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over a finite field, I [☆]Eda Cesaratto ^{a,b}, Guillermo Matera ^{a,b}, Mariana Pérez ^a,
Melina Privitelli ^{b,c}^a Instituto del Desarrollo Humano, Universidad Nacional de General Sarmiento,
J.M. Gutiérrez 1150 (B1613GSX), Los Polvorines, Buenos Aires, Argentina^b National Council of Science and Technology (CONICET), Argentina^c Instituto de Ciencias, Universidad Nacional de General Sarmiento,
J.M. Gutiérrez 1150 (B1613GSX), Los Polvorines, Buenos Aires, Argentina

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ABSTRACT

We obtain an estimate on the average cardinality of the value set of any family of monic polynomials of $\mathbb{F}_q[T]$ of degree d for which s consecutive coefficients a_{d-1}, \dots, a_{d-s} are fixed. Our estimate holds without restrictions on the characteristic of \mathbb{F}_q and asserts that $\mathcal{V}(d, s, \mathbf{a}) = \mu_d q + \mathcal{O}(1)$, where $\mathcal{V}(d, s, \mathbf{a})$ is such an average cardinality, $\mu_d := \sum_{r=1}^d (-1)^{r-1}/r!$ and $\mathbf{a} := (a_{d-1}, \dots, a_{d-s})$. We provide an explicit upper bound for the constant underlying the \mathcal{O} -notation in terms of d and s with “good” behavior. Our approach reduces the question to estimate the number of \mathbb{F}_q -rational points with pairwise-distinct coordinates of a certain family of complete intersections defined over \mathbb{F}_q . We show that the polynomials defining such complete intersections are invariant under the action of the symmetric group of permutations of the coordinates. This allows us to obtain critical information concerning the singular locus of the varieties under consideration, from which

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E-mail addresses: ecesarat@ungs.edu.ar (E. Cesaratto), gmatera@ungs.edu.ar (G. Matera), vperez@ungs.edu.ar (M. Pérez), mprivitelli@conicet.gov.ar (M. Privitelli).

a suitable estimate on the number of \mathbb{F}_q -rational points is established.

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1. Introduction

Let \mathbb{F}_q be the finite field of q elements, let T be an indeterminate over \mathbb{F}_q and let $f \in \mathbb{F}_q[T]$. We define the value set $\mathcal{V}(f)$ of f as $\mathcal{V}(f) := |\{f(c) : c \in \mathbb{F}_q\}|$ (cf. [20]). Birch and Swinnerton-Dyer established the following significant result [2]: for fixed $d \geq 1$, if f is a generic polynomial of degree d , then

$$\mathcal{V}(f) = \mu_d q + \mathcal{O}(q^{1/2}),$$

where $\mu_d := \sum_{r=1}^d (-1)^{r-1} / r!$ and the constant underlying the \mathcal{O} -notation depends only on d .

Results on the average value $\mathcal{V}(d, 0)$ of $\mathcal{V}(f)$ when f ranges over all monic polynomials in $\mathbb{F}_q[T]$ of degree d with $f(0) = 0$ were obtained by Uchiyama [24] and improved by Cohen [9]. More precisely, in [9, §2] it is shown that

$$\mathcal{V}(d, 0) = \sum_{r=1}^d (-1)^{r-1} \binom{q}{r} q^{1-r} = \mu_d q + \mathcal{O}(1).$$

However, if some of the coefficients of f are fixed, the results on the average value of $\mathcal{V}(f)$ are less precise. In fact, Uchiyama [25] and Cohen [8] obtained the following result. Given s with $1 \leq s \leq d-2$ and $\mathbf{a} := (a_{d-1}, \dots, a_{d-s}) \in \mathbb{F}_q^s$, for every $\mathbf{b} := (b_{d-s-1}, \dots, b_1)$ we denote

$$f_{\mathbf{b}} := f_{\mathbf{b}}^{\mathbf{a}} := T^d + \sum_{i=1}^s a_{d-i} T^{d-i} + \sum_{i=s+1}^{d-1} b_{d-i} T^{d-i}.$$

Then for $p := \text{char}(\mathbb{F}_q) > d$,

$$\mathcal{V}(d, s, \mathbf{a}) := \frac{1}{q^{d-s-1}} \sum_{\mathbf{b} \in \mathbb{F}_q^{d-s-1}} \mathcal{V}(f_{\mathbf{b}}) = \mu_d q + \mathcal{O}(q^{1/2}), \tag{1.1}$$

where the constant underlying the \mathcal{O} -notation depends only on d and s .

This paper is devoted to obtain a strengthened explicit version of (1.1), which holds without any restriction on p . More precisely, we shall show the following result (see Theorem 4.4 below).

Theorem 1.1. *With notations as above, for $q > d$ and $1 \leq s \leq \frac{d}{2} - 1$ we have*

$$\left| \mathcal{V}(d, s, \mathbf{a}) - \mu_d q - \frac{1}{2e} \right| \leq \frac{(d-2)^5 e^{2\sqrt{d}}}{2^{d-2}} + \frac{7}{q}.$$

This result strengthens (1.1) in several aspects. The first one is that it holds without any restriction on the characteristic p of \mathbb{F}_q , while (1.1) holds for $p > d$. The second aspect is that we show that $\mathcal{V}(d, s, \mathbf{a}) = \mu_d q + \mathcal{O}(1)$, while (1.1) only asserts that $\mathcal{V}(d, s, \mathbf{a}) = \mu_d q + \mathcal{O}(q^{1/2})$. Finally, we obtain an explicit expression for the constant underlying the \mathcal{O} -notation with a good behavior, in the sense that we prove that $\mathcal{V}(d, s, \mathbf{a}) = \mu_d q + \frac{1}{2e} + \mathcal{O}(\rho^{-d}) + \mathcal{O}(q^{-1})$ for any ρ with $\frac{1}{2} < \rho < 1$.

On the other hand, it must be said that our result holds for $s \leq d/2 - 1$, while (1.1) holds for s varying in a larger range of values. This aspect shall be addressed in the second paper, where we obtain an explicit estimate showing that $\mathcal{V}(d, s, \mathbf{a}) = \mu_d q + \mathcal{O}(q^{1/2})$ which is valid for $1 \leq s \leq d - 4$ and $p > 2$. We shall also exhibit estimates on the second moment of the value set of the families of polynomials under consideration.

In order to obtain our estimate, we express the quantity $\mathcal{V}(d, s, \mathbf{a})$ in terms of the number $\chi_r^\mathbf{a}$ of certain “interpolating sets” with $d - s + 1 \leq r \leq d$ (see Theorem 2.1 below). More precisely, for $f^\mathbf{a} := T^d + a_{d-1}T^{d-1} + \dots + a_{d-s}T^{d-s}$, we define $\chi_r^\mathbf{a}$ as the number of r -element subsets of \mathbb{F}_q at which $f^\mathbf{a}$ can be interpolated by a polynomial of degree at most $d - s - 1$.

Then we express $\chi_r^\mathbf{a}$ in terms of the number of q -rational solutions with pairwise-distinct coordinates of a polynomial system $\{R_{d-s}^\mathbf{a} = 0, \dots, R_{r-1}^\mathbf{a} = 0\}$, where $R_{d-s}^\mathbf{a}, \dots, R_{r-1}^\mathbf{a}$ are certain polynomials in $\mathbb{F}_q[X_1, \dots, X_r]$. A critical point for our approach is that $R_{d-s}^\mathbf{a}, \dots, R_{r-1}^\mathbf{a}$ are symmetric polynomials, namely invariant under any permutation of the variables X_1, \dots, X_r . More precisely, we prove that each $R_j^\mathbf{a}$ can be expressed as a polynomial in the first s elementary symmetric polynomials of $\mathbb{F}_q[X_1, \dots, X_r]$ (Proposition 2.4). This allows us to establish a number of facts concerning the geometry of the set $V_r^\mathbf{a}$ of solutions of such a polynomial system (see, e.g., Corollary 3.4 and Theorems 3.6 and 3.7). Combining these results with estimates on the number of \mathbb{F}_q -rational points of singular complete intersections of [5], we obtain our main result.

We finish this introduction by stressing on the methodological aspects. As mentioned before, a key point is the invariance of the family of sets $V_r^\mathbf{a}$ under the action of the symmetric group of r elements. In fact, our results on the geometry of $V_r^\mathbf{a}$ and the estimates on the number of \mathbb{F}_q -rational points can be extended *mutatis mutandis* to any symmetric complete intersection whose projection on the set of primary invariants (using the terminology of invariant theory) defines a nonsingular complete intersection. This might be seen as a further source of interest of our approach, since symmetric polynomials arise frequently in combinatorics, coding theory and cryptography (for example, in the study of deep holes in Reed–Solomon codes, almost perfect nonlinear polynomials or differentially uniform mappings; see, e.g., [1,4,22]).

2. Value sets in terms of interpolating sets

Let notations and assumptions be as in the previous section. In this section we fix s with $1 \leq s \leq d - 2$, an s -tuple $\mathbf{a} := (a_{d-1}, \dots, a_{d-s}) \in \mathbb{F}_q^s$ and denote

$$f^{\mathbf{a}} := T^d + a_{d-1}T^{d-1} + \dots + a_{d-s}T^{d-s}.$$

For every $\mathbf{b} := (b_{d-s-1}, \dots, b_1) \in \mathbb{F}_q^{d-s-1}$, we denote by $f_{\mathbf{b}} := f_{\mathbf{b}}^{\mathbf{a}} \in \mathbb{F}_q[T]$ the following polynomial

$$f_{\mathbf{b}} := f^{\mathbf{a}} + b_{d-s-1}T^{d-s-1} + \dots + b_1T.$$

For a given $\mathbf{b} \in \mathbb{F}_q^{d-s-1}$, the value set $\mathcal{V}(f_{\mathbf{b}})$ of $f_{\mathbf{b}}$ equals the number of elements $b_0 \in \mathbb{F}_q$ for which the polynomial $f_{\mathbf{b}} + b_0$ has at least one root in \mathbb{F}_q . Let $\mathbb{F}_q[T]_d$ denote the set of polynomials of $\mathbb{F}_q[T]$ of degree at most d , let $\mathcal{N} : \mathbb{F}_q[T]_d \rightarrow \mathbb{Z}_{\geq 0}$ be the counting function of the number of roots in \mathbb{F}_q and let $\mathbf{1}_{\{\mathcal{N} > 0\}} : \mathbb{F}_q[T]_d \rightarrow \{0, 1\}$ be the characteristic function of the set of elements of $\mathbb{F}_q[T]_d$ having at least one root in \mathbb{F}_q . From our previous assertion we deduce the following identity:

$$\begin{aligned} \sum_{\mathbf{b} \in \mathbb{F}_q^{d-s-1}} \mathcal{V}(f_{\mathbf{b}}) &= \sum_{b_0 \in \mathbb{F}_q} \sum_{\mathbf{b} \in \mathbb{F}_q^{d-s-1}} \mathbf{1}_{\{\mathcal{N} > 0\}}(f_{\mathbf{b}} + b_0) \\ &= |\{g \in \mathbb{F}_q[T]_{d-s-1} : \mathcal{N}(f^{\mathbf{a}} + g) > 0\}|. \end{aligned}$$

For a set $\mathcal{X} \subset \mathbb{F}_q$, we define $\mathcal{S}_{\mathcal{X}}^{\mathbf{a}} \subset \mathbb{F}_q[T]_{d-s-1}$ as the set of polynomials of $\mathbb{F}_q[T]$ of degree at most $d - s - 1$ which interpolate $-f^{\mathbf{a}}$ at all the points of \mathcal{X} , namely

$$\mathcal{S}_{\mathcal{X}}^{\mathbf{a}} := \{g \in \mathbb{F}_q[T]_{d-s-1} : (f^{\mathbf{a}} + g)(x) = 0 \text{ for any } x \in \mathcal{X}\}.$$

Finally, for $r \in \mathbb{N}$ we shall use the symbol \mathcal{X}_r to denote a subset of \mathbb{F}_q of r elements.

