the current where its direction depends on the inertial term and the amount of friction.

In this paper, we also include the problem of quantum transport in periodic, asymmetric, ratchet potential in the underdamped regime that is in contact with a dissipative environment. This system is related to the study of the dynamics of cold atoms in an optical lattice, which is a realization of ratchet phenomena in quantum systems [Carlo *et al.*, 2005].

The present work is organized as follows: in Sec. 2 we introduce the classical model under study, with some emphasis on the study of the transport regime. In Sec. 3, we discuss transport in the quantum analog of the system. Conclusions are discussed in the last section.

2. The Classical System

We consider noninteracting, massive (m) particles placed in a one-dimensional ring of radius $R = 1$, subject to a periodic potential and driven by an external force. The system is assumed to be in contact with a dissipative environment. The time evolution is given by the equation:

$$m\ddot{x} = -\mu\dot{x} - \frac{\partial V_\alpha(x)}{\partial x} + F^{dr}(x,t). \tag{1}$$

In Eq. (1), $0 \leq x(t) \leq 2\pi$ represents the coordinate of the particle, μ the damping coefficient and $V(x)_\alpha$ is the one-dimensional, periodic potential

$$V_\alpha(x) = \begin{cases} V_o \cos\left(\dfrac{\pi(\alpha+1)x}{a\alpha}\right), & \text{for } 0 \leq x \leq \dfrac{a\alpha}{(\alpha+1)} \\ -V_o \cos\left(\dfrac{\pi(\alpha+1)x}{a} - \alpha\pi\right), & \text{for } \dfrac{a\alpha}{(\alpha+1)} \leq x \leq a \end{cases} \tag{2}$$

This fulfills the periodicity condition $V_\alpha(x + a) = V_\alpha(x)$ where $a = 2\pi/N$, N being the number of wells (or sites) along the ring. The parameter α ($\alpha > 0$) controls the left–right asymmetry of the potential. For $\alpha > 1 (< 1)$ the minimum in each well of the ratchet is displaced towards the right (left) while $\alpha = 1$ corresponds to symmetric potential.

Particles are driven by the external periodic driving force $F^{dr}(x,t)$. We take this to be the gradient of a time dependent potential with a spatial periodicity that is twice that of V_α in order that consecutive wells alternate in time as absolute minima. This condition restricts N to be even. We thus consider

$$F^{dr}(x,t) = -\varepsilon\frac{\partial V^{dr}(x,t)}{\partial x} = -\varepsilon\sin(\omega t)\frac{d\sin\left(\dfrac{Nx}{2}\right)}{dx}, \tag{3}$$

that represents a longitudinal stationary wave along the ring. Time is measured in units of the period $\tau = 2\pi/\omega$ of the external driving and ε is the coupling strength. There is a critical value $\varepsilon = \varepsilon_c$ such that if $\varepsilon < \varepsilon_c$ the potential $(V_\alpha(x) + V^{dr}(x,t))$ always has a minimum per site. For $\varepsilon > \varepsilon_c$ the potential may lose the minima that correspond to alternate sites as time changes. In our model, we restrict to consider $N = 4$, $\alpha = 5$ and $\varepsilon < \varepsilon_c$.

This equation takes the following dimensionless form:

$$\ddot{x} = \beta\left[-\dot{x} + \gamma\left(F_\alpha(Nx) - \frac{\varepsilon'}{2}\cos\left(\frac{Nx}{2}\right)\sin(2\pi t)\right)\right], \tag{4}$$

where $\beta = 2\pi\mu/m\omega$ is the specific frictional parameter, $\gamma = 2\pi V_o N/\mu\omega$ is the ratio between the external force and the dissipation, $F(Nx) = -(1/NV_o)(\partial V_\alpha(Nx)/\partial x)$ and $\varepsilon' = \varepsilon/V_o$.

The hamiltonian system presents two symmetries:

$$S_1 : x \to x + (2k+1)\frac{2\pi}{N}, \quad t \to -t, \ \ v \to -v; \tag{5}$$

$$S_2 : x \to (2k+1)\frac{2\pi}{N} - x, \quad t \to t, \ \ v \to -v, \tag{6}$$

which need to be broken to fulfill the necessary conditions to obtain a directional dc-current. If $\alpha \neq 1$, the S_2 symmetry is removed but the S_1 symmetry still holds. When $\mu > 0$ the only symmetry removed is S_1. Therefore, in order to obtain directional transport the values of α and μ must be $\alpha \neq 1$ and $\mu > 0$. Taking this into account, we claim that the current is induced by damping.

