Communications in Contemporary Mathematics
Vol. 18, No. 4 (2016) 1550053 (35 pages)

DOI: 10.1142/S0219199715500534

Finite-dimensional pointed Hopf algebras over finite simple groups of Lie type II: Unipotent classes in symplectic groups

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Received 23 December 2014
Accepted 3 April 2015
Published 19 May 2015

We show that Nichols algebras of most simple Yetter–Drinfeld modules over the projective symplectic linear group over a finite field, corresponding to unipotent orbits, have infinite dimension. We give a criterion to deal with unipotent classes of general finite simple groups of Lie type and apply it to regular classes in Chevalley and Steinberg groups.

Keywords: Nichols algebra; Hopf algebra; rack; finite group of Lie type; conjugacy class.

Mathematics Subject Classification 2010: 16T05

1. Introduction

This is the second paper of a series intended to determine the finite-dimensional pointed Hopf algebras with group of group-likes isomorphic to a finite simple group of Lie type. An introduction to the whole series was given in Part I [1]. The base field is $\mathbb{C}$. Let p be a prime number, $m \in \mathbb{N}$, $q = p^m$ and $\mathbb{F}_q$ the field with q elements. In this paper we consider Nichols algebras associated to unipotent conjugacy classes in *symplectic groups* $\boldsymbol{G} = \mathbf{PSp}_{2n}(q)$, $n \geq 2$, see, e.g., [15, 9]. We consider also here the non-simple group $\mathbf{PSp}_4(2) \simeq \mathbb{S}_6$ for convenience.

Let $\mathcal{O}$ be a conjugacy class of $\boldsymbol{G}$. We seek to determine all $\mathcal{O}$ that *collapse* [2, 2.2], that is, the dimension of the Nichols algebra $\mathfrak{B}(\mathcal{O}, q)$ is infinite for every finite faithful 2-cocycle q. Our main result says the following.

Theorem 1.1. *Let $\mathcal{O}$ be a unipotent conjugacy class in $\boldsymbol{G}$. If $\mathcal{O}$ is not listed in Table* 1, *then it collapses.*

In a subsequent paper of the series we will deal with the non-semisimple classes in $\boldsymbol{G}$; for this we will also need to consider the unipotent classes in the finite unitary groups.

Notation. We denote the cardinal of a set X by $|X|$. If $k < \ell$ are positive integers, then we set $\mathbb{I}_{k,\ell} = \{i \in \mathbb{N} : k \leq i \leq \ell\}$ and simply $\mathbb{I}_\ell = \mathbb{I}_{1,\ell}$.

Let G be a group; $N < G$, respectively $N \lhd G$, means that N is a subgroup, respectively a normal subgroup, of G. The centralizer, respectively the normalizer, of $x \in G$ is denoted by $C_G(x)$, respectively $N_G(x)$; the inner automorphism defined by conjugation by x is denoted by $\operatorname{Ad} x$. If $\mathcal{F} \in \operatorname{Aut} G$, then $G^{\mathcal{F}}$ denotes the subgroup consisting of points fixed by $\mathcal{F}$.

2. Preliminaries on Racks

Recall that a rack is a non-empty set X with a self-distributive binary operation $\triangleright$ such that $x \triangleright \_$ is bijective for all $x \in X$. The archetype of a rack is a conjugacy class in a group with the conjugation operation. This notion allows considerable flexibility in the treatment of the conjugacy classes.

Table 1.

$\mathbf{PSp}_{2n}(q)$			
n	Type	q	Remark
≥ 2	$W(1)^a \oplus V(2)$	even	cthulhu, Lemma 4.22
	$(1^{r_1}, 2)$	odd, 9 or not a square	cthulhu, Lemma 4.22
3	$W(1) \oplus W(2)$	2	cthulhu, Lemma 4.25
2	$W(2)$	even	cthulhu, Lemma 4.26
	$(2, 2)$	3	one class cthulhu Lemma 4.5
	$V(2)^2$	2	cthulhu, Lemma 4.24

2.1. *Collapsing criteria*

We use criteria from [1, 2] to prove Theorem 1.1; see [1] for more details. Let X be a rack. One says that

- X is *of type D* provided that there is a decomposable subrack $Y = R\coprod S$ with elements $r \in R$, $s \in S$ such that
$$r \triangleright (s \triangleright (r \triangleright s)) \neq s. \tag{2.1}$$
- X is *of type F* if it has a family of mutually disjoint subracks $(R_a)_{a\in\mathbb{I}_4}$ and a family $(r_a)_{a\in\mathbb{I}_4}$ such that for all $a, b \in \mathbb{I}_4$
  - $R_a \triangleright R_b = R_b$;
  - $r_a \in R_a$ and $r_a \triangleright r_b \neq r_b$ when $a \neq b$.
- X is *cthulhu* if it is neither of type D nor of type F.
- X is *sober* if every subrack is either abelian or indecomposable.

Theorem 2.1 ([2, Theorem 3.6; 1, Theorem 2.2]). *A rack X of type D (respectively, F) collapses.*

Lemma 2.2. *Let G be a finite group, $P < G$, $\pi \colon P \to L$ a quotient map and $x \in P$. If $\mathcal{O}^L_{\pi(x)}$ is of type D, respectively F, then $\mathcal{O}^G_x$ is again so.*

Proof. The class $\mathcal{O}^G_x$ contains the subrack $\mathcal{O}^P_x$ and π induces a rack epimorphism $\mathcal{O}^P_x \to \mathcal{O}^L_{\pi(x)}$. The statement follows from [1, Remark 2.9]. □

Recall the convention in [1]: all racks considered in this series are crossed sets.

Lemma 2.3 ([1, Lemma 2.10(i)]). *Let X and Y be racks. Assume that there are $y_1 \neq y_2 \in Y$, $x_1 \neq x_2 \in X$ such that $x_1 \triangleright (x_2 \triangleright (x_1 \triangleright x_2)) \neq x_2$, $y_1 \triangleright y_2 = y_2$. Then $X \times Y$ is of type D.*

2.2. *Conjugacy classes and subgroups*

For recursive reasoning, we need to consider how a conjugacy class splits when intersected with a subgroup. Let G be a finite group, $N < G$ and $x \in N$. Let $\mathcal{C}(N, x)$ be the set of N-conjugacy classes contained in $\mathcal{O}^G_x$. We start with the case N normal.

Remark 2.4 ([1, Remark 2.1]). If $N \lhd G$, then $\mathcal{O}^G_x$ is a union of N-conjugacy classes isomorphic to each other as racks.

Next relevant case is $N = G^{\mathcal{F}}$, where $\mathcal{F} \in \operatorname{Aut} G$. Recall that G acts on itself by $x \rightharpoonup y = xy\mathcal{F}(x^{-1})$, $x, y \in G$. Let $H^1(\mathcal{F}, G)$ be the set of $\mathcal{F}$-twisted conjugacy classes in G, i.e. the orbits with respect to the action $\rightharpoonup$.

Remark 2.5. Let $M < G$ be $\mathcal{F}$-stable, $g, h \in G$, $x \in G^{\mathcal{F}}$. Set $z := g^{-1}\mathcal{F}(g)$, $w := h^{-1}\mathcal{F}(h)$. Then

(a) $gxg^{-1} \in G^{\mathcal{F}} \Leftrightarrow z \in C_G(x)$.

(b) gMg^{-1} is $\mathcal{F}$-stable $\Leftrightarrow z \in N_G(M)$.
(c) Assume that $z \in N_G(M)$. Then $(gMg^{-1})^{\mathcal{F}} = g(M^{\mathrm{Ad}\, z\circ\mathcal{F}})g^{-1}$.
(d) Assume that $z \in C_G(M)$. Then $(gMg^{-1})^{\mathcal{F}} = g(M^{\mathcal{F}})g^{-1}$, and hence $[(gMg^{-1})^{\mathcal{F}}, (gMg^{-1})^{\mathcal{F}}] = g[M^{\mathcal{F}}, M^{\mathcal{F}}]g^{-1}$.
(e) Assume that $z \in C_G(M)$ and $x \in M^{\mathcal{F}}$, respectively $x \in [M^{\mathcal{F}}, M^{\mathcal{F}}]$. Then $gxg^{-1} \in (gMg^{-1})^{\mathcal{F}}$, respectively $\in [(gMg^{-1})^{\mathcal{F}}, (gMg^{-1})^{\mathcal{F}}]$, and there are rack isomorphisms

$$\mathcal{O}_x^{M^{\mathcal{F}}} \simeq \mathcal{O}_{gxg^{-1}}^{(gMg^{-1})^{\mathcal{F}}}, \quad \text{respectively } \mathcal{O}_x^{[M^{\mathcal{F}},M^{\mathcal{F}}]} \simeq \mathcal{O}_{gxg^{-1}}^{[(gMg^{-1})^{\mathcal{F}},(gMg^{-1})^{\mathcal{F}}]}.$$

(f) Assume that $z \in C_G(M)$, $x \in M^{\mathcal{F}}$ and $\mathcal{O}_x^{M^{\mathcal{F}}}$ is of type D, respectively F, then $\mathcal{O}_{gxg^{-1}}^{G^{\mathcal{F}}}$ is of the same type.
(g) Assume that $z \in C_G(M)$, $x \in [M^{\mathcal{F}}, M^{\mathcal{F}}]$ and $\mathcal{O}_x^{[M^{\mathcal{F}},M^{\mathcal{F}}]}$ is of type D, respectively F, then $\mathcal{O}_{gxg^{-1}}^{[G^{\mathcal{F}},G^{\mathcal{F}}]}$ is of the same type.
(h) If $z, w \in C_G(x)$, then $\mathcal{O}_{gxg^{-1}}^{G^{\mathcal{F}}} = \mathcal{O}_{hxh^{-1}}^{G^{\mathcal{F}}}$ if and only if z and w belong to the same $\mathcal{F}$-twisted conjugacy classes in $C_G(x)$.

Proof. Conditions (a), (b), (c) and (d) are straightforward; Condition (e) follows from (a) and (d), while (f) and (g) follow from (e). (h): Assume that $\mathcal{O}_{gxg^{-1}}^{G^{\mathcal{F}}} = \mathcal{O}_{hxh^{-1}}^{G^{\mathcal{F}}}$; take $k \in G^{\mathcal{F}}$ such that $kgxg^{-1}k^{-1} = hxh^{-1}$ and $u = h^{-1}kg$. Then $u \in C_G(x)$ and $u \rightharpoonup z = w$. Conversely, if $u \in C_G(x)$ satisfies $u \rightharpoonup z = w$, then $k = hug^{-1} \in G^{\mathcal{F}}$ and $kgxg^{-1}k^{-1} = hxh^{-1}$. $\square$

By Remark 2.5(h), the map $\varphi : \mathcal{C}(G^{\mathcal{F}}, x) \to H^1(\mathcal{F}, C_G(x))$ sending $\mathcal{O}_{gxg^{-1}}^{G^{\mathcal{F}}}$ to the class of z, is well defined and injective.

Lemma 2.6. *Let $M < G$ be $\mathcal{F}$-stable, such that $x \in M$. Assume that every element in the image of φ has a representative in $C_G(M) \subset C_G(x)$.*

(1) *If $x \in M^{\mathcal{F}}$ and $\mathcal{O}_x^{M^{\mathcal{F}}}$ is of type D, respectively F, then $\mathcal{O}$ is so for every $\mathcal{O} \in \mathcal{C}(G^{\mathcal{F}}, x)$.*
(2) *If $x \in [M^{\mathcal{F}}, M^{\mathcal{F}}]$ and $\mathcal{O}_x^{[M^{\mathcal{F}},M^{\mathcal{F}}]}$ is of type D, respectively F, then $\mathcal{O}$ is so for every $\mathcal{O} \in \mathcal{C}([G^{\mathcal{F}}, G^{\mathcal{F}}], x)$.*

Proof. (1). By Remark 2.5(a) and the assumption, there exists $g \in G$ such that $gxg^{-1} \in G^{\mathcal{F}}$, $\mathcal{O} = \mathcal{O}_{gxg^{-1}}^{G^{\mathcal{F}}}$ and $z = g^{-1}\mathcal{F}(g) \in C_G(M)$. Then Remark 2.5(f) applies. The proof of (2) is similar, using Remark 2.5(g). $\square$

3. Preliminaries on Finite Simple Groups of Lie Type

3.1. *Algebraic groups*

We mainly follow [9] as a source on algebraic groups and finite groups of Lie type, with exceptions signaled along the text.

Let $\Bbbk = \overline{\mathbb{F}_q}$ be the algebraic closure of $\mathbb{F}_q$. All algebraic groups are affine and defined over $\Bbbk$. If $\mathbb{H}$ is an algebraic group, then $\mathbb{H}^\circ$ indicates the connected component of $\mathbb{H}$ containing the identity. Also, $X(\mathbb{H}) = \operatorname{Mor}(\mathbb{H}, \Bbbk^\times)$ is the group of characters of $\mathbb{H}$, and $X_*(\mathbb{H}) = \operatorname{Mor}(\Bbbk^\times, \mathbb{H})$ is the set of multiplicative one-parameter subgroups in $\mathbb{H}$.

Let $\mathbb{G}$ be a simple algebraic group, $\mathbb{G}_{\mathrm{ad}}$ its adjoint quotient, $\mathbb{G}_{\mathrm{sc}}$ its simply connected cover, with projection $\boldsymbol{\pi} : \mathbb{G}_{\mathrm{sc}} \to \mathbb{G}$. We fix a maximal torus $\mathbb{T}$ of $\mathbb{G}$ and a Borel subgroup $\mathbb{B}$ containing it. The unipotent radical of $\mathbb{B}$ is denoted by $\mathbb{U}$. We add a subscript ad or sc for the maximal torus and Borel of $\mathbb{G}_{\mathrm{ad}}$ or $\mathbb{G}_{\mathrm{sc}}$; our choices are compatible with projections, e.g., $\boldsymbol{\pi}(\mathbb{T}_{\mathrm{sc}}) = \mathbb{T}$.

The root system of $\mathbb{G}$ is denoted by Φ, identified as a subset of $X(\mathbb{T})$; the set of positive roots relative to $\mathbb{T}$ and $\mathbb{B}$ is denoted by Φ^+ and the simple roots by $\alpha_1, \dots, \alpha_n$, numbered as in [4]. The Weyl group $N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$ is denoted by W; $(-,-)$ is the W-invariant bilinear form on the $\mathbb{R}$-span of Φ. Let $\langle\, , \rangle : X(\mathbb{T}) \times X_*(\mathbb{T}) \to \mathbb{Z}$ be given by $\langle \chi, \lambda \rangle = m$ if $(\chi \circ \lambda)(x) = x^m$. The coroot system of $\mathbb{G}$ is denoted by $\Phi^\vee = \{\beta^\vee : \beta \in \Phi\} \subset X_*(\mathbb{T})$, where $\langle \alpha, \beta^\vee \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)}$, for all $\alpha \in \Phi$. Hence

$$\alpha(\beta^\vee(\zeta)) = \zeta^{\frac{2(\alpha,\beta)}{(\beta,\beta)}}, \quad \alpha, \beta \in \Phi, \;\; \zeta \in \Bbbk^\times.$$

For $\alpha \in \Phi$, there is a monomorphism of abelian groups $x_\alpha : \Bbbk \to \mathbb{U}$; we set $\mathbb{U}_\alpha$ for the image of x_α, called a root subgroup. We adopt the normalization of x_α and the notation for the elements in $\mathbb{T}$ from [10, 8.1.4]. We recall the commutation rule: $t x_\alpha(a) t^{-1} = x_\alpha(\alpha(t)a)$, for $t \in \mathbb{T}$ and $\alpha \in \Phi$. The group $\mathbb{U}$ is generated by the root subgroups $\mathbb{U}_\alpha$, for $\alpha \in \Phi^+$. More precisely, let us fix an arbitrary ordering on Φ^+; then every $u \in \mathbb{U}$ has a unique expression as a product (with respect to the fixed ordering)

$$u = \prod_{\alpha \in \Phi^+} x_\alpha(c_\alpha), \quad c_\alpha \in \Bbbk, \;\; \alpha \in \Phi^+. \tag{3.1}$$

Let $\operatorname{supp}(u) = \{\alpha \in \Phi^+ \mid c_\alpha \neq 0\}$, that of course depends on the ordering. In the sequel we will use frequently the Chevalley's commutator formula (3.2) below, see [12, Lemma 15, p. 22 and Corollary, p. 24]. Let $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta \in \Phi^+$. Fix a total order in the set Γ of pairs (i, j) of positive integers such that $i\alpha + j\beta \in \Phi$. Then there exist integers $c_{ij}^{\alpha\beta}$ such that

$$x_\alpha(\xi) x_\beta(\eta) x_\alpha(\xi)^{-1} x_\beta(\eta)^{-1} = \prod_{(i,j) \in \Gamma} x_{i\alpha + j\beta}(c_{ij}^{\alpha\beta} \xi^i \eta^j), \quad \forall\, \xi, \eta \in \Bbbk. \tag{3.2}$$

(Clearly, (3.2) also holds when $\alpha + \beta$ is not a root, as $\mathbb{U}_\alpha$ and $\mathbb{U}_\beta$ commute in this case.) Let m, respectively M, be the maximum integer for which $\beta - m\alpha \in \Phi$, respectively $\beta + M\alpha \in \Phi$. Then the α-string through β is the set of roots of the form $\beta - m\alpha, \dots, \beta + M\alpha$, and $m - M = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. Up to a non-zero scalar, $c_{11}^{\alpha\beta} = m + 1$ [13, Lemma 15, Theorem 1]. If the Dynkin diagram of $\mathbb{G}$ is simply laced, then $m + 1 = 1$; otherwise, $|m + 1| \in \{1, 2, 3\}$. Then $c_{11}^{\alpha\beta} \neq 0$ except in the cases listed in Table 2.