Theorem 2.1. For given $s, d \in \mathbb{N}$ with $d < q$ and $1 \leq s \leq d - 2$, we have

$$\mathcal{V}(d, s, \mathbf{a}) = \sum_{r=1}^{d-s} (-1)^{r-1} \binom{q}{r} q^{1-r} + \frac{1}{q^{d-s-1}} \sum_{r=d-s+1}^d (-1)^{r-1} \chi_r^{\mathbf{a}}, \quad (2.1)$$

where $\mathcal{V}(d, s, \mathbf{a})$ is defined as in (1.1) and $\chi_r^{\mathbf{a}}$ is the number of subsets \mathcal{X}_r of \mathbb{F}_q of r elements such that there exists $g \in \mathbb{F}_q[T]_{d-s-1}$ with $(f^{\mathbf{a}} + g)|_{\mathcal{X}_r} \equiv 0$.

Proof. Given a subset $\mathcal{X}_r := \{x_1, \dots, x_r\} \subset \mathbb{F}_q$, we consider the corresponding set $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}} \subset \mathbb{F}_q[T]_{d-s-1}$ defined as above. It is easy to see that $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}} = \bigcap_{i=1}^r \mathcal{S}_{\{x_i\}}^{\mathbf{a}}$ and

$$\{g \in \mathbb{F}_q[T]_{d-s-1} : \mathcal{N}(f^{\mathbf{a}} + g) > 0\} = \bigcup_{x \in \mathbb{F}_q} \mathcal{S}_{\{x\}}^{\mathbf{a}}.$$

Therefore, by the inclusion–exclusion principle we obtain

$$\mathcal{V}(d, s, \mathbf{a}) = \frac{1}{q^{d-s-1}} \left| \bigcup_{x \in \mathbb{F}_q} \mathcal{S}_{\{x\}}^{\mathbf{a}} \right| = \frac{1}{q^{d-s-1}} \sum_{r=1}^q (-1)^{r-1} \sum_{\mathcal{X}_r \subset \mathbb{F}_q} |\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}|. \quad (2.2)$$

Now we estimate $|\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}|$ for a given set $\mathcal{X}_r := \{x_1, \dots, x_r\} \subset \mathbb{F}_q$. Let $g := b_{d-s-1}T^{d-s-1} + \dots + b_1T + b_0$ be an arbitrary element of $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}$. Then we have $f^{\mathbf{a}}(x_i) + g(x_i) = 0$ for $1 \leq i \leq r$. These identities can be expressed in matrix form as follows:

$$\mathcal{M}(\mathcal{X}_r) \cdot \mathbf{b}_0 + f^{\mathbf{a}}(\mathcal{X}_r) = 0$$

where $\mathcal{M}(\mathcal{X}_r) := (m_{i,j}) \in \mathbb{F}_q^{r \times (d-s)}$ is the Vandermonde matrix defined by $m_{i,j} := x_i^{d-s-j}$ for $1 \leq i \leq r$ and $1 \leq j \leq d-s$, $\mathbf{b}_0 := (b_{d-s-1}, \dots, b_0) \in \mathbb{F}_q^{d-s}$ and $f^{\mathbf{a}}(\mathcal{X}_r) := (f^{\mathbf{a}}(x_1), \dots, f^{\mathbf{a}}(x_r)) \in \mathbb{F}_q^r$.

Since $x_i \neq x_j$ for $i \neq j$, it follows that

$$\text{rank}(\mathcal{M}(\mathcal{X}_r)) = \min\{r, d-s\}. \tag{2.3}$$

We conclude that $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}$ is an \mathbb{F}_q -linear variety and either $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}} = \emptyset$ or

$$\text{rank}(\mathcal{M}(\mathcal{X}_r)) + \dim \mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}} = d-s. \tag{2.4}$$

Suppose first that $r \leq d-s$. Then (2.3) implies $\text{rank}(\mathcal{M}(\mathcal{X}_r)) = r$, and hence, $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}$ is not empty. From (2.4) we infer that $\dim \mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}} = d-s-r$, and thus

$$|\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}| = q^{d-s-r}. \tag{2.5}$$

Next we suppose that $r \geq d-s+1$. On one hand, if $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}$ is nonempty, then (2.4) implies $\dim \mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}} = 0$, and hence $|\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}| = 1$. On the other hand, if $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}$ is empty, then $|\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}| = 0$.

For $r > d$ we have that, if $g \in \mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}$, then $g \in \mathbb{F}_q[T]_{d-s-1}$ and $f^{\mathbf{a}}(x_i) + g(x_i) = 0$ holds for $1 \leq i \leq r$. As a consequence, the (nonzero) polynomial $f^{\mathbf{a}} + g$ has degree d and r different roots, which contradicts the hypothesis $r > d$. We conclude that $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}$ is empty, and thus,

$$|\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}| = 0. \tag{2.6}$$

Finally, for $d-s+1 \leq r \leq d$ any of the cases $|\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}| = 0$ or $|\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}| = 1$ can arise.

Now we are able to obtain the expression for $\mathcal{V}(d, s, \mathbf{a})$ of the statement of the theorem. Indeed, combining (2.2), (2.5) and (2.6) we obtain

$$q^{d-s-1}\mathcal{V}(d, s, \mathbf{a}) = \sum_{r=1}^{d-s} (-1)^{r-1} \binom{q}{r} q^{d-s-r} + \sum_{r=d-s+1}^d (-1)^{r-1} \sum_{\mathcal{X}_r \subset \mathbb{F}_q} |\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}|.$$

From this identity we immediately deduce the statement of the theorem. \square

By definition we have $0 \leq \chi_r^{\mathbf{a}} \leq \binom{q}{r}$. We shall show that $\chi_r^{\mathbf{a}} = \frac{1}{r!}q^{d-s} + \mathcal{O}(q^{d-s-1})$, with an explicit upper bound for the constant underlying the \mathcal{O} -notation in terms of d , s and r (see Theorem 4.1 below).

2.1. An algebraic approach to estimate the number of interpolating sets

According to [Theorem 2.1](#), the asymptotic behavior of $\mathcal{V}(d, s, \mathbf{a})$ is determined by that of $\chi_r^{\mathbf{a}}$ for $d - s + 1 \leq r \leq d$. In order to determine the latter, we follow an approach inspired in [\[7\]](#), and further developed in [\[4\]](#), which we now describe.

Fix a set $\mathcal{X}_r := \{x_1, \dots, x_r\} \subset \mathbb{F}_q$ of r elements and $g \in \mathbb{F}_q[T]_{d-s-1}$. Then g belongs to $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}$ if and only if $(T - x_1) \cdots (T - x_r)$ divides $f^{\mathbf{a}} + g$ in $\mathbb{F}_q[T]$. Since $\deg g \leq d - s - 1 < r$, we have that the latter is equivalent to the condition that $-g$ is the remainder of the division of $f^{\mathbf{a}}$ by $(T - x_1) \cdots (T - x_r)$. In other words, the set $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}$ is not empty if and only if the remainder of the division of $f^{\mathbf{a}}$ by $(T - x_1) \cdots (T - x_r)$ has degree at most $d - s - 1$.

Let X_1, \dots, X_r be indeterminates over $\overline{\mathbb{F}}_q$, let $\mathbf{X} := (X_1, \dots, X_r)$ and let $Q \in \mathbb{F}_q[\mathbf{X}][T]$ be the polynomial

$$Q = (T - X_1) \cdots (T - X_r).$$

Observe that there exists $R^{\mathbf{a}} \in \mathbb{F}_q[\mathbf{X}][T]$ with $\deg R^{\mathbf{a}} \leq r - 1$ such that the following congruence relation holds:

$$f^{\mathbf{a}} \equiv R^{\mathbf{a}} \pmod{Q}. \tag{2.7}$$

Let $R^{\mathbf{a}} := R_{r-1}^{\mathbf{a}}(\mathbf{X})T^{r-1} + \cdots + R_0^{\mathbf{a}}(\mathbf{X})$. Then $R^{\mathbf{a}}(x_1, \dots, x_r, T) \in \mathbb{F}_q[T]$ is the remainder of the division of $f^{\mathbf{a}}$ by $(T - x_1) \cdots (T - x_r)$. As a consequence, the set $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}$ is not empty if and only if the following identities hold:

$$R_j^{\mathbf{a}}(x_1, \dots, x_r) = 0 \quad (d - s \leq j \leq r - 1). \tag{2.8}$$

On the other hand, suppose that there exists $\mathbf{x} := (x_1, \dots, x_r) \in \mathbb{F}_q^r$ with pairwise-distinct coordinates such that [\(2.8\)](#) holds and set $\mathcal{X}_r := \{x_1, \dots, x_r\}$. Then the remainder of the division of $f^{\mathbf{a}}$ by $Q(\mathbf{x}, T) = (T - x_1) \cdots (T - x_r)$ is a polynomial $R^{\mathbf{a}}(\mathbf{x}, T)$ of degree at most $d - s - 1$. This shows that $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}$ is not empty. We summarize the conclusions of the argumentation above in the following result.

Lemma 2.2. *For $s, d \in \mathbb{N}$ with $1 \leq s \leq d - 2$, let $R_j^{\mathbf{a}}$ ($d - s \leq j \leq r - 1$) be the polynomials of [\(2.8\)](#) and let $\mathcal{X}_r := \{x_1, \dots, x_r\} \subset \mathbb{F}_q$ be a set with r elements. Then $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}$ is not empty if and only if [\(2.8\)](#) holds.*

It follows that the number $\chi_r^{\mathbf{a}}$ of sets $\mathcal{X}_r \subset \mathbb{F}_q$ of r elements such that $\mathcal{S}_{\mathcal{X}_r}^{\mathbf{a}}$ is not empty equals the number of points $\mathbf{x} := (x_1, \dots, x_r) \in \mathbb{F}_q^r$ with pairwise-distinct coordinates satisfying [\(2.8\)](#), up to permutations of coordinates, namely equals $1/r!$ times the number of solutions $\mathbf{x} \in \mathbb{F}_q^r$ of the following system of equalities and non-equalities:

$$R_j^{\mathbf{a}}(X_1, \dots, X_r) = 0 \quad (d - s \leq j \leq r - 1), \quad \prod_{1 \leq i < j \leq r} (X_i - X_j) \neq 0. \tag{2.9}$$

2.2. R^a in terms of the elementary symmetric polynomials

Fix r with $d-s+1 \leq r \leq d$. Assume that $2(s+1) \leq d$ holds and consider the elementary symmetric polynomials Π_1, \dots, Π_r of $\mathbb{F}_q[X_1, \dots, X_r]$. For convenience of notation, we shall denote $\Pi_0 := 1$. In Section 2.1 we obtain polynomials $R_j^a \in \mathbb{F}_q[X_1, \dots, X_r]$ ($d-s \leq j \leq r-1$) with the following property: for a given set $\mathcal{X}_r := \{x_1, \dots, x_r\} \subset \mathbb{F}_q$ of r elements, the set $\mathcal{S}_{\mathcal{X}_r}^a$ is not empty if and only if (x_1, \dots, x_r) is a common zero of $R_{d-s}^a, \dots, R_{r-1}^a$.