In order to discuss the features of the directional transport and of the dynamics of the ratchet, as a function of the specific frictional coefficient μ/m, we set $N = 4$, $V_o = 10$, $\omega = 6$, $\mu = 1$ (that is $\gamma = 40\pi/3$) and $\varepsilon = 6.5$ greater than $\varepsilon_c = 6.26007$. Since we fix all the parameters except m, we study the system as β changes.

We calculated the mean velocity $\bar{v}$ when S_1 and S_2 symmetries are broken and measure it for each site taking as unit the period of the driving force, as:

$$\bar{v} = \left(\frac{N}{2\pi}\right) \lim_{t\to\infty} \frac{(x(t) + 2\pi k)}{t}, \tag{7}$$

where k is the winding number (i.e. the number of rotation that the particle makes around the ring).

Then, the ensamble average of the velocity is given by:

$$\langle \bar{v} \rangle = \frac{1}{N_{\rm in}} \sum_{j=1}^{N_{\rm in}} \bar{v}_j. \tag{8}$$

In computing the average maximal Lyapunov exponent $L^j_{\max}$ as [Wolf, 1984]:

$$\langle L_{\max} \rangle = \frac{1}{N_{\rm in}} \sum_{j=1}^{N_{\rm in}} L^j_{\max}, \tag{9}$$

we found diverse and complementary information to the mean velocity. In Figs. 1(a) and 1(b), the average mean velocity $\langle \bar{v} \rangle$ and the average maximal Lyapunov exponent $L^j_{\max}$ are plotted as a function of β. Three different regions can be recognized:

(I) The region of overdamping (great values of β) with an average current $\langle \bar{v} \rangle_{\max} = 2$ corresponding to the regular orbits with chaotic intrawell dynamics reflected in the $L^j_{\max}$ fluctuations.

(II) The hamiltonian limit ($\beta \sim 0$) with $\langle \bar{v} \rangle_{\max} \sim 0$ and positive $L^j_{\max}$.

(III) An intermediate region of β with a very rich and complex dynamical structure with coexistence of regular and chaotic attractors and the appearance and disappearance of different kinds of orbits [Carusela *et al.*, 2008].

In this last region, the dynamics is strongly dependent on the value of β. In some regions, a slight variation of this parameter can destroy stationary orbits and/or can create new ones. There are regions where two or three orbits with quite distinct characteristics coexist. This fact is reflected in the transport properties of the system, since the mean velocity of the orbits involved should be quite different not only in their magnitude but also in their directions. As can be seen in panel (a) of Fig. 1 the $\langle \bar{v} \rangle_{\max}$ fluctuates from −2 to 2 in some regions while vanishes in others.

We now turn to study the transport current assisted by noise, when the dynamics of the

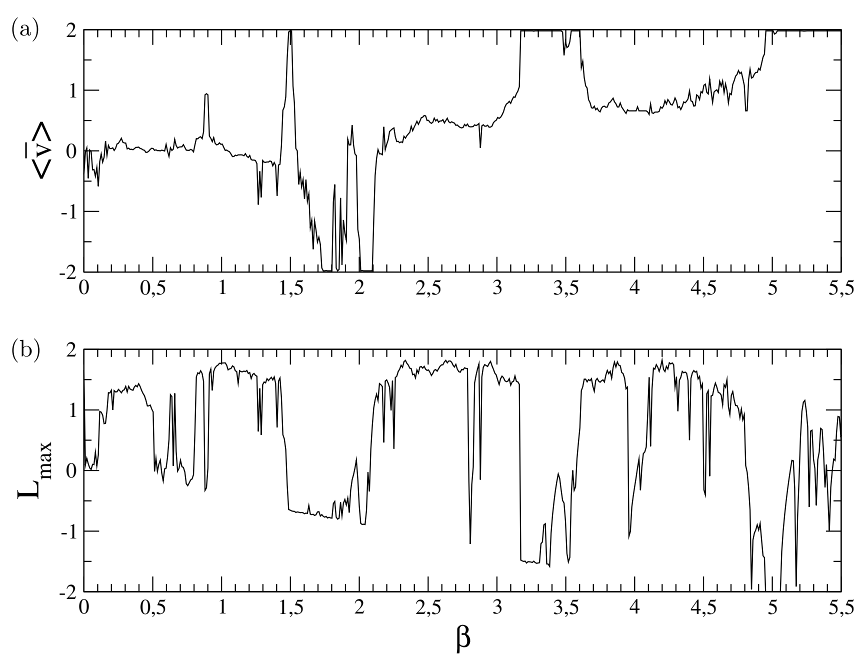

Fig. 1. (a) Average transport velocity $\langle \bar{v} \rangle$ and (b) average of the maximum Lyapunov exponent $L_{\max}$ as a function of the specific frictional coefficient β. The velocity is measured in terms of sites per period of the driven force. The points are joined by a line to help the eye.