Table 2.

p	Type of Φ	α	β
3	G_2	α_1	$2\alpha_1 + \alpha_2$
		$2\alpha_1 + \alpha_2$	α_1
		$\alpha_1 + \alpha_2$	$2\alpha_1 + \alpha_2$
		$2\alpha_1 + \alpha_2$	$\alpha_1 + \alpha_2$
2	B_n, C_n, F_4	orthogonal	to each other
	G_2	α_1	$\alpha_1 + \alpha_2$
		$\alpha_1 + \alpha_2$	α_1

Let $\Sigma_\alpha = \{\beta \in \Phi^+ : \alpha + \beta \in \Phi$ but (α, β) does not appear in Table 2$\}$, for $\alpha \in \Phi^+$. If $\beta \in \Sigma_\alpha$, then $x_\alpha(\xi)$ and $x_\beta(\eta)$ do not commute for $\xi, \eta \in \Bbbk^\times$.

3.2. *Conjugacy classes in finite simple groups of Lie type*

3.2.1. *Finite simple groups of Lie type*

Let $\mathbb{H}$ be a semisimple algebraic group defined over $\mathbb{F}_q$. A Steinberg endomorphism $F : \mathbb{H} \to \mathbb{H}$ is an abstract group automorphism having a power equal to a Frobenius map [9, Definition 21.3]. We may assume that F is the product of a Frobenius endomorphism with an automorphism of $\mathbb{H}$ induced by a non-trivial Dynkin diagram automorphism. The subgroup $\mathbb{H}^F$ is called a *finite group of Lie type* [9, Definition 21.6].

Let $\mathbb{G}$ be a simple algebraic group and let F be a Steinberg endomorphism of $\mathbb{G}_{\mathrm{sc}}$. Assume that it descends to a Steinberg endomorphism of $\mathbb{G}$ (again called F), that happens when $\ker \boldsymbol{\pi}$ is F-stable, see [9, Example 22.8] for precise conditions. In particular, F descends to $\mathbb{G}_{\mathrm{ad}} \simeq \mathbb{G}/Z(\mathbb{G})$ always, and to $\mathbb{G}$ when it is $\mathbb{F}_q$-split. It is well known that $\mathbb{G}_{\mathrm{ad}}$ is a simple abstract group [9, Proposition 12.5] but $\mathbb{G}_{\mathrm{ad}}^F$ is not simple in general. However $\boldsymbol{G} := \mathbb{G}_{\mathrm{sc}}^F / Z(\mathbb{G}_{\mathrm{sc}}^F)$ is a finite simple group except for the following eight examples [9, Theorem 24.17]:

- $\mathbf{PSL}_2(2) \simeq \mathbb{S}_3$; $\mathbf{PSL}_2(3) \simeq \mathbb{A}_4$; $\mathbf{PSp}_4(2) \simeq \mathbb{S}_6$;
- $\mathbf{PSU}_3(2)$, ${}^2B_2(2^2)$ (both solvable);
- $G_2(2) \simeq \operatorname{Aut} \mathbf{PSU}_3(3)$, ${}^2G_2(3) \simeq \operatorname{Aut} \mathbf{PSU}_2(8)$ (almost simple);
- ${}^2F_4(2)$, that contains a normal subgroup isomorphic to the Tits group, with index 2.

Unless otherwise stated, we assume that $\boldsymbol{G} = \mathbb{G}_{\mathrm{sc}}^F / Z(\mathbb{G}_{\mathrm{sc}}^F)$ is not one of these eight groups and call it a *finite simple group of Lie type*. Notice that $\boldsymbol{\pi}(\mathbb{G}_{\mathrm{sc}}^F) = [\mathbb{G}^F, \mathbb{G}^F]$ and there is the alternative useful description $\boldsymbol{G} \simeq [\mathbb{G}^F, \mathbb{G}^F]/\boldsymbol{\pi}(Z(\mathbb{G}_{\mathrm{sc}}^F))$. In particular, $\boldsymbol{G} \simeq [\mathbb{G}_{\mathrm{ad}}^F, \mathbb{G}_{\mathrm{ad}}^F]$ [9, Proposition 24.21].

3.2.2. *Conjugacy classes*

There is a huge literature on the description of the conjugacy classes in $\boldsymbol{G}$, see for instance the bibliography in [7–9]. We shall give precise references as they are needed. To start with, we recall the following arguments:

- ◇ Every F-stable $\mathbb{G}$-conjugacy class $\mathcal{O}$ meets $\mathbb{G}^F$ [9, Theorem 21.11 (a)], a consequence of the Lang–Steinberg theorem [9, Theorem 21.7].
- ◇ Let $\mathcal{O}$ be an F-stable $\mathbb{G}$-conjugacy class, $x \in \mathcal{O} \cap \mathbb{G}^F$ and

$$A(x) := C_{\mathbb{G}}(x)/C_{\mathbb{G}}(x)^{\circ}. \tag{3.3}$$

Then $\mathcal{C}(\mathbb{G}^F, x)$ is in bijection with $H^1(F, A(x))$ [7, 8.5; 9, Theorem 21.11(b)].

From the preceding two facts, we see that to determine the conjugacy classes in $\boldsymbol{G}$, one possible way is to consider the following questions:

(a) Describe the F-stable $\mathbb{G}$-conjugacy classes.
(b) For a given F-stable $\mathbb{G}$-conjugacy class $\mathcal{O}$, describe the $\mathbb{G}^F$-conjugacy classes in $\mathcal{O} \cap \mathbb{G}^F$.
(c) Pass this information to $\boldsymbol{G}$.

These questions were treated in extent in the literature. We will recall the known answers for different kinds of conjugacy classes along the way. Now we state some other useful facts.

- ◇ The Borel subgroup $\mathbb{B}$ and the maximal torus $\mathbb{T}$ are chosen F-stable, which is possible by [9, Corollary 21.12]. Hence so is $\mathbb{U} = [\mathbb{B}, \mathbb{B}]$.
- ◇ The subgroup W^F of F-fixed points in the Weyl group W is isomorphic to $N_{\mathbb{G}^F}(\mathbb{T})/\mathbb{T}^F$ by [9, Proposition 23.2]; clearly, $N_{\mathbb{G}^F}(\mathbb{T}) = N_{\mathbb{G}}(\mathbb{T}) \cap \mathbb{G}^F$. Even more, every element in W^F has a representative in $N_{[\mathbb{G}^F, \mathbb{G}^F]}(\mathbb{T})$ [9, Corollary 24.2].

3.3. *Unipotent classes in finite simple groups of Lie type*

We need to describe the unipotent conjugacy classes in finite simple groups of Lie type. We keep the notations and assumptions from Sec. 3.2.1 for $\mathbb{G}$, F and $\boldsymbol{G}$; let $\pi : \mathbb{G}_{\mathrm{sc}}^F \to \boldsymbol{G} = \mathbb{G}_{\mathrm{sc}}^F / Z(\mathbb{G}_{\mathrm{sc}}^F)$ be the natural projection. Every $x \in \mathbb{G}_{\mathrm{sc}}$ has a Chevalley–Jordan decomposition $x = x_s x_u = x_u x_s$, with x_s semisimple and x_u unipotent. This decomposition boils down to the group $\mathbb{G}$ and to the finite groups $\mathbb{G}^F$, $[\mathbb{G}^F, \mathbb{G}^F]$ and $\boldsymbol{G}$, where it agrees with the decomposition in the p-part, namely x_u, and the p-regular part, namely x_s.

3.3.1. *Unitary groups*

In some inductive arguments we use the unitary groups $\mathbf{PSU}_n(q)$. When dealing with them we will use the following matrix description. Let $\mathtt{J}_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} = \mathtt{J}_n^{-1} \in \mathbf{GL}_n(\Bbbk)$. Let Fr_q, respectively F, be the Frobenius endomorphism of $\mathbf{GL}_n(\Bbbk)$ raising all entries of the matrix to the qth power, respectively given by $F(X) = \mathtt{J}_n{}^t(\mathrm{Fr}_q(X))^{-1}\mathtt{J}_n$, $X \in \mathbf{GL}_n(\Bbbk)$. Following [9, Examples 21.14(2), 23.10(2)], the unitary and special unitary groups are $\mathbf{GU}_n(q) = \mathbf{GL}_n(\Bbbk)^F$, $\mathbf{SU}_n(q) = \mathbf{SL}_n(\Bbbk)^F$.

Also, $\mathbf{SU}_n(q)$ can be realized as a subgroup of $\mathbf{SL}_n(q^2)$ [15]. If $h \in \mathbb{N}$, then

$$F^{2h}(X) = \mathrm{Fr}_{q^{2h}}(X), \quad F^{2h+1}(X) = \mathtt{J}_n{}^t(\mathrm{Fr}_{q^{2h+1}}(X))^{-1}\mathtt{J}_n.$$

Hence $\mathbf{GU}_n(q)$, respectively $\mathbf{SU}_n(q)$, $\mathbf{PSU}_n(q)$, can be identified with a subgroup of $\mathbf{GU}_n(q^{2h+1})$, respectively $\mathbf{SU}_n(q^{2h+1})$, $\mathbf{PSU}_n(q^{2h+1})$.

The unipotent conjugacy classes in $\mathbf{SU}_n(q)$ are described as the unipotent conjugacy classes in $\mathbf{SL}_n(q)$. Indeed,

- ◇ Every unipotent class in $\mathbf{SU}_n(q)$ has a type: $u \in \mathbf{SU}_n(q)$ is of type $\lambda = (\lambda_1, \dots, \lambda_k)$ if the elementary factors of its characteristic polynomial equal $(X-1)^{\lambda_1}, (X-1)^{\lambda_2}, \dots, (X-1)^{\lambda_k}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. Conversely, since all unipotent classes in $\mathbb{G} = \mathbf{SL}_n(\Bbbk)$ are F-stable (by a direct computation), for any type there is a unipotent class in $\mathbf{SU}_n(q)$, by the Lang–Steinberg theorem.
- ◇ By [7, 8.5] every unipotent class in $\mathbf{GL}_n(\Bbbk)$ meets $\mathbf{GU}_n(q)$ in exactly one class, since $C_{\mathbf{GL}_n(\Bbbk)}(x)$ is connected for every x [11, I.3.5].
- ◇ Since $\mathbf{SU}_n(q)$ is normal in $\mathbf{GU}_n(q)$, Remark 2.4 says that all unipotent classes in $\mathbf{SU}_n(q)$ with the same type are isomorphic as racks.

3.3.2. *The isogeny argument*

Section 4 is devoted to unipotent classes in Chevalley groups. By the isogeny argument, Lemma 3.1 below, it is enough to treat the unipotent classes in $\mathbb{G}_{\mathrm{sc}}^F$ or $[\mathbb{G}^F, \mathbb{G}^F]$. This takes care of Question (c) in Sec. 3.2.2 and gives flexibility to choose $\mathbb{G}$ in a suitable form, e.g., in matrix form. Let $\mathcal{G}$ be a semisimple algebraic, respectively finite, group and $\mathcal{G}_u$ the set of unipotent, respectively p-elements, in $\mathcal{G}$.

Lemma 3.1 ([1, Lemma 1.2]). *Let $\mathcal{Z}$ be a central subgroup of $\mathcal{G}$ whose elements are all semisimple, respectively p-regular. Then the quotient map $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{Z}$ induces a rack isomorphism $\pi : \mathcal{G}_u \to (\mathcal{G}/\mathcal{Z})_u$ and a bijection between the sets of $\mathcal{G}$-conjugacy classes in $\mathcal{G}_u$ and in $(\mathcal{G}/\mathcal{Z})_u$.*

3.3.3. *A reduction argument*

The determination of the unipotent conjugacy classes in $\mathbb{G}$ and those that are F-stable, Question (a) in Sec. 3.2.2, is well known, see [7, Chap. 7; 8, Chaps. 7, 17, 22]. But the description of the $\mathbb{G}^F$-conjugacy classes in $\mathcal{O} \cap \mathbb{G}^F$ for an F-stable $\mathbb{G}$-conjugacy class $\mathcal{O}$, Question (b) in Sec. 3.2.2, is more delicate; for example there is a class in $\mathbf{Sp}_4(\Bbbk)$ which splits into two classes in $\mathbf{Sp}_4(q)$ of different size [8, Table 8.1]. Similar examples occur for other groups. This was not the case when $\mathbb{G} = \mathbf{SL}_n(\Bbbk)$ and F is $\mathbb{F}_q$-split by Remark 2.4. To start with, observe that $\mathbb{U}^F$, which is a p-Sylow subgroup of $\mathbb{G}^F$ and $[\mathbb{G}^F, \mathbb{G}^F]$ [9, Corollary 24.11], is isomorphic to its image in $\boldsymbol{G}$ by Lemma 3.1. Hence, every unipotent element in $\mathbb{G}^F$, or in $\boldsymbol{G}$, is conjugated to an element in $\mathbb{U}^F$. Also, the $[\mathbb{G}^F, \mathbb{G}^F]$-classes into which a $\mathbb{G}^F$-class in $[\mathbb{G}^F, \mathbb{G}^F]$ splits are all isomorphic as racks, see Remark 2.4.

The following result combines Remark 2.5, Lemma 2.6 and [7, 8.5].

Lemma 3.2. *Let $\mathbb{M}$ be an F-stable subgroup of $\mathbb{G}$ and $x \in \mathbb{M}^F$.*

Let $g \in \mathbb{G}$ such that $z = g^{-1}F(g) \in C_{\mathbb{G}}(x)$. Assume that the class of z in $H^1(F, A(x))$ has a representative in $C_{\mathbb{G}}(\mathbb{M})$.

(a) *If $\mathcal{O}_x^{\mathbb{M}^F}$ is of type D, respectively F, then $\mathcal{O}_{gxg^{-1}}^{\mathbb{G}^F}$ is so.*
(b) *If $x \in [\mathbb{M}^F, \mathbb{M}^F]$ and $\mathcal{O}_x^{[\mathbb{M}^F,\mathbb{M}^F]}$ is of type D, respectively F, then $\mathcal{O}_{gxg^{-1}}^{[\mathbb{G}^F,\mathbb{G}^F]}$ is so.*

Assume that every element in $H^1(F, A(x))$ has a representative in $C_{\mathbb{G}}(\mathbb{M})$. This happens for instance if $C_{\mathbb{G}}(x) = C_{\mathbb{G}}(\mathbb{M})\mathbb{H}$ with $\mathbb{H}$ connected.

(c) *If $\mathcal{O}_x^{\mathbb{M}^F}$ is of type D, respectively F, then $\mathcal{O}$ is so for every $\mathcal{O} \in \mathcal{C}(\mathbb{G}^F, x)$.*
(d) *If $x \in [\mathbb{M}^F, \mathbb{M}^F]$ and $\mathcal{O}_x^{[\mathbb{M}^F,\mathbb{M}^F]}$ is of type D, respectively F, then $\mathcal{O}$ is so for every $\mathcal{O} \in \mathcal{C}([\mathbb{G}^F, \mathbb{G}^F], x)$.*

3.4. *Criteria to collapse for unipotent classes*

Let $\boldsymbol{G}$ be a finite simple group of Lie type and $\mathcal{O}$ a unipotent conjugacy class in $\boldsymbol{G}$. We realize $\mathcal{O}$ as a unipotent conjugacy class in $[\mathbb{G}^F, \mathbb{G}^F]$, where as above, $\mathbb{G}$ is a simple algebraic group and F is a Steinberg endomorphism of $\mathbb{G}$.

Definition 3.3. Let $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta \in \Phi$ but the pair α, β does not appear in Table 2. We fix an ordering of Φ^+. We say that $\mathcal{O}$ has the *$\alpha\beta$-property* if there exists $u \in \mathcal{O} \cap \mathbb{U}^F$ such that $\alpha, \beta \in \operatorname{supp} u$ and

$$\alpha + \beta = \sum_{1 \le i \le r} \gamma_i, \quad \text{with } r > 1, \ \gamma_i \in \operatorname{supp} u$$
$$\Rightarrow r = 2, \quad \{\gamma_1, \gamma_2\} = \{\alpha, \beta\}. \tag{3.4}$$

Remark 3.4. Let $u \in \mathcal{O} \cap \mathbb{U}^F$.