The main purpose of this section is to show how the polynomials R_j^a can be expressed in terms of Π_1, \dots, Π_s . In order to do this, we first obtain a recursive expression for the remainder of the division of T^j by $Q := (T - X_1) \cdots (T - X_r)$ for $r \leq j \leq d$.

Lemma 2.3. *For $r \leq j \leq d$, the following congruence relation holds:*

$$T^j \equiv H_{r-1,j}T^{r-1} + H_{r-2,j}T^{r-2} + \cdots + H_{0,j} \pmod{Q}, \tag{2.10}$$

where each $H_{i,j}$ is equal to zero or a homogeneous element of $\mathbb{F}_q[X_1, \dots, X_r]$ of degree $j - i$. Furthermore, for $j - i \leq r$, the polynomial $H_{i,j}$ is a monic element of $\mathbb{F}_q[\Pi_1, \dots, \Pi_{j-i-1}][\Pi_{j-i}]$, up to a nonzero constant of \mathbb{F}_q , of degree 1 in Π_{j-i} .

Proof. We argue by induction on $j \geq r$. Taking into account that

$$T^r \equiv \Pi_1 T^{r-1} - \Pi_2 T^{r-2} + \cdots + (-1)^{r-1} \Pi_r \pmod{Q}, \tag{2.11}$$

we immediately deduce the statement of the lemma for $j = r$.

Next assume that (2.10) holds for a given j with $r \leq j$. Multiplying both sides of (2.10) by T and combining with (2.11) we obtain:

$$\begin{aligned} T^{j+1} &\equiv H_{r-1,j}T^r + H_{r-2,j}T^{r-1} + \cdots + H_{0,j}T \\ &\equiv (\Pi_1 H_{r-1,j} + H_{r-2,j})T^{r-1} + \cdots + ((-1)^{r-2} \Pi_{r-1} H_{r-1,j} + H_{0,j})T \\ &\quad + (-1)^{r-1} \Pi_r H_{r-1,j}, \end{aligned}$$

where all congruences are taken modulo Q .

Define

$$\begin{aligned} H_{k,j+1} &:= (-1)^{r-1-k} \Pi_{r-k} H_{r-1,j} + H_{k-1,j} \quad \text{for } 1 \leq k \leq r-1, \\ H_{0,j+1} &:= (-1)^{r-1} \Pi_r H_{r-1,j}. \end{aligned}$$

Then we have

$$T^{j+1} \equiv H_{r-1,j+1}T^{r-1} + H_{r-2,j+1}T^{r-2} + \cdots + H_{0,j+1} \pmod{Q}.$$

There remains to prove that the polynomials $H_{k,j+1}$ have the form asserted.

Fix k with $1 \leq k \leq r - 1$. Then $H_{k,j+1} = (-1)^{r-1-k} \Pi_{r-k} H_{r-1,j} + H_{k-1,j}$. By the inductive hypothesis we have that $H_{r-1,j}$ and $H_{k-1,j}$ are equal to zero or homogeneous polynomials of degree $j - r + 1$ and $j - k + 1$, respectively. We easily conclude that $H_{k,j+1}$ is equal to zero or homogeneous of degree $j - k + 1$. Further, for $j + 1 - k \leq r$, since $\max\{r - k, j - r + 1\} \leq j - k < r$ we see that $\Pi_{r-k} H_{r-1,j}$ is an element of the polynomial ring $\mathbb{F}_q[\Pi_1, \dots, \Pi_{j-k}]$. On the other hand, $H_{k-1,j}$ is a monic element of $\mathbb{F}_q[\Pi_1, \dots, \Pi_{j-k}][\Pi_{j-k+1}]$, up to a nonzero constant of \mathbb{F}_q , which implies that so is $H_{k,j+1}$.

Finally, for $k = 0$ we have $H_{0,j+1} := (-1)^{r-1} \Pi_r H_{r-1,j}$, which shows that $H_{0,j+1}$ is equal to zero or a homogeneous polynomial of $\mathbb{F}_q[X_1, \dots, X_r]$ of degree $r + j - r + 1 = j + 1$. This finishes the proof of the lemma. \square

We observe that an explicit expression of the polynomials $H_{i,j}$ can be obtained following the approach of [4, Proposition 2.2]. As we do not need such an explicit expression we shall not pursue this point any further.

Finally we obtain an expression of the polynomials $R_j^\alpha \in \mathbb{F}_q[X_1, \dots, X_r]$ ($d - s \leq j \leq r - 1$) in terms of the polynomials $H_{i,j}$.

Proposition 2.4. *For $s, d \in \mathbb{N}$ with $1 \leq s \leq d - 2$ and $2(s + 1) \leq d$, and for $d - s \leq j \leq r - 1$, the following identity holds:*

$$R_j^\alpha = a_j + \sum_{i=r}^d a_i H_{j,i}, \tag{2.12}$$

where the polynomials $H_{j,i}$ are defined in Lemma 2.3. In particular, R_j^α is a monic element of $\mathbb{F}_q[\Pi_1, \dots, \Pi_{d-1-j}][\Pi_{d-j}]$, up to a nonzero constant of \mathbb{F}_q , of degree $d - j \leq s$ for $d - s \leq j \leq r - 1$, and thus of degree 1 in Π_{d-j} .

Proof. By Lemma 2.3 we have the following relation for $r \leq j \leq d$:

$$T^j \equiv H_{r-1,j} T^{r-1} + H_{r-2,j} T^{r-2} + \dots + H_{0,j} \pmod{Q}.$$

Hence we obtain

$$\begin{aligned} \sum_{j=d-s}^d a_j T^j &= \sum_{j=d-s}^{r-1} a_j T^j + \sum_{j=r}^d a_j T^j \\ &\equiv \sum_{j=d-s}^{r-1} a_j T^j + \sum_{j=r}^d a_j \sum_{i=d-s}^{r-1} H_{i,j} T^i + \mathcal{O}(T^{d-s-1}) \pmod{Q} \\ &\equiv \sum_{j=d-s}^{r-1} \left(a_j + \sum_{i=r}^d a_i H_{j,i} \right) T^j + \mathcal{O}(T^{d-s-1}) \pmod{Q}, \end{aligned}$$

where $\mathcal{O}(T^{d-s-1})$ represents a sum of terms of $\mathbb{F}_q[X_1, \dots, X_r][T]$ of degree at most $d - s - 1$ in T . This shows that the polynomials R_j^α have the form asserted in the statement of the proposition. Furthermore, we observe that, for each $H_{j,i}$ occurring in (2.12), we have $i - j \leq s \leq d - s - 2 \leq r$. This implies that each $H_{j,i}$ is a monic element of $\mathbb{F}_q[\Pi_1, \dots, \Pi_{i-j-1}][\Pi_{i-j}]$ of degree $i - j$. As a consequence, we see that R_j^α is a monic element of $\mathbb{F}_q[\Pi_1, \dots, \Pi_{d-1-j}][\Pi_{d-j}]$ of degree $d - j$ for $d - s \leq j \leq r - 1$. This finishes the proof. \square

3. The geometry of the set of zeros of $R_{d-s}^\alpha, \dots, R_{r-1}^\alpha$

For positive integers s, d with $q < d, 1 \leq s \leq d - 2$ and $2(s + 1) \leq d$, we fix as in the previous section an s -tuple $\alpha := (a_{d-1}, \dots, a_{d-s}) \in \mathbb{F}_q^s$ and consider the polynomial $f^\alpha := T^d + a_{d-1}T^{d-1} + \dots + a_{d-s}T^{d-s}$. For fixed r with $d - s + 1 \leq r \leq d$, in Section 2.1 we associate to f^α polynomials $R_j^\alpha \in \mathbb{F}_q[X_1, \dots, X_r]$ ($d - s \leq j \leq r - 1$), whose sets of common \mathbb{F}_q -rational zeros are relevant for our purposes.

According to Proposition 2.4, we may express each R_j^α as a polynomial in the first s elementary symmetric polynomials Π_1, \dots, Π_s of $\mathbb{F}_q[X_1, \dots, X_r]$. More precisely, let Y_1, \dots, Y_s be new indeterminates over $\overline{\mathbb{F}}_q$. Then we have that

$$R_j^\alpha = S_j^\alpha(\Pi_1, \dots, \Pi_{d-j}) \quad (d - s \leq j \leq r - 1),$$

where each $S_j^\alpha \in \mathbb{F}_q[Y_1, \dots, Y_{d-j}]$ is a monic element of $\mathbb{F}_q[Y_1, \dots, Y_{d-1-j}][Y_{d-j}]$, up to a nonzero constant of \mathbb{F}_q , of degree 1 in Y_{d-j} .

In this section we obtain critical information on the geometry of the set of common zeros of the polynomials R_j^α that will allow us to establish estimates on the number of common \mathbb{F}_q -rational zeros of $R_{d-s}^\alpha, \dots, R_{r-1}^\alpha$.

3.1. Notions of algebraic geometry

Since our approach relies on tools of algebraic geometry, we briefly collect the basic definitions and facts that we need in the sequel. We use standard notions and notations of algebraic geometry, which can be found in, e.g., [18,23].

We denote by \mathbb{A}^n the affine n -dimensional space $\overline{\mathbb{F}}_q^n$ and by \mathbb{P}^n the projective n -dimensional space over $\overline{\mathbb{F}}_q^{n+1}$. Both spaces are endowed with their respective Zariski topologies, for which a closed set is the zero locus of polynomials of $\overline{\mathbb{F}}_q[X_1, \dots, X_n]$ or of homogeneous polynomials of $\overline{\mathbb{F}}_q[X_0, \dots, X_n]$. For $\mathbb{K} := \mathbb{F}_q$ or $\mathbb{K} := \overline{\mathbb{F}}_q$, we say that a subset $V \subset \mathbb{A}^n$ is an affine \mathbb{K} -variety if it is the set of common zeros in \mathbb{A}^n of polynomials $F_1, \dots, F_m \in \mathbb{K}[X_1, \dots, X_n]$. Correspondingly, a projective \mathbb{K} -variety is the set of common zeros in \mathbb{P}^n of a family of homogeneous polynomials $F_1, \dots, F_m \in \mathbb{K}[X_0, \dots, X_n]$. We shall frequently denote by $V(F_1, \dots, F_m)$ the affine or projective \mathbb{K} -variety consisting of the common zeros of polynomials F_1, \dots, F_m . The set $V(\mathbb{F}_q) := V \cap \mathbb{F}_q^n$ is the set of \mathbb{F}_q -rational (or simply q -rational) points of V .

A \mathbb{K} -variety V is \mathbb{K} -irreducible if it cannot be expressed as a finite union of proper \mathbb{K} -subvarieties of V . Further, V is absolutely irreducible if it is $\overline{\mathbb{F}}_q$ -irreducible as an $\overline{\mathbb{F}}_q$ -variety. Any \mathbb{K} -variety V can be expressed as an irredundant union $V = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_s$ of irreducible (absolutely irreducible) \mathbb{K} -varieties, unique up to reordering, which are called the irreducible (absolutely irreducible) \mathbb{K} -components of V .

For a \mathbb{K} -variety V contained in \mathbb{A}^n or \mathbb{P}^n , we denote by $I(V)$ its defining ideal, namely the set of polynomials of $\mathbb{K}[X_1, \dots, X_n]$, or of $\mathbb{K}[X_0, \dots, X_n]$, vanishing on V . The coordinate ring $\mathbb{K}[V]$ of V is defined as the quotient ring $\mathbb{K}[X_1, \dots, X_n]/I(V)$ or $\mathbb{K}[X_0, \dots, X_n]/I(V)$. The dimension $\dim V$ of a \mathbb{K} -variety V is the length r of the longest chain $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_r$ of nonempty irreducible \mathbb{K} -varieties contained in V . A \mathbb{K} -variety V is called equidimensional if all the irreducible \mathbb{K} -components of V are of the same dimension.

