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On basic chordal graphs and some of its subclasses

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ABSTRACT

Basic chordal graphs arose when comparing clique trees of chordal graphs and compatible trees of dually chordal graphs. They were defined as those chordal graphs whose clique trees are exactly the compatible trees of its clique graph.

In this work, we consider some subclasses of basic chordal graphs. One of them is the class of hereditary basic chordal graphs, which will turn out to have many possible characterizations. Those characterizations will show that the class was already studied, but under different names and in different contexts.

We also study the connection between basic chordal graphs and some subclasses of chordal graphs with special clique trees, like *DV* graphs and *RDV* graphs. As a result, it will be possible to define the classes of basic *DV* graphs and basic *RDV* graphs.

Additionally, we study the behavior of the clique operator over all the considered subclasses.

1. Introduction

1.1. Definitions

For a graph G, we denote the set of its *vertices* by $V(G)$ and the set of its *edges* by $E(G)$. The subgraph *induced* by a subset A of $V(G)$, denoted by $G[A]$, has A as vertex set, and two vertices are adjacent in $G[A]$ if and only if they are adjacent in G. We say that A is a *complete set* of G if $G[A]$ is a complete graph, i.e., all its vertices are pairwise adjacent. A *clique* is a maximal subset of pairwise adjacent vertices, that is, a maximal complete set. The family of cliques of G is denoted by $\mathcal{C}(G)$. For $v \in V(G)$, the family of cliques containing v is denoted by $\mathcal{C}_v$. The reader must be aware of the fact that many papers use the term clique to refer to complete (not necessarily maximal) sets. Thus, a clique in this paper is equivalent to a maximal clique in those other papers.

For a vertex $v \in V(G)$, the *open neighborhood* of v, denoted by $N(v)$ or $N_G(v)$, is the set of all the vertices adjacent to v in G. The *degree* $deg(v)$ of v is the number $|N(v)|$. The *closed neighborhood* of v, denoted by $N[v]$ or $N_G[v]$, is the set $N(v) \cup \{v\}$. Vertex v is said to be *simplicial* if $N[v]$ is complete. This is equivalent to $N[v]$ being a clique. Any clique that is the closed neighborhood of some vertex is called *simplicial clique*.

Given two nonadjacent vertices u and v in the same connected component of G, a uv*-separator* is a set S contained in $V(G)$ such that u and v are in different connected components of $G - S$, where $G - S$ denotes the induced subgraph $G[V(G) \setminus S]$. This separator S is *minimal* if no proper subset of S is also a uv-separator. We will just say *minimal vertex separator* to refer to a set S that is a uv-minimal separator for some pair of nonadjacent vertices u and v in G. The family of all minimal vertex separators of G will be denoted by $\mathcal{S}(G)$.

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0166-218X/

Let T be a tree. For $v, w \in V(T)$, the notation $T[v, w]$ is used to denote either the path in T from v to w or the vertices of that path, depending on the context. The set of inner vertices of this path is denoted by $T(v, w)$.

Let $\mathcal{F}$ be a family of nonempty sets of vertices of G. If $F \in \mathcal{F}$, then F is called a *member* of $\mathcal{F}$. If $v \in \bigcup_{F\in\mathcal{F}} F$, then we say that v is a *vertex* of $\mathcal{F}$. The family $\mathcal{F}$ is *Helly* if the intersection of all the members of every subfamily of pairwise intersecting sets is not empty. If $\mathcal{C}(G)$ is a Helly family, then we say that G is a *clique-Helly graph*. We say that $\mathcal{F}$ is *separating* if, for every ordered pair (v, w) of vertices of $\mathcal{F}$, there exists $F \in \mathcal{F}$ such that $v \in F$ and $w \notin F$. The *intersection graph* of $\mathcal{F}$, denoted $L(\mathcal{F})$, has the members of $\mathcal{F}$ as vertices, two of them being adjacent if and only if they are not disjoint. The *clique graph* $K(G)$ of G is the intersection graph of $\mathcal{C}(G)$. The *two-section graph* $S(\mathcal{F})$ of $\mathcal{F}$ is another graph whose vertices are the vertices of $\mathcal{F}$, in such a way that two vertices v and w are adjacent in $S(\mathcal{F})$ if and only if there exists $F \in \mathcal{F}$ such that $\{v, w\} \subseteq F$.

For every vertex v of $\mathcal{F}$, let $D_v = \{F \in \mathcal{F} : v \in F\}$. The *dual family* $D\mathcal{F}$ of $\mathcal{F}$ consists of all the sets D_v. When $\mathcal{F} = \mathcal{C}(G)$, we have that $D_v = \mathcal{C}_v$. An even more general notation will also be used: given a set A of vertices, $\mathcal{C}_A$ is defined to be equal to $\{C \in \mathcal{C}(G) : A \subseteq C\}$.

All graphs considered will be assumed to be connected, unless stated otherwise.

1.2. Chordal graphs, basic chordal graphs and goals

Chordal graphs were originally defined as those graphs for which every cycle of length greater than or equal to four has a *chord*, i.e., an edge connecting two nonconsecutive vertices of the cycle. It was later found that they can be characterized in many other ways. One of them involves the clique tree. A *clique tree* of a graph G is a tree T whose vertex set is $\mathcal{C}(G)$ and such that, for every $v \in V(G)$, the set $\mathcal{C}_v$ induces a subtree of T. Alternatively, it can be defined as a tree T whose vertices are the cliques of G and such that, for all $C_1, C_2 \in \mathcal{C}(G)$ and C_3 in $T[C_1, C_2]$, we have $C_1 \cap C_2 \subseteq C_3$. Chordal graphs can be characterized using clique trees as follows,

Theorem 1.1 (*[9]*). *A graph is chordal if and only if it has a clique tree.*

It is interesting to note that every clique tree of a chordal graph G is a spanning tree of $K(G)$. To prove it, suppose to the contrary that T has an edge CC' that is not an edge of $K(G)$, that is, C and C' are such that $C \cap C' = \emptyset$. Let T_1 and T_2 be the two connected components of $T - CC'$, with $C \in V(T_1)$ and $C' \in V(T_2)$. Since no $v \in V(G)$ is such that $\{C, C'\} \subseteq \mathcal{C}_v$, we can partition $V(G)$ into two sets A and B, where $A = \{v \in V(G) : \mathcal{C}_v \subseteq V(T_1)\}$ and $B = \{v \in V(G) : \mathcal{C}_v \subseteq V(T_2)\}$. Both sets are not empty because C is contained in A and C' is contained in B. Thus, no vertex of A is adjacent to a vertex of B because there is no clique in G containing vertices of both A and B. As a consequence, G would be disconnected, contrary to our initial assumption that we would work with connected graphs only.

Another classical characterization of chordal graphs states that a graph is chordal if and only if every minimal separator of two nonadjacent vertices is a complete set [7]. However, no minimal vertex separator of a chordal graph is a clique. There is an important connection between minimal vertex separators and clique trees that will be reflected in the next three theorems, which are stated here due to their ulterior usefulness.

Given a graph G, two cliques C_1 and C_2 are a *separating pair* if $C_1 \cap C_2$ is a separator of every pair v, w of vertices such that $v \in C_1 \setminus C_2$ and $w \in C_2 \setminus C_1$.

Theorem 1.2 (*[13]*). *Let G be a chordal graph and $S \in \mathcal{S}(G)$. Then, there exists a separating pair C_1, C_2 such that $S = C_1 \cap C_2$.*

Theorem 1.3 (*[13]*). *Let C_1 and C_2 be two distinct cliques of a chordal graph G. Then, there exists a clique tree T of G such that $C_1C_2 \in E(T)$ if and only if C_1 and C_2 form a separating pair.*

Finally, it is interesting to note that, when just one clique tree of a graph is known, it is possible to determine what the edges of the other clique trees (if any) can be:

Theorem 1.4 (*[13]*). *Let G be a chordal graph, T be a clique tree of G and $C_1, C_2 \in \mathcal{C}(G)$, with $C_1 \neq C_2$. Then, there exists a clique tree of G having C_1C_2 as an edge if and only if there exist two cliques C_3 and C_4 that are adjacent in $T[C_1, C_2]$ and with $C_3 \cap C_4 = C_1 \cap C_2$. In that case. $T - C_3C_4 + C_1C_2$ is also a clique tree of G.*

The clique graphs of chordal graphs form an also well known class: *dually chordal graphs*. Dually chordal graphs also have a representative tree structure. A *compatible tree* of a graph G is a spanning tree T of G such that every clique of G induces a subtree in T. The compatible tree can also be defined using the condition that every closed neighborhood of G induces a subtree of T. A graph is dually chordal if and only if it has a compatible tree [2].

In case that we use the definition of compatible tree involving closed neighborhoods, it can be proved that the clique graph of a chordal graph G is dually chordal by showing that any clique tree of G is a compatible tree of $K(G)$.