(i) If there exist *simple* roots α and $\beta \in \operatorname{supp} u$ adjacent in the Dynkin diagram of Φ (so that $\alpha + \beta$ is a root), then $\mathcal{O}$ has the $\alpha\beta$-property.
(ii) Let $\alpha, \beta \in \Phi^+$ such that $\mathcal{O}$ has the $\alpha\beta$-property. By (3.4), neither α nor β can be decomposed as a sum of roots in $\operatorname{supp}(u)$. Using the Chevalley commutator formula (3.2), we infer that α and β lie in the support of u for every ordering on Φ^+, and the $\alpha\beta$-property is independent of the ordering.

3.4.1. *Unipotent classes of type D in Chevalley and Steinberg groups*

We give a criterion to determine if unipotent classes in Chevalley and Steinberg groups are of type D. *In this subsection, we assume that q is odd.* Recall that the only groups corresponding to very twisted Steinberg endomorphisms in odd characteristic are the Ree groups ${}^2G_2(3^{2h+1})$. See [5, Sec. 12.4].

Proposition 3.5. *Let $\boldsymbol{G}$ be a finite simple group of Lie type. Assume $\mathcal{O}$ has the $\alpha\beta$-property, for some $\alpha, \beta \in \Phi^+$ and that $q > 3$ when $(\alpha, \beta) = 0$. Then $\mathcal{O}$ is of type D.*

Proof. *Step* 1. If there exists $t \in \mathbb{T} \cap [\mathbb{G}^F, \mathbb{G}^F]$ such that $1 \neq \alpha(t) \neq \beta(t)$, then $\mathcal{O}$ is of type D.

Fix an ordering of the positive roots ending with $\alpha+\beta < \beta < \alpha$. Since $\mathcal{O}$ has the $\alpha\beta$-property, there exists $u \in \mathcal{O}$ with

$$u = \prod_{\gamma \in \mathrm{supp}(u)} x_\gamma(a_\gamma) \in \left(\prod_{\substack{\gamma \in \mathrm{supp}(u) \\ \gamma \neq \alpha, \beta, \alpha+\beta}} \mathbb{U}_\gamma \right) x_{\alpha+\beta}(a_{\alpha+\beta}) x_\beta(a_\beta) x_\alpha(a_\alpha),$$

and $a_\alpha a_\beta \neq 0$. Let $r = u$, $s = trt^{-1} \in \mathcal{O}$. Then

$$\langle r, s\rangle \subseteq H := \langle \mathbb{U}_\gamma \,|\, \gamma \in \mathrm{supp}(u)\rangle.$$

Also $s \in (\prod_{\gamma\in\mathrm{supp}(u), \gamma\neq\alpha} \mathbb{U}_\gamma) x_\alpha(\alpha(t)a_\alpha)$; we see using (3.2) and (3.4) that

$$\mathcal{O}_r^H \subseteq \left(\prod_{\substack{\delta = \gamma_1 + \cdots + \gamma_l \\ \delta \neq \alpha, \gamma_i \in \mathrm{supp}(u)}} \mathbb{U}_\delta \right) x_\alpha(a_\alpha), \quad \mathcal{O}_s^H \subseteq \left(\prod_{\substack{\delta = \gamma_1 + \cdots + \gamma_l \\ \delta \neq \alpha, \gamma_i \in \mathrm{supp}(u)}} \mathbb{U}_\delta \right) x_\alpha(\alpha(t)a_\alpha).$$

Since $\alpha(t) \neq 1$, $\mathcal{O}_r^H \neq \mathcal{O}_s^H$, hence $\mathcal{O}_r^{\langle r,\, s\rangle} \neq \mathcal{O}_s^{\langle r,\, s\rangle}$. Since rs, $sr \in \mathbb{U}^F$ and $p \neq 2$, $(rs)^2 \neq (sr)^2$ if and only if $rs \neq sr$. To prove the last inequality, and conclude that $\mathcal{O}$ is of type D, let $V := \langle \mathbb{U}_\gamma \,|\, \gamma \in \mathrm{supp}(u), \gamma \neq \alpha, \beta, \alpha+\beta\rangle$. Observe that if a right coclass Vw of some $w \in \mathbb{U}$ contains an element of the form $x_{\alpha+\beta}(z)x_\beta(y)x_\alpha(x)$, then x, y, z are unique by (3.4) and Remark 3.4(ii), using (3.2). Again by (3.4) and Remark 3.4(ii), using (3.2), we see that the coclass Vrs contains $x_{\alpha+\beta}(z)x_\beta((1+\beta(t))a_\beta)x_\alpha((1+\alpha(t))a_\alpha)$, with

$$z = \beta(t)c_{11}^{\alpha,\beta} a_\alpha a_\beta + (1 + (\alpha+\beta)(t))a_{\alpha+\beta},$$

while Vsr contains $x_{\alpha+\beta}(z')x_\beta((1+\beta(t))a_\beta)x_\alpha((1+\alpha(t))a_\alpha)$, with

$$z' = \alpha(t)c_{11}^{\alpha,\beta} a_\alpha a_\beta + (1 + (\alpha+\beta)(t))a_{\alpha+\beta}.$$

Since $\alpha(t) \neq \beta(t)$ and $c_{11}^{\alpha,\beta} a_\alpha a_\beta \neq 0$ by assumption, we get that $rs \neq sr$.

Step 2. If $\boldsymbol{G}$ is a Chevalley group, then there exists $t \in \mathbb{T} \cap [\mathbb{G}^F, \mathbb{G}^F]$ such that $1 \neq \alpha(t) \neq \beta(t)$.

Without loss of generality, if α and β have different lengths, we choose β to be the longest one. Take $t = \beta^\vee(\zeta) \in \mathbb{T}$, where ζ is a generator of $\mathbb{F}_q^\times$. Then $t \in [\mathbb{G}^F, \mathbb{G}^F]$ by [10, 8.1.4], and $\alpha(t) = \zeta^{\frac{2(\alpha,\beta)}{(\beta,\beta)}}$, $\beta(t) = \zeta^2$. If Φ is simply laced then $r = \frac{2(\alpha,\beta)}{(\beta,\beta)} = -1$ and $1 \neq \alpha(t) \neq \beta(t)$. If Φ is of type G_2, then $r \in \{-1, 1\}$ and the same assertion follows. If Φ is doubly laced, then $r \in \{-1, 0\}$. But if $r = 0$, then $\beta(t) = \zeta^2 \neq 1$ since by assumption $q > 3$. Thus $1 \neq \beta(t) \neq \alpha(t)$ and the claim follows by interchanging α and β.

Hence the proposition for Chevalley groups follows from Steps 1 and 2.

Step 3. If $\boldsymbol{G}$ is a Steinberg group, then there exists $t \in \mathbb{T} \cap [\mathbb{G}^F, \mathbb{G}^F]$ such that $1 \neq \alpha(t) \neq \beta(t)$.

Here Φ is simply laced so $\frac{2(\alpha,\beta)}{(\beta,\beta)} = -1$. Assume first that the Dynkin diagram automorphism θ associated with F is an involution. Then the $\langle\theta\rangle$-orbit of β is either $\{\beta\}$ or $\{\beta, \theta(\beta)\}$. In the former case, take $t = \beta^\vee(\zeta) \in \mathbb{T}$ for a generator ζ of $\mathbb{F}_q^\times$ and conclude as in Step 2. In the latter, take $t = \beta^\vee(\xi)(\theta\beta)^\vee(\xi^q) \in \mathbb{T}$ for a generator ξ of $\mathbb{F}_{q^2}^\times$. Then $t \in \boldsymbol{\pi}(\mathbb{G}_{sc}^F) = [\mathbb{G}^F, \mathbb{G}^F]$; $\alpha(t) \in \{\xi^{-1}, \xi^{-1\pm q}, \xi^{-1+2q}\}$ and $\beta(t) \in \{\xi^2, \xi^{2-q}\}$. Hence $\alpha(t) \neq 1, \beta(t)$ unless $q = 3$ and either

$$(\alpha, \beta) = -1, \quad (\alpha, \theta\beta) = 0 \quad \text{and} \quad (\beta, \theta\beta) = -1, \quad \text{or} \tag{3.5}$$

$$(\alpha, \beta) = -1, \quad (\alpha, \theta\beta) = 1 \quad \text{and} \quad (\beta, \theta\beta) = 0. \tag{3.6}$$

Let $q = 3$. If Φ is of type D_n or E_6, then case (3.5) never occurs because $(\beta, \theta\beta) = 0$ whenever $\beta \neq \theta\beta$. If Φ is of type A_n, then case (3.5) occurs only if $\beta = \varepsilon_i - \varepsilon_j$, for $i < j$ and either: $\alpha = \varepsilon_l - \varepsilon_i$ for $l < i$, $2j = n + 2$, and $l \neq j, n + 2 - i$, or $\alpha = \varepsilon_j - \varepsilon_l$ for $j < l$, $2i = n + 2$; and $l \neq i, n + 2 - j$. In both situations we take $t = \alpha^\vee(\xi)(\theta\alpha)^\vee(\xi^3)$ for ξ a generator of $\mathbb{F}_9^\times$. This gives the claim in case (3.5).

By applying θ we observe, using Remark 3.4(ii) that if α and β satisfy condition (3.4), then $\theta\alpha$ and $\theta\beta$ also lie in $\mathrm{supp}(u)$. Therefore, (3.4) forces $\theta(\alpha + \beta) \neq \alpha + \beta$.

If Φ is of type A_n, then a pair of roots satisfies (3.6) only if $\beta = \varepsilon_i - \varepsilon_j$ with $\{i, j\} \cap \{n - i + 2, n - j + 2\} = \emptyset$ and either $\alpha = \varepsilon_j - \varepsilon_{n-i+2}$ or $\alpha = \varepsilon_{n-j+2} - \varepsilon_i$ with $|\{i, j, n - j + 2, n - i + 2\}| = 4$. Since in this case $\alpha + \beta$ would be θ-invariant, such pairs are discarded.

If Φ is of type D_n, case (3.6) occurs only if $\beta = \varepsilon_i \pm \varepsilon_n$, $\alpha = \varepsilon_j \mp \varepsilon_n$ with $n \neq i \neq j \neq n$. We discard such pairs as we did for type A_n.

Let Φ be of type E_6. If a pair (α, β) satisfies (3.6) and (β, α) does not, we interchange α and β. We verify by inspection that there are no pairs of roots α and β such that (α, β) and (β, α) are both in case (3.6) and such that $\theta(\alpha+\beta) \neq \alpha+\beta$. This gives the claim when $\theta^2 = 1$.

Assume now Φ is of type D_4 and θ has order 3. We will show that $\alpha_2 \in \{\alpha, \beta\}$. Let us fix an ordering of the roots in increasing height ht and let $u \in \mathcal{O} \cap \mathbb{U}^F$ be as in Definition 3.3. We consider the support of u with respect to this ordering. The outer automorphism θ of order 3 permutes α_1, α_3 and α_4 and fixes α_2. By inspection, for simple roots we have $\alpha \in \mathrm{supp}(u)$ if and only if $\theta(\alpha) \in \mathrm{supp}(u)$. In addition, $\gamma + \gamma' \notin \Phi$ if $\mathrm{ht}(\gamma) = \mathrm{ht}(\gamma') \geq 2$ or if $\mathrm{ht}(\gamma) = \mathrm{ht}(\gamma') = 1$ and $\gamma, \gamma' \neq \alpha_2$. So, if $\{\alpha_1, \alpha_2\} \not\subset \mathrm{supp}(u)$ we have $\alpha \in \mathrm{supp}(u)$ if and only if $\theta(\alpha) \in \mathrm{supp}(u)$ for every $\alpha \in \Phi^+$. Thus, if $\alpha_2 \notin \mathrm{supp}(u)$, condition (3.4) is not verified for any pair $\alpha, \beta \in \mathrm{supp}(u)$ such that $\alpha + \beta \in \Phi$. So, $\alpha_2 \in \mathrm{supp}(u)$. If $\alpha_1 \in \mathrm{supp}(u)$ then we take $\alpha = \alpha_2$, $\beta = \alpha_1$. If, instead, $\alpha_1 \notin \mathrm{supp}(u)$, then u has the $\alpha\beta$-property if and only if $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \in \mathrm{supp}(u)$ and $\{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3 + \alpha_4\} \not\subset \mathrm{supp}(u)$. In this case, we have $\alpha = \alpha_2$, $\beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. In both cases, we take $t = \alpha_1^\vee(\xi)(\theta\alpha_1)^\vee(\xi^q)(\theta^2\alpha_1)^\vee(\xi^{q^2})$ for ξ a generator of $\mathbb{F}_{q^3}^\times$. Then, $t \in \boldsymbol{\pi}(\mathbb{G}_{sc}^F) = [\mathbb{G}^F, \mathbb{G}^F]$, and $\alpha(t) = \xi^{-(1+q+q^2)}$ and $\beta(t) \in \{\xi^2, \xi^{(1+q+q^2)}\}$. Then, $1 \neq \alpha(t) \neq \beta(t)$ unless $q = 3$ and $\beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. In this case, we replace t by $t\alpha_2^\vee(-1)$.

Hence the proposition for Steinberg groups follows from Steps 1 and 3.

Step 4. If $\boldsymbol{G} = {}^2G_2(3^{2h+1})$, $h \geq 1$, then there exists $t \in \mathbb{T} \cap [\mathbb{G}^F, \mathbb{G}^F]$ such that $1 \neq \alpha(t) \neq \beta(t)$.

In this case the possible (unordered) pairs $\{\alpha, \beta\}$ are

$$\{\alpha_1, \alpha_2\}, \quad \{\alpha_1, \alpha_1 + \alpha_2\}, \quad \{\alpha_2, 3\alpha_1 + \alpha_2\}.$$

The last two pairs are interchanged by the non-standard graph automorphism θ such that $x_{\alpha_1}(\zeta) \mapsto x_{\alpha_2}(\zeta^3)$ and $x_{\alpha_2}(\zeta) \mapsto x_{\alpha_1}(\zeta)$ for every $\zeta \in \Bbbk$, [5, 12.4]. Using invariance of $u \in \mathcal{O}$ as in Definition 3.3 with respect to $\mathrm{Fr}_{3^h} \circ \theta$ we see that $\mathcal{O}$ has the $\alpha\beta$-property for $\{\alpha_1, \alpha_1 + \alpha_2\}$ if and only if it has it for $\{\alpha_2, 3\alpha_1 + \alpha_2\}$. So, it is enough to consider $\{\alpha_1, \alpha_2\}$ and $\{\alpha_1, \alpha_1 + \alpha_2\}$. Let ζ be a generator of $\mathbb{F}_{3^{2h+1}}^\times$ and let $t = \alpha_1^\vee(\zeta^{3^h})\alpha_2^\vee(\zeta)$. Then $t \in \mathbb{T}^F \cap [\mathbb{G}^F, \mathbb{G}^F]$ and

$$1 \neq \alpha_1(t) = \zeta^{2\cdot 3^h - 1} \neq \alpha_2(t) = \zeta^{-3^{h+1}+2},$$
$$\alpha_1(t) = \zeta^{2\cdot 3^h - 1} \neq (\alpha_1 + \alpha_2)(t) = \zeta^{3^h - 1}.$$

Hence the proposition for Ree groups follows from Steps 1 and 4. □

3.4.2. *Unipotent classes of type F in Chevalley and Steinberg groups*

In this subsection we address the case when q is even albeit some results are valid more generally. We give criteria to determine when a unipotent class is of type F in Chevalley or Steinberg groups. The groups related to the very twisted Steinberg endomorphisms [9] can be dealt with in a similar way.

Proposition 3.6. *Assume that one of the following conditions holds*:

- $\boldsymbol{G}$ *is a Chevalley group and* $q \notin \{2, 3, 4, 5, 7\}$;
- $\boldsymbol{G} = \mathbf{PSU}_3(q)$ *and* $q \notin \{2, 5, 8\}$;
- $\boldsymbol{G}$ *is a Steinberg group and* $q > 8$.

If $\mathcal{O}$ *is a unipotent class in* $\boldsymbol{G}$ *and has the* $\alpha\beta$*-property, for some* $\alpha, \beta \in \Phi^+$, *then it is of type F.*

Proof. *Step* 1. If there exists a family $(t_a)_{a \in \mathbb{I}_4}$ in $\mathbb{T} \cap [\mathbb{G}^F, \mathbb{G}^F]$ such that

$$\alpha(t_a)\beta(t_b) \neq \alpha(t_b)\beta(t_a) \quad \text{for every } a \neq b, \tag{3.7}$$

then $\mathcal{O}$ is of type F.

Notice that (3.7) implies

$$(\alpha(t_a), \beta(t_a)) \neq (\alpha(t_b), \beta(t_b)) \quad \text{for every } a \neq b. \tag{3.8}$$

Let $r_a := t_a u t_a^{-1}$ and $R_a := \mathbb{U}^F \triangleright r_a$, $a \in \mathbb{I}_4$. We claim that (3.7) ensures $r_a \triangleright r_b \neq r_b$ for every $a \neq b$, and that (3.8) ensures that $R = \coprod_{a \in \mathbb{I}_4} R_a$ is a subrack with $R_a \triangleright R_b = R_b$.