The degree $\deg V$ of an irreducible \mathbb{K} -variety V is the maximum number of points lying in the intersection of V with a linear space L of codimension $\dim V$, for which $V \cap L$ is a finite set. More generally, following [17] (see also [13]), if $V = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_s$ is the decomposition of V into irreducible \mathbb{K} -components, we define the degree of V as

$$\deg V := \sum_{i=1}^s \deg \mathcal{C}_i.$$

With this definition of degree, we have the following *Bézout inequality* (see [17,13,26]): if V and W are \mathbb{K} -varieties, then

$$\deg(V \cap W) \leq \deg V \cdot \deg W. \tag{3.1}$$

We shall also make use of the following well-known identities relating the degree of an affine \mathbb{K} -variety $V \subset \mathbb{A}^n$, the degree of its projective closure (with respect to the projective Zariski \mathbb{K} -topology) $\overline{V} \subset \mathbb{P}^n$ and the degree of the affine cone \widetilde{V} of \overline{V} (see, e.g., [6, Proposition 1.11]):

$$\deg V = \deg \overline{V} = \deg \widetilde{V}.$$

Elements F_1, \dots, F_{n-r} in $\mathbb{K}[X_1, \dots, X_n]$ or in $\mathbb{K}[X_0, \dots, X_n]$ form a regular sequence if F_1 is nonzero and each F_i is not a zero divisor in the quotient ring $\mathbb{K}[X_1, \dots, X_n]/(F_1, \dots, F_{i-1})$ or $\mathbb{K}[X_0, \dots, X_n]/(F_1, \dots, F_{i-1})$ for $2 \leq i \leq n - r$. In such a case, the (affine or projective) \mathbb{K} -variety $V := V(F_1, \dots, F_{n-r})$ they define is equidimensional of dimension r , and is called a set-theoretic complete intersection. In particular, if the ideal (F_1, \dots, F_{n-r}) generated by F_1, \dots, F_{n-r} is radical, then V is an ideal-theoretic complete intersection. If $V \subset \mathbb{P}^n$ is an ideal-theoretic complete intersection defined over \mathbb{K} , of dimension r and degree δ , and F_1, \dots, F_{n-r} is a system of generators of $I(V)$, the degrees d_1, \dots, d_{n-r} depend only on V and not on the system of generators. Arranging the d_i in such a way that $d_1 \geq d_2 \geq \dots \geq d_{n-r}$, we call $\mathbf{d} := (d_1, \dots, d_{n-r})$ the multidegree of V . It follows that $\delta = \prod_{i=1}^{n-r} d_i$ holds (see, e.g., [16, Theorem 18.3]).

Let $V \subset \mathbb{A}^n$ be a variety and let $I(V) \subset \overline{\mathbb{F}}_q[X_1, \dots, X_n]$ be the defining ideal of V . Let \mathbf{x} be a point of V . The dimension $\dim_{\mathbf{x}} V$ of V at \mathbf{x} is the maximum of the dimensions of the irreducible components of V that contain \mathbf{x} . If $I(V) = (F_1, \dots, F_m)$, the tangent space $\mathcal{T}_{\mathbf{x}} V$ to V at \mathbf{x} is the kernel of the Jacobian matrix $(\partial F_i / \partial X_j)_{1 \leq i \leq m, 1 \leq j \leq n}(\mathbf{x})$ of the polynomials F_1, \dots, F_m with respect to X_1, \dots, X_n at \mathbf{x} . The point \mathbf{x} is regular if $\dim \mathcal{T}_{\mathbf{x}} V = \dim_{\mathbf{x}} V$ holds. Otherwise, the point \mathbf{x} is called singular. The set of singular points of V is the singular locus $\text{Sing}(V)$ of V . A variety is called nonsingular if its singular locus is empty. For a projective variety, the concepts of tangent space, regular and singular point can be defined by considering an affine neighborhood of the point under consideration.

Let V and W be irreducible affine \mathbb{K} -varieties of the same dimension and let $f : V \rightarrow W$ be a regular map for which $\overline{f(V)} = W$ holds, where $\overline{f(V)}$ denotes the closure of $f(V)$ with respect to the Zariski topology of W . Such a map is called dominant. Then f induces a ring extension $\mathbb{K}[W] \hookrightarrow \mathbb{K}[V]$ by composition with f . We say that the dominant map f is a finite morphism if this extension is integral, namely if each element $\eta \in \mathbb{K}[V]$ satisfies a monic equation with coefficients in $\mathbb{K}[W]$. A basic fact is that a dominant finite morphism is necessarily closed. Another fact concerning dominant finite morphisms we shall use in the sequel is that the preimage $f^{-1}(S)$ of an irreducible closed subset $S \subset W$ is equidimensional of dimension $\dim S$ (see, e.g., [10, §4.2, Proposition]).

3.2. The singular locus of symmetric complete intersections

With the notations and assumptions of the beginning of Section 3, let $V_r^{\alpha} \subset \mathbb{A}^r$ be the affine \mathbb{F}_q -variety defined by the polynomials $R_{d-s}^{\alpha}, \dots, R_{r-1}^{\alpha} \in \mathbb{F}_q[X_1, \dots, X_r]$. In this section we shall establish several facts concerning the geometry of V_r^{α} . For this purpose, we consider the somewhat more general framework that we now introduce. This will allow us to make more transparent the facts concerning the algebraic structure of the family of polynomials $R_{d-s}^{\alpha}, \dots, R_{r-1}^{\alpha}$ which are important at this point.

Let Y_1, \dots, Y_s be new indeterminates over $\overline{\mathbb{F}}_q$ and let be given polynomials $S_j \in \mathbb{F}_q[Y_1, \dots, Y_s]$ for $d-s \leq j \leq r-1$. Let $(\partial \mathbf{S} / \partial \mathbf{Y}) := (\partial S_j / \partial Y_k)_{d-s \leq j \leq r-1, 1 \leq k \leq s}$ be the Jacobian matrix of S_{d-s}, \dots, S_{r-1} with respect to Y_1, \dots, Y_s . Our assumptions on s , d and r imply $r-d+s \leq s$ and thus, $(\partial \mathbf{S} / \partial \mathbf{Y})$ has full rank if and only if $\text{rank}(\partial \mathbf{S} / \partial \mathbf{Y}) = r-d+s$ holds. Assume that S_{d-s}, \dots, S_{r-1} satisfy the following conditions:

- (H1) S_{d-s}, \dots, S_{r-1} form a regular sequence of $\mathbb{F}_q[Y_1, \dots, Y_s]$;
- (H2) $(\partial \mathbf{S} / \partial \mathbf{Y})(\mathbf{y})$ has full rank $r-d+s$ for every $\mathbf{y} \in \mathbb{A}^s$.

From (H1) and (H2) we immediately conclude that the affine variety $W_r \subset \mathbb{A}^s$ defined by S_{d-s}, \dots, S_{r-1} is a nonsingular set-theoretic complete intersection of dimension $d-r$. Furthermore, as a consequence of [11, Theorem 18.15] we conclude that S_{d-s}, \dots, S_{r-1} define a radical ideal, and hence W_r is an ideal-theoretic complete intersection.

Denote by Π_1, \dots, Π_s the first s elementary symmetric polynomials of $\mathbb{F}_q[X_1, \dots, X_r]$ and let $R_j := S_j(\Pi_1, \dots, \Pi_s)$ for $d - s \leq j \leq r - 1$. We denote by $V_r \subset \mathbb{A}^r$ the affine variety defined by R_{d-s}, \dots, R_{r-1} . In what follows we shall establish several facts concerning the geometry of V_r .

For this purpose, we consider the following surjective morphism of \mathbb{F}_q -varieties:

$$\begin{aligned} \Pi^r : \mathbb{A}^r &\rightarrow \mathbb{A}^r \\ \mathbf{x} &\mapsto (\Pi_1(\mathbf{x}), \dots, \Pi_r(\mathbf{x})). \end{aligned}$$

It is easy to see that Π^r is a dominant finite morphism (see, e.g., [23, §5.3, Example 1]). In particular, the preimage $(\Pi^r)^{-1}(Z)$ of an irreducible affine variety $Z \subset \mathbb{A}^r$ of dimension m is equidimensional and of dimension m .

We now consider S_{d-s}, \dots, S_{r-1} as elements of $\mathbb{F}_q[Y_1, \dots, Y_r]$. Since they form a regular sequence, the affine variety $W_j^r = V(S_{d-s}, \dots, S_j) \subset \mathbb{A}^r$ is equidimensional of dimension $r - j + d - s - 1$. This implies that the affine variety $V_j^r = (\Pi^r)^{-1}(W_j^r)$ defined by R_{d-s}, \dots, R_j is equidimensional of dimension $r - j + d - s - 1$. We conclude that the polynomials R_{d-s}, \dots, R_{r-1} form a regular sequence of $\mathbb{F}_q[X_1, \dots, X_r]$ and deduce the following result.

Lemma 3.1. *Let $V_r \subset \mathbb{A}^r$ be the \mathbb{F}_q -variety defined by R_{d-s}, \dots, R_{r-1} . Then V_r is a set-theoretic complete intersection of dimension $d - s$.*

Next we analyze the dimension of the singular locus of V_r . For this purpose, we consider the following surjective morphism of \mathbb{F}_q -varieties:

$$\begin{aligned} \Pi : V_r &\rightarrow W_r \\ \mathbf{x} &\mapsto (\Pi_1(\mathbf{x}), \dots, \Pi_s(\mathbf{x})). \end{aligned}$$

For $\mathbf{x} \in V_r$ and $\mathbf{y} := \Pi(\mathbf{x})$, we denote by $\mathcal{T}_{\mathbf{x}}V_r$ and $\mathcal{T}_{\mathbf{y}}W_r$ the tangent spaces to V_r at \mathbf{x} and to W_r at \mathbf{y} . We also consider the differential map of Π at \mathbf{x} , namely

$$\begin{aligned} d_{\mathbf{x}}\Pi : \mathcal{T}_{\mathbf{x}}V_r &\rightarrow \mathcal{T}_{\mathbf{y}}W_r \\ \mathbf{v} &\mapsto A(\mathbf{x}) \cdot \mathbf{v}, \end{aligned}$$

where $A(\mathbf{x})$ stands for the $(s \times r)$ -matrix

$$A(\mathbf{x}) := \left(\frac{\partial \Pi}{\partial \mathbf{X}} \right) (\mathbf{x}) := \left(\frac{\partial \Pi_i}{\partial X_j} (\mathbf{x}) \right)_{\substack{1 \leq i \leq s \\ 1 \leq j \leq r}}. \tag{3.2}$$

In order to prove our result about the singular locus of V_r , we first make a few remarks concerning the Jacobian matrix of the elementary symmetric polynomials that will be useful in the sequel.