Proposition 1.5 (*[6]*). *Let G be a chordal graph. Then, every clique tree of G is compatible with $K(G)$.*

However, it is not necessarily true that every compatible tree of $K(G)$ is a clique tree of G. Consider for example the graph of Fig. 1, which has cliques $A = \{1, 2, 3\}$, $B = \{2, 3, 5\}$, $C = \{2, 4, 5\}$ and $D = \{3, 5, 6\}$. Thus $K(G)$ is the complete graph

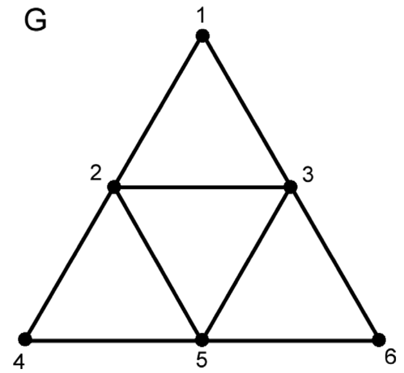

Fig. 1. A graph G such that not every compatible tree of $K(G)$ is a clique tree of G.

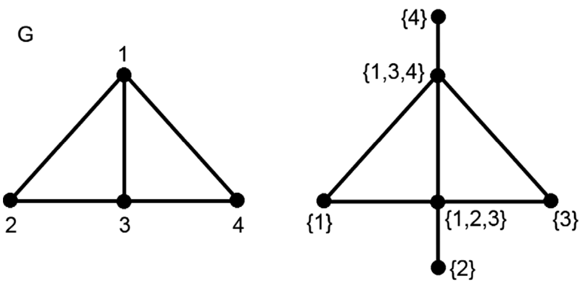

Fig. 2. For the dually chordal graph G, set $\mathcal{F} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2, 3\}, \{1, 3, 4\}\}$. The family is separating, every member of $\mathcal{F}$ is in $\mathcal{SDC}(G)$ and $S(\mathcal{F}) = G$. Thus $L(\mathcal{F})$, shown at right, is a basic chordal graph whose clique graph equals G.

on four vertices. Hence every spanning tree of $K(G)$ is a compatible tree. However, the path $ABCD$ is not a clique tree of G because $\mathcal{C}_3 = \{A, B, D\}$ and it does not induce a subpath.

Basic chordal graphs were defined as those chordal graphs whose clique trees are exactly the compatible trees of its clique graph [6].

One of the aspects about basic chordal graphs that has been studied is the recognition problem. Let S be a minimal vertex separator of a graph G. Remember that $\mathcal{C}_S$ denotes the family of cliques of G containing S. Furthermore, let B_S be the family of cliques of G that intersect every $C \in \mathcal{C}(G)$ such that $C \cap S \neq \emptyset$. The following characterization leads to a polynomial-time recognition algorithm of basic chordal graphs.

Theorem 1.6 ([6]). *A chordal graph G is basic chordal if and only if $B_S = \mathcal{C}_S$ for all $S \in \mathcal{S}(G)$.*

Another important fact about basic chordal graphs is that, despite being a strict subclass of chordal graphs, their clique graphs also form the class of dually chordal graphs. Following the notation in [6], define for a dually chordal graph G the family $\mathcal{SDC}(G)$ as the one consisting of all the subsets F of $V(G)$ such that, for every compatible tree T of G, the subgraph $T[F]$ is a subtree of T. Then we have:

Theorem 1.7 ([6]). *The following statements are true:*

- *The class of clique graphs of basic chordal graphs is equal to the class of dually chordal graphs. In other words, K(BASIC CHORDAL) = DUALLY CHORDAL.*
- *Let G be a dually chordal graph and H be a chordal graph. Then H is basic chordal and $K(H) = G$ if and only if $H = L(\mathcal{F})$, for some separating family $\mathcal{F}$ such that the two section graph of $\mathcal{F}$ equals G and every $F \in \mathcal{F}$ is in $\mathcal{SDC}(G)$.*

See an example of Theorem 1.7 in Fig. 2.

Note that each $F \in \mathcal{F}$ is a complete set of $\mathcal{S}(\mathcal{F})$, that is, a complete set of G. Thus, not any member of $\mathcal{SDC}(G)$ can be a member of $\mathcal{F}$. We have instead that the members of $\mathcal{F}$ are cliques or subsets of them. Cliques are always in $\mathcal{SDC}(G)$ by definition. However, not every subset of a clique is necessarily in $\mathcal{SDC}(G)$.

In this paper, we define and begin the study of some subclasses of basic chordal graphs. In Section 2, we introduce the class of hereditary basic chordal graphs and find several characterizations for them (see Theorem 2.5). Particularly, we find the family of minimal forbidden induced subgraphs for the class of hereditary basic chordal graphs, which allows to show that this class is equivalent to some others that have arisen in significantly different contexts, like strictly chordal graphs and (4, 6)-leaf powers.

In Section 3, we show that the class of clique graphs of the hereditary basic chordal graphs is equal to the class of block graphs, among other properties.

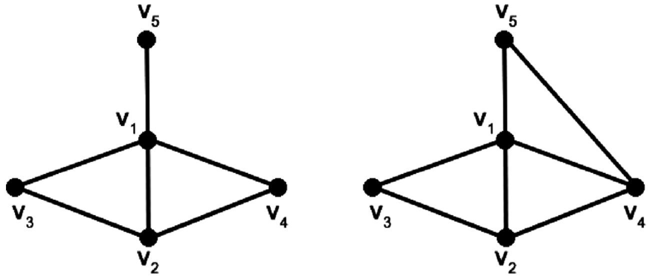

Fig. 3. A dart (left) and a gem (right).

In Section 4, we study the correspondence between special classes of clique trees and compatible trees, namely, the $DV(RDV)$-clique tree and the $DV(RDV)$-compatible tree, thus giving rise to the classes of basic DV and basic RDV graphs, which can be defined similarly to basic chordal graphs. We also find simple characterizations (see Theorem 4.3) and find the clique graphs of these classes. Much like the clique graphs of basic chordal graphs are the dually chordal graphs, we prove that the clique graphs of basic $DV(RDV)$ graphs are the dually $DV(RDV)$ graphs (see Theorems 4.4 and 4.5). Finally, we show that hereditary basic chordal graphs form a subclass of basic RDV graphs and that they can be characterized as a special type of RDV graphs (see Theorem 4.9).

2. Hereditary basic chordal graphs

We already know that not every chordal graph is basic chordal. However, every chordal graph that is not basic chordal can be transformed into a basic chordal graph by adding some vertices and edges.

Proposition 2.1 (*[6]*). *Let G be a chordal graph and V' be the set of vertices of G that are not simplicial. Let G' be the graph constructed from G by adding, for each $v \in V'$, a vertex v^* and the edge vv^*. Then, G' is basic chordal.*

In the previous proposition, G is an induced subgraph of G'. Hence, every chordal graph is the induced subgraph of some basic chordal graph. The most important conclusion from this fact is that the class of basic chordal graphs is not hereditary.

We are consequently motivated to define a graph G to be *hereditary basic chordal* if G and all its induced subgraphs are basic chordal. The previous paragraphs imply that hereditary basic chordal graphs form a strict subclass of basic chordal graphs, which will be the subject of this section.

The given definition automatically makes the class of hereditary basic chordal graphs be hereditary, so it has a family of minimal forbidden induced subgraphs.

In order to get an idea of what that family could be, suppose that G is a chordal graph that is not basic chordal. Hence, by Theorem 1.6, we can take $S \in \mathcal{S}(G)$ such that $B_S \neq \mathcal{C}_S$. Let C_1, C_2 be a separating pair (existing because of Theorem 1.2) such that $C_1 \cap C_2 = S$ and C_3 be in $B_S \setminus \mathcal{C}_S$. Then, the intersection between C_3 and S is necessarily not empty.

In order to prove the latter, suppose to the contrary that $C_3 \cap S = \emptyset$. Since $C_3 \in B_S$, it holds that $C_1 \cap C_3 \neq \emptyset$ and $C_2 \cap C_3 \neq \emptyset$. Let $w_1 \in C_1 \cap C_3$ and $w_2 \in C_2 \cap C_3$. The fact that both w_1 and w_2 are in C_3 implies that they are adjacent. On the other hand, w_1 and w_2 cannot be in S by our assumption and C_1, C_2 is a separating pair. Thus w_1 and w_2 are in different connected components of $G - S$, which contradicts that the two vertices are adjacent. Therefore, $C_3 \cap S \neq \emptyset$.

Let v_1 be a vertex in $C_3 \cap S$. Since $C_3 \notin \mathcal{C}_S$, we can take a vertex v_2 in $S \setminus C_3$. Furthermore, $C_1 \setminus C_2$ and $C_2 \setminus C_1$ must be nonempty sets. Let v_3 and v_4 be in those sets, respectively. As $v_2 \notin C_3$, we can take a vertex $v_5 \in C_3 \setminus N[v_2]$.