As in Step 1 of Proposition 3.5, we fix an ordering of Φ^+ ending with $\alpha + \beta < \beta < \alpha$. Let $\mathbb{V} = \langle \mathbb{U}_\gamma \,|\, \gamma \in \mathrm{supp}(r), \gamma \neq \alpha, \beta, \alpha + \beta \rangle$. Since $\mathcal{O}$ has the $\alpha\beta$-property, there exists $r \in \mathcal{O}$ with $r \in \mathbb{V} x_{\alpha+\beta}(a_{\alpha+\beta}) x_\beta(a_\beta) x_\alpha(a_\alpha)$ and $a_\alpha a_\beta \neq 0$. Then

$r_a \in \mathbb{V} x_{\alpha+\beta}((\alpha+\beta)(t_a)a_{\alpha+\beta}) x_\beta(\beta(t_a)a_\beta) x_\alpha(\alpha(t_a)a_\alpha)$. By (3.4) and Remark 3.4(ii), using (3.2), we see that the coclass $\mathbb{V} r_a r_b$ contains $x_{\alpha+\beta}(x) x_\beta(y) x_\alpha(z)$ with

$$x = (\alpha+\beta)(t_a)a_{\alpha+\beta} + c_{11}^{\alpha,\beta}\alpha(t_a)\beta(t_b)a_\beta a_\alpha, \tag{3.9}$$

$$y = (\beta(t_a)+\beta(t_b))a_\beta, \tag{3.10}$$

$$z = (\alpha(t_a)+\alpha(t_b))a_\alpha. \tag{3.11}$$

Arguing as in the proof of Step 1 of Proposition 3.5, we see that $r_a \triangleright r_b \neq r_b$ and that $R_a \subset \mathbb{V}' x_\beta(\beta(t_a)y) x_\alpha(\alpha(t_a)x)$ with $\mathbb{V}' = \langle \mathbb{U}_\gamma \mid \gamma \neq \alpha, \beta \rangle$. By a direct computation; $\coprod_{a\in\mathbb{I}_4} R_a$ is subrack of $\mathcal{O}$, hence $\mathcal{O}$ is of type F.

Step 2. If $\boldsymbol{G}$ is a Chevalley group and $q > 7$, then there exists a family $(t_a)_{a\in\mathbb{I}_4}$ in $\mathbb{T} \cap [\mathbb{G}^F, \mathbb{G}^F]$ satisfying (3.7).

Let ζ be a generator of $\mathbb{F}_q^\times$. By assumption on q, for $e_a := a-1$ with $a \in \mathbb{I}_4$ we have $re_a \not\equiv re_b \mod (q-1)$ for all pairs $a \neq b$ and $1 \leq r \leq 3$. If α and β have different lengths, we assume that α is the longest one. Set $t_a = \alpha^\vee(\zeta^{e_a}) \in \mathbb{T}$; by [10, 8.1.4], $t_a \in [\mathbb{G}^F, \mathbb{G}^F]$. Then $\alpha(t_a)\beta(t_b) = \zeta^{2e_a + me_b}$ with $m \in \{-1, 0, 1\}$ and a direct verification gives the claim.

Hence the proposition for Chevalley groups follows from Steps 1 and 2.

Step 3. If $\boldsymbol{G} = \mathbf{PSU}_3(q)$, for $q \notin \{2, 5, 8\}$, then there exists $t_a \in \mathbb{T} \cap [\mathbb{G}^F, \mathbb{G}^F]$ for $a \in \mathbb{I}_4$ satisfying (3.7).

The only classes with the $\alpha\beta$-property are the regular ones. In this case we have $\alpha = \alpha_1$, $\beta = \theta(\alpha_1) = \alpha_2$ and, for ζ a generator in $\mathbb{F}_{q^2}^\times$ we set

$$t_a = \alpha^\vee(\zeta^{a-1})\beta^\vee(\zeta^{(a-1)q}), \quad \text{for } a \in \mathbb{I}_4. \tag{3.12}$$

Then $(\alpha(t_a), \beta(t_a)) = (\zeta^{(a-1)(2-q)}, \zeta^{(a-1)(2q-1)})$ and the claim follows from a direct computation.

Hence the proposition for $\mathbf{PSU}_3(q)$ follows from Steps 1 and 3. We assume in the remaining Steps 4, 5 and 6 that $\boldsymbol{G}$ is a Steinberg group, $\boldsymbol{G} \neq {}^{(3)}D_4(q)$; by the preceding Step, we also assume that $\boldsymbol{G} \neq \mathbf{PSU}_3(q)$.

Step 4. If $q > 5$ and $\{\alpha, \beta\} \cap \{\theta(\alpha), \theta(\beta)\} = \emptyset$, then there exists $t_a \in \mathbb{T} \cap [\mathbb{G}^F, \mathbb{G}^F]$ for $a \in \mathbb{I}_4$ satisfying (3.7). The same holds if $q = 4$, except when $(\alpha, \theta(\alpha)) = -1$ and $(\theta(\alpha), \beta) = 1$.

Since θ preserves positivity of roots, we have $\theta(\alpha) \neq -\alpha$, $\theta(\beta) \neq -\beta$. Hence, for $m := (\alpha, \theta(\alpha)), m' := (\theta(\alpha), \beta)$, we have $m, m' \in \{-1, 0, 1\}$. Let t_a be as in (3.12) with $\beta = \theta(\alpha)$. Then $\alpha(t_a) = \zeta^{(a-1)(2+mq)}$, $\beta(t_a) = \zeta^{(a-1)(m'q-1)}$. Therefore, (3.7), follows if $|\zeta^{(3+q(m-m'))}| \geq 4$. A direct estimate making use of the equalities

$$\gcd(q^2-1, q \pm 3) = \gcd(8, q \pm 3), \quad \text{for } q \neq 3$$

shows that (3.7) holds for $q > 5$, or for $q = 4$ provided that $(m, m') \neq (-1, 1)$.

Step 5. If $q > 7$ and $\beta \neq \theta(\alpha)$, then there exists $t_a \in \mathbb{T} \cap [\mathbb{G}^F, \mathbb{G}^F]$ for $a \in \mathbb{I}_4$ satisfying (3.7).

For $\alpha = \theta(\alpha)$, or $\alpha \neq \theta(\alpha)$ but $\beta = \theta(\beta)$, and $q > 7$, the proof is as in Step 2. If $\alpha \neq \theta(\alpha)$ and $\beta \neq \theta(\beta)$, then this is Step 4.

Step 6. If $q \notin \{2, 5, 8\}$ and $\beta = \theta(\alpha)$, then there exists $t_a \in \mathbb{T} \cap [\mathbb{G}^F, \mathbb{G}^F]$ for $a \in \mathbb{I}_4$ satisfying (3.7).

This step is proved as Step 3.

Hence the proposition for a Steinberg group $\boldsymbol{G}$ different from ${}^{(3)}D_4(q)$ and $\mathbf{PSU}_3(q)$ follows from Steps 1, 4, 5 and 6.

Step 7. Assume $G = {}^{(3)}D_4(q)$ and $q \neq 2, 3, 4, 7$. Then $\mathcal{O}$ is of type F.

By the proof of Step 3 in Proposition 3.5 we can always assume that $\alpha = \alpha_2$ and $\beta \in \{\alpha_1, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$. Let $\zeta \in \mathbb{F}_q^\times$ be a generator. Let $e_a = a - 1$, with $a \in \mathbb{I}_4$. In the first case we take $t_a = \beta^\vee(\zeta)\theta(\beta)^\vee(\zeta)\theta^2(\beta)^\vee(\zeta)\alpha^\vee(\zeta^{e_a})$. Since $\zeta^q = \zeta$, we have $t_a \in \mathbb{T}^F$. Further, since $(\alpha, \theta^i(\beta)) = -1$ and $(\theta^i(\beta), \theta^i(\beta)) = 2$ we have $\alpha(t_a)\beta(t_b) = \zeta^{-1+2e_a-e_b}$. As in Step 2, (3.7) are satisfied if $q \neq 2, 3, 4, 7$. In the second case we take $\alpha = \alpha_2$, $\beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. Then, the proof follows as in Step 2. Indeed, define $t_a = \alpha^\vee(\zeta^{e_a}) \in \mathbb{T}^F \cap [\mathbb{G}^F, \mathbb{G}^F]$. Then $\alpha(t_a)\beta(t_b) = \zeta^{2e_a-e_b}$ and a direct verification gives the claim. Thus, $\mathcal{O}$ is of type F by Step 1. □

3.5. *Regular unipotent classes*

A *regular* unipotent conjugacy class in a reductive algebraic group is the unique unipotent class with maximal dimension. Then we say that $\mathcal{O}$ is *regular* if it is contained in the regular unipotent class in $\mathbb{G}$. We shall prove often that a class is of type D or F by considering the intersection with a smaller group, containing a regular class of the latter. Thus, we need to see when regular classes are of type D or F.

Proposition 3.7. *Assume that q is odd. Let $\boldsymbol{G}$ be a finite simple group of Lie type not of type A_1. If $\mathcal{O}$ is regular, then it is of type D.*

Proof. By [13, 3.2, 3.3], every regular unipotent element u in $\mathbb{U}$ can be written as $u = \mathbb{U}' x_{\alpha_1}(a_1) \cdots x_{\alpha_n}(a_n)$ where $\mathbb{U}'$ is the product of root subgroups of height at least 2 and each $a_i \in \Bbbk^\times$. If the rank of $\mathbb{G}$ is not 1, this ensures that for every $u \in \mathcal{O} \cap \mathbb{U}^F$, there are α, β simple adjacent roots in supp(u); hence $\mathcal{O}$ has the $\alpha\beta$-property. Now Proposition 3.5 applies. □

Proposition 3.8. *Assume $\boldsymbol{G}$ is either*

(a) *a Chevalley group with $q > 7$ and $\mathbb{G} \neq \mathbf{SL}_2(\Bbbk)$, or*
(b) $\mathbf{PSU}_3(q)$, *with $q \notin \{2, 5, 8\}$, or*
(c) $\mathbf{PSU}_n(q)$, *with $n \geq 5$, or ${}^{(2)}E_6(q)$, and $q \notin \{2, 3, 5\}$, or*
(d) ${}^{(2)}D_n(q)$ *for $n \geq 4$, or $\mathbf{PSU}_4(q)$, and $q > 7$, or*
(e) ${}^{(3)}D_4(q)$ *and $q \neq 2, 3, 4, 7$.*

Then every regular unipotent class in $\boldsymbol{G}$ is of type F.

Proof. Arguing as in Proposition 3.7 there exists $u \in \mathcal{O} \cap \mathbb{U}^F$ and $\alpha, \beta \in \Phi^+$ such that $\mathcal{O}$ has the $\alpha\beta$-property, hence we may invoke Proposition 3.6. If $\boldsymbol{G} = \mathbf{PSU}_n(q)$, with $n \geq 5$, or $\boldsymbol{G} = {}^{(2)}E_6(q)$ we can always find adjacent simple roots α and β such that $\{\alpha, \beta\} \cap \{\theta(\alpha), \theta(\beta)\} = \emptyset$, so Step 4 applies. If $\boldsymbol{G} = {}^{(2)}D_n(q)$ for $n \geq 4$ or $\mathbf{PSU}_4(q)$, then we can always find adjacent simple roots α and β with $\beta \neq \theta(\alpha)$ and Step 5 applies. $\square$

Remark 3.9. If p is good (see [11, I.4.3] for the list of bad primes) then all regular unipotent classes in $\mathbb{G}^F$ are isomorphic as racks [14, Lemma 4.1]. But this is not always the case for p bad. Let, for instance, $p = 2$, $\mathbb{G}^F = \mathbf{Sp}_4(2) \cong \mathbb{S}_6$. The regular unipotent classes $\mathcal{O}$ and $\mathcal{O}'$ correspond to the partitions $(1^2, 4)$ and $(2, 4)$, and have isomorphic centralizers. We compute the inner groups of them, see [3, Definition 1.3], using [3, Lemma 1.9]: $\mathrm{Inn}_{\mathcal{O}} = \mathbb{S}_6$ and $\mathrm{Inn}_{\mathcal{O}'} = \mathbb{A}_6$. Thus $\mathcal{O}$ and $\mathcal{O}'$ are not isomorphic as racks.

We next deal with some specific groups. See Sec. 4 for the needed notation of symplectic groups.

Lemma 3.10. *Let $q > 2$ be even. The regular unipotent classes in $\mathbf{Sp}_{2n}(q)$ are of type F.*

Proof. There are exactly two regular unipotent classes in $\mathbf{Sp}_{2n}(q)$ [8, Theorem 6.2.1]. Both are treated similarly, so fix one of them, say $\mathcal{C}$. There is an upper-triangular matrix $u \in \mathcal{C}$. By Jordan theory u is regular in $\mathbf{SL}_{2n}(q)$ [11, IV.2.15.9(ii)], so all its coefficients in the upper subdiagonal are $\neq 0$.

Assume first that $n = 2$. Then we may assume that

$$u = \begin{pmatrix} 1 & x & 0 & p \\ 0 & 1 & y & xy \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Indeed, if

$$u = \begin{pmatrix} 1 & x & l & p \\ 0 & 1 & y & m \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then since $u \in \mathbf{Sp}_4(q)$ and it is a regular element we have that $x = z$, $m = l + xy$ and $xyz \neq 0$. Conjugating by

$$v = \begin{pmatrix} 1 & ly^{-1} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & ly^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

we obtain that $(vuv^{-1})_{13} = 0$ and $(vuv^{-1})_{24} = xy \neq 0$. Let ζ be a generator of $\mathbb{F}_q^\times$. Conjugation by the diagonal matrices $(\zeta^a, \zeta^b, \zeta^{-b}, \zeta^{-a})$, $a, b \in \{0, 1\}$, provides four different representatives $x_{a,b}$ of $\mathcal{O}_x^{\mathbf{Sp}_4(q)}$. We check that the subracks $R_{a,b} := \mathbb{U}^F \triangleright x_{a,b}$ are disjoint for $(a,b) \neq (c,d)$ as in [1, 3.1]. A direct computation looking at $(x_{a,b}x_{c,d})_{13}$ and $(x_{a,b}x_{c,d})_{14}$ shows that $x_{a,b} \triangleright x_{c,d} \neq x_{c,d}$ if $(a,b) \neq (c,d)$ for all $q > 2$.

Assume now $n > 2$. Then Lemma 2.2 applies with $P = \mathbb{P}^F$ such that $\mathbb{P}$ is the standard parabolic subgroup containing $\mathbb{L} = \mathbb{T}\mathbf{Sp}_4(\Bbbk)$ as a Levi factor and $L = \mathbb{L}^F$, the $\mathbb{F}_q$-points of $\mathbb{L}$. Indeed since u is regular unipotent, it is a p-element so its image $\overline{u}$ is contained in $\mathbf{Sp}_4(q) \subset L$ and it is regular therein. $\square$

For further use, we treat here regular classes in other groups. Recall the notation in Sec. 3.3.1.

Lemma 3.11. *Let $q = 2^{2h+1}$, where $h \in \mathbb{N}_0$. The regular unipotent classes in $\mathbf{GU}_n(q)$ for $1 < n$ odd are of type D.*

Proof. Assume $q = 2$, $n = 3$. By Sec. 3.3.1, there is a unique unipotent regular conjugacy class $\mathcal{O}$ in $\mathbf{GU}_3(2)$. Let $\zeta, \eta \in \mathbb{F}_4^\times - 1$. Then $r = \begin{pmatrix} 1 & 1 & \zeta \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{O}$. By [7, 6.22], $C_{\mathbf{SL}_3(\Bbbk)}(u) = Z(\mathbf{SL}_3(\Bbbk))C_{\mathbf{SL}_3(\Bbbk)}(u)^\circ$. It is not hard to verify that, for F the twisted Steinberg endomorphism on $\mathbf{SL}_3(\Bbbk)$, the F-twisted action of $Z(\mathbf{SL}_3(\Bbbk)) \simeq \mathbb{Z}/3$ on itself is trivial, see [9, Example 21.14]. Thus, there are exactly three regular unipotent conjugacy classes in $\mathbf{SU}_3(2)$. Let $x \in \Bbbk$ be such that $x^3 = \eta^{-1}$. Then for $g = \begin{pmatrix} x^4 & & \\ & x & \\ & & x^4 \end{pmatrix}$ we have $g^{-1}F(g) = \eta \mathrm{id}_3$ so, $grg^{-1} \in \mathcal{O} \backslash \mathcal{O}_r^{\mathbf{SU}_3(2)}$. Clearly $\mathtt{J}_3 \in \mathbf{SU}_3(2) < \mathbf{GU}_3(2)$, so also $s = \mathtt{J}_3 grg^{-1}\mathtt{J}_3 \in \mathcal{O} \backslash \mathcal{O}_r^{\mathbf{SU}_3(2)}$. By a direct computation, $(rs)^2 \neq (sr)^2$. Since $\langle r, s\rangle \subset \mathbf{SU}_3(2)$, $\mathcal{O}$ is of type D.