It is well known that the first partial derivatives of the elementary symmetric polynomials Π_i satisfy the following equalities (see, e.g., [19]) for $1 \leq i, j \leq r$:

$$\frac{\partial \Pi_i}{\partial X_j} = \Pi_{i-1} - X_j \Pi_{i-2} + X_j^2 \Pi_{i-3} + \dots + (-1)^{i-1} X_j^{i-1}. \tag{3.3}$$

As a consequence, denoting by A_r the $(r \times r)$ -Vandermonde matrix

$$A_r := (X_j^{i-1})_{1 \leq i, j \leq r}, \tag{3.4}$$

we deduce that the Jacobian matrix $(\partial \Pi^r / \partial \mathbf{X})$ of $\Pi^r := (\Pi_1, \dots, \Pi_r)$ with respect to X_1, \dots, X_r can be factored as follows:

$$\left(\frac{\partial \Pi^r}{\partial \mathbf{X}} \right) := B_r \cdot A_r := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \Pi_1 & -1 & 0 & & \\ \Pi_2 & -\Pi_1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \Pi_{r-1} & -\Pi_{r-2} & \Pi_{r-3} & \dots & (-1)^{r-1} \end{pmatrix} \cdot A_r. \tag{3.5}$$

We observe that the left factor B_r is a square, lower-triangular matrix whose determinant is equal to $(-1)^{(r-1)r/2}$. This implies that the determinant of the matrix $(\partial \Pi^r / \partial \mathbf{X})$ is equal, up to a sign, to the determinant of A_r , i.e.,

$$\det \left(\frac{\partial \Pi^r}{\partial \mathbf{X}} \right) = (-1)^{\binom{r}{2}} \prod_{1 \leq i < j \leq r} (X_i - X_j).$$

Let $(\partial \mathbf{R} / \partial \mathbf{X}) := (\partial R_j / \partial X_k)_{d-s \leq j \leq r-1, 1 \leq k \leq r}$ be the Jacobian matrix of the polynomials R_{d-s}, \dots, R_{r-1} with respect to X_1, \dots, X_r .

Theorem 3.2. *The set of points $\mathbf{x} \in \mathbb{A}^r$ for which $(\partial \mathbf{R} / \partial \mathbf{X})(\mathbf{x})$ has not full rank, has dimension at most $s - 1$. In particular, the singular locus Σ_r of V_r has dimension at most $s - 1$.*

Proof. Since $\mathbf{R} = \mathbf{S} \circ \Pi$, by the chain rule we obtain

$$\left(\frac{\partial \mathbf{R}}{\partial \mathbf{X}} \right) = \left(\frac{\partial \mathbf{S}}{\partial \mathbf{Y}} \circ \Pi \right) \cdot \left(\frac{\partial \Pi}{\partial \mathbf{X}} \right).$$

Fix an arbitrary point \mathbf{x} for which $(\partial \mathbf{R} / \partial \mathbf{X})(\mathbf{x})$ has not full rank. Let $\mathbf{v} \in \mathbb{A}^{r-d+s}$ be a nonzero vector in the left kernel of $(\partial \mathbf{R} / \partial \mathbf{X})(\mathbf{x})$. Then

$$\mathbf{0} = \mathbf{v} \cdot \left(\frac{\partial \mathbf{R}}{\partial \mathbf{X}} \right) (\mathbf{x}) = \mathbf{v} \cdot \left(\frac{\partial \mathbf{S}}{\partial \mathbf{Y}} \right) (\Pi(\mathbf{x})) \cdot A(\mathbf{x}),$$

where $A(\mathbf{x})$ is the matrix defined in (3.2). Since by (H2) the Jacobian matrix $(\partial\mathbf{S}/\partial\mathbf{Y})(\Pi(\mathbf{x}))$ has full rank, $\mathbf{w} := \mathbf{v} \cdot (\partial\mathbf{S}/\partial\mathbf{Y})(\Pi(\mathbf{x})) \in \mathbb{A}^s$ is nonzero and

$$\mathbf{w} \cdot A(\mathbf{x}) = \mathbf{0}.$$

Hence, all the maximal minors of $A(\mathbf{x})$ must be zero.

The matrix $A(\mathbf{x})$ is the $(s \times r)$ -submatrix of $(\partial\Pi^r/\partial\mathbf{X})(\mathbf{x})$ consisting of the first s rows of the latter. Therefore, from (3.5) we conclude that

$$A(\mathbf{x}) = B_{s,r}(\mathbf{x}) \cdot A_r(\mathbf{x}),$$

where $B_{s,r}(\mathbf{x})$ is the $(s \times r)$ -submatrix of $B_r(\mathbf{x})$ consisting of the first s rows of $B_r(\mathbf{x})$. Since the last $r - s$ columns of $B_{s,r}(\mathbf{x})$ are zero, we may rewrite this identity in the following way:

$$A(\mathbf{x}) = B_s(\mathbf{x}) \cdot \left(x_j^{i-1}\right)_{\substack{1 \leq i \leq s \\ 1 \leq j \leq r}}, \tag{3.6}$$

where $B_s(\mathbf{x})$ is the $(s \times s)$ -submatrix of $B_r(\mathbf{x})$ consisting on the first s rows and the first s columns of $B_r(\mathbf{x})$.

Fix $1 \leq l_1 < \dots < l_s \leq r$, set $I := (l_1, \dots, l_s)$ and consider the $(s \times s)$ -submatrix $M_I(\mathbf{x})$ of $A(\mathbf{x})$ consisting of the columns l_1, \dots, l_s of $A(\mathbf{x})$, namely $M_I(\mathbf{x}) := (\partial\Pi_i/\partial X_{l_j})_{1 \leq i, j \leq s}(\mathbf{x})$. From (3.5) and (3.6) we easily see that $M_I(\mathbf{x}) = B_s(\mathbf{x}) \cdot A_{s,I}(\mathbf{x})$, where $A_{s,I}(\mathbf{x})$ is the Vandermonde matrix $A_{s,I}(\mathbf{x}) := (x_{l_j}^{i-1})_{1 \leq i, j \leq s}$. Therefore, we obtain

$$\det(M_I(\mathbf{x})) = (-1)^{\binom{s}{2}} \det A_{s,I}(\mathbf{x}) = (-1)^{\binom{s}{2}} \prod_{1 \leq m < n \leq s} (x_{l_m} - x_{l_n}) = 0. \tag{3.7}$$

Since (3.7) holds for every $I := (l_1, \dots, l_s)$ as above, we conclude that \mathbf{x} has at most $s - 1$ pairwise-distinct coordinates. In particular, the set of points \mathbf{x} with $\text{rank}(\partial\mathbf{R}/\partial\mathbf{X})(\mathbf{x}) < r - d + s$ is contained in a finite union of linear varieties of \mathbb{A}^r of dimension $s - 1$, and thus is an affine variety of dimension at most $s - 1$.

Now let \mathbf{x} be an arbitrary point Σ_r . By Lemma 3.1 we have $\dim \mathcal{T}_{\mathbf{x}}V_r > d - s$. This implies that $\text{rank}(\partial\mathbf{R}/\partial\mathbf{X})(\mathbf{x}) < r - d + s$, for otherwise we would have $\dim \mathcal{T}_{\mathbf{x}}V_r \leq d - s$, contradicting thus the fact that \mathbf{x} is a singular point of V_r . This finishes the proof of the theorem. \square

From Lemma 3.1 and Theorem 3.2 we obtain further algebraic and geometric consequences concerning the polynomials R_j and the variety V_r . By Theorem 3.2 we have that the set of points $\mathbf{x} \in \mathbb{A}^r$ for which the Jacobian matrix $(\partial\mathbf{R}/\partial\mathbf{X})(\mathbf{x})$ has not full rank, has dimension at most $s - 1$. Since R_{d-s}, \dots, R_{r-1} form a regular sequence and $s - 1 < d - s$ holds, from [11, Theorem 18.15] we conclude that R_{d-s}, \dots, R_{r-1} define a radical ideal of $\mathbb{F}_q[X_1, \dots, X_r]$. On the other hand, by the Bézout inequality (3.1) we have $\deg V_r \leq \prod_{j=d-s}^{r-1} \deg R_j$. In other words, we have the following statement.

Corollary 3.3. *The polynomials R_{d-s}, \dots, R_{r-1} define a radical ideal and the variety V_r has degree $\deg V_r \leq \prod_{j=d-s}^{r-1} \deg R_j$.*

3.3. The geometry of V_r^α

Now we consider the affine \mathbb{F}_q -variety $V_r^\alpha \subset \mathbb{A}^r$ defined by the polynomials $R_{d-s}^\alpha, \dots, R_{r-1}^\alpha \in \mathbb{F}_q[X_1, \dots, X_r]$ associated to $\alpha := (a_{d-1}, \dots, a_{d-s}) \in \mathbb{F}_q^s$ and the polynomial $f^\alpha := T^d + a_{d-1}T^{d-1} + \dots + a_{d-s}T^{d-s}$. According to Proposition 2.4, we may express each R_j^α in the form $R_j^\alpha = S_j^\alpha(\Pi_1, \dots, \Pi_{d-j})$, where $S_j^\alpha \in \mathbb{F}_q[Y_1, \dots, Y_{d-j}]$ is a monic polynomial in Y_{d-j} , up to a nonzero constant, of degree 1 in Y_{d-j} . In particular, by a recursive argument it is easy to see that

$$\overline{\mathbb{F}}_q[Y_1, \dots, Y_s] / (S_{d-s}^\alpha, \dots, S_j^\alpha) \simeq \overline{\mathbb{F}}_q[Y_1, \dots, Y_{d-j-1}] \tag{3.8}$$

for $d - s \leq j \leq r - 1$. We conclude that $S_{d-s}^\alpha, \dots, S_{r-1}^\alpha$ form a regular sequence of $\mathbb{F}_q[Y_1, \dots, Y_s]$, namely they satisfy (H1). Furthermore, by the isomorphism (3.8), for $j = r - 1$ we deduce that $S_{d-s}^\alpha, \dots, S_{r-1}^\alpha$ define a radical ideal of $\overline{\mathbb{F}}_q[Y_1, \dots, Y_s]$ and the variety $W_r^\alpha \subset \mathbb{A}^s$ defined by $S_{d-s}^\alpha, \dots, S_{r-1}^\alpha$ is nonsingular. As a consequence, we have that $(\partial S^\alpha / \partial Y)(\mathbf{y})$ has full rank for every $\mathbf{y} \in \mathbb{A}^s$, that is, $S_{d-s}^\alpha, \dots, S_{r-1}^\alpha$ satisfy (H2). Then the results of Section 3.2 can be applied to V_r^α . In particular, we have the following immediate consequence of Lemma 3.1, Theorem 3.2 and Corollary 3.3.

Corollary 3.4. *Let $V_r^\alpha \subset \mathbb{A}^r$ be the \mathbb{F}_q -variety defined by $R_{d-s}^\alpha, \dots, R_{r-1}^\alpha$. Then V_r^α is an ideal-theoretic complete intersection of dimension $d - s$, degree at most $s! / (d - r)!$ and singular locus Σ_r^α of dimension at most $s - 1$.*

3.3.1. The projective closure of V_r^α

In order to obtain estimates on the number of q -rational points of V_r^α we also need information concerning the behavior of V_r^α “at infinity”. For this purpose, we consider the projective closure $\text{pcl}(V_r^\alpha) \subset \mathbb{P}^r$ of V_r^α , whose definition we now recall. Consider the embedding of \mathbb{A}^r into the projective space \mathbb{P}^r which assigns to any $\mathbf{x} := (x_1, \dots, x_r) \in \mathbb{A}^r$ the point $(1 : x_1 : \dots : x_r) \in \mathbb{P}^r$. The closure $\text{pcl}(V_r^\alpha) \subset \mathbb{P}^r$ of the image of V_r^α under this embedding in the Zariski topology of \mathbb{P}^r is called the projective closure of V_r^α . The points of $\text{pcl}(V_r^\alpha)$ lying in the hyperplane $\{X_0 = 0\}$ are called the points of $\text{pcl}(V_r^\alpha)$ at infinity.