Let us study what type of subgraph $\{v_1.v_2, v_3, v_4, v_5\}$ induces. It is clear that $v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_3$ and v_2v_4 are edges in G, whereas v_3v_4 is not. In case there are not more edges connecting these vertices, $\{v_1.v_2, v_3, v_4, v_5\}$ induces the graph known as *dart* (see Fig. 3). Note that v_3v_5 and v_4v_5 cannot be both edges of G, otherwise $v_2v_4v_5v_3v_2$ would be a chordless cycle of G. In case that v_4v_5 is in $E(G)$, the set $\{v_1.v_2, v_3, v_4, v_5\}$ induces a graph known as *gem* (see also Fig. 3). The same can be said in the case that $v_3v_5 \in E(G)$.

Therefore, every hereditary basic chordal graph G has no induced cycle C_n, with $n \geq 4$, because G is chordal, and has no induced dart and no induced gem by the previous reasoning. Actually, it is true that the hereditary basic chordal graphs are the (dart, gem)-free chordal graphs. Before we prove this characterization of hereditary basic chordal graphs, we look into some properties of gem-free graphs and dart-free graphs.

Proposition 2.2. *Let G be a chordal graph. Then, G is gem-free if and only if every edge of $K(G)$ is in some clique tree of G.*

Proof. Suppose that there exists an edge CC' of $K(G)$ that is in no clique tree of G. By Theorem 1.3, C, C' is not a separating pair. Let v be a vertex in $C \cap C'$ and P be a path in $G - C \cap C'$ of minimum length among all the paths in $G - C \cap C'$ with initial vertex in $C \setminus C'$ and final vertex in $C' \setminus C$.

If the length of P were 1, let v_1, v_2 be the vertices of P, with $v_1 \in C$ and $v_2 \in C'$. Let v_3 be a vertex of C not adjacent to v_2 and v_4 be a vertex of C' not adjacent to v_1. Thus, $\{v, v_1, v_2, v_3, v_4\}$ induces a gem.

If the length of P were 2, let v_1, v_2, v_3 be the vertices of P, with $v_1 \in C$ and $v_3 \in C'$. The cycle $vv_1v_2v_3v$ must have a chord and, by the construction of P, that chord is vv_2. The definition of P also implies that $v_2 \notin C$, so we can take a vertex $v_4 \in C$ that is not adjacent to v_2. The vertex v_4 is not adjacent to v_3, otherwise the definition of P would be contradicted. Thus, $\{v, v_1, v_2, v_3, v_4\}$ induces a gem.

If the length of P were larger than 2, let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of P, with $v_1 \in C$ and $v_n \in C'$. By the definition of P, the chords of the cycle $vv_1v_2v_3\ldots v_nv$ must have v as an endpoint, so we can infer that v is adjacent to all the vertices of P. Thus, $\{v, v_1, v_2, v_3, v_4\}$ induces a gem.

Therefore, G is not gem-free.

Conversely, suppose that G is not gem-free. Let v_1, v_2, v_3, v_4, v_5 be vertices of G inducing a gem like the one of Fig. 3, C be a clique of G containing $\{v_1, v_2, v_3\}$ and C' be another clique containing $\{v_1, v_4, v_5\}$. Thus, CC' is an edge of $K(G)$ because $v_1 \in C \cap C'$ but it is not a separating pair because $v_2 \in C \setminus C'$, $v_4 \in C' \setminus C$ and these two vertices are adjacent. Consequently, by Theorem 1.3, CC' is the edge of no clique tree of G. □

Proposition 2.3. *Let G be a chordal (dart, gem)-free graph. Then, for every three different cliques C_1, C_2 and C_3 in G such that $C_1 \cap C_2 \neq \emptyset$ and $C_2 \cap C_3 \neq \emptyset$, there exists a clique tree of G such that $C_1C_2C_3$ is a path of that tree.*

Proof. Since G is gem-free, there exists by Proposition 2.2 a clique tree T of G such that $C_1C_2 \in E(T)$. We now consider several cases:

Case 1: $C_2 \in T[C_1, C_3]$

If $C_1C_2C_3$ is a path of T, then we are done. Otherwise, C_2C_3 is the edge of another clique tree of G due to Proposition 2.2. Therefore, there exists by Theorem 1.4 an edge e in $T[C_2, C_3]$ such that $T + C_2C_3 - e$ is a clique tree of G, which has $C_1C_2C_3$ as a path.

Case 2: $C_1 \in T[C_2, C_3]$

Case 2.a: $C_1 \cap C_2 \neq C_2 \cap C_3$.

Since $C_2 \cap C_3 \neq \emptyset$, the edge C_2C_3 is by Proposition 2.2 in some clique tree of G. Hence there exists by Theorem 1.4 an edge e in $T[C_2, C_3]$ such that $T + C_2C_3 - e$ is a clique tree of G, call it T'. As $C_1 \cap C_2 \neq C_2 \cap C_3$, edge e cannot be equal to C_1C_2. Therefore, T' is a clique tree of G having $C_1C_2C_3$ as a path.

Case 2.b: $C_1 \cap C_2 = C_2 \cap C_3$ and $C_1 \cap C_3 = C_2 \cap C_3$.

Since $C_1 \cap C_3 \neq \emptyset$, the edge C_1C_3 is by Proposition 2.2 in some clique tree of G. Hence there exists by Theorem 1.4 an edge e in $T[C_1, C_3]$ such that $T + C_1C_3 - e$ is a clique tree of G. As $C_1 \cap C_3 = C_2 \cap C_3$, it follows that $T + C_2C_3 - e$ is also a clique tree of G. Furthermore, it has $C_1C_2C_3$ as a path.

Case 2.c: $C_1 \cap C_2 = C_2 \cap C_3$ and $C_1 \cap C_3 \neq C_2 \cap C_3$.

Since T is a clique tree and $C_1 \in T[C_2, C_3]$, we have that $C_2 \cap C_3 \subseteq C_1 \cap C_3$, this inclusion being strict by the assumption in this case.

Let v_1 be a vertex in $C_1 \setminus C_3$, v_2 be a vertex in $C_3 \setminus C_1$, v_3 be a vertex in $C_2 \setminus (C_1 \cap C_3)$, v_4 be a vertex in $C_1 \cap C_2 \cap C_3$ and v_5 be a vertex in $(C_1 \cap C_3) \setminus C_2$. It is clear that v_1v_4, v_1v_5, v_2v_4, v_2v_5, v_3v_4 and v_4v_5 are edges of G. Furthermore, by Proposition 2.2, we have that C_1, C_2 and C_1, C_3 and C_2, C_3 are separating pairs. Thus, v_1v_2, v_1v_3, v_2v_3 and v_3v_5 are not edges of G. Therefore, $\{v_1, v_2, v_3, v_4.v_5\}$ induces a dart, which is a contradiction.

It follows that case 2.c is not possible. We could find a clique tree having $C_1C_2C_3$ as a path in every other case, so the proof is complete. □

Proposition 2.4. *Let G be a chordal dart-free graph. Then, no minimal vertex separator of G contains another.*

Proof. Suppose to the contrary that there are two distinct minimal vertex separators S and S' such that $S \subseteq S'$. Let $v_1 \in S$, $v_2 \in S' \setminus S$, C_1, C_2 and C_3, C_4 be separating pairs such that $C_1 \cap C_2 = S$ and $C_3 \cap C_4 = S'$, $v_3 \in C_3 \setminus S'$ and $v_4 \in C_4 \setminus S'$. The vertices v_2, v_3 and v_4 are contained in the same connected component of $G-S$. We can assume without loss of generality that $C_1 \setminus S$ is contained in a connected component of $G-S$ different from that of $\{v_2, v_3, v_4\}$. Let $v_5 \in C_1 \setminus S$. Thus, $\{v_1, v_2, v_3, v_4, v_5\}$ induces a dart, which is a contradiction.

Therefore, no minimal vertex separator of G contains another. □

Now we proceed to give several characterizations for hereditary basic chordal graphs. Since the previous propositions and theorems involve darts, gems, clique trees, minimal vertex separators and intersections of cliques, the characterizations will be in terms of them.

Theorem 2.5. *Let G be a chordal graph. Then, the following conditions are equivalent:*

(i) *G is hereditary basic chordal.*
(ii) *G is a (dart, gem)-free graph.*
(iii) *for every three different cliques C_1, C_2 and C_3 in G such that $C_1 \cap C_2 \neq \emptyset$ and $C_2 \cap C_3 \neq \emptyset$, there exists a clique tree of G such that $C_1C_2C_3$ is a path of that tree.*
(iv) *Every edge of $K(G)$ is in some clique tree of G and no minimal vertex separator of G contains another.*
(v) *For every triple C_1, C_2, C_3 of pairwise intersecting cliques of G, it holds that $C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3$.*
(vi) *For all $C \in \mathcal{C}(G)$ and $S \in \mathcal{S}(G)$, we have that $S \cap C \neq \emptyset$ implies $S \subseteq C$.*
(vii) *The minimal vertex separators of G are pairwise disjoint.*

Proof. (i) $\rightarrow$ (ii) It suffices to prove that the dart and the gem are not basic chordal graphs. Let G' be any of the two graphs of Fig. 3, $C_1 = \{v_1, v_2, v_3\}$, $C_2 = \{v_1, v_2, v_4\}$ and C_3 be the clique containing $\{v_1, v_5\}$. Thus, the path $C_1C_3C_2$ is a compatible tree of $K(G')$ but not a clique tree of G' because $\mathcal{C}_{v_2} = \{C_1, C_2\}$.