Assume $q = 2$, $n = 2l + 1 > 3$. Let $\mathbb{P}$ be the standard F-stable parabolic subgroup associated to the simple roots α_l, α_{l+1} and let $\mathbb{L}$ be the corresponding standard F-stable Levi subgroup; $\mathbb{L}$ contains a subgroup isomorphic to $\mathbf{GL}_3(\Bbbk)$. Then, $\mathbb{L}^F$ contains a subgroup isomorphic to $\mathbf{GU}_3(q)$ and Lemma 2.2 applies with $P = \mathbb{P}^F$ and $L = \mathbb{L}^F$, by the case $n = 3$. The claim for general q follows since $\mathbf{GU}_n(2) < \mathbf{GU}_n(2^{2h+1})$. $\square$

3.6. *Further remarks*

We shall often invoke the following result.

Lemma 3.12 ([1, Sec. 3.5]). *Let $\mathcal{O}$ be a unipotent class in $\mathbf{SL}_n(q)$, with partition $(\lambda_1, \ldots, \lambda_k)$. Table 3 summarizes when $\mathcal{O}$ is of type D or F.*

We end this subsection with another useful observation.

Remark 3.13. Let $\mathbb{P}$ be an F-stable parabolic subgroup of $\mathbb{G}$, let $\mathbb{L}$ be an F-stable Levi subgroup and let $\pi : \mathbb{P} \to \mathbb{L}$ be the projection associated with the Levi

Table 3.

n	q	Type $(\lambda_1, \ldots, \lambda_k)$	Type
2	odd square > 9	(2)	D
>2	odd	$\lambda_1 \geq 3$	D
		$(2, 2, \ldots)$	D
		$(2, 1, \ldots)$	D
>2	even	$\lambda_1 \geq 5$	F
		$\lambda_1 = 4$	D
		$(3, 3, \ldots)$	D
		$(3, 2, \ldots)$	F
		$(3, 1, \ldots)$	D
		$(2, 2, \ldots)$	D
		$(2, 1, 1, 1, \ldots)$	F
3	even ≥ 8	(3)	F
	4	(3)	D

decomposition $\mathbb{P} = \mathbb{L}\mathsf{Q}$. Let $G = \mathbb{G}^F$, $P = \mathbb{P}^F$, $Q = \mathsf{Q}^F$, $L = \mathbb{L}^F$.

(1) Let $r, s \in P$ with $s \in Q \not\ni r$. Then $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$ because $Q \lhd P$.
(2) If moreover $\mathcal{O}_r^G = \mathcal{O}_s^G$ and $\pi(rs)^2 \neq \pi(sr)^2$, then $\mathcal{O}_r^G$ is of type D.

4. Unipotent Classes in Finite Symplectic Groups

In this section, $\mathbb{G}$ is the symplectic group $\mathbf{Sp}_{2n}(\Bbbk)$, that is the subgroup of $\mathbf{GL}_{2n}(\Bbbk)$ leaving invariant the bilinear form $\left(\begin{smallmatrix} 0 & \mathtt{J}_n \\ -\mathtt{J}_n & 0 \end{smallmatrix}\right)$, for $\mathtt{J}_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$. We assume $n \geq 2$, since $\mathbf{Sp}_2(\Bbbk) = \mathbf{SL}_2(\Bbbk)$. Let $\mathbb{B}$ be the Borel subgroup of $\mathbb{G}$ consisting of upper triangular matrices. Since $\mathbb{G}$ is simply connected, $\mathbb{G}^F = [\mathbb{G}^F, \mathbb{G}^F] = \mathbf{Sp}_{2n}(q) =: G$ and $\boldsymbol{G} = \mathbf{PSp}_{2n}(q) = \mathbb{G}^F/Z(\mathbb{G}^F)$. By the isogeny argument, Lemma 3.1, it suffices to consider unipotent classes in G.

4.1. *Symplectic groups for q odd*

To a unipotent class $\mathcal{O}$ in $\mathbf{Sp}_{2n}(\Bbbk)$ we attach the partition of $2n$ determined by the Jordan form of $\mathcal{O}$ in $\mathbf{GL}_{2n}(\Bbbk)$. If q is odd, then the partition uniquely determines $\mathcal{O}$. The partitions corresponding to unipotent classes in $\mathbb{G}$ are of the form $(1^{r_1}, 2^{r_2}, \ldots, 2n^{r_{2n}})$ where r_i is even for every odd i [11, IV.2.15.9(ii)]. We call them *symplectic* partitions.

Let $u \in \mathbb{G}$ unipotent. There is a reductive subgroup $\mathbb{J}$ of $\mathbb{G}$ containing u as a regular unipotent element, such that $C_{\mathbb{G}}(u) = C_{\mathbb{G}}(\mathbb{J})\mathbb{V}$ where $\mathbb{V}$ is a connected normal subgroup of $C_{\mathbb{G}}(u)$ [8, Lemmas 3.14, 3.17]. Namely, $\mathbb{J}$ is given in [8, (3.4), p. 48]: if $\mathcal{O}_u$ corresponds to $(1^{r_1}, 2^{r_2}, \ldots, 2n^{r_{2n}})$, then

$$\mathbb{J} \cong \prod_{i \text{odd}} \mathbf{O}_i(\Bbbk) \times \prod_{i \text{even}} \mathbf{Sp}_i(\Bbbk), \tag{4.1}$$

where the product is taken over those i such that $r_i \neq 0$. We can always assume that $\mathbb{J}$ is F-stable and that F induces an $\mathbb{F}_q$-split morphism on each of its simple factors [8, p. 113]. Recall that $\mathcal{C}(G,u)$ denotes the set of G-conjugacy classes contained in $\mathcal{O}_u^{\mathbb{G}}$, when $u \in G$.

Lemma 4.1. *Let u be a non-trivial unipotent element in G associated with the partition $(1^{r_1}, \dots, n^{r_n})$. Assume that one of these conditions holds*:

(1) *there exists $i > 3$ for which $r_i \neq 0$;*
(2) *$9 < q$ is a square and the partition is $(1^{r_1}, 2^{r_2})$ with $r_2 > 0$.*

Then $\mathcal{O}$ is of type D for every $\mathcal{O} \in \mathcal{C}(G,u)$.

Proof. Since u is unipotent, it lies in the following subgroup of $\mathbb{J}$

$$\mathbb{M} = \prod_{i\text{odd}} \mathbf{SO}_i(\Bbbk) \times \prod_{i\text{even}} \mathbf{Sp}_i(\Bbbk)$$

and each component of u in $\mathbb{M}$ is regular in its factor. We show that $\mathcal{O}_u^{\mathbb{M}^F}$ is of type D. Case (1) follows from Proposition 3.7. In Case (2), $r_2 > 0$ and $\mathbb{M}^F$ is a group of type A_1; hence [1, Lemma 3.6] applies. For the other classes in $\mathcal{C}(G,u)$, we apply Lemma 3.2(c); indeed $C_{\mathbb{G}}(u) = C_{\mathbb{G}}(\mathbb{J})\mathbb{V}$ and $\mathbb{V}$ is connected, so representatives of $A(u)$ can be found in $C_{\mathbb{G}}(\mathbb{J}) < C_{\mathbb{G}}(\mathbb{M})$. $\square$

By Lemma 4.1(1), it remains to consider the partitions $(1^{r_1}, 2^{r_2}, 3^{r_3})$. We start by $(1^{r_1}, 3^{r_3})$; the argument in Lemma 4.1(2) also applies for it, but there is an alternative without the restrictions in the parameters.

Lemma 4.2. *Let $u \in G$ be a unipotent element corresponding to a partition of the form $(1^{r_1}, 3^{r_3})$, with $r_3 > 0$. Then $\mathcal{O}_u^G$ is of type D.*

Proof. By [8, Theorem 3.1(v)] we have $C_{\mathbb{G}}(u) = C_{\mathbb{G}}(u)^\circ$ so $\mathcal{C}(G,u)$ consists of a single class $\mathcal{O}$. We set $r_3 = 2e$. Let $v = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. A representative of $\mathcal{O}$ is $\begin{pmatrix} v \otimes \mathrm{id}_e & 0 & 0 \\ 0 & \mathrm{id}_{2n-3r_3} & 0 \\ 0 & 0 & (\mathrm{J}_3{}^t v^{-1} \mathrm{J}_3) \otimes \mathrm{id}_e \end{pmatrix}$. Now we may assume that $e = 1$; in this case the injective morphism $\iota : \mathbf{SL}_3(q) \to G$, $X \mapsto \mathrm{diag}(X, \mathrm{id}_{2n-6}, \mathrm{J}_3{}^t X^{-1} \mathrm{J}_3)$ induces a rack embedding $\mathcal{O}_v^{\mathbf{SL}_3(q)} \hookrightarrow \mathcal{O}_u^G$ and by the isogeny argument, Proposition 3.7 applies. $\square$

Lemma 4.3. *Let $\mathcal{O}$ be a unipotent class in G with partition $(1^{r_1}, 2^{r_2})$, with $r_2 > 1$. If $q > 3$ or $n > 2$, then the class is of type D.*

Proof. In this case $A(u) \simeq \mathbb{Z}/2$ [8, Theorem 3.1(v)], so $\mathcal{C}(G,u)$ consists of two classes. One of them is represented by $u = \begin{pmatrix} \mathrm{id}_{r_2} & 0 & \mathrm{J}_{r_2} \\ 0 & \mathrm{id}_{r_1} & 0 \\ 0 & 0 & \mathrm{id}_{r_2} \end{pmatrix}$. We find a representative for the other. Recall the notation (4.1); it can be shown that $C_{\mathbb{G}}(\mathbb{J}) \simeq \mathbf{O}_{r_2}(\Bbbk) \times$

$\mathbf{Sp}_{r_1}(\Bbbk)$ is the subgroup of matrices $g_{A,M} = \begin{pmatrix} A & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & \mathrm{J}_{r_2} A \mathrm{J}_{r_2} \end{pmatrix}$ with $A^t A = \mathrm{id}_{r_2}$ and $M \in \mathbf{Sp}_{r_1}(\Bbbk)$. The non-trivial element in $A(u)$ is represented by $g_{L,\mathrm{id}_{r_1}}$ for $L = \mathrm{diag}(-1, 1, \dots, 1)$. Let $\xi \in \mathbb{F}_{q^2} \backslash \mathbb{F}_q$ be such that $\xi^{q-1} = -1$, so $\zeta = \xi^2 \in \mathbb{F}_q$ is not a square in $\mathbb{F}_q$. Let $g = \mathrm{diag}(\xi, 1, \dots, 1, \xi^{-1}) \in \mathbf{Sp}_{2n}(\Bbbk)$. Then $g^{-1}F(g) = g_{L,\mathrm{id}_{r_1}}$, so $\mathcal{C}(G, u) = \{\mathcal{O}_u^G, \mathcal{O}_v^G\}$ where

$$v = gug^{-1} = \begin{pmatrix} 1 & & & & \zeta \\ & \mathrm{id}_{r_2-1} & & \mathrm{J}_{r_2-1} & \\ & & \mathrm{id}_{r_1} & & \\ & & & \mathrm{id}_{r_2-1} & \\ & & & & 1 \end{pmatrix}.$$

Let $\mathbb{M}$ be the F-stable subgroup of $\mathbf{Sp}_{2n}(\Bbbk)$ of matrices $\begin{pmatrix} a & 0 & b \\ 0 & M & 0 \\ c & 0 & d \end{pmatrix}$ with $ad - bc = 1$ and $M \in \mathbf{Sp}_{2n-2}(\Bbbk)$. Clearly, $\mathbb{M} \simeq \mathbf{SL}_2(\Bbbk) \times \mathbf{Sp}_{2n-2}(\Bbbk)$. Then $\mathcal{O}_u^{\mathbb{M}^F} \simeq \mathcal{O}_v^{\mathbb{M}^F} \simeq \mathcal{O}_{x_1}^{\mathbf{SL}_2(q)} \times \mathcal{O}_{y_1}^{\mathbf{Sp}_{2n-2}(q)}$ with

$$x_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad y_1 = \begin{pmatrix} \mathrm{id}_{r_2-1} & 0 & \mathrm{J}_{r_2-1} \\ 0 & \mathrm{id}_{r_1} & 0 \\ 0 & 0 & \mathrm{id}_{r_2-1} \end{pmatrix}.$$

We show that this subrack is of type D by application of Lemma 2.3. First, $x_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in \mathcal{O}_{x_1}^{\mathbf{SL}_2(q)}$ satisfies $(x_1 x_2)^2 \neq (x_2 x_1)^2$. So, we have to find $y_2 \in \mathcal{O}_{y_1}^{\mathbf{Sp}_{2n-2}(q)}$ commuting with y_1.

Assume $q > 3$, and let $\eta \in \mathbb{F}_q^\times \backslash \{1, -1\}$. Then we take

$$\begin{aligned} y_2 &= \begin{pmatrix} \eta \mathrm{id}_{r_2-1} & & \\ & \mathrm{id}_{r_1} & \\ & & \eta^{-1} \mathrm{id}_{r_2-1} \end{pmatrix} \triangleright y_1 \\ &= \begin{pmatrix} \mathrm{id}_{r_2-1} & 0 & \eta^2 \mathrm{J}_{r_2-1} \\ 0 & \mathrm{id}_{r_1} & 0 \\ 0 & 0 & \mathrm{id}_{r_2-1} \end{pmatrix}. \end{aligned}$$

Assume now that $n > 2$. If $r_2 > 2$, then

$$y_2 = \begin{pmatrix} 1 & 1 & & & \\ & 1 & & & \\ & & \mathrm{id}_{2n-6} & & \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix} \triangleright y_1 \in \mathcal{O}_{y_1}^{\mathbf{Sp}_{2n-2}(q)}$$

commutes with y_1. If $r_2 = 2$, then necessarily $r_1 > 1$. In this case we take

$$y_2 = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \mathrm{id}_{2n-6} & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix} \triangleright y_1 = \begin{pmatrix} 1 & & & 0 & 0 \\ & 1 & & 1 & 0 \\ & & \mathrm{id}_{2n-6} & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}. \qquad \square$$

Lemma 4.4. *Let u be a unipotent element in G with partition $(1^{r_1}, 2^{r_2}, 3^{r_3})$, such that $r_2 r_3 > 0$. Then $\mathcal{O}_u^G$ is of type D.*

Proof. Here $r_3 = 2a$ is even and $C_{\mathbb{G}}(\mathbb{J}) \simeq \mathbf{Sp}_{r_1}(\Bbbk) \times \mathbf{O}_{r_2}(\Bbbk) \times \mathbf{Sp}_{r_3}(\Bbbk)$, so $A(u) \simeq \mathbb{Z}/2$ [8, Theorem 3.1(v)] and $\mathcal{C}(G,u) = \{\mathcal{O}_u^G, \mathcal{O}_v^G\}$ has two elements. Let $x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and w in $\mathbf{Sp}_{2n-3r_3}(q)$ unipotent with partition $(1^{r_1}, 2^{r_2})$. Then we choose

$$u = \begin{pmatrix} x \otimes \mathrm{id}_a & & \\ & w & \\ & & \mathrm{J}_3\, {}^t x^{-1} \mathrm{J}_3 \otimes \mathrm{id}_a \end{pmatrix}.$$

Let $\mathbb{M} < \mathbb{G}$ consisting of matrices of the form $\mathrm{diag}(\mathrm{id}_{3a}, Y, \mathrm{id}_{3a})$, with Y in $\mathbf{Sp}_{2n-3r_3}(\Bbbk)$, and let $\mathbb{J}_w$ be the reductive group containing w as a regular element, cf. [8, 3.2.3]. Then $C_{\mathbb{M}}(\mathbb{J}_w) \simeq \mathbf{Sp}_{r_1}(\Bbbk) \times \mathbf{O}_{r_2}(\Bbbk) < C_{\mathbb{G}}(\mathbb{J}) < C_{\mathbb{G}}(u)$ and a representative b for the non-trivial class in $A(u)$ may be chosen in $\mathbb{M}$. By Lang–Steinberg's theorem, there exists $g \in \mathbb{M}$ such that $g^{-1}F(g) = b$, so we may pick $v = gug^{-1}$. In other terms, $v = \begin{pmatrix} x \otimes \mathrm{id}_a & & \\ & z & \\ & & \mathrm{J}_3^t x^{-1} \mathrm{J}_3 \otimes \mathrm{id}_a \end{pmatrix}$ with $\mathcal{C}(\mathbf{Sp}_{2n-3r_3}(q), w) = \{\mathcal{O}_w^{\mathbf{Sp}_{2n-3r_3}(q)}, \mathcal{O}_z^{\mathbf{Sp}_{2n-3r_3}(q)}\}$. Then $\mathcal{O}_u^G$ and $\mathcal{O}_v^G$ contain a subrack isomorphic to $\mathcal{O}_x^{\mathbf{SL}_3(q)}$, and by the isogeny argument, Proposition 3.7 applies. $\square$

Lemma 4.5. *There are two unipotent classes in $\mathbb{G}^F = \mathbf{Sp}_4(3)$ of type (2^2); one of them is of type D and the other is cthulhu.*

Proof. As in the proof of Lemma 4.3, there are two classes of type (2^2), represented by $z = \begin{pmatrix} 1 & & & 1 \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 & & & -1 \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$. It can be verified that $x = x_{\alpha_1}(1) = \begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix} \in \mathcal{O}_w^{\mathbf{Sp}_4(3)}$. The discussion in [1, 3.1] shows that $\mathcal{O}_x^{\mathbb{U}^F} \neq \mathcal{O}_w^{\mathbb{U}^F}$, and that $xw \neq wx$. In addition, xw and $w(xw)w^{-1} = wx$ lie in $\mathbb{U}^F$ so they have odd order, hence $(xw)^2 \neq (wx)^2$. The claim on the other class was verified with GAP. $\square$

4.2. *Symplectic groups for q even*

In this section q is even, so the symplectic group $\mathbb{G}$ is the subgroup of $\mathbf{GL}_{2n}(\Bbbk)$ leaving invariant the bilinear form $\mathtt{J}_{2n}$. Here symplectic partitions do not distinguish conjugacy classes.