It is well-known that $\text{pcl}(V_r^\alpha)$ is the \mathbb{F}_q -variety of \mathbb{P}^r defined by the homogenization $F^h \in \mathbb{F}_q[X_0, \dots, X_r]$ of each polynomial F belonging to the ideal $(R_{d-s}^\alpha, \dots, R_{r-1}^\alpha) \subset \mathbb{F}_q[X_1, \dots, X_r]$ (see, e.g., [18, §I.5, Exercise 6]). Denote by $(R_{d-s}^\alpha, \dots, R_{r-1}^\alpha)^h$ the ideal generated by all the polynomials F^h with $F \in (R_{d-s}^\alpha, \dots, R_{r-1}^\alpha)$. Since $(R_{d-s}^\alpha, \dots, R_{r-1}^\alpha)$ is radical it turns out that $(R_{d-s}^\alpha, \dots, R_{r-1}^\alpha)^h$ is also a radical ideal (see, e.g., [18, §I.5, Exercise 6]). Furthermore, $\text{pcl}(V_r^\alpha)$ is an equidimensional variety of dimension $d - s$ (see, e.g., [18, Propositions I.5.17 and II.4.1]) and degree at most $s! / (d - r)!$ (see, e.g., [6, Proposition 1.11]).

Now we discuss the behavior of $\text{pcl}(V_r^\alpha)$ at infinity. By [Proposition 2.4](#), for $d - s \leq j \leq r - 1$ we have

$$R_j^\alpha = a_j + \sum_{i=r}^d a_i H_{j,i},$$

where the polynomials $H_{j,i}$ are homogeneous of degree $i - j$. Hence, the homogenization of each R_j^α is the following polynomial of $\mathbb{F}_q[X_0, \dots, X_r]$:

$$R_j^{\alpha,h} = a_j X_0^{d-j} + \sum_{i=r}^d a_i H_{j,i} X_0^{d-i}. \tag{3.9}$$

It follows that $R_j^{\alpha,h}(0, X_1, \dots, X_r) = H_{j,d}$ ($d - s \leq j \leq r - 1$), which are the polynomials associated to the polynomial $T^d \in \mathbb{F}_q[T]$ in the sense of [Lemma 2.2](#).

Lemma 3.5. *$\text{pcl}(V_r^\alpha)$ has singular locus at infinity of dimension at most $s - 2$.*

Proof. Let $\Sigma_{r,\infty}^\alpha \subset \mathbb{P}^r$ denote the singular locus of $\text{pcl}(V_r^\alpha)$ at infinity, namely the set of singular points of $\text{pcl}(V_r^\alpha)$ lying in the hyperplane $\{X_0 = 0\}$, and let $\mathbf{x} := (0 : x_1 : \dots : x_r)$ be an arbitrary point of $\Sigma_{r,\infty}^\alpha$. Since the polynomials $R_j^{\alpha,h}$ vanish identically in $\text{pcl}(V_r^\alpha)$, we have $R_j^{\alpha,h}(\mathbf{x}) = H_{j,d}(x_1, \dots, x_r) = 0$ for $d - s \leq j \leq r - 1$. Let $(\partial \mathbf{H}_d / \partial \mathbf{X})$ be the Jacobian matrix of $\{H_{j,d} : d - s \leq j \leq r - 1\}$ with respect to X_1, \dots, X_r . We have

$$\text{rank} \left(\frac{\partial \mathbf{H}_d}{\partial \mathbf{X}} \right) (\mathbf{x}) < r - d + s, \tag{3.10}$$

for if not, we would have that $\dim \mathcal{T}_{\mathbf{x}}(\text{pcl}(V_r^\alpha)) \leq d - s$, which implies that \mathbf{x} is a nonsingular point of $\text{pcl}(V_r^\alpha)$, contradicting thus the hypothesis on \mathbf{x} .

By [Lemma 2.3](#) the polynomials $H_{j,d}$ ($d - s \leq j \leq r - 1$) satisfy the hypotheses of [Theorem 3.2](#). Then [Theorem 3.2](#) shows that the set of points satisfying (3.10) is an affine equidimensional cone of dimension at most $s - 1$. We conclude that the projective variety $\Sigma_{r,\infty}^\alpha$ has dimension at most $s - 2$. \square

Now we are able to completely characterize the behavior of $\text{pcl}(V_r^\alpha)$ at infinity.

Theorem 3.6. *$\text{pcl}(V_r^\alpha) \cap \{X_0 = 0\} \subset \mathbb{P}^{r-1}$ is an absolutely irreducible ideal-theoretic complete intersection of dimension $d - s - 1$, degree $s!/(d - r)!$, and singular locus of dimension at most $s - 2$.*

Proof. From (3.9) it is easy to see that the polynomials $H_{j,d}$ vanish identically in $\text{pcl}(V_r^\alpha) \cap \{X_0 = 0\}$ for $d - s \leq j \leq r - 1$. [Lemma 2.3](#) shows that $\{H_{j,d} : d - s \leq j \leq r - 1\}$ satisfy the conditions (H1) and (H2). Then [Corollary 3.4](#) shows that the variety of \mathbb{A}^r

defined by $H_{j,d}$ ($d - s \leq j \leq r - 1$) is an affine equidimensional cone of dimension $d - s$, degree at most $s!/(d - r)!$ and singular locus of dimension at most $s - 1$. It follows that the projective variety of \mathbb{P}^{r-1} defined by these polynomials is equidimensional of dimension $d - s - 1$, degree at most $s!/(d - r)!$ and singular locus of dimension at most $s - 2$.

Observe that $V(H_{j,d}: d - s \leq j \leq r - 1) \subset \mathbb{P}^{r-1}$ is a set-theoretic complete intersection, whose singular locus has codimension at least $d - s - 1 - (s - 2) \geq 3$. Therefore, the Hartshorne connectedness theorem (see, e.g., [18, Theorem 4.2]) shows that $V(H_{j,d}: d - s \leq j \leq r - 1)$ is absolutely irreducible.

On the other hand, since $\text{pcl}(V_r^\alpha)$ is equidimensional of dimension $d - s$, we have that each irreducible component of $\text{pcl}(V_r^\alpha) \cap \{X_0 = 0\}$ has dimension at least $d - s - 1$. Furthermore, $\text{pcl}(V_r^\alpha) \cap \{X_0 = 0\}$ is contained in the projective variety $V(H_{j,d}: d - s \leq j \leq r - 1)$, which is absolutely irreducible of dimension $d - s - 1$. We conclude that $\text{pcl}(V_r^\alpha) \cap \{X_0 = 0\}$ is also absolutely irreducible of dimension $d - s - 1$, and hence

$$\text{pcl}(V_r^\alpha) \cap \{X_0 = 0\} = V(H_{j,d}: d - s \leq j \leq r - 1).$$

Finally, by [11, Theorem 18.15] we deduce that the polynomials $H_{j,d}$ ($d - s \leq j \leq r - 1$) define a radical ideal. As a consequence, we see that $\deg(\text{pcl}(V_r^\alpha) \cap \{X_0 = 0\}) = \prod_{j=d-s}^{r-1} \deg H_{j,d} = s!/(d - r)!$. This finishes the proof of the theorem. \square

We conclude this section with a statement that summarizes all the facts we shall need concerning the projective closure $\text{pcl}(V_r^\alpha)$.

Theorem 3.7. *The projective variety $\text{pcl}(V_r^\alpha) \subset \mathbb{P}^r$ is an absolutely irreducible ideal-theoretic complete intersection of dimension $d - s$, degree $s!/(d - r)!$ and singular locus of dimension at most $s - 1$.*

Proof. We have already shown that $\text{pcl}(V_r^\alpha)$ is an equidimensional variety of dimension $d - s$ and degree at most $s!/(d - r)!$. According to Corollary 3.4, the singular locus of $\text{pcl}(V_r^\alpha)$ lying in the open set $\{X_0 \neq 0\}$ has dimension at most $s - 1$, while Lemma 3.5 shows that the singular locus at infinity has dimension at most $s - 2$. This shows that the singular locus of $\text{pcl}(V_r^\alpha)$ has dimension at most $s - 1$.

On the other hand, we observe that $\text{pcl}(V_r^\alpha)$ is contained in the projective variety $V(R_j^{\alpha,h}: d - s \leq j \leq r - 1)$. We have the inclusions

$$V(R_j^{\alpha,h}: d - s \leq j \leq r - 1) \cap \{X_0 \neq 0\} \subset V(R_j^\alpha: d - s \leq j \leq r - 1),$$

$$V(R_j^{\alpha,h}: d - s \leq j \leq r - 1) \cap \{X_0 = 0\} \subset V(H_{d,j}: d - s \leq j \leq r - 1).$$

Both $\{R_j^\alpha: d - s \leq j \leq r - 1\}$ and $\{H_{j,d}: d - s \leq j \leq r - 1\}$ satisfy the conditions (H1) and (H2). Then Corollary 3.4 shows that $V(R_j^\alpha: d - s \leq j \leq r - 1) \subset \mathbb{A}^r$ is equidimensional of dimension $d - s$ and $V(H_{d,j}: d - s \leq j \leq r - 1) \subset \mathbb{P}^{r-1}$ is equidimensional of dimension $d - s - 1$. We conclude that $V(R_j^{\alpha,h}: d - s \leq j \leq r - 1) \subset \mathbb{P}^r$ has dimension at most

$d - s$. Taking into account that it is defined by $r - d + s$ polynomials, we deduce that it is a set-theoretic complete intersection of dimension $r - (r - d + s) = d - s$. This implies that it is equidimensional of dimension $d - s$ and therefore has no irreducible component contained in the hyperplane at infinity. In particular, it agrees with the projective closure of its restriction to \mathbb{A}^r (see, e.g., [18, Proposition I.5.17]). As such a restriction is the affine variety $V_r^\alpha = V(R_j^\alpha: d - s \leq j \leq r - 1)$, we deduce that

$$\text{pcl}(V_r^\alpha) = V(R_j^{\alpha,h}: d - s \leq j \leq r - 1).$$

Since its singular locus has dimension at most $s - 1$ and $d - s - (s - 1) \geq 3$, the Hartshorne connectedness theorem (see, e.g., [18, Theorem 4.2]) proves that $V(R_j^{\alpha,h}: d - s \leq j \leq r - 1)$ is absolutely irreducible. Finally, arguing as in the proof of Theorem 3.6, by [11, Theorem 18.15] we see that the polynomials $R_j^{\alpha,h}$ ($d - s \leq j \leq r - 1$) define a radical ideal. This in turn implies that $\deg \text{pcl}(V_r^\alpha) = \prod_{j=d-s}^{r-1} \deg R_j^{\alpha,h} = s!/(d - r)!$ and finishes the proof of the theorem. \square

4. The number of q -rational points of V_r^α

As before, we fix integers d and s with $d < q$, $1 \leq s \leq d - 2$ and $2(s + 1) \leq d$. For a given $\alpha := (a_{d-1}, \dots, a_{d-s})$, set $f^\alpha := T^d + a_{d-1}T^{d-1} + \dots + a_{d-s}T^{d-s} \in \mathbb{F}_q[T]$. As asserted before, our objective is to determine the asymptotic behavior of the average value set $\mathcal{V}(d, s, \alpha)$ of (1.1).