Therefore, the dart and the gem are not basic chordal graphs.

(ii) $\rightarrow$ (iii) See Proposition 2.3.

(iii) $\rightarrow$ (iv) Let C_1C_2 be any edge of $K(G)$. If it is the only edge of $K(G)$, then C_1C_2 itself is a clique tree. Otherwise, the fact that $K(G)$ is connected implies that there exists another clique C_3 that is adjacent to C_1 or to C_2 in $K(G)$. Assume without loss of generality that C_3 is adjacent to C_2 in $K(G)$. Hence, by the hypothesis, there exists a clique tree having $C_1C_2C_3$ as a path. Particularly, there exists a clique tree having C_1C_2 as an edge.

Therefore, every edge of $K(G)$ is in some clique tree of G.

Now suppose that S and S' are minimal vertex separators of G such that $S \subseteq S'$. Let C_1, C_2, C_3 and C_4 be cliques of G such that $S = C_1 \cap C_2$ and $S' = C_3 \cap C_4$.

If $C_3 = C_1$ or $C_4 = C_1$, then it is clear that $C_3 \cap C_4$ is contained in C_1. Otherwise, there exists by the hypothesis a clique tree T such that $C_3C_1C_4$ is a path in T. We also conclude from this that $C_3 \cap C_4$ is contained in C_1.

Similarly, we can prove that $C_3 \cap C_4 \subseteq C_2$. Therefore, $C_3 \cap C_4 \subseteq C_1 \cap C_2$, that is, $S' \subseteq S$. We infer that $S = S'$.

Therefore, no minimal vertex separator of G contains another.

(iv) $\rightarrow$ (v) Let C_1, C_2, C_3 be a triple of pairwise intersecting cliques of G. By (iv), there exists a clique tree T of G such that $C_1C_2 \in E(T)$. Suppose without loss of generality that $C_2 \in T[C_1, C_3]$. As a consequence, $C_1 \cap C_3 \subseteq C_1 \cap C_2$ and $C_1 \cap C_3 \subseteq C_2 \cap C_3$. Since intersecting cliques are in the edge of some clique tree by the hypothesis and the cliques of an edge of a clique tree form a separating pair by Theorem 1.3, we have that $C_1 \cap C_2$, $C_1 \cap C_3$ and $C_2 \cap C_3$ are minimal vertex separators of G. As no minimal vertex separator contains another, the inclusions in this paragraph imply that $C_1 \cap C_3 = C_1 \cap C_2$ and $C_1 \cap C_3 = C_2 \cap C_3$.

(v) $\rightarrow$ (vi) Let $S \in \mathcal{S}(G)$ and $C \in \mathcal{C}(G)$ be such that $S \cap C \neq \emptyset$. Take cliques C_1 and C_2 such that $S = C_1 \cap C_2$. If $C = C_1$ or $C = C_2$, then it is clear that $S \subseteq C$. If not, then C, C_1, C_2 is a triple of pairwise intersecting cliques. Therefore, by (v), $C \cap C_1 = C_1 \cap C_2 = S$, from which the inclusion $S \subseteq C$ follows.

(vi) $\rightarrow$ (vii) Let S and S' be minimal vertex separators of G such that $S \cap S' \neq \emptyset$. We will prove that $S = S'$.

Let S be the intersection of two cliques C_1 and C_2 of G. The fact that $S \cap S' \neq \emptyset$ implies that $C_1 \cap S' \neq \emptyset$ and $C_2 \cap S' \neq \emptyset$. By (vi), $S' \subseteq C_1$ and $S' \subseteq C_2$. Hence, $S' \subseteq C_1 \cap C_2$, that is, $S' \subseteq S$.

Similarly, we can prove that $S \subseteq S'$. Therefore, $S = S'$.

(vii) $\rightarrow$ (i) Suppose that G is not hereditary basic chordal and let G' be an induced subgraph of G that is not basic chordal. We need to prove that there exist two different minimal vertex separators of G that are not disjoint.

Consider a minimal vertex separator S of G' such that B_S and $\mathcal{C}_S$, both computed with respect to G', are different. Let C, C_1 and C_2 be cliques of G' such that $C \in B_S \setminus \mathcal{C}_S$, C_1, C_2 is a separating pair and $C_1 \cap C_2 = S$. Next we will prove that $C \cap S \neq \emptyset$.

As $C \in B_S$, it follows that $C \cap C_1 \neq \emptyset$ and $C \cap C_2 \neq \emptyset$. The clique C cannot intersect both $C_1 \setminus S$ and $C_2 \setminus S$ because otherwise $C_1 \setminus S$ and $C_2 \setminus S$ would not be separated by S, contradicting that C_1, C_2 is a separating pair. Thus, $C \cap C_1 \subseteq S$ or $C \cap C_2 \subseteq S$. In either case, we conclude that $C \cap S \neq \emptyset$.

Let v_1 be a vertex of $C \cap S$. Since $C \notin \mathcal{C}_S$, we can also take a vertex $v_2 \in S \setminus C$. There also exists a vertex v_3 in C that is not adjacent to v_2. The vertex v_3 is clearly different from v_1 because S is complete.

Let S' be a minimal v_2v_3-separator in G. Then, $S \neq S'$ because $v_2 \in S \setminus S'$ and $S \cap S' \neq \emptyset$ because $v_1 \in S \cap S'$. □

It is interesting to note that part (vi) of Theorem 2.5 implies that the vertices of a given minimal vertex separator of a hereditary basic chordal have the same closed neighborhood, i.e., are twins, since they are contained in the same cliques.

On another side, part (vii) ensures the existence of a quite simple procedure to determine whether a given chordal graph is a hereditary basic chordal graph. When a clique tree of the chordal graph is found, the intersections of the endpoints of each edge of the tree give all the minimal vertex separators of the graph and it can be tested whether they are pairwise disjoint.

Finally, the characterization of hereditary basic chordal graphs via minimal forbidden induced subgraphs reveals that this class appeared before in some other works under very different contexts and names.

First, in 2005, William Kennedy studied strictly chordal graphs in his Masters Thesis [15]. There, a graph G is said to be *strictly chordal* if G is chordal and for every subfamily $\mathcal{F} \subseteq \mathcal{C}(G)$ satisfying that $I(\mathcal{F}) := \bigcap_{C \in \mathcal{C}(G)} C \neq \emptyset$ and $\mathcal{F} = \{C \in \mathcal{C}(G) : C \cap I(\mathcal{F}) \neq \emptyset\}$, we have that $C \cap C' = I(\mathcal{F})$ for all $C, C' \in \mathcal{F}$. Moreover, the thesis includes a proof of the fact that strictly chordal graphs are just the (dart–gem)-free chordal graphs.

Note that, for the cliques of a family $\mathcal{F}$ like the one of the previous paragraph, the conditions $I(\mathcal{F}) := \bigcap_{C \in \mathcal{C}(G)} C \neq \emptyset$ and $C \cap C' = I(\mathcal{F})$ for all $C, C' \in \mathcal{F}$, imply that the members of $\mathcal{F}$ are pairwise intersecting and that the intersection of every two of them is always the same. This bears some resemblance with condition (v) of Theorem 2.5. A difference is that condition (v) involves three cliques only. However, it is possible to get a very similar condition using more cliques.

Proposition 2.6. *Let G be a chordal graph. Then G is hereditary basic chordal graph if and only if for every tuple $C_1, C_2, \ldots, C_n$ of pairwise intersecting cliques of G we have that $C_i \cap C_j = \bigcap_{m=1}^{n} C_m$, for $1 \leq i, j \leq n$, $i \neq j$.*

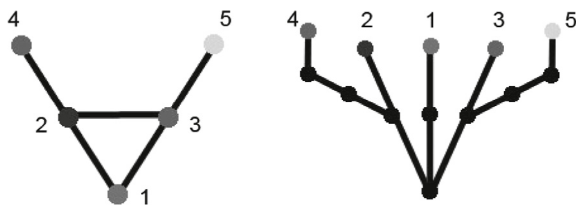

Fig. 4. The bull graph (left) and a representation of it as a (4, 6)-leaf power (right).

Proof. Suppose that for every tuple $C_1, C_2, \ldots, C_n$ of pairwise intersecting cliques of G we have that $C_i \cap C_j = \bigcap_{k=1}^{n} C_k$, for $1 \leq i, j \leq n, i \neq j$. Thus condition (v) of Theorem 2.5 is satisfied and hence G is hereditary basic chordal.

Conversely, suppose that G is hereditary basic chordal and let $C_1, C_2, \ldots, C_n$ be any tuple of pairwise intersecting cliques of G. Let i, j, k and l be four integers between 1 and n, with $i \neq j$ and $k \neq l$.