4.2.1. *Unipotent conjugacy classes*

We parametrize the unipotent conjugacy classes in $\mathbb{G}$ as in [8, 6.1, cf. Lemma 6.2]. Let V be the natural representation of $\mathbb{G}$ and let $u \in \mathbb{G}$ unipotent. Then V decomposes, as an u-module by restriction, into an orthogonal direct sum of indecomposable submodules (where $k, r \in \mathbb{N}_0$, the m_i's are distinct, ditto for the k_j's)

$$V = \bigoplus_{i=1}^{k} W(m_i)^{a_i} \oplus \bigoplus_{j=1}^{r} V(2k_j)^{b_j}, \quad 0 < a_i, \ \ 0 < b_j \leq 2. \tag{4.2}$$

We describe the summands in the right-hand side:

- $\dim W(m_i) = 2m_i$ and $u_{|W(m_i)}$ is regular in a subgroup $\mathbb{H}_{m_i}$, that is the image of $\mathbf{SL}_{m_i}(\Bbbk)$ by the embedding of $\mathbf{GL}_{m_i}(\Bbbk)$ in $\mathbf{Sp}(W(m_i))$ given by
$$X \mapsto \operatorname{diag}(X, \mathtt{J}_{m_i}{}^{t}X^{-1}\mathtt{J}_{m_i}) \tag{4.3}$$
and thus $u_{|W(m_i)}$ is of partition (m_i, m_i) in $\mathbf{Sp}(W(m_i))$;
- $\dim V(2k_j) = 2k_j$ and $u_{|V(2k_j)}$ is regular in a subgroup $\mathbb{J}_{2k_j} \simeq \mathbf{Sp}_{2k_j}(\Bbbk)$ and thus $u_{|V(2k_j)}$ is of partition $(2k_j)$.

  Set $\mathbb{H} = \prod_{i=1}^{k} \mathbb{H}_{m_i}^{a_i}$, $\mathbb{J} = \prod_{j=1}^{r} \mathbb{J}_{2k_j}^{b_j}$. Then u is regular in $\mathbb{M} := \mathbb{H} \times \mathbb{J}$. Let $\mathcal{W} = \bigoplus_{i=1}^{k} W(m_i)^{a_i}$ and $\mathcal{V} = \bigoplus_{j=1}^{r} V(2k_j)^{b_j}$. Then
$$\begin{aligned} \mathbb{M} &< \prod_{i=1}^{k} \mathbf{Sp}(W(m_i)^{a_i}) \times \prod_{j=1}^{r} \mathbf{Sp}(V(2k_j)^{b_j}) \\ &< \mathbf{Sp}(\mathcal{W}) \times \mathbf{Sp}(\mathcal{V}) < \mathbb{G}. \end{aligned} \tag{4.4}$$

  By the description in [8, p. 91], there is $u \in G = \mathbb{G}^F$ such that all subgroups $\mathbb{H}_{m_i}$, $\mathbb{J}_{2k_j}$, $\mathbb{H}$, $\mathbb{J}$, $\mathbb{M}$, $\mathbf{Sp}(\mathcal{W})$, $\mathbf{Sp}(\mathcal{V})$ are F-stable and F acts on each of them by a split Frobenius automorphism. In particular,
$$\mathbb{H}^F \simeq \prod_{i=1}^{k} \mathbf{SL}_{m_i}(q)^{a_i}, \quad \mathbb{J}^F \simeq \prod_{j=1}^{r} \mathbf{Sp}_{2k_j}(q)^{b_j}.$$
  We fix this u in the rest of this subsection.

4.2.2. *Representatives of classes in $\mathcal{C}(G, u)$*

We now address the problem of finding suitable subracks for $\mathcal{O} \in \mathcal{C}(G, u)$, that we recall again is the set of G-conjugacy classes contained in $\mathcal{O}_u^{\mathbb{G}}$. First we need some information on $A(u)$, cf. (3.3).

Lemma 4.6. *There is a set of representatives Ξ of $A(u)$ in $C_{\mathbb{G}}(u)$ such that for every $x \in \Xi$ there is $g \in \mathbb{G}$ with $x = g^{-1}F(g)$ satisfying*:

(a) $F(g\mathbb{M}g^{-1}) = g\mathbb{M}g^{-1}$ *and* $F(g\mathbb{J}_{2k_j}g^{-1}) = g\mathbb{J}_{2k_j}g^{-1}$ *for every* j.

(b) $(g\mathbb{M}g^{-1})^F \simeq \prod_{i=1}^k (\mathbf{SL}_{m_i}(q)^{a_i-1} \times G_i) \times \mathbb{J}^F$, *where G_i is $\mathbf{SL}_{m_i}(q)$ or $\mathbf{SU}_{m_i}(q)$; $\mathbf{SU}_{m_i}(q)$ may occur only if $m_i > 1$ is odd.*

Proof. By the proof of [8, Theorem 6.21], there is a maximal torus $\mathbb{T}_0$ of $C_{\mathbb{G}}(u)$ such that

$$C_{\mathbb{G}}(u) = C_{\mathbb{G}}(u)^\circ NH, \tag{4.5}$$

where $N = N_{\mathbb{G}}(\mathbb{T}_0) \cap C_{\mathbf{Sp}(\mathcal{W})}(u)$, $H = C_{\mathbf{Sp}(\mathcal{V})}(u)$, $NH \simeq H \times N$. Also,

$$(NH) \cap C_{\mathbb{G}}(u)^\circ = (N \cap C_{\mathbb{G}}(u)^\circ)(H \cap C_{\mathbb{G}}(u)^\circ), \quad H/H \cap C_{\mathbb{G}}(u)^\circ = H/H^\circ.$$

Hence, we may construct a set of representatives Ξ for $A(u)$ as a product $\Sigma\Sigma' \in NH$ where Σ, respectively, Σ', is a set of representatives of $N/N \cap C_{\mathbb{G}}(u)^\circ$, respectively of H/H°. We claim that there are $\Sigma, \Sigma' \subset N_{\mathbb{G}}(\mathbb{M}) \cap N_{\mathbb{G}}(\mathbb{J}_{2k_j})$ for every j. First H/H° is generated by images of the components u_j of u in some factors $\mathbb{J}_{2k_j}$ [8, Lemmas 6.13, 6.14], so any $\Sigma' \subset \mathbb{J}$ will do. For Σ we need additional information from the proof of [8, Theorem 6.21]:

- $\mathbb{T}_0$ is a product of subtori $\mathbb{T}_i$ of dimension a_i acting on a single summand $W(m_i)^{a_i}$ without fixed points;
- $N = \prod_{i=1}^k N_i$ where $N_i = N_{\mathbb{G}}(\mathbb{T}_i) \cap C_{\mathbf{Sp}(W(m_i)^{a_i})}(u)$.

Then $\prod_{i=1}^k N_i/(N_i \cap C_{\mathbb{G}}(u)^\circ)$ maps onto $N/(N \cap C_{\mathbb{G}}(u)^\circ)$ and $N_i \subset N_{\mathbb{G}} \times (\mathbf{Sp}(W(m_j)^{a_j}) \cap N_{\mathbb{G}}(\mathbb{J}_{2k_j})$ for every i, j. In order to describe the action of an $x \in N_i$ on $\mathbf{Sp}(W(m_i)^{a_i})$ we analyze the component of u lying in this subgroup. We may assume it is $\begin{pmatrix} v \otimes \mathrm{id}_{a_i} & \\ & \mathtt{J}_{m_i}{}^t v^{-1}\mathtt{J}_{m_i} \otimes \mathrm{id}_{a_i}\end{pmatrix}$ where v is a regular unipotent matrix in $\mathbf{SL}_{m_i}(\Bbbk)$, and so $\mathbb{T}_i$ is the subgroup of diagonal matrices

$$(\lambda_1 \mathrm{id}_{m_i}, \ldots, \lambda_{a_i}\mathrm{id}_{m_i}, \lambda_{a_i}^{-1}\mathrm{id}_{m_i}, \ldots, \lambda_1^{-1}\mathrm{id}_{m_i}).$$

Then $N_{\mathbf{Sp}_{2m_ia_i}(\Bbbk)}(\mathbb{T}_i)$ normalizes $[C_{\mathbf{Sp}_{2m_ia_i}(\Bbbk)}(\mathbb{T}_i), C_{\mathbf{Sp}_{2m_ia_i}(\Bbbk)}(\mathbb{T}_i)] = \mathbb{H}_{m_i}^{a_i}$, and the claim follows. The claim implies (a) by Remark 2.5(b).

Let $x = sh \in \Xi$, $s \in \Sigma$, $h \in \Sigma'$. By construction and Lang–Steinberg's theorem applied to $s \in \prod_i \mathbf{Sp}(W(m_i)^{a_i})$ and $h \in \mathbb{J}$, we may choose g such that $g^{-1}F(g) = x$ as

$$\begin{aligned} g = yz, \quad \text{where } y \in \prod_i \mathbf{Sp}(W(m_i)^{a_i}), \quad y^{-1}F(y) = s; \\ z \in \mathbb{J}, \ \ z^{-1}F(z) = h. \end{aligned} \tag{4.6}$$

Since $\Sigma' \subset C_{\mathbb{G}}(\mathbb{H}) \cap \mathbb{J}$, $\Sigma \subset C_{\mathbb{G}}(\mathbb{J})$,

$$(yz\mathbb{J}z^{-1}y^{-1})^F = (y\mathbb{J}y^{-1})^F = \mathbb{J}^F.$$

Also, $N_i/N_i \cap C_{\mathbb{G}}(u)^\circ$ is either trivial or $\simeq \mathbb{Z}/2$ [8, Theorem 6.21]. A representative of the non-trivial element is $x_i = x_i' x_i''$, where

$$x_i' = \begin{pmatrix} \mathrm{id}_{(a_i-1)m_i} & & & \\ & 0_{m_i} & \mathrm{id}_{m_i} & \\ & \mathrm{id}_{m_i} & 0_{m_i} & \\ & & & \mathrm{id}_{(a_i-1)m_i} \end{pmatrix},$$

$$x_i'' = \begin{pmatrix} \mathrm{id}_{m_i(a_i-1)} & & & \\ & X & & \\ & & \widetilde{X} & \\ & & & \mathrm{id}_{m_i(a_i-1)} \end{pmatrix};$$

here $X \in \mathbf{GL}_{m_i}(\Bbbk)$ satisfies $XvX^{-1} = \mathtt{J}_{m_i}{}^t v^{-1} \mathtt{J}_{m_i}$, and $\widetilde{X} = \mathtt{J}_{m_i}{}^t X^{-1} \mathtt{J}_{m_i}$. Then x_i normalizes each factor in $\mathbf{SL}_{m_i}(\Bbbk)^{a_i}$, centralizes the first $a_i - 1$ factors and induces a non-trivial graph automorphism on the last one. Thus,

$$\begin{aligned}(zy\mathbb{H}_{m_i}^{a_i} y^{-1} z^{-1})^F &= (z\mathbb{H}_{m_i}^{a_i} z^{-1})^F \overset{\text{Remark 2.5(c)}}{=} z(\mathbb{H}_{m_i}^{a_i})^{\mathrm{Ad}(x)F} z^{-1} \\ &= z((\mathbb{H}_{m_i}^{a_i-1})^F \times \mathbb{H}_{m_i}^{\mathrm{Ad}(x)F}) z^{-1}\end{aligned}$$

and the first part of (b) follows from Remark 2.5(c). Finally, $N_i = N_i \cap C_{\mathbb{G}}(u)^\circ$ when m_i is even or $m_i = 1$, hence the last restriction in (b). $\square$

Corollary 4.7. *Let $u \in G$ with decomposition* (4.2). *If $\mathbb{J}^F \neq 1$ then every $\mathcal{O} \in \mathcal{C}(G, u)$ contains a subrack that is a regular unipotent class in $\mathbb{J}^F$.*

Proof. We choose the representatives of elements in $A(u)$ in NH by (4.5). For $x \in NH$, there is $g \in \mathbf{Sp}(\mathcal{W}) \times \mathbb{J}$ such that $g^{-1}F(g) = x$, see (4.6). Then the component $gug^{-1}_{|\mathcal{V}}$ of gug^{-1} on $\mathcal{V}$ is regular in $g\mathbb{J}g^{-1} = \mathbb{J}$ and $\mathcal{O}^{\mathbb{J}^F}_{gug^{-1}_{|\mathcal{V}}}$ is a subrack of $\mathcal{O}^{\mathbb{G}^F}_{gug^{-1}}$. $\square$

4.2.3. *Preliminary results*

Before starting the analysis of the various classes, we state two results needed for the application of Lemma 2.3.

Lemma 4.8. *Let $\mathcal{O}$ be a regular unipotent class in either $\mathbf{SL}_n(q)$, $\mathbf{SU}_n(q)$ or $\mathbf{Sp}_{2n}(q)$. Then there are $x_1, x_2 \in \mathcal{O}$ such that $(x_1x_2)^2 \neq (x_2x_1)^2$.*

Proof. Let $\mathtt{J} = \mathtt{J}_n$.

Case 1. $\mathbf{SL}_n(q)$ for $n \geq 2$. By Remark 2.4 we may assume that $\mathcal{O} \ni x_1 = \begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$; then take $x_2 = \mathtt{J}x_1\mathtt{J}^{-1}$.

Case 2. $\mathbf{SU}_n(q)$, n even. By Remark 2.4 we assume that $\mathcal{O} \ni x_1 = \begin{pmatrix} \mathtt{u} & \mathtt{v} \\ 0 & \mathtt{u}^{-1} \end{pmatrix}$, with $\mathtt{u} = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 1 \\ & & & & 1 \end{pmatrix}$ and $\mathtt{v} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$. Let $x_2 := {}^t(\mathrm{Fr}_q(x_1)) = {}^t x_1$; we claim that $x_2 \in \mathcal{O}$. Indeed, by definition of $\mathbf{SU}_n(q)$, $x_2 = \mathtt{J} x_1^{-1} \mathtt{J} \in \mathcal{O}^{-1} = \mathcal{O}$, the last equality by [14, 1.4(ii)]. Since x_1 is regular, $C_{\mathbf{SL}_n(\Bbbk)}(x_1)$ is contained in the Borel subgroup of upper triangular matrices; as $(x_2x_1x_2)_{21} = 1$, $x_2x_1x_2 \notin C_{\mathbf{SL}_n(\Bbbk)}(x_1)$.

Case 3. $\mathbf{SU}_n(q)$, n odd. Let $\xi \in \mathbb{F}_{q^2}$ satisfy $\xi^q + \xi + 1 = 0$. By Remark 2.4 we assume that $\mathcal{O} \ni x_1 = \begin{pmatrix} \mathtt{u} & d & \xi\mathtt{v} \\ 0 & 1 & b \\ 0 & 0 & \mathtt{u}^{-1} \end{pmatrix}$, with $\mathtt{u}, \mathtt{v}$ as in Case 2, $d = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, and $b = (1\ \ 0\ \ \cdots\ \ 0)$. Now take $x_2 := {}^t\,\mathrm{Fr}_q(x_1) \in \mathcal{O}$ and repeat the argument for Case 2.