deterministic system is unbound. We solve the following stochastic differential equation:

$$\ddot{x} = \beta\left[-\dot{x} + \gamma\left(F_\alpha(Nx) - \frac{\varepsilon'}{2}\cos\left(\frac{Nx}{2}\right)\sin(2\pi t)\right) + \xi(t)\right], \quad (10)$$

where the external random force $\xi(t)$ fulfills $\langle \xi(t)\xi(t')\rangle = \sigma^2\delta(t-t')$.

It is interesting to note that in the deterministic case, the current $\langle \overline{v}\rangle$ in Fig. 1(a) shows abrupt inversions as β changes in a similar way as other ratchets. Since the variation of β depends only on the mass m, identical particles with different masses will have different velocities. This behavior is the basis of a mass separating device. An appealing feature is the role of noise as the ultimate effect of mass separation. In Fig. 3, the mean velocity is plotted as a function of σ for $\beta = 1.95$ and $\beta = 1.98$. For these two values of β, the deterministic system presents the coexistence of regular and chaotic orbits and similar $\langle \overline{v}\rangle$ (see panel (a) of Fig. 1). When noise is added, an orbit is not robust and disappears, while the other (with $\overline{v} = -2$) persists. Correspondingly, the current not only has an inversion, but also increases its absolute value. We emphasize the fact that noise, in the present case, stabilizes the dynamics as can be seen from the reduction of average maximal Lyapunov exponent $\langle L_{\max}\rangle$ (see Fig. 2). This is due to the fact that the destroyed orbit has a positive Lyapunov exponent [Carusela *et al.*, 2008].

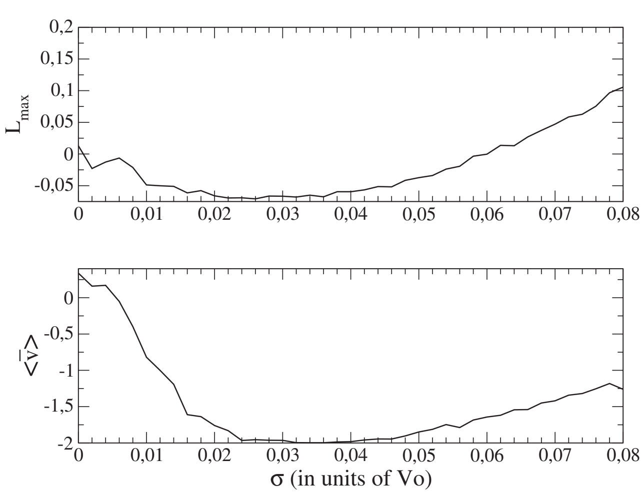

Fig. 2. Top panel: Average of the maximum Lyapunov exponent $L_{\max}$ as a function of the noise intensity σ for $\beta = 1.987$. Lower panel: Average transport velocity $\langle \overline{v}\rangle$ as a function of the noise intensity σ for the same β.

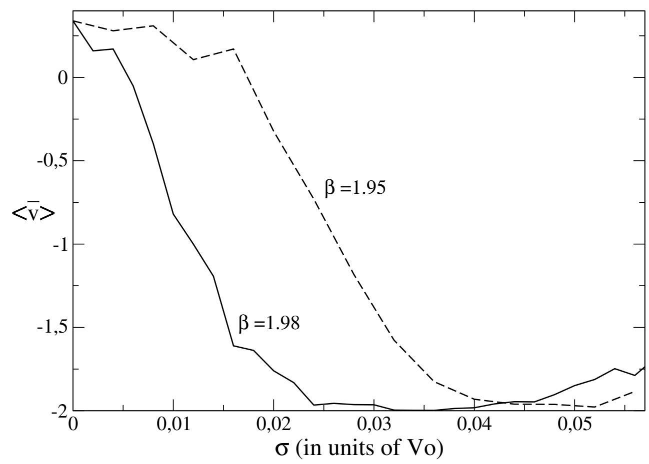

Fig. 3. Average transport velocity $\langle \overline{v}\rangle$ as a function of the noise intensity σ. The values of β are indicated in the figure.

Regular and chaotic attractors coexist in this region, the last being less robust against noise. Likewise, in the deterministic case, a mass separation can also be achieved but for smaller intensities of noise than the optimun for which a minimal current is achieved.