If $|\{i, j, k, l\}| = 2$, then $\{i, j\} = \{k, l\}$. It is thus clear that $C_i \cap C_j = C_k \cap C_l$.

If $|\{i, j, k, l\}| = 3$, then we can suppose without loss of generality that $i = k$ and $j \neq l$. Apply condition (v) of Theorem 2.5 to C_i, C_j, C_l to obtain that $C_i \cap C_j = C_i \cap C_l$. Since $i = k$, it follows that $C_i \cap C_j = C_k \cap C_l$.

If $|\{i, j, k, l\}| = 4$, then apply condition (v) of Theorem 2.5 to C_i, C_j, C_k and to C_j, C_k, C_l to obtain that $C_i \cap C_j = C_j \cap C_k$ and $C_j \cap C_k = C_k \cap C_l$. Therefore, $C_i \cap C_j = C_k \cap C_l$.

As the intersection of every two cliques of the tuple is always the same, we have that $C_i \cap C_j = \bigcap_{m=1}^{n-1} (C_i \cap C_j) = \bigcap_{m=1}^{n-1} (C_m \cap C_{m+1}) = \bigcap_{m=1}^{n} C_m$. □

We could use Proposition 2.6 to simplify the definition of strictly chordal graphs given in [15].

A connection with leaf powers was found later. Let G be a graph and k, l be integers such that $2 \leq k < l$. The graph G is defined to be a *(k,l)-leaf power* if there exists a tree T whose set of leaves is $V(G)$ and such that $d_T(u, v) \leq k$ for all $uv \in E(G)$ and $d_T(u, v) \geq l$ for all $uv \notin E(G)$ (see example in Fig. 4). Brandstädt and Wagner proved that strictly chordal graphs are exactly the (4, 6)-leaf powers [4].

We present even two more previous characterizations of (dart, gem)-free chordal graphs.

- A graph G is the *k-simplicial power* of a graph H if $V(G)$ equals the set of simplicial vertices of H, and for every two distinct vertices x and y of G, it holds that $xy \in E(G)$ if and only if the distance in H between x and y is at most k. The (dart, gem)-free chordal graphs are the 2-simplicial powers of block graphs [3].
- A subset S of $V(G)$ is said to be *convex* if for every two vertices v and w in S, the vertices of every minimum length path between v and w in G are contained in S.

  For $u \in S$, define $ecc_S(u)$ to be equal to $max\{d(u, v) : v \in S\}$. We say that u is a *contour vertex* of S if $ecc_S(u) \geq ecc_S(v)$ for every neighbor v of u in S.

  G is a (dart, gem)-free chordal graph if and only if the set of contour vertices of S equals the set of simplicial vertices of $G[S]$, for every convex set S [5].

The previous paragraphs can be summarized as follows:

Theorem 2.7. *Let G be a graph. The following conditions are equivalent:*

1. *G is hereditary basic chordal.*
2. *G is strictly chordal.*
3. *G is a (4, 6)-leaf power.*
4. *G is the 2-simplicial power of a block graph.*
5. *The set of contour vertices of S equals the set of simplicial vertices of G[S], for every convex set S of G.*

3. Clique graphs of hereditary basic chordal graphs

In this section, we show that the class of clique graphs of hereditary basic chordal graphs is the class of block graphs and we find some applications of this result. We will also obtain new characterizations for hereditary basic chordal graphs.

As a first step, note that hereditary basic chordal graphs are sun-free because every k-sun has the gem as an induced subgraph, $k \geq 3$. Chordal sun-free graphs are exactly the strongly chordal graphs [8], so every hereditary basic chordal graph is strongly chordal. Since the clique graph of a strongly chordal graph is also strongly chordal [1], we conclude that the clique graph of every hereditary basic chordal graph is chordal.

For the next steps, we need the following proposition:

Proposition 3.1. *Let G be a hereditary basic chordal graph and C, C' be two cliques of G such that $C \cap C' \neq \emptyset$. Then, the edge CC' is contained in only one clique of $K(G)$.*

Proof. Let C'' be another clique of G adjacent to both C and C' in $K(G)$. Thus, C, C', C'' is a triple of pairwise intersecting cliques and, by Theorem 2.5, $C \cap C'' = C \cap C'$. Therefore, $C \cap C' \subseteq C''$.

We infer that the family of cliques of G containing $C \cap C'$ forms a complete set of $K(G)$ that contains all the cliques adjacent to both C and C' in $K(G)$. Therefore, this family is the only clique of $K(G)$ containing the edge CC'. □

As a remark on the previous proof, note that, by Proposition 2.2, $C \cap C'$ is in $\mathcal{S}(G)$, so the only clique of $K(G)$ containing the edge CC' consists of all the cliques of G containing the minimal vertex separator $C \cap C'$. This fact can be used to derive a new characterization of basic chordal graphs.

Proposition 3.2. *Let G be a chordal graph. Then G is hereditary basic chordal if and only if G is complete or $\mathcal{C}(K(G)) = \{\mathcal{C}_S : S \in \mathcal{S}(G)\}$.*

Proof. Suppose that G is a hereditary basic chordal graph that is not complete. Thus G has more than one clique. Let D be a clique of $K(G)$. Assuming the connectedness of G, we have that D contains an edge CC'. Let $S = C \cap C'$. Apply the reasonings in the proof of Proposition 3.1 and the remark after it to conclude that $D = \mathcal{C}_S$. Therefore, every clique of $K(G)$ is of the form $\mathcal{C}_S$, with $S \in \mathcal{S}(G)$.

Furthermore, no two different sets of the form $\mathcal{C}_S$ can be such that one is contained in the other because the minimal vertex separators of G are pairwise disjoint due to Theorem 2.5. This is sufficient to conclude that $\mathcal{C}(K(G)) = \{\mathcal{C}_S : S \in \mathcal{S}(G)\}$.

In order to prove the converse, it is trivial that a complete graph is hereditary basic chordal, so we can assume that G is not complete. If $|\mathcal{S}(G)| = 1$, then condition (vii) of Theorem 2.5 trivially holds, so G is hereditary basic chordal. From now on, assume that $|\mathcal{S}(G)| \geq 2$.

Let S and S' be any two different minimal vertex separators of G. We will prove that $S \cap S' = \emptyset$. Suppose to the contrary that there exists a vertex $v \in S \cap S'$. Thus $\mathcal{C}_S \cup \mathcal{C}_{S'} \subseteq \mathcal{C}_v$. Since S and S' are different, it follows that $\mathcal{C}_S \subsetneq \mathcal{C}_v$ or $\mathcal{C}_{S'} \subsetneq \mathcal{C}_v$. Suppose without loss of generality that $\mathcal{C}_S \subsetneq \mathcal{C}_v$. As $\mathcal{C}_v$ is complete in $K(G)$, we infer that $\mathcal{C}_S$ is not a clique of $K(G)$, which contradicts the hypothesis. Therefore, $S \cap S' = \emptyset$.

We conclude that the minimal vertex separators of G are pairwise disjoint, which implies by Theorem 2.5 that G is hereditary basic chordal. □

A graph G is *biconnected* if $G - v$ is connected for every $v \in V(G)$. A *block* of a graph G is a maximal biconnected component of G. We say that G is a *block graph* if it is the intersection graph of the blocks of some graph H.

We will basically work with two other characterizations of block graphs:

- A graph G is a block graph if and only if every block of G is a clique.
- A graph G is a block graph if and only if G is a diamond-free chordal graph.

Now we prove that the clique graphs of hereditary basic chordal graphs are the block graphs.

Theorem 3.3. *Let G be a graph. Then, G is the clique graph of a hereditary basic chordal graph if and only if G is a block graph.*

Proof. Let G be a hereditary basic chordal graph. Reason like in the second paragraph of this section to conclude that $K(G)$ is chordal.

Now we prove that $K(G)$ is diamond-free. It suffices to verify that for all four cliques C_1, C_2, C_3, C_4 of G, if C_1C_2, C_1C_3, C_1C_4, C_2C_3 and C_2C_4 are all edges of $K(G)$, then C_3C_4 is also an edge of $K(G)$, thus avoiding the formation of an induced diamond. Proving it requires Proposition 3.1. Since C_1C_2 is contained in only one clique of $K(G)$, the clique of $K(G)$ containing $\{C_1, C_2, C_3\}$ is equal to the one containing $\{C_1, C_2, C_4\}$. Thus, C_3 and C_4 are in the same clique and hence are adjacent.

Therefore $K(G)$ is a chordal diamond-free graph, i.e., it is a block graph.

This proves that the clique graph of every basic chordal graph is a block graph. Furthermore, trees form a subclass of hereditary basic chordal graphs and their clique graphs are all block graphs [14].

Therefore the class of clique graphs of hereditary basic chordal graphs is equal to the class of block graphs. □

Block graphs not only appear as the clique graphs of hereditary basic chordal graphs. They also allow to characterize the structure of basic chordal graphs. To further explain this point, we need to introduce the notion of critical clique of a graph.