Case 4. $\mathbf{Sp}_{2n}(q)$, $n \geq 2$. There are two regular classes, $\mathcal{O}$ and $\mathcal{O}'$. Each of them is represented by a triangular matrix whose terms in the upper subdiagonal are $\neq 0$. If x_1 is a representative like this, then $x_2 = \sigma \triangleright x_1$, where $\sigma = \begin{pmatrix} \mathtt{J}_2 & 0 & 0 \\ 0 & \mathrm{id}_{2n-4} & 0 \\ 0 & 0 & \mathtt{J}_2 \end{pmatrix}$, does the job. □

Lemma 4.9. *Let $n > 2$ or $q > 2$ and $\mathcal{O}$ be a regular unipotent class in $\mathbf{SL}_n(q)$, $\mathbf{SU}_n(q)$, or $\mathbf{Sp}_n(q)$. Then there are $y_1, y_2 \in \mathcal{O}$ with $y_1 \neq y_2$, $y_1y_2 = y_2y_1$.*

Proof. By [14, 1.4(ii)] for $\mathbf{SL}_n(q)$ or $\mathbf{SU}_n(q)$, and [6] for $\mathbf{Sp}_n(q)$, $\mathcal{O} = \mathcal{O}^{-1}$. If $n > 2$ no regular element is an involution, so $y_1 = y_2^{-1}$ will do. If $n = 2$ and $q > 2$, then take $y_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $y_2 = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$ for $1 \neq \xi \in \mathbb{F}_q^{\times}$. □

4.2.4. *Analysis of the different classes*

We now assume that $u \in G$ is unipotent with decomposition (4.2). Let $\mathcal{O}$ be an arbitrary class in $\mathcal{C}(G, u)$.

Lemma 4.10. *If* (4.2) *contains $W(4)$, then $\mathcal{O}$ is of type D.*

Proof. By Lemma 4.6, $\mathcal{O}$ contains a subrack isomorphic to the regular class in $\mathbf{SL}_4(q)$. Then Lemma 3.12 applies. □

Lemma 4.11. *If* (4.2) *contains any of these terms, then $\mathcal{O}$ is of type F.*

(a) *$V(2k_j)$ with $k_j > 1$ and $q > 2$.*
(b) *$W(m_i)$ either with $m_i > 4$, $q > 4$; or else with $m_i = 3$, $q > 8$; or else with $m_i > 4$ even.*

Proof. (a) By Corollary 4.7, $\mathcal{O}$ contains a subrack isomorphic to a regular class in $\mathbf{Sp}_{2k_j}(q)$; then Lemma 3.10 applies. (b) By Lemma 4.6, $\mathcal{O}$ contains a subrack isomorphic to a regular class in $\mathbf{SL}_{m_i}(q)$, or $\mathbf{SU}_{m_i}(q)$ (the last occurs only when $m_i > 1$ is odd); then either Lemma 3.12, or else Proposition 3.8(b), (c), or (d), applies. $\square$

In the next lemma, we use that $u_{|W(3)}$ is regular in the image of $\mathbf{GL}_3(\Bbbk)$ via (4.3), hence $\mathcal{O}$ may contain a subrack isomorphic to a regular class in a subgroup $\simeq$ $\mathbf{GU}_3(8)$ when appropriate. We do so because the regular unipotent class in $\mathbf{SU}_3(8)$ is not known to be of type D or F.

Lemma 4.12. *If* (4.2) *contains any of these terms, then $\mathcal{O}$ is either of type D or else of type F.*

(a) $W(m_i)$ *with* $m_i > 1$ *odd and* $q = 4$.
(b) $W(3)$ *and* $q = 8$.

Proof. We apply Lemma 4.6. (a) $\mathcal{O}$ contains a subrack isomorphic to the regular class in $\mathbf{SL}_{m_i}(4)$ or in $\mathbf{SU}_{m_i}(4)$. Then Lemma 3.12 or Proposition 3.8(b) or (c) applies. (b) $\mathcal{O}$ contains a subrack isomorphic to the regular class in $\mathbf{SL}_3(8)$ or in $\mathbf{GU}_3(8)$. Then Lemma 3.12 or Lemma 3.11 applies. $\square$

Lemma 4.13. *If* (4.2) *contains any of these terms, then $\mathcal{O}$ is of type D.*

(a) $V(2k_i) \oplus V(2k_j)$ *with* $k_i k_j > 1$ *or* $q > 2$.
(b) $W(m_i) \oplus V(2k_j)$ *with* $m_i > 2$; *or* $m_i = 2$ *and either* $q > 2$ *or* $k_j > 1$.
(c) $W(m_i) \oplus W(m_j)$ *with either* $q > 2$ *and* $m_i > 1$, $m_j > 1$; *or else* $q = 2$ *and* $m_i > 1$ *and* $m_j > 2$; *or else* $q = 2$, $m_i > 2$ *and* $m_j > 1$.

Proof. By Lemma 4.6, $\mathcal{O}$ contains a subrack $\mathcal{O}_{u_i}^{L_i} \times \mathcal{O}_{u_j}^{L_j}$ with each factor regular, where $L_i \times L_j$ in each case is

$$\begin{aligned}
&\text{(a)}: \mathbf{Sp}_{2k_j}(q) \times \mathbf{Sp}_{2k_i}(q),\\
&\text{(b)}: \mathbf{SL}_{m_i}(q) \times \mathbf{Sp}_{2k_j}(q), \quad \text{or} \quad \mathbf{SU}_{m_i}(q) \times \mathbf{Sp}_{2k_j}(q),\\
&\text{(c)}: \mathbf{SL}_{m_i}(q) \times \mathbf{SL}_{m_j}(q), \quad \text{or} \quad \mathbf{SL}_{m_i}(q) \times \mathbf{SU}_{m_j}(q), \quad \text{or}\\
&\qquad \mathbf{SU}_{m_i}(q) \times \mathbf{SU}_{m_j}(q).
\end{aligned}$$

The claim follows by Lemmas 2.3, 4.8 and 4.9. $\square$

Lemma 4.14. *If* $q = 2$ *and* (4.2) *contains a term of the form* $W(m_i)$, $m_i > 1$ *odd, then $\mathcal{O}$ is of type D.*

Proof. *Step* 1: If (4.2) is of the form $W(m)$ with $m > 1$ odd, then $\mathcal{O}$ is of type D. Indeed, there are two classes of this type [8, Theorem 6.21]. One class contains a subrack isomorphic to the regular unipotent class in $\mathbf{GU}_m(2)$, and we invoke Lemma 3.11. We consider next the second class. Assume first that $m = 3$. Then this class is represented by

$$v = \begin{pmatrix} 1 & 1 & 1 & & & \\ & 1 & 1 & & & \\ & & 1 & & & \\ & & & 1 & 1 & 0 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix} = x_{\alpha_1}(1)x_{\alpha_2}(1).$$

Let

$$s_2 := \begin{pmatrix} 1 & & & & & \\ & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & 1 \end{pmatrix}, \quad r = s_2 \triangleright v = \begin{pmatrix} 1 & 1 & 1 & & & \\ & 1 & 0 & & & \\ & 1 & 1 & & & \\ & & & 1 & 0 & 1 \\ & & & 1 & 1 & 0 \\ & & & & & 1 \end{pmatrix}$$

$$= x_{\alpha_1+\alpha_2}(1)x_{-\alpha_2}(1),$$

$$s_3 = \begin{pmatrix} \mathrm{id}_2 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \mathrm{id}_2 \end{pmatrix}, \quad s = s_3 \triangleright v = \begin{pmatrix} 1 & 1 & 0 & 1 & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ & & & 1 & 0 & 0 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}$$

$$= x_{\alpha_1}(1)x_{\alpha_2+\alpha_3}(1).$$

Then $(rs)^2 \neq (sr)^2$ by (3.2). In addition, $r, s \in \mathbb{P}^F$ where $\mathbb{P}$ is the standard parabolic subgroup of $\mathbb{G}$ associated with the simple root α_2 so Remark 3.13(1) applies. Assume $m > 3$. Then the class is represented by

$$v = \begin{pmatrix} 1 & 1 & \cdots & 1 & & & & & \\ & 1 & \cdots & 1 & & & & & \\ & & 1 & 1 & & & & & \\ & & & 1 & & & & & \\ & & & & 1 & 1 & 0 & \cdots & 0 \\ & & & & & 1 & 1 & 0 & \cdots \\ & & & & & & 1 & 1 & 0 \\ & & & & & & & 1 & 1 \\ & & & & & & & & 1 \end{pmatrix} = x_{\alpha_1}(1)x_{\alpha_2}(1)\cdots x_{\alpha_{n-1}}(1).$$

We apply Lemma 2.2 to $\mathbb{P}^F$, where $\mathbb{P}$ is the standard parabolic associated with the simple roots $\alpha_n, \alpha_{n-1}, \alpha_{n-2}$, using the case $m = n = 3$.

Step 2: We now prove the lemma. Let u_i be the component of u in $\mathbb{M}_i := \mathbf{Sp}(W(m_i))$. Choosing each representative x of $A(u)$ in Ξ and the corresponding element g as in Lemma 4.6, we have $g\mathbb{M}_i g^{-1} = \mathbb{M}_i$, and $(g\mathbb{M}_i g^{-1})^F = g\mathbb{M}_i^{\mathrm{Ad}(x)\circ F} g^{-1}$. So, $\mathcal{O}^G_{gug^{-1}}$ will contain a subrack isomorphic to $\mathcal{O}_{u_i}^{\mathbb{M}_i^{\mathrm{Ad}(x)\circ F}}$. The component in $\mathbb{M}_i$ of each term in Ξ is either trivial or, possibly, $x'_i x''_i$, with notation as in Lemma 4.6. Then, the two possible subracks are isomorphic to those in Case 1, whence the statement. □

Remark 4.15. By the previous lemmas, it remains to consider the following forms of (4.2), see Table 4 for details:

$$\begin{aligned} q > 2:\ & V = W(1)^a \oplus W(2), && 0 \le a, \\ & V = W(1)^a \oplus V(2), && 0 \le a, \\ q = 2:\ & V = W(1)^a \oplus W(2)^b \oplus V(2)^c, && 0 \le a, b;\ \ 0 \le c \le 2, \\ & V = W(1)^a \oplus V(2k), && 0 \le a;\ \ 1 < k. \end{aligned} \tag{4.7}$$

Recall the conventions in (4.2) on a_i, b_j.

Lemma 4.16. *If* (4.2) *is of either of the following forms, then* $\mathcal{C}(G, u)$ *consists of only one class which is of type D*:

(a) $W(1)^{a_1} \oplus W(2)^{a_2}$ *or* $W(2)^{a_2}$, $a_2 > 1$.
(b) $W(1)^a \oplus W(2)^{a_2} \oplus V(2)^{b_1}$, $0 \le a$.

Table 4.

(4.2) $\supseteq$	k_j	m_i	q	Criterion
$V(2k_j)$	>1	—	>2	F, Lemma 4.11(a)
$W(m_i)$	—	>4	>4	F, Lemma 4.11(b)
		>4 even	all	F, Lemma 4.11(b)
		>4 odd	4	F or D, Lemma 4.12
		4	all	D, Lemma 4.10
		3	>8	F, Lemma 4.11(b)
			8, 4	F or D, Lemma 4.12
		>1 odd	2	D, Lemma 4.14
$V(2k_i) \oplus V(2k_j)$			>2	D, Lemma 4.13(a)
	$k_ik_j > 1$		2	
$W(m_i) \oplus V(2k_j)$		>2	all	
		2	>2	D, Lemma 4.13(b)
	>1	2	2	
$W(m_i) \oplus W(m_j)$		$m_i, m_j > 1$	>2	
		$m_i > 1$, $m_j > 2$	2	D, Lemma 4.13(c)
		$m_i > 2$, $m_j > 1$	2	

Proof. In all cases $\mathcal{C}(G,u)$ has only one class $\mathcal{O}$ by [8, Theorem 6.21].

(a): Let $x = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Let v be the block diagonal matrix

$$(x \otimes \mathrm{id}_{a_2}, \mathrm{id}_{2a_1}, \mathrm{J}_2{}^t x^{-1} \mathrm{J}_2 \otimes \mathrm{id}_{a_2});$$

then $v \in \mathcal{O}$ because its decomposition (4.2) is $W(1)^{a_1} \oplus W(2)^{a_2}$. Now v lies in the subgroup $H \simeq \mathbf{SL}_n(q)$ of matrices $\left(\begin{smallmatrix} y & 0 \\ 0 & \mathrm{J}_n\, {}^t y^{-1} \mathrm{J}_n \end{smallmatrix}\right)$, with $y \in \mathbf{SL}_n(q)$. If $a_2 > 1$, then $\mathcal{O}$ contains a subrack isomorphic to a unipotent class of type $(2, \ldots, 2)$ (a_2 times) in $\mathbf{SL}_{2a_2}(q)$, and Lemma 3.12 applies.

(b): It is enough to consider $W(2) \oplus V(2)$. Let

$$v = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix} = x_{\alpha_2}(1) x_{2\alpha_1 + 2\alpha_2 + \alpha_3}(1) \in \mathcal{O}.$$

We set

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 1 & 0 & 0 & & & \\ & & & 0 & 0 & 1 \\ & & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 1 & 1 & 0 \\ & & 1 & 0 & 1 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix},$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 1 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix},$$

and

$$r = (z\sigma) \triangleright v = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad s = y \triangleright v = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The discussion in [1, 3.1] implies that $\mathcal{O}_r^{\mathbb{U}^F} \neq \mathcal{O}_s^{\mathbb{U}^F}$. A direct computation shows that $(rs)^2 \neq (sr)^2$. $\square$

Lemma 4.17. *If* (4.2) *is equal to* $W(1)^{a_1} \oplus W(2)$, $a_1 > 1$ *then* $\mathcal{C}(G,u)$ *consists of only one class which is of type F.*

Proof. In all cases $\mathcal{C}(G,u)$ has only one class $\mathcal{O}$ by [8, Theorem 6.21]. It is enough to prove the statement for $a_1 = 2$. Let x and v be as in Lemma 4.16(a). Then $v \in \mathcal{O}$ because it has decomposition (4.2) equal to $W(2) \oplus W(1)^2$. We consider the following elements of G

$$\sigma = \begin{pmatrix} 0&0&1&0&&&& \\ 1&0&0&0&&&& \\ 0&1&0&0&&&& \\ 0&0&0&1&&&& \\ &&&&1&0&0&0 \\ &&&&0&0&1&0 \\ &&&&0&0&0&1 \\ &&&&0&1&0&0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1&0&0&0&&&& \\ 0&0&0&1&&&& \\ 0&1&0&0&&&& \\ 0&0&1&0&&&& \\ &&&&0&1&0&0 \\ &&&&0&0&1&0 \\ &&&&1&0&0&0 \\ &&&&0&0&0&1 \end{pmatrix},$$

$$\omega = \begin{pmatrix} \mathrm{id}_2 &&& \\ & 0 & \mathrm{id}_2 & \\ & \mathrm{id}_2 & 0 & \\ &&& \mathrm{id}_2 \end{pmatrix}$$

and the following elements of $\mathcal{O}$:

$r_1 = v, \quad r_2 = \sigma \triangleright r_1 = (1, x, \mathrm{id}_2, x, 1),$

$$r = \tau \triangleright r_2 = (\mathrm{id}_2, x \otimes \mathrm{id}_2, \mathrm{id}_2), \quad r_3 = (r_2\tau) \triangleright r_2 = \begin{pmatrix} 1&0&0&0&&&& \\ 0&1&0&1&&&& \\ 0&0&1&1&&&& \\ 0&0&0&1&&&& \\ &&&&1&1&1&0 \\ &&&&0&1&0&0 \\ &&&&0&0&1&0 \\ &&&&0&0&0&1 \end{pmatrix},$$

$$r_4 = (r_1 r \omega) \triangleright r_2 = \begin{pmatrix} 1&0&0&0&1&1&0&0 \\ 0&1&0&0&1&1&0&0 \\ 0&0&1&0&0&0&1&1 \\ 0&0&0&1&0&0&1&1 \\ &&&&1&0&0&0 \\ &&&&0&1&0&0 \\ &&&&0&0&1&0 \\ &&&&0&0&0&1 \end{pmatrix}.$$

Then, $H := \langle r_1, r_2, r_3, r_4\rangle \subset \mathbb{U}^F$ and $\mathcal{O}_{r_i}^{\mathbb{U}^F} \neq \mathcal{O}_{r_j}^{\mathbb{U}^F}$, for $i \neq j$, hence $\mathcal{O}_{r_i}^{H} \neq \mathcal{O}_{r_j}^{H}$. A direct computation shows that $r_i \triangleright r_j \neq r_j$ for $i \neq j$. □

Lemma 4.18. *Assume $q > 2$. If* (4.2) *is equal to $W(1) \oplus W(2)$, then $\mathcal{C}(G, u)$ consists of only one class which is of type F.*

Proof. There is only one class $\mathcal{O}$ with (4.2) equal to $W(2) \oplus W(1)$, which is represented by $r_1 = \mathrm{id}_6 + (e_{2,3} + e_{4,5})$, [8, Theorem 6.21].