3. The Quantum System

To describe the quantum evolution of the classical system given by Eq. (1) we solve the master equation in the Lindblad form [Lindblad, 1976] for the statistical operator $\hat{\rho}$:

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] - \frac{1}{2}\sum_{j=1}^{j=2}\{\hat{L}_j^\dagger \hat{L}_j, \hat{\rho}\} + \sum_{j=1}^{j=2}\hat{L}_j\hat{\rho}\hat{L}_j^\dagger. \quad (11)$$

Here, $\hat{L}_j$ are Lindblad operators, $\{,\}$ stands for anticommutators and $\hat{H} = \hat{H}_o + V^{dr}(\hat{x}, t)$, where:

$$\hat{H}_o = \frac{\hat{p}^2}{2m} + V_\alpha(\hat{x}) \quad (12)$$

and $V^{dr}(x,t)$ is defined by Eq. (3). The Hamiltonian system evolves following H which is periodic in time (with period $\tau = 2\pi/\omega$). For Eq. (11), we choose the base of the Floquet's states [Holder & Reichl, 2005] defined by (setting $\hbar = 1$):

$$\left(H - i\frac{\partial}{\partial t}\right)\Psi_k(x,t) = \epsilon_k\Psi_k(x,t). \quad (13)$$

We write the Floquet states as:

$$\Psi_k(x,t) = \sum_{n,l} A_{n,l}^k \exp(in\omega t)\phi_l(x) \quad (14)$$

where $\phi_l(x)$ are the eigenstates of the static hamiltonian H_o given by Eq. (12). (In the present case, the number of sites $N = 4$, are linear combinations of Mathieu functions of integer and half-integer orders.) Replacing Eq. (14) in Eq. (13) we obtain the coefficients $A^k_{n,l}$ and the corresponding quasienergies ϵ_k. To build the appropriate base, we select the Floquet state of quasienergy ϵ_m in the middle of the spectrum and we consider all states k such that $|\epsilon_k - \epsilon_m| < \omega/2$.

The Lindblad operators are:

$$\begin{aligned}\hat{L}_1 &= d\sum_{n=0}\sqrt{n+1}|n\rangle\langle n+1|,\\ \hat{L}_2 &= d\sum_{n=0}\sqrt{n+1}|-n\rangle\langle -n-1|,\end{aligned} \tag{15}$$

where d is related to the classical friction coefficient. Since the eigenstates of angular momentum in the x representation are $\langle x|n\rangle = \exp-(inx)/\sqrt{2\pi}$, the evaluation of the matrix elements of the Lindblad operators among Floquet states is reduced to calculate some integrals involving products of Mathieu functions and a single trigonometric function.

We solve the problem following [Carlo *et al.*, 2005], using the quantum trajectories approach. The first and the second terms of Eq. (11) can be seen as an effective non-hermitian hamiltonian $\hat{H}_e = \hat{H} - (i/2)\sum_{j=1}^{j=2}\hat{L}_j^\dagger\hat{L}_j$ that evolves the system. In addition, the third term induces quantum leaps. Starting from a pure Floquet state, $|\Psi_k(t_i)\rangle$, the probability of a jump to the state $|\Phi_j\rangle = \hat{L}_j|\Psi_k(t_i)\rangle/|\hat{L}_j|\Psi_k(t_i)|$ in an infinitesimal time interval δt is $\delta p_j = \langle\Psi_k(t_i)|\hat{L}_j^\dagger\hat{L}_j|\Psi_k(t_i)\rangle\delta t$. The probability of no jump (which is the evolution given by $\hat{H}_e$) in this interval is $1-\delta p_1-\delta p_2$. Representing the evolution from t_i to t as a sucession of $K \gg 1$ infinitesimal evolutions of $\delta t = (t-t_i)/K$, it is possible to develop a Montecarlo simulation for the time evolution. The mean velocity for a given initial state is calculated as:

$$\bar{v} = \frac{1}{K}\sum_{j=1}^{j=K}\left\langle \Phi(j\delta t)\left| i\frac{\partial}{\partial x}\right|\Phi(j\delta t)\right\rangle \tag{16}$$

where $|\Phi(j\delta t)\rangle$ is the evolved state at $t = j\delta t$. Averaging over $P = 100$ initial states, we obtain the average velocity:

$$\langle\bar{v}\rangle = \frac{1}{P}\sum_{n=1}^{n=P}\bar{v}_j \tag{17}$$

To analyze whether this formalism verifies the quantum-classical correspondence, we calculate the average and the mean velocity in the limit of strong dissipation. Unlike the classical case where the maximun current ($\langle\bar{v}\rangle = 2$) is reached, the average velocity in the quantum limit takes zero value. To illustrate this point, we calculate $\langle\bar{v}\rangle$ in a range of parameters where the classical system presents an overdamped regime. In the first five panels of Fig. 4,