A subset C of $V(G)$ is a *critical clique* of G if all the vertices of C have the same closed neighborhood and C is maximal in this sense. That is, C is a critical clique if $N[v] = N[w]$ for every $v, w \in C$ and $N[v] \neq N[u]$ for every $u \notin C$ and $v \in C$. The critical cliques of G form a partition of $V(G)$, since they are the equivalence classes of the relation where x and y in $V(G)$ are related if and only if $N[x] = N[y]$.

Now we find the critical cliques of a hereditary basic chordal graph.

Proposition 3.4. *Let G be a hereditary basic chordal graph and C be a subset of vertices of G. Then C is a critical clique of G if and only if $C \in \mathcal{S}(G)$ or C is the set of simplicial vertices of a simplicial clique D of G.*

Proof. It is trivial that C is a critical clique of G if it is the set of simplicial vertices of a simplicial clique D of G.

Now suppose that C is a minimal vertex separator of G. We know from the remark after Theorem 2.5 that all the vertices of C have the same closed neighborhood. Let v be any vertex of G not in C. Thus, vertex v is not adjacent to any vertex outside the connected component of $G-C$ containing v. We cannot say the same about the vertices of C. Therefore, v does not have the same closed neighborhood as the vertices of C, so C is a critical clique of G.

Conversely, suppose that C is a critical clique of G. Let v be a vertex of C. If v is simplicial, then let D be the only clique of G containing v. Thus D is a simplicial clique and the set of simplicial vertices of D is a critical clique intersecting C. Since critical cliques are pairwise disjoint, C is equal to the set of simplicial vertices of D.

If v is not simplicial, then there exists a minimal vertex separator S such that $v \in S$. Thus S is a critical clique intersecting C. Therefore $C = S$. □

Define the *critical clique graph* $\mathcal{CC}(G)$ of G as the graph whose vertices are the critical cliques of G and such that two critical cliques C and C' are adjacent in $\mathcal{CC}(G)$ if and only if every vertex of C is adjacent to every vertex of C'. Thus we have the following characterization of hereditary basic chordal graphs.

Theorem 3.5 ([4]). *Let G be a graph. Then G is hereditary basic chordal if and only if $\mathcal{CC}(G)$ is a block graph.*

Given a graph G, we say that G' is obtained from G by *replacing the vertex $v \in V(G)$ by the complete set C*, where $C \cap (V(G) \setminus \{v\}) = \emptyset$, if $V(G') = (V(G) \setminus \{v\}) \cup C$ and $E(G')$ is obtained from $E(G)$ by removing all the edges containing v and adding all the edges between vertices of C and all the edges between vertices of C and $N_G(v)$.

We say that G' is obtained from G by *adding the simplicial vertex v to the clique $C \in \mathcal{C}(G)$* if G' results from G after adding the vertex v and all the edges between v and the vertices of C.

The characterization of the structure of hereditary basic chordal graphs that had been mentioned before can be deduced from Theorem 3.5 and is as follows:

Theorem 3.6 ([4]). *Let G be a graph. Then G is chordal and (dart, gem)-free if and only if G can be obtained from some block graph by replacing vertices by complete sets.*

We finally find, for a block graph G, all the hereditary basic chordal graphs with clique graph equal to G. All the concepts that we have been using are still useful.

Lemma 3.7. *Let G be a hereditary basic chordal graph and G' be a graph obtained from G by replacing a vertex by a complete set or by adding a simplicial vertex to a clique. Then G' is also hereditary basic chordal.*

Proof. Suppose that G' is obtained from G by replacing a vertex by a complete set. It is easy to verify that $\mathcal{CC}(G) = \mathcal{CC}(G')$. Since G is hereditary basic chordal, $\mathcal{CC}(G)$ is a block graph. Hence $\mathcal{CC}(G')$ is also a block graph. By Theorem 3.5, G' is hereditary basic chordal.

Now suppose that G' is obtained by adding the simplicial vertex v to the clique C. It is easy to verify that G and G' have the same clique trees. Since the minimal vertex separators of a chordal graph are determined by the intersections of the endpoints of the edges of a clique tree, $\mathcal{S}(G) = \mathcal{S}(G')$. As G is a hereditary basic chordal graph, the members of $\mathcal{S}(G)$ are pairwise disjoint, so the same can be said about $\mathcal{S}(G')$. Therefore, G' is hereditary basic chordal by Theorem 2.5. □

Lemma 3.8 ([12]). *Let $\mathcal{F}$ be a Helly and separating family. Then, $\mathcal{C}(L(\mathcal{F})) = D\mathcal{F}$.*

Proposition 3.9. *Let G be a block graph. For each simplicial vertex v of G, add the vertex v^* and the edge vv^* to G to obtain the graph G'. Let $H = K(G')$ and let J be another graph. Then J is hereditary basic chordal and $K(J) = G$ if and only if $J = H$ or J can be obtained from H by successive applications of the two following operations: replacing a vertex by a complete set and adding a simplicial vertex to a clique.*

Proof. Suppose that J is a hereditary basic chordal graph such that $K(J) = G$. By Theorem 1.7, J is the intersection graph of a separating family $\mathcal{F}$, where $\mathcal{F} = \{F_i\}_{i\in I}$, such that every F in $\mathcal{F}$ is in $\mathcal{SDC}(G)$ and $S(G) = \mathcal{F}$. In other words, J is the intersection graph of complete sets of G that are in $\mathcal{SDC}(G)$ and cover the edges of G. Since G is a block graph, every of these complete sets is either a clique of G or it is a unit set. As a consequence, $\mathcal{C}(G)$ must necessarily be contained in $\mathcal{F}$ to cover all the edges of G. Furthermore, the fact that $\mathcal{F}$ is separating implies that every simplicial vertex v of G satisfies that $\{v\}$ is in $\mathcal{F}$.

Let $\mathcal{F}'$ be the subfamily of $\mathcal{F}$ consisting of the cliques of G and the sets of the form $\{v\}$, where v is a simplicial vertex of G. Let I' be a subset of I such that $\mathcal{F}' = \{F_i\}_{i\in I'}$. If $\mathcal{F} = \mathcal{F}'$, then it is easy to verify that $J = H$. Otherwise, let $i \in I \setminus I'$. If $F_i = F_j$ for some $j \in I'$, then the intersection graph of $\{F_i\}_{i\in I'\cup\{j\}}$ can be obtained from the intersection graph of $\{F_i\}_{i\in I'}$ by replacing vertex F_j by the complete set $\{F_i, F_j\}$.

If $F_i \neq F_j$ for every $j \in I'$, then we necessarily have that $F_i = \{v\}$, where v is a nonsimplicial vertex of G, that is, a cut vertex of G. Since $\mathcal{F}'$ is a Helly and separating family, we infer from Lemma 3.8 that $\mathcal{C}(L(\mathcal{F}')) = D\mathcal{F}'$. Thus the sets of $\mathcal{F}'$ that contain v form a clique of $L(\mathcal{F}')$. Therefore, the intersection graph of $\{F_i\}_{i\in I'\cup\{j\}}$ can be obtained from the intersection graph of $\{F_i\}_{i\in I'}$ by adding the simplicial vertex $\{v\}$ to the clique $\{F_i : i \in I', v \in F_i\}$.

Applying similar reasonings to the remaining elements of $I \setminus I'$ in a successive way, we conclude that J can be obtained from H through the operations of replacing a vertex by a complete set and adding a simplicial vertex to a clique.

Conversely, we can infer from the first part of the proof that $K(H) = G$. Note that H is a block graph because it is the clique graph of the block graph G', so H is hereditary basic chordal. Furthermore, replacing a vertex by a complete set and adding a simplicial vertex to a clique are operations that do not change the clique graph. Therefore, every graph obtained from H through the operations of replacing a vertex by a complete set and adding a simplicial vertex to a clique has its clique graph equal to G and is hereditary basic chordal because of Lemma 3.7. □

Note that the first part of the proof of Proposition 3.9 does not require that J is a hereditary basic chordal graph. Actually, it is enough that J is basic chordal to be able to apply Theorem 1.7. Hence, the following result can be proved.

Proposition 3.10. *Let J be a basic chordal graph such that $K(J)$ is a block graph. Then J is a hereditary basic chordal graph.*

Proof. Let $G = K(J)$ and G' and H be like in Proposition 3.9. Once again H is a hereditary basic chordal graph. Reason like in the first part of the proof of Proposition 3.9 to conclude that $J = H$ or that J can be obtained from H through the operations of replacing a vertex by a complete set and adding a simplicial vertex to a clique. Therefore, Lemma 3.7 yields that J is a hereditary basic chordal graph. □

Thus we have a new characterization of hereditary basic chordal graphs.

Corollary 3.11. *Let G be a graph. Then G is hereditary basic chordal if and only if it is basic chordal and $K(G)$ is a block graph.*

3.1. Clique trees of hereditary basic chordal graphs

In this last section about hereditary basic chordal graphs, we characterize their clique trees and use them to further characterize hereditary basic chordal graphs.