Let $\zeta \in F_q^{\times}\backslash 1$ and let us consider the following elements of G:

$$s_1 := \begin{pmatrix} 0 & 1 & 0 & & & \\ 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & 1 & 0 & 0 \\ & & & 0 & 0 & 1 \\ & & & 0 & 1 & 0 \end{pmatrix}, \quad s_2 := \begin{pmatrix} 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 1 & 0 & & & \\ & & & 0 & 1 & 0 \\ & & & 1 & 0 & 0 \\ & & & 0 & 0 & 1 \end{pmatrix},$$

$$s_3 := \begin{pmatrix} \mathrm{id}_2 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \mathrm{id}_2 \end{pmatrix}.$$

We construct the following elements in $\mathcal{O}$:

$$r_2 = (r_1 s_2 s_1) \triangleright r_1 = \mathrm{id}_6 + (e_{1,2} + e_{5,6}) + (e_{1,3} + e_{4,6}),$$

$$r_3 = ((\mathrm{id}_6 + (e_{2,1} + e_{6,5}))s_3 s_1) \triangleright r_1 = \mathrm{id}_6 + (e_{1,4} + e_{3,6}) + (e_{2,4} + e_{3,5}),$$

$$r_4 = (\mathrm{diag}(1, \zeta, 1, 1, \zeta^{-1}, 1)(\mathrm{id}_6 + e_{3,4})(\mathrm{id}_6 + e_{1,2} + e_{5,6})) \triangleright r_1$$

$$= \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ & 1 & \zeta & \zeta & 0 & 0 \\ & & 1 & 0 & \zeta & 1 \\ & & & 1 & \zeta & 1 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}.$$

A direct computation shows that $r_i \triangleright r_j \neq r_j$ for $i \neq j$. Moreover, as $H := \langle r_1, r_2, r_3, r_4\rangle \subset \mathbb{U}^F \subset \mathbf{SL}_6(q)$, the usual argument shows that $\mathcal{O}_{r_i}^{H} \neq \mathcal{O}_{r_j}^{H}$ for $i \neq j$. □

Lemma 4.19. *If $q = 2$ and* (4.2) *is of the form $W(1)^a \oplus V(4)$, $0 \leq a$, then $\mathcal{O}$ is of type D.*

Proof. There are two classes like this [8, Theorems 6.6, 6.12]; both contain a subrack isomorphic to one of the regular classes in $\mathbf{Sp}_4(2) \simeq \mathbb{S}_6$, which corresponds either to the partition $(4, 2)$ or else to $(4, 1^2)$. These are of type D by [2, 4.1]. $\square$

Lemma 4.20. *If $q = 2$ and (4.2) contains the term $V(2k)$, $k \geq 3$, then $\mathcal{O}$ is of type D.*

Proof. By Corollary 4.7, $\mathcal{O}$ contains a subrack isomorphic to one of the two regular unipotent classes in $\mathbf{Sp}_{2k}(2)$. So, we may assume $k = n$ for notational purposes. For both classes, there is a representative v lying in $x_{\alpha_1}(1) \cdots x_{\alpha_n}(1)\mathbb{U}'$, $\mathbb{U}'$ as in Proposition 3.7. Let $\mathbb{P}$ be the standard parabolic subgroup of $\mathbb{G}$ corresponding to the simple roots α_{n-1} and α_n, and let $\mathbb{L}$ be its standard Levi subgroup, whose derived subgroup is isomorphic to $\mathbf{Sp}_4(\Bbbk)$. Recall the notation in Remark 3.13. Then $v \in \mathbb{U}^F < \mathbb{P}^F$ and if $v = v_L v_P$ is its decomposition according to $\mathbb{P}^F = \mathbb{L}^F \ltimes \mathsf{Q}^F$, then u_L is regular unipotent in $\mathbb{L}^F$. The result follows from Lemmas 2.2 and 4.19. $\square$

Lemma 4.21. *If $q = 2$ and (4.2) is of the form $W(1)^{a_1} \oplus V(2)^2$, then $\mathcal{O}$ is of type D.*

Proof. There is only one class in $\mathcal{C}(G, u)$ by [8, Theorem 6.21]. We may assume $a_1 = 1$, $n = 3$ as in the proof of Lemma 4.16(a). Then $\mathcal{O}$ is represented by $xy = x_{2(\alpha_1+\alpha_2)+\alpha_3}(1)x_{2\alpha_2+\alpha_3}(1)$. It contains the subrack $X \times Y$ for $X = \mathcal{O}_x^H$, $Y = \mathcal{O}_y^K$ with $H \simeq \mathbf{SL}_2(2)$ being the subgroup corresponding to the root $2(\alpha_1 + \alpha_2) + \alpha_3$ and $K \simeq \mathbf{Sp}_4(2) \simeq \mathbb{S}_6$ the subgroup corresponding to the roots α_2 and α_3. Since all conjugacy classes of involutions in $\mathbb{S}_6$ contain distinct commuting elements, we apply Lemmas 2.3 and 4.8. $\square$

4.3. *Proof of Theorem* 1.1

We first study classes that do not collapse.

Lemma 4.22. *Let $u \in G$ unipotent with partition $(1^{2n-2}, 2)$.*

(1) *If q is odd and either not a square or 9, then $\mathcal{C}(G, u)$ consists of two cthulhu classes.*
(2) *If q is even, then $\mathcal{C}(G, u)$ consists of a unique cthulhu class.*

Proof. If q is even, the decomposition (4.2) of any element with Jordan form $(1^{2n-2}, 2)$, is necessarily $W(1)^{2n-2} \oplus V(2)$. Thus we have only one conjugacy class in $\mathbb{G}$ with this form.

For any q, we fix $u = \begin{pmatrix} 1 & 0 & 1 \\ 0 & \mathrm{id}_{2n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = x_\beta(1)$, where $\beta \in \Phi^+$ is the highest root. If q is even, $\mathcal{C}(G, u)$ has a unique class by [8, Theorems 6.6, 6.12]. If q is odd, then, by [8, Theorem 3.1(v)] and arguing as in Lemma 4.3, $\mathcal{C}(G, u)$ consists of two classes represented by u and $\begin{pmatrix} 1 & 0 & \zeta \\ 0 & \mathrm{id}_{2n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = x_\beta(\zeta)$, for $\zeta \in \mathbb{F}_q^\times$ not a square. We show (for any q) that every subrack of $\mathcal{O}_u^G$ generated by two elements is either abelian or indecomposable, implying that $\mathcal{O}_u^G$ is cthulhu. The same argument applies to the other class, when q is odd.

Assume there is $g \in G$ such that $v = gug^{-1} \in \mathcal{O}_u^G$ and $uv \neq vu$. We claim that the rack generated by u and v is indecomposable. Consider the Bruhat decomposition $g = ytn_w z$, with $y, z \in \mathbb{U}^F$, $t \in \mathbb{T}^F$ and $n_w \in N_G(\mathbb{T})$ with class $w \in W$. By (3.2), $u \in Z(\mathbb{U}^F)$, so that $v = huh^{-1}$ with $h = ytn_w$. Now the subrack generated by u and v is isomorphic to the subrack generated by $u = y^{-1}uy$ and $y^{-1}vy = kuk^{-1}$ with $k = tn_w$, so we may assume that $v = kuk^{-1}$. Now a direct computation gives $v = x_{w\beta}(\eta)$ for some $\eta \in \mathbb{F}_q^\times$ [9, Theorems 24.10; 8.17(e)]. The assumption $uv \neq vu$ forces $w\beta \in -\Phi^+$ and $w\beta + \beta \in \Phi \cup \{0\}$. As the root system is of type C_n, this is possible only if $w\beta = -\beta$. An element in $N_G(\mathbb{T})$ mapping $u = x_\beta(1)$ to $v = x_{-\beta}(\eta)$ is of the form

$$\begin{pmatrix} 0 & 0 & \xi \\ 0 & X & 0 \\ -\xi^{-1} & 0 & 0 \end{pmatrix} = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \mathrm{id}_{2n-2} & 0 \\ 0 & 0 & \xi^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{id}_{2n-2} & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \beta^\vee(\xi) n_\beta Y,$$

for $X \in \mathbf{Sp}_{2n-2}(q)$ and $\xi \in \mathbb{F}_q^\times$. Hence $\eta = -\xi^{-2}$ and Y commutes with u.

Let

$$H = \left\{ \begin{pmatrix} a & & b \\ & \mathrm{id}_{2n-2} & \\ c & & d \end{pmatrix} \in G : ad - bc = 1 \right\} \simeq \mathbf{SL}_2(q).$$

Then $u, v, \beta^\vee(\xi)$, $n_\beta \in H$ and $v \in \mathcal{O}_u^H$. By [1, Lemma 3.5], $\mathcal{O}_u^H$ is sober, hence the rack generated by u and v is indecomposable. $\square$

Remark 4.23. Let q be odd, $u = x_\beta(1)$, $\mathsf{Sp}_n = \mathcal{O}_u^G$ and let $u' = x_\beta(\zeta)$, for $\zeta \in \mathbb{F}_q^\times$ not a square. Then $\mathsf{Sp}_n \simeq \mathcal{O}_{u'}^G$ as racks because the outer automorphism of $G = \mathbf{Sp}_{2n}(q)$ given by conjugation by the matrix $\mathrm{diag}(\mathrm{id}_n, \zeta^{-1}\mathrm{id}_n)$ maps u to u'. Thus for q even or $q = 9$, or q odd and not a square, we have a family of cthulhu racks $(\mathsf{Sp}_n)_{n\in\mathbb{N}}$, with Sp_1 the sober rack $\mathcal{O}_x^{\mathbf{SL}_2(q)}$ with x non-trivial unipotent. Note that $\mathsf{Sp}_n \subset \mathsf{Sp}_{n+1}$ and

$$|\mathsf{Sp}_n| = \begin{cases} \dfrac{(q^{2n}-1)}{2} & \text{if } q \text{ is odd,} \\ (q^{2n}-1) & \text{if } q \text{ is even.} \end{cases} \tag{4.8}$$

Lemma 4.24. *If $q = 2$ and (4.2) is of the form $V(2)^2$, then $\mathcal{O}$ is cthulhu.*

Proof. There is only one class in $\mathcal{C}(G,u)$ by [8, Theorem 6.21]. Here $\mathbb{G}^F = \mathbf{Sp}_4(2) \simeq \mathbb{S}_6$ and $\mathcal{O}$ corresponds to the partition $(1^2, 2^2)$. By [2, Remark 4.2(e)], $\mathcal{O}$ is not of type D. We will show that it cannot be of type F either. For $i \in \mathbb{I}_4$, let $r_i \in \mathcal{O}$ with $[r_i, r_j] \neq 1$ and $\mathcal{O}_{r_i}^{\langle r_i, r_j\rangle} \neq \mathcal{O}_{r_j}^{\langle r_i, r_j\rangle}$ for $i \neq j$. Then for every $i \neq j$, the permutations r_i and r_j may not have a 2-cycle in common, and $\langle r_i, r_j\rangle$ cannot be contained in a standard subgroup isomorphic to $\mathbb{S}_4, \mathbb{S}_5$, or $\mathbb{S}_3 \times \mathbb{S}_3$. If $r_4 = (12)(34)$, then for $i \in \mathbb{I}_3$ we necessarily have r_i either in $A = \{(13)(56), (14)(56), (23)(56), (24)(56)\}$ or in $B = \{(15)(26), (16)(25), (35)(46), (45)(36)\}$. However, if $r_2 \in A$, respectively B, then r_3, r_4 must lie in B, respectively A, leading to a contradiction. $\square$

Lemma 4.25. *Assume q is even. If (4.2) is equal to $W(1)^{a_1} \oplus W(2)$, then $\mathcal{C}(G,u)$ consists of only one class $\mathcal{O}$ which is not of type D. If $a_1 = 1$ and $q = 2$, then $\mathcal{O}$ is cthulhu.*

Proof. $\mathcal{C}(G,u) = \{\mathcal{O}\}$ by [8, Theorem 6.21]. We shall prove that for any two elements r, $s \in \mathcal{O}$ such that $(rs)^2 \neq (sr)^2$, it holds $\mathcal{O}_r^{\langle r,s\rangle} = \mathcal{O}_s^{\langle r,s\rangle}$. Let $\gamma = \varepsilon_1 + \varepsilon_2$ be the highest short root in the root system of $\mathbb{G}$. The class $\mathcal{O}$ is represented by $r = x_\gamma(1) = \mathrm{id}_{2n} + e_{1,2n-1} + e_{2,2n}$, which is central in $\mathbb{U}^F$ by (3.2) and Table 2. Let $s = g \triangleright r \in \mathcal{O}$ satisfy $(sr)^2 \neq (rs)^2$ and let $g = u\dot{w}v \in \mathbb{U}^F N_G(\mathbb{T})\mathbb{U}^F$ be the Bruhat decomposition of g. Then $s = (u\dot{w}) \triangleright r = u \triangleright x_{w(\gamma)}(\eta)$ for some $\eta \in \mathbb{F}_q^\times$. Conjugating by u^{-1} we may assume $s = x_{w(\gamma)}(\eta)$. Now, as $sr \neq rs$, we necessarily have $w(\gamma) \in \{-\gamma, -\varepsilon_1 \pm \varepsilon_k, -\varepsilon_2 \pm \varepsilon_k, k \neq 1,2\}$. We claim that $w(\gamma) = -\gamma$. Assume indeed that $w(\gamma) \in \{-\varepsilon_1 \pm \varepsilon_k, -\varepsilon_2 \pm \varepsilon_k, k \neq 1,2\}$. By (3.2), we have $rsrs \in \mathbb{U}_{\gamma + w(\gamma)}$, so it is an involution, leading to a contradiction. Thus,

$$H := \langle r, s\rangle \simeq \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \eta & 1 \end{pmatrix} \right\rangle \leq \mathbf{SL}_2(q).$$

Since the non-trivial unipotent rack in $\mathbf{SL}_2(q)$ is sober, we have the first statement. The second one follows from a computation with GAP. $\square$

Lemma 4.26. *Let $G = \mathbf{Sp}_4(q)$ for q even and let $\mathcal{O}_u^G$ be a class corresponding to $W(2)$. Then $\mathcal{C}(G,u)$ contains a unique class which is cthulhu.*

Proof. The root system of $\mathbb{G} = \mathbf{Sp}_4(\Bbbk)$ is of type C_2, so there exists a non-standard graph automorphism θ interchanging long and short roots [5, 12.1], commuting with F. Thus, θ induces an automorphism on $\mathbb{G}^F$ mapping the class of type $W(1)^2 \oplus V(2)$, represented by $x_{\alpha_1}(1)$, onto the class of type $W(2)$, represented by $x_{\alpha_2}(1)$. The claim follows from Lemma 4.22(2). $\square$

We next show that the classes not listed in Table 1 collapse. Let $\mathcal{O}$ be a unipotent class in G. We summarize in Table 5 the results in Sec. 4.1 proving the claim for q odd.

Table 5.

q, n	Type $(1^{r_1}, 2^{r_2}, \dots, n^{r_n})$	Criterion
	$\exists i > 3 : r_i \neq 0$	type D, Lemma 4.1
> 9 square	$(1^{r_1}, 2^{r_2})$, $r_2 > 0$	type D, Lemma 4.1
$q > 3$, or $n > 2$	$(1^{r_1}, 2^{r_2})$, $r_2 > 1$	type D, Lemma 4.3
	$(1^{r_1}, 3^{r_3})$, $r_3 > 0$	type D, Lemma 4.2
	$(1^{r_1}, 2^{r_2}, 3^{r_3})$, $r_2 r_3 > 0$	type D, Lemma 4.4
3	(2^2)	one of type D, Lemma 4.5

Assume that q is even. We show how the results in Sec. 4.2 imply the claim. By Remark 4.15 we may assume that (4.2) has the form (4.7). For all q even, we have

- ◇ $V = W(2)$: cthulhu, Lemma 4.26.
- ◇ $V = W(1)^a \oplus V(2)$, $0 \leq a$: cthulhu, Lemma 4.22(2).

Case $2 < q$ *even*. The remaining cases are disposed as follows:

- ◇ $V = W(1)^a \oplus W(2)$, $1 \leq a$: type F, Lemmas 4.17 and 4.18.

Case $q = 2$. Here we invoke the following statements:

- ◇ $V = W(1)^a \oplus W(2)^b \oplus V(2)^c$, $0 < bc$: type D, Lemma 4.16(b).
- ◇ $V = W(1)^a \oplus W(2)^b$, $1 < ab$; or $W(2)^b$, $1 < b$: type D or F, Lemmas 4.16(a), 4.17.
- ◇ $V = W(1)^a \oplus V(2)^2$, $0 < a$: type D, Lemma 4.21.
- ◇ $V = V(2)^2$: cthulhu, Lemma 4.24.
- ◇ $V = W(1)^a \oplus V(2k)$, $0 \leq a$, $k \geq 2$: type D, Lemmas 4.19 and 4.20.
- ◇ $V = W(2) \oplus W(1)$, cthulhu, Lemma 4.25.

Acknowledgments

This work was partially supported by ANPCyT-Foncyt, CONICET, Secyt (UNC), the Grant CPDA125818/12 of the University of Padova, GNSAGA, the Visiting Scientist Program of the University of Padova and the bilateral agreement between the Universities of Córdoba and Padova.

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