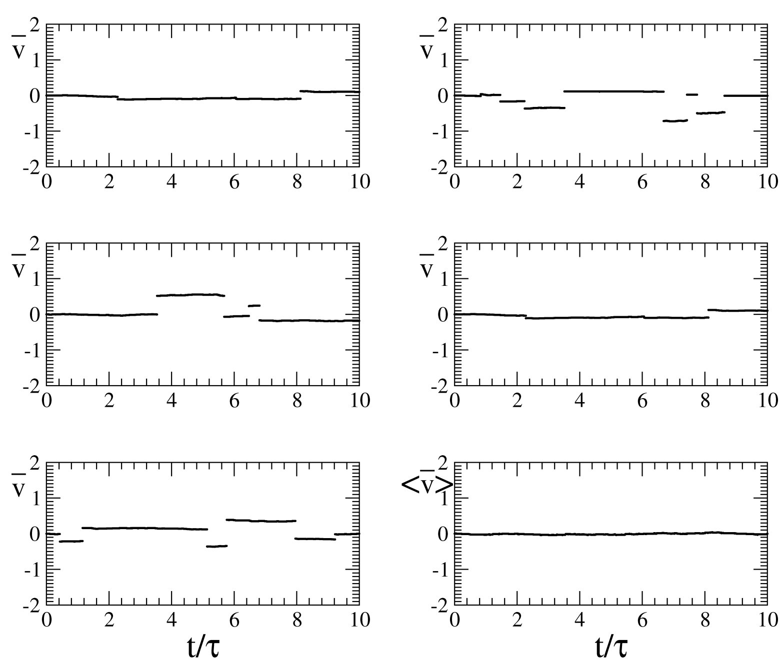

Fig. 4. Left column and two upper panels of the right column: Mean velocity $\bar{v}$ as a function of time for five random initial states. Bottom panel of the right column: Average transport velocity $\langle\bar{v}\rangle$ as a function of time. Time is measured in units of the period $\tau = 2\pi/\omega$, with $\omega = 24$ and for $d = 0.447$.

we plot the mean velocity $\bar{v}$ as a function of time for an initial condition randomly selected from an ensemble of 100 initial states. The parameters are set as follows: $\omega = 24$ and $d = 0.447$ ($\beta \sim 145$), where d is related to the specific frictional coefficient β in the following way:

$$\beta = (1 - e^{-d^2})\frac{2\pi}{\omega\delta t} \tag{18}$$

In the last panel of Fig. 4 we show that $\langle\bar{v}\rangle$ as a function of time goes to zero (averaging over the ensemble of initial conditions), contrary to the classical expected result.

We summarize the results that we have obtained.

(1) For weak dissipation, the average velocity $\langle\bar{v}\rangle \sim 0$. This result is consistent with the classical limit. However, to test this correspondence, we should calculate the Husimi representation [Takahashi & Saitô, 1985] of the final states to verify if the classical structures in the Poincaré section are reproduced.

(2) Unexpectedly, for stronger dissipation, the average current is still zero unlike what is expected for the classical results. This indicates that for such a limit of dissipation, the present approach is not appropriate. As already mentioned, the directional velocity is induced by the damping, therefore when a nonvanishing current is present, the current formalism is inadequate. In an alternative approach, the dissipation could be modelized by an approach inspired in the Legget–Caldeira model [Caldeira & Legget, 1983] as in [Denisov *et al.*, 2008].

4. Conclusions

We have studied the dynamics in the stationary regime of an inertial ratchet as a function of the specific frictional coefficient β. The fact that the dynamics is strongly dependent on such coefficient is reflected in the transport properties of the system, and because the mean velocity of the orbits involved, presents multiple and abrupt reversals with very different magnitudes. This property provides the conceptual basis of a mass separation device.

We have also explored the effect of noise when a random force is added in the deterministic system where the dynamics is unbound. We observe that in some regions of the parameter space, noise has a stabilizing effect producing current inversions. In particular, in some regions noise produces mass separation.

We have also studied the quantum ratchet analogue using the Master equation for the statistical operator in the Lindblad form. Using the Floquet states of the Hamiltonian system, we have simulated the temporal evolution through the quantum trajectory method. For weak dissipation, we obtain no transport according to the classical counterpart. However, for a stronger dissipation — which is necessary to ensure the transport — the formalism fails.

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