There are two facts that simplify this task: that the clique graph of a hereditary basic chordal graph is a block graph and that the clique trees of the graph are the compatible trees of that block graph.

Proposition 3.12. *Let G be a noncomplete hereditary basic chordal graph and T be a graph with $V(T) = \mathcal{C}(G)$. Let $\mathcal{S}(G) = \{S_i\}_{1\le i\le n}$. Then T is a clique tree of G if and only if $E(T) = \bigcup_{i=1}^{n} E(T_i)$, where T_i is a tree such that $V(T_i) = \mathcal{C}_{S_i}$, for $1 \le i \le n$.*

Proof. Suppose that T is a clique tree of G. Then $\mathcal{C}_{S_i}$ induces a subtree of T for $1 \le i \le n$. Let $T_i = T[\mathcal{C}_{S_i}]$. It is clear that $\bigcup_{i=1}^{n} E(T_i) \subseteq E(T)$.

Let CC' be an edge of T and let j be such that $C \cap C' = S_j$. Thus $CC' \in E(T_j)$. It follows that $E(T) \subseteq \bigcup_{i=1}^{n} E(T_i)$. Therefore, $E(T) = \bigcup_{i=1}^{n} E(T_i)$.

Conversely, suppose that $T_1, \ldots, T_n$ are trees such that $V(T_i) = \mathcal{C}_{S_i}$ for $1 \le i \le n$ and that $E(T) = \bigcup_{i=1}^{n} E(T_i)$. We now prove that $T[\mathcal{C}_{S_i}] = T_i$ for $1 \le i \le n$.

Let CC' be an edge of $T[\mathcal{C}_{S_i}]$ and suppose that there exists $j \ne i$ such that CC' is an edge of T_j. Thus C and C' are both in $\mathcal{C}_{S_i}$ and $\mathcal{C}_{S_j}$, so $S_i \subseteq C \cap C'$ and $S_j \subseteq C \cap C'$. Consequently, either $S_i \ne C \cap C'$ or $S_j \ne C \cap C'$. In either case, we would have two different minimal vertex separators such that one is contained in the other, thus contradicting that G is hereditary basic chordal. Therefore, such number j cannot exists and every edge of $T[\mathcal{C}_{S_i}]$ must be an edge of T_i. We conclude that $T[\mathcal{C}_{S_i}] = T_i$.

Now we prove that T is a tree. To prove that T is connected it is enough to demonstrate that every two adjacent vertices of $K(G)$ are connected by T.

Let CC' be an edge of $K(G)$. Let D be the clique of $K(G)$ containing that edge. By Proposition 3.2 there exists i such that $D = \mathcal{C}_{S_i}$. Hence C and C' are connected by T_i.

Suppose that C is a cycle in T. Then C is a biconnected graph. Let D be the block of $K(G)$ containing C. Once again we conclude that D is of the form $\mathcal{C}_{S_i}$, so C is a cycle in $T[\mathcal{C}_{S_i}]$, that is, a cycle in T_i, which is a contradiction.

Hence T contains no cycle and it is a tree.

Since $T[\mathcal{C}_{S_i}] = T_i$ for $1 \le i \le n$, it follows from Proposition 3.2 that every clique of $K(G)$ induces a subtree of T. Hence T is a compatible tree of $K(G)$. As G is basic chordal, T is also a clique tree of G. □

Theorem 3.13. *Let G be a graph. Then G is hereditary basic chordal if and only if every spanning tree of $K(G)$ is a clique tree of G.*

Proof. Suppose that G is hereditary basic chordal. Then $K(G)$ is a block graph. Now we prove that every spanning tree T of $K(G)$ is compatible with it. For that purpose, it is sufficient to prove that each block of $K(G)$ induces a subtree in T.

Let B be a block of $K(G)$ and suppose that $T[B]$ is not a subtree of T. Thus there exist C_1 and C_2 in B such that $T[C_1, C_2]$ is not contained in B. In that case, the subgraph of $K(G)$ induced by $B \cup T[C_1, C_2]$ is biconnected, thus contradicting that B is a block. Hence we necessarily have that $T[B]$ is a subtree.

Therefore, every block of $K(G)$ induces a subtree of T, so T is compatible with $K(G)$. Since G is basic chordal, T is also a clique tree of G.

Consequently, every spanning tree of $K(G)$ is a clique tree of G.

Suppose now that G is not hereditary basic chordal. Then, by Theorem 2.5, there exist cliques $C1$, C_2 and C_3 such that $C_1 \cap C_2 \ne \emptyset$, $C_2 \cap C_3 \ne \emptyset$ and the path $C_1C_2C_3$ does not appear in any clique tree of G. Note that $C_1C_2C_3$ is a path in $K(G)$, so there exists a spanning tree T of $K(G)$ containing that path. Hence, T is a spanning tree of $K(G)$ that is not a clique tree of G. □

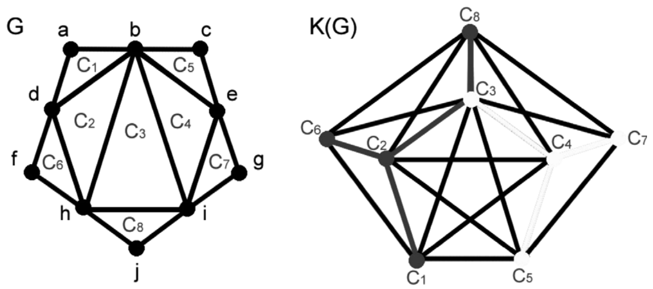

Fig. 5. A *UV* graph *G* and its clique graph.

Going back to condition (iii) of Theorem 2.5, Theorem 3.13 now allows to strengthen it to cover the case of longer paths.

Theorem 3.14. *Let G be a graph. Then G is hereditary basic chordal if and only if every path of $K(G)$ appears in at least one clique tree of G.*

Proof. Suppose that G satisfies that every path of $K(G)$ appears in at least one clique tree of G. Thus, condition (iii) of Theorem 2.5 holds, so G is hereditary basic chordal.

Conversely, suppose that G is hereditary basic chordal and let P be a path of $K(G)$. Let T be a spanning tree of $K(G)$ containing P. By Theorem 3.13, T is a clique tree of G, thus completing the proof. □

4. Basic DV and basic RDV graphs

This section focuses on the existence of special types of clique trees and compatible trees and their connection with basic chordal graphs. Three subclasses of clique trees have been studied among others for more than twenty years [16] and we describe them below.

A *UV-clique tree* of a graph G is a clique tree such that, for every $v \in V(G)$, we have that $T[\mathcal{C}_v]$ is a path off T. A chordal graph is *UV* if it has a *UV*-clique tree. A *DV-clique tree* of G is a clique tree such that its edges have been directed and, for every $v \in V(G)$, we have that $T[\mathcal{C}_v]$ is a directed path of T. A chordal graph is *DV* if it has a *DV*-clique tree. An *RDV-clique tree* of G is a *DV*-clique tree that is rooted at a vertex w. A chordal graph is *RDV* if it has an *RDV*-clique tree.

It is clear from the definition that *RDV* graphs form a subclass of *DV* graphs and *DV* graphs form in turn a subclass of *UV* graphs. The class of clique graphs of *UV* graphs is that of dually chordal graphs. This and other reasons justify that a dual class was not defined for *UV* graphs.

On the other hand, *DV* and *RDV* graphs do have each a dual class. *Dually DV* graphs are the clique graphs of *DV* graphs and *dually RDV* graphs are the clique graphs of *RDV* graphs [17]. This duality is also reflected by the existence of characteristic trees. The *DV(RDV)-compatible tree* characterizing a dually *DV*(*RDV*) graph is a (rooted) directed spanning tree such that every clique of the graph induces a directed path.

Once again our focus will be on finding the relationship between clique trees and compatibles trees, but the concepts that have just been introduced will allow to do it at a deeper level. We had mentioned in the introduction that every clique tree of a chordal graph G is a compatible tree of $K(G)$. There is a similar result for *DV* and *RDV* graphs.

Proposition 4.1. *Let G be a DV(RDV) graph. Then, every DV(RDV)-clique tree of G is a DV(RDV)-compatible tree of $K(G)$.*

Proof. Let T be a *DV*(*RDV*)-clique tree of G. Since the 3-sun is not a *DV*(*RDV*) graph, it is the induced subgraph of no *DV*(*RDV*) graph. It is known that every chordal graph without induced 3-suns is clique-Helly [11]. Hence, G is a clique-Helly graph. This implies that every clique of $K(G)$ is of the form $\mathcal{C}_v$, for some $v \in V(G)$. These sets induce directed paths of T because T is a *DV*(*RDV*)-clique tree. Therefore, T is a *DV*(*RDV*)-compatible tree. □

Note that there is not a similar result for *UV* graphs. Consider for example the *UV* graph G in Fig. 5. The spanning tree of $K(G)$ with edges C_1C_2, C_2C_3, C_3C_4, C_4C_5, C_2C_6, C_3C_8 and C_4C_7 is a *UV*-clique tree of G. However $\{C_3, C_4, C_5, C_7\}$ is a clique of $K(G)$ not inducing a path in the tree.