For this purpose, according to Theorem 2.1, we have to determine, for $d - s + 1 \leq r \leq d$, the number χ_r^α of subsets $\mathcal{X}_r \subset \mathbb{F}_q$ of r elements such that there exists $g \in \mathbb{F}_q[T]$ of degree at most $d - s - 1$ interpolating $-f^\alpha$ at all the elements of \mathcal{X}_r . In Section 2.1 we associate to α certain polynomials $R_j^\alpha \in \mathbb{F}_q[X_1, \dots, X_r]$ ($d - s \leq j \leq r - 1$) with the property that the number of common q -rational zeros of $R_{d-s}^\alpha, \dots, R_{r-1}^\alpha$ with pairwise-distinct coordinates equals $r! \chi_r^\alpha$, namely

$$\chi_r^\alpha = \frac{1}{r!} \left| \left\{ \mathbf{x} \in \mathbb{F}_q^r: R_j^\alpha(\mathbf{x}) = 0 \ (d - s \leq j \leq r - 1), \ x_k \neq x_l \ (1 \leq k < l \leq r) \right\} \right|.$$

The results of Section 3 are fundamental for establishing the asymptotic behavior of χ_r^α . Fix r with $d - s + 1 \leq r \leq d$, let $V_r^\alpha \subset \mathbb{A}^r$ be the affine variety defined by $R_{d-s}^\alpha, \dots, R_{r-1}^\alpha \in \mathbb{F}_q[X_1, \dots, X_r]$ and denote by $\text{pcl}(V_r^\alpha) \subset \mathbb{P}^r$ the projective closure of V_r^α . According to Theorems 3.6 and 3.7, both $\text{pcl}(V_r^\alpha) \cap \{X_0 = 0\} \subset \mathbb{P}^{r-1}$ and $\text{pcl}(V_r^\alpha) \subset \mathbb{P}^r$ are projective, absolutely irreducible, ideal-theoretic complete intersections defined over \mathbb{F}_q , of dimension $d - s - 1$ and $d - s$, respectively, both of degree $s!/(d - r)!$, having a singular locus of dimension at most $s - 2$ and $s - 1$, respectively.

4.1. Estimates on the number of q -rational points of complete intersections

In what follows, we shall use an estimate on the number of q -rational points of a projective complete intersection defined over \mathbb{F}_q due to [5] (see [14,15] for further explicit

estimates of this type). In [5, Corollary 8.4] the authors prove that, for an absolutely irreducible ideal-theoretic complete intersection $V \subset \mathbb{P}^m$ of dimension $n := m - r$, degree $\delta \geq 2$, which is defined over \mathbb{F}_q by polynomials of degree $d_1 \geq \dots \geq d_r \geq 2$, and having singular locus of dimension at most $s \leq n - 3$, the number $|V(\mathbb{F}_q)|$ of q -rational points of V satisfies the estimate

$$||V(\mathbb{F}_q)| - p_n| \leq 14D^3\delta^2q^{n-1}, \tag{4.1}$$

with $p_n := q^n + q^{n-1} + \dots + q + 1 = |\mathbb{P}^n(\mathbb{F}_q)|$ and $D := \sum_{i=1}^r (d_i - 1)$.

From (4.1) we obtain the following result.

Theorem 4.1. *With notations and assumptions as above, for $d - s + 1 \leq r \leq d$ we have*

$$\left| \chi_r^{\mathbf{a}} - \frac{q^{d-s}}{r!} \right| \leq \binom{r}{2} \frac{\delta_r}{r!} q^{d-s-1} + \frac{14}{r!} D_r^3 \delta_r^2 (q+1) q^{d-s-2},$$

where $D_r := \sum_{j=d-r+1}^s (j-1)$ and $\delta_r := \prod_{j=d-r+1}^s j = s!/(d-r)!$.

Proof. First we obtain an estimate on the number of q -rational points of $V_r^{\mathbf{a}}$. Let $V_{r,\infty}^{\mathbf{a}} := \text{pcl}(V_r^{\mathbf{a}}) \cap \{X_0 = 0\}$. Combining Theorems 3.6 and 3.7 with (4.1), we obtain

$$\begin{aligned} ||\text{pcl}(V_r^{\mathbf{a}})(\mathbb{F}_q)| - p_{d-s}| &\leq 14D_r^3\delta_r^2q^{d-s-1}, \\ ||V_{r,\infty}^{\mathbf{a}}(\mathbb{F}_q)| - p_{d-s-1}| &\leq 14D_r^3\delta_r^2q^{d-s-2}. \end{aligned}$$

As a consequence,

$$\begin{aligned} ||V_r^{\mathbf{a}}(\mathbb{F}_q)| - q^{d-s}| &= ||\text{pcl}(V_r^{\mathbf{a}})(\mathbb{F}_q)| - |V_{r,\infty}^{\mathbf{a}}(\mathbb{F}_q)| - p_{d-s} + p_{d-s-1}| \\ &\leq ||\text{pcl}(V_r^{\mathbf{a}})(\mathbb{F}_q)| - p_{d-s}| + ||V_{r,\infty}^{\mathbf{a}}(\mathbb{F}_q)| - p_{d-s-1}| \\ &\leq 14D_r^3\delta_r^2(q+1)q^{d-s-2}. \end{aligned} \tag{4.2}$$

Next we obtain an upper bound on the number of q -rational points of $V_r^{\mathbf{a}}$ which are not useful for our purposes, namely those with at least two distinct coordinates taking the same value.

Let $V_{r,=}^{\mathbf{a}}(\mathbb{F}_q)$ be the subset of $V_r^{\mathbf{a}}(\mathbb{F}_q)$ consisting of all such points, namely

$$V_{r,=}^{\mathbf{a}}(\mathbb{F}_q) := \bigcup_{1 \leq i < j \leq r} V_r^{\mathbf{a}}(\mathbb{F}_q) \cap \{X_i = X_j\},$$

and set $V_{r,\neq}^{\mathbf{a}}(\mathbb{F}_q) := V_r^{\mathbf{a}}(\mathbb{F}_q) \setminus V_{r,=}^{\mathbf{a}}(\mathbb{F}_q)$. Let $\mathbf{x} := (x_1, \dots, x_r) \in V_{r,=}^{\mathbf{a}}(\mathbb{F}_q)$. Without loss of generality we may assume that $x_{r-1} = x_r$ holds. Then \mathbf{x} is a q -rational point of the affine variety $W_{r-1,r} \subset \{X_{r-1} = X_r\}$ defined by the polynomials $S_{d-s}^{\mathbf{a}}(\Pi_1^*, \dots, \Pi_s^*), \dots, S_{r-1}^{\mathbf{a}}(\Pi_1^*, \dots, \Pi_s^*) \in \mathbb{F}_q[X_1, \dots, X_{r-1}]$, where $\Pi_i^* := \Pi_i(X_1, \dots, X_{r-1}, X_{r-1})$

is the polynomial of $\mathbb{F}_q[X_1, \dots, X_{r-1}]$ obtained by substituting X_{r-1} for X_r in the i th elementary symmetric polynomial of $\mathbb{F}_q[X_1, \dots, X_r]$. Taking into account that Π_1^*, \dots, Π_s^* are algebraically independent elements of $\overline{\mathbb{F}}_q[X_1, \dots, X_{r-1}]$, we conclude that $S_{d-s}^{\mathbf{a}}(\Pi_1^*, \dots, \Pi_s^*), \dots, S_{r-1}^{\mathbf{a}}(\Pi_1^*, \dots, \Pi_s^*)$ form a regular sequence of $\mathbb{F}_q[X_1, \dots, X_{r-1}]$. This implies that $W_{r-1,r}$ is of dimension $d - s - 1$, and hence, [14, Proposition 12.1] or [3, Proposition 3.1] show that

$$|W_{r-1,r}(\mathbb{F}_q)| \leq \deg W_{r-1,r} q^{d-s-1} \leq \deg V_r^{\mathbf{a}} q^{d-s-1}.$$

Therefore, we obtain

$$|V_{r,=}^{\mathbf{a}}(\mathbb{F}_q)| \leq \binom{r}{2} \delta_r q^{d-s-1}.$$

Combining (4.2) with this upper bound we have

$$||V_{r,\neq}^{\mathbf{a}}(\mathbb{F}_q)| - q^{d-s}| \leq \binom{r}{2} \delta_r q^{d-s-1} + 14D_r^3 \delta_r^2 (q+1) q^{d-s-2}.$$

From this inequality we easily deduce the statement of the theorem. \square

The estimate of Theorem 4.1 is the essential step in order to determine the behavior of the average value set $\mathcal{V}(d, s, \mathbf{a})$. More precisely, we have the following result.

Corollary 4.2. *With assumptions and notations as in Theorem 4.1, we have*

$$\left| \mathcal{V}(d, s, \mathbf{a}) - \mu_d q - \frac{1}{2e} \right| \leq \frac{s^2 + 1}{(d - s - 1)!} + \frac{21}{8} \frac{s^6 (s!)^2}{d!} \sum_{k=0}^{s-1} \binom{d}{k} \frac{1}{k!} + \frac{7}{q}. \tag{4.3}$$

Proof. According to Theorem 2.1, we have

$$\begin{aligned} & \mathcal{V}(d, s, \mathbf{a}) - \mu_d q \\ &= \sum_{r=1}^{d-s} (-q)^{1-r} \left(\binom{q}{r} - \frac{q^r}{r!} \right) + \frac{1}{q^{d-s-1}} \sum_{r=d-s+1}^d (-1)^{r-1} \left(\chi_r^{\mathbf{a}} - \frac{q^{d-s}}{r!} \right). \end{aligned} \tag{4.4}$$

First we obtain an upper bound for the absolute value $A(d, s)$ of the first term in the right-hand side of (4.4). For this purpose, given positive integers k, n with $k \leq n$, we shall denote by $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ the unsigned Stirling number of the first kind, namely the number of permutations of n elements with k disjoint cycles. The following properties of the Stirling numbers are well-known (see, e.g., [12, §A.8]):

$$\left[\begin{smallmatrix} r \\ r \end{smallmatrix} \right] = 1, \quad \left[\begin{smallmatrix} r \\ r-1 \end{smallmatrix} \right] = \binom{r}{2}, \quad \sum_{k=0}^r \left[\begin{smallmatrix} r \\ k \end{smallmatrix} \right] = r!.$$

Taking into account the identity $\binom{q}{r} = \sum_{k=0}^r \frac{(-1)^{r-k}}{r!} \left[\begin{smallmatrix} r \\ k \end{smallmatrix} \right] q^k$, we obtain

$$\begin{aligned}
 A(d, s) &:= \sum_{r=2}^{d-s} (-q)^{1-r} \left(\binom{q}{r} - \frac{q^r}{r!} \right) \\
 &= \sum_{r=2}^{d-s} q^{1-r} \sum_{k=0}^{r-1} \frac{(-1)^{k+1}}{r!} \begin{bmatrix} r \\ k \end{bmatrix} q^k \\
 &= \sum_{r=0}^{d-s-2} \frac{(-1)^r}{2r!} + \sum_{r=2}^{d-s} q^{1-r} \sum_{k=0}^{r-2} \frac{(-1)^{k+1}}{r!} \begin{bmatrix} r \\ k \end{bmatrix} q^k.
 \end{aligned}$$

In order to bound the second term in the right-hand side of the previous expression, we have

$$\begin{aligned}
 \sum_{k=0}^{r-2} \frac{1}{r!} \begin{bmatrix} r \\ k \end{bmatrix} q^k &\leq \sum_{k=0}^{r-3} \frac{1}{r!} \begin{bmatrix} r \\ k \end{bmatrix} q^k + \frac{1}{r!} \begin{bmatrix} r \\ r-2 \end{bmatrix} q^{r-2} \leq q^{r-3} + \frac{8}{r^2} q^{r-2} \\
 &\leq \left(\frac{1}{d} + \frac{8}{r^2} \right) q^{r-2}.
 \end{aligned}$$

As a consequence, we obtain

$$\left| A(d, s) - \frac{1}{2e} \right| \leq \frac{1}{2(d-s-1)!} + \sum_{r=2}^{d-s} \left(\frac{1}{d} + \frac{8}{r^2} \right) \frac{1}{q} \leq \frac{1}{2(d-s-1)!} + \frac{7}{q}.$$

Next we consider the absolute value of the second term in the right-hand side of (4.4). From Theorem 4.1 we have that

$$\begin{aligned}
 B(d, s) &:= \frac{1}{q^{d-s-1}} \sum_{r=d-s+1}^d \left| \chi_r^a - \frac{q^{d-s}}{r!} \right| \\
 &\leq \sum_{r=d-s+1}^d \binom{r}{2} \frac{\delta_r}{r!} + \sum_{r=d-s+1}^d \frac{14}{r!} D_r^3 \delta_r^2 \left(1 + \frac{1}{q} \right).
 \end{aligned}$$

Concerning the first term in the right-hand side, we see that

$$\begin{aligned}
 \sum_{r=d-s+1}^d \binom{r}{2} \frac{\delta_r}{r!} &= \frac{s!}{2(d-2)!} \sum_{r=d-s+1}^d \binom{d-2}{r-2} \\
 &\leq \frac{s \cdot s!}{2(d-2)! (s-1)!} = \frac{s^2}{2(d-s-1)!}.
 \end{aligned}$$

On the other hand,

$$\sum_{r=d-s+1}^d \frac{14}{r!} D_r^3 \delta_r^2 \leq \frac{7}{4} \sum_{r=d-s+1}^d \frac{s^3 (s-1)^3 (s!)^2}{r! ((d-r)!)^2} = \frac{7}{4} \sum_{k=0}^{s-1} \frac{s^6 (s!)^2}{(d-k)! (k!)^2}.$$

Therefore, we obtain

$$B(d, s) \leq \frac{s^2}{2(d-s-1)!} + \frac{21}{8} \frac{s^6 (s!)^2}{d!} \sum_{k=0}^{s-1} \binom{d}{k} \frac{1}{k!}.$$

Combining the upper bounds for $A(d, s)$ and $B(d, s)$ the statement of the corollary follows. \square

4.2. On the behavior of (4.3)

In this section we analyze the behavior of the right-hand side of (4.3). Such an analysis consists of elementary calculations, which shall only be sketched.

Fix k with $0 \leq k \leq s-1$ and denote $h(k) := \binom{d}{k} \frac{1}{k!}$. Analyzing the sign of the differences $h(k+1) - h(k)$ for $0 \leq k \leq s-2$, we deduce the following remark, which is stated without proof.

Remark 4.3. Let $k_0 := -1/2 + \sqrt{5 + 4d}/2$. Then h is a unimodal function in the integer interval $[0, s-1]$ which reaches its maximum at $\lfloor k_0 \rfloor$.

From Remark 4.3 we see that

$$\frac{s^6 (s!)^2}{d!} \sum_{k=0}^{s-1} \binom{d}{k} \frac{1}{k!} \leq \frac{s^7 (s!)^2}{d!} \binom{d}{\lfloor k_0 \rfloor} \frac{1}{\lfloor k_0 \rfloor!} = \frac{s^7 (s!)^2}{(d - \lfloor k_0 \rfloor)! (\lfloor k_0 \rfloor!)^2}. \tag{4.5}$$

In order to obtain an upper bound for the right-hand side of (4.5) we shall use the Stirling formula (see, e.g., [12, p. 747]): for $m \in \mathbb{N}$, there exists θ with $0 \leq \theta < 1$ such that $m! = (m/e)^m \sqrt{2\pi m} e^{\theta/12m}$ holds.

Applying the Stirling formula, and taking into account that $2(s+1) \leq d$, we see that there exist θ_i ($i = 1, 2, 3$) with $0 \leq \theta_i < 1$ such that

$$C(d, s) := \frac{s^7 (s!)^2}{(d - \lfloor k_0 \rfloor)! (\lfloor k_0 \rfloor!)^2} \leq \frac{(\frac{d}{2} - 1)^8 (\frac{d}{2} - 1)^{d-2} e^{2 + \lfloor k_0 \rfloor + \frac{\theta_1}{3d-6} - \frac{\theta_2}{12(d - \lfloor k_0 \rfloor)} - \frac{\theta_3}{6\lfloor k_0 \rfloor}}{(d - \lfloor k_0 \rfloor)^{d - \lfloor k_0 \rfloor} \sqrt{2\pi(d - \lfloor k_0 \rfloor)} \lfloor k_0 \rfloor^{2\lfloor k_0 \rfloor + 1}}.$$

By elementary calculations we obtain

$$\begin{aligned} (d - \lfloor k_0 \rfloor)^{-d + \lfloor k_0 \rfloor} &\leq d^{-d + \lfloor k_0 \rfloor} e^{\lfloor k_0 \rfloor (d - \lfloor k_0 \rfloor) / d}, \\ \frac{d^{\lfloor k_0 \rfloor}}{\lfloor k_0 \rfloor^{2\lfloor k_0 \rfloor}} &\leq e^{(d - \lfloor k_0 \rfloor)^2 / \lfloor k_0 \rfloor}, \\ \left(\frac{d}{2} - 1\right)^{d-2} &\leq \left(\frac{d}{2}\right)^{d-2} e^{4/d-2}. \end{aligned}$$

It follows that

$$C(d, s) \leq \left(\frac{d}{2} - 1\right)^8 \frac{e^{\lfloor k_0 \rfloor + \frac{1}{3d-6} + \frac{4}{d} + \frac{\lfloor k_0 \rfloor}{d}(d - \lfloor k_0 \rfloor) + \frac{1}{\lfloor k_0 \rfloor}(d - \lfloor k_0 \rfloor)^2}}{d^2 2^{d-2} \sqrt{2\pi(d - \lfloor k_0 \rfloor)} \lfloor k_0 \rfloor}.$$

By the definition of $\lfloor k_0 \rfloor$, it is easy to see that

$$\begin{aligned} \lfloor k_0 \rfloor + \frac{\lfloor k_0 \rfloor}{d}(d - \lfloor k_0 \rfloor) &\leq 2\lfloor k_0 \rfloor - \frac{1}{5}, \\ \frac{1}{\lfloor k_0 \rfloor}(d - \lfloor k_0 \rfloor)^2 &\leq 4, \\ \frac{\left(\frac{d}{2} - 1\right)^3}{d^2 \lfloor k_0 \rfloor \sqrt{d - \lfloor k_0 \rfloor}} &\leq \frac{3}{20}. \end{aligned}$$

Therefore, taking into account that $d \geq 2$, we conclude that

$$C(d, s) \leq \frac{3\left(\frac{d}{2} - 1\right)^5 e^{\frac{1}{3d-6} + \frac{4}{d} - \frac{1}{5} + 3 + \sqrt{5+4d}}}{5\sqrt{2\pi}2^d}. \tag{4.6}$$

Combining this bound with [Corollary 4.2](#) we obtain the main result of this section, namely [Theorem 1.1](#) of the introduction, which is restated here for the sake of readability.

Theorem 4.4. For $q > d$ and $1 \leq s \leq \frac{d}{2} - 1$, we have

$$\left| \mathcal{V}(d, s, \mathbf{a}) - \mu_{dq} - \frac{1}{2e} \right| \leq \frac{(d-2)^5 e^{2\sqrt{d}}}{2^{d-2}} + \frac{7}{q}. \tag{4.7}$$

Proof. From [\(4.6\)](#) and the fact that $\sqrt{5+4d} \leq 4/5 + 2\sqrt{d}$ holds for $d \geq 2$, we conclude that

$$\frac{21}{8} \frac{s^6 (s!)^2}{d!} \sum_{k=0}^{s-1} \binom{d}{k} \frac{1}{k!} \leq 3 \frac{(d-2)^5 e^{2\sqrt{d}}}{2^d}.$$

On the other hand, it is not difficult to see that

$$\frac{s^2 + 1}{2(d-s-1)!} \leq \frac{(d-2)^5 e^{2\sqrt{d}}}{2^d}.$$

From these inequalities the statement of the theorem easily follows. \square

We make several remarks concerning the upper bound of [\(4.7\)](#).

Remark 4.5. Let $f : \mathbb{Z}_{\geq 4} \rightarrow \mathbb{R}$, $f(d) := e^{2\sqrt{d}}(d-2)^5 2^{-d}$. Then f is a unimodal function which reaches its maximum value at $d_0 := 14$, namely $f(d_0) \approx 1.08 \cdot 10^5$. Furthermore, it is easy to see that $\lim_{d \rightarrow +\infty} f(d) = 0$, and indeed for $d \geq 51$, we have $f(d) < 1$.

An obvious upper bound for the left-hand side of (4.7) is $|\mathcal{V}(d, s, \mathbf{a}) - \mu_d q - (2e)^{-1}| \leq (1 - \mu_d)q$. Direct computations show that the upper bound of Theorem 4.4 is not interesting for small values of q if $d \leq 44$.

On the other hand, with a slight further restriction for the range of values admissible for s , namely for $1 \leq s \leq \frac{d}{2} - 3$, it is possible to obtain significant improvements of the upper bound of Theorem 4.4. More precisely, arguing as in the proof of Theorem 4.4 we obtain the upper bound

$$\left| \mathcal{V}(d, s, \mathbf{a}) - \mu_d q - \frac{1}{2e} \right| \leq \frac{9(d-6)e^{2\sqrt{d}}}{2^{d-2}} + \frac{7}{q}. \tag{4.8}$$

Let $g := \mathbb{Z}_{\geq 7} \rightarrow \mathbb{R}$, $g(d) := 9(d-6)e^{2\sqrt{d}}2^{-d+2}$. Then g is a unimodal function reaching at $d_1 := 9$ its maximum value, namely $g(d_1) := 85$. Furthermore, we have that $\lim_{d \rightarrow +\infty} g(d) = 0$ and $g(d) < 1$ for $d \geq 24$. In particular, (4.8) is nontrivial for $d \geq 19$.

Remark 4.6. It may be worthwhile to discuss the asymptotic behavior of the right-hand side of (4.3). Let

$$H(d, s) := \frac{s^6 (s!)^2}{d!} \sum_{k=0}^{s-1} \binom{d}{k} \frac{1}{k!}.$$

Let $a_d(k) := \binom{d}{k} \frac{1}{k!}$ for $0 \leq k \leq d$. In [21] it is shown that a_d is a unimodal function in the integer interval $[0, d]$ reaching its maximum at $\lfloor k_0 \rfloor$, where k_0 is defined as in Remark 4.3. Furthermore, for $\epsilon > 1/4$ it is proved that

$$\sum_{k=0}^d a_d(k) \sim \sum_{k \in (k_0 - d^\epsilon, k_0 + d^\epsilon)} a_d(k) \sim \frac{1}{2\sqrt{\pi e}} d^{-1/4} e^{2\sqrt{d}},$$

where the symbol \sim denotes equal asymptotic behavior. Assume that $s > \lfloor k_0 \rfloor + d^\epsilon$ with $\epsilon > 1/4$. Then by the Stirling formula we obtain

$$H(d, s) \sim \frac{1}{\sqrt{2e}} \left(\frac{e}{d}\right)^d \left(\frac{s}{e}\right)^{2s} s^7 e^{2(s-\sqrt{d})} d^{-3/4}.$$

We finally observe that, if $s \leq \lfloor k_0 \rfloor + d^\epsilon$ with $\epsilon > 1/4$, then the right-hand side of this expression is an upper bound for $H(d, s)$ for d sufficiently large. This shows that $H(d, s)$ converges to 0 with a double exponential rate $d^{-(1-2\lambda)d}$ for $s \leq \lambda d$ with $\lambda \in [0, 1/2[$.

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