Once again, the converse is not necessarily true. Given a *DV*(*RDV*) graph, there could be a *DV*(*RDV*)-compatible tree of $K(G)$ that is not a *DV*(*RDV*)-clique tree of G. See an example in Fig. 6. However, the case that G is basic chordal is special.

Proposition 4.2. *Let G be a basic chordal graph. Then every DV(RDV)-compatible tree of $K(G)$ is a DV(RDV)-clique tree of G.*

Proof. Let T be a *DV*(*RDV*)-compatible tree of $K(G)$ and v be any vertex of G. Since G is basic chordal, T is a clique tree of G, so $\mathcal{C}_v$ induces a subtree of T. Furthermore, $\mathcal{C}_v$ is a complete set of $K(G)$. Let D be a clique of $K(G)$ containing $\mathcal{C}_v$. As T is *DV*(*RDV*) compatible with $K(G)$, we have that D induces a directed path of T. Hence, $\mathcal{C}_v$ induces a subtree of T contained in a directed path of T. Therefore, $T[\mathcal{C}_v]$ itself is a directed path. This completes the proof. □

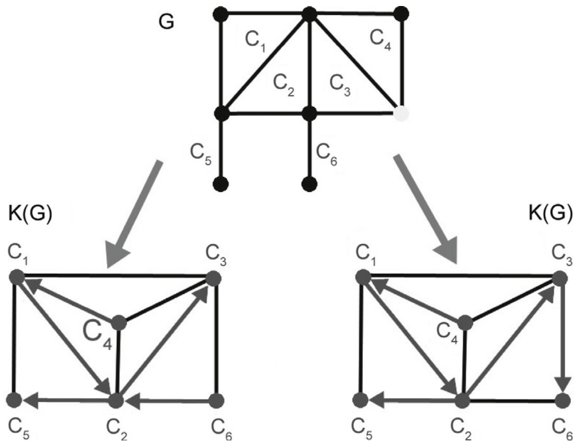

Fig. 6. For the graph G on top, the lower left shows a DV-compatible tree of $K(G)$ and the lower right shows an RDV-compatible tree of $K(G)$. Neither is a clique tree of G because one of the sets $\mathcal{C}_v$ is $\{C_3, C_4\}$, which does not induce a subtree.

The most important thing about Proposition 4.2 is that it shows that, for a basic chordal graph G, the correspondence is not only between the clique trees of G and the compatible trees of $K(G)$, but it is stronger in a way that also the $DV(RDV)$-clique trees of G correspond to the $DV(RDV)$-compatible trees of $K(G)$, when they exist.

As we are now comparing $DV(RDV)$-clique trees and $DV(RDV)$-compatible trees, it would be interesting to define new subclasses of basic chordal graphs that we could call basic DV and basic RDV graphs. In our opinion, the most natural definitions that we can give at the moment would be as a follows:

A chordal graph is *basic DV* if it is a DV basic chordal graph such that the DV-clique trees of G are exactly the DV-compatible trees of $K(G)$. Similarly, a *basic RDV* graph is defined to be an RDV basic DV graph whose RDV-clique trees are exactly the RDV-compatible trees of $K(G)$.

Based on Proposition 4.2, we find that the definitions can be simplified, since it is not necessary to require the correspondence between $DV(RDV)$-clique trees and $DV(RDV)$-compatible trees.

Theorem 4.3. *Let G be a chordal graph. Then,*

(a) *G is basic DV if and only if G is basic chordal and DV.*
(b) *G is basic RDV if and only if G is basic chordal and RDV.*

Proof. The definition clearly implies that a basic $DV(RDV)$ graph is basic chordal and $DV(RDV)$. Conversely, if G is basic chordal and $DV(RDV)$, then Proposition 4.2 implies that the $DV(RDV)$-clique trees of G are exactly the $DV(RDV)$-compatible trees of $K(G)$. □

Now we will study the clique graphs of these classes. For that purpose, given a dually DV graph G, we define $\mathcal{X}(G)$ to be the family of sets of vertices of G inducing a directed path of every DV-compatible tree of G. If G is RDV, then $\mathcal{Y}(G)$ will denote the family of sets of vertices of G inducing a directed path of every RDV compatible tree of G. The connection between these families and clique graphs is the following:

Theorem 4.4. *Let H be a chordal graph.*

(a) *Let G be dually DV. Then, $K(H) = G$ and H is basic DV if and only if H is the intersection graph of a separating family $\mathcal{F}$ where every $F \in \mathcal{F}$ is in $\mathcal{SDC}(G) \cap \mathcal{X}(G)$ and such that $\mathcal{S}(\mathcal{F}) = G$.*
(b) *Let G be dually RDV. Then, $K(H) = G$ and H is basic RDV if and only if H is the intersection graph of a separating family $\mathcal{F}$ where every $F \in \mathcal{F}$ is in $\mathcal{SDC}(G) \cap \mathcal{Y}(G)$ and such that $\mathcal{S}(\mathcal{F}) = G$.*

Proof. (a) Suppose that H is basic DV. We know that H is equal to the intersection graph of $\{\mathcal{C}_v\}_{v\in V(H)}$, which is a subfamily of $\mathcal{SDC}(G)$ because H is basic chordal. Every member of this family induces a directed path of every DV-clique tree of H, i.e, of every DV compatible tree of G. Hence, every member of $\{\mathcal{C}_v\}_{v\in V(H)}$ is in $\mathcal{X}(G)$. It is not difficult to verify that the two section of the family is $K(H)$, which equals G, and that it is separating.

Conversely, suppose that H is the intersection graph of a separating family $\mathcal{F}$ whose members are in $\mathcal{SDC}(G) \cap \mathcal{X}(G)$ and such that $\mathcal{S}(\mathcal{F}) = G$. Then, H can be seen as the intersection graph of directed paths of any DV-compatible tree of G and hence H is DV. Furthermore, Theorem 1.7 implies that H is basic chordal and $K(H) = G$. Hence, H is basic DV and $K(H) = G$.

(b) The proof is the same provided that we replace DV by RDV and $\mathcal{X}(G)$ by $\mathcal{Y}(G)$. □

By the definition of the different types of compatible trees, a family $\mathcal{F}$ satisfying the conditions of Theorem 4.4 is the family of cliques and unit sets of vertices. Thus, much like the clique graphs of basic chordal graphs are all dually chordal graphs, the clique graphs of basic $DV(RDV)$ graphs are all dually $DV(RDV)$ graphs.

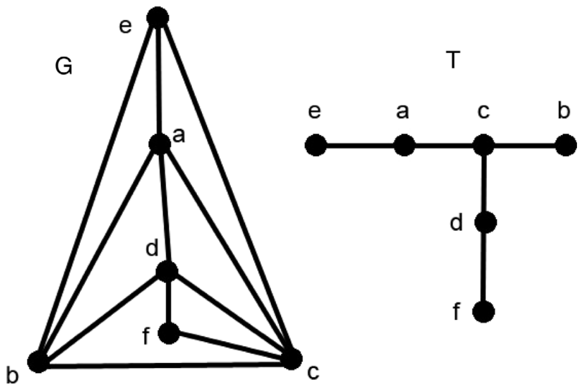

Fig. 7. A dually chordal graph G and a compatible tree T.

Theorem 4.5. *$K(\text{Basic } DV) = \text{Dually } DV$ and $K(\text{Basic } RDV) = \text{Dually } RDV$.*

Let us now consider the case where the family $\mathcal{F}$ of Theorem 4.4 not necessarily satisfies that each of its members is in $\mathcal{SDC}(G)$. In that scenario, we still have a correspondence between some clique trees and some compatible trees, but it is weaker.

Proposition 4.6. (a) *Let G be a dually DV graph. Let $\mathcal{F}$ be a separating family such that every $F \in \mathcal{F}$ is in $\mathcal{X}(G)$ and $\mathcal{S}(\mathcal{F}) = G$. Then $L(\mathcal{F})$ is DV, $K(L(\mathcal{F})) = G$ and every DV-compatible tree of G is a DV-clique tree of $L(\mathcal{F})$.*
(b) *Let G be a dually RDV graph. Let $\mathcal{F}$ be a separating family such that every $F \in \mathcal{F}$ is in $\mathcal{Y}(G)$ and $\mathcal{S}(\mathcal{F}) = G$. Then $L(\mathcal{F})$ is RDV, $K(L(\mathcal{F})) = G$ and every RDV-compatible tree of G is an RDV-clique tree of $L(\mathcal{F})$.*

Proof. We only prove part (a), because the proof of part (b) is very similar.

As every member of $\mathcal{F}$ is in $\mathcal{X}(G)$, we can see $\mathcal{F}$ as a family of directed paths of a fixed DV-compatible tree T of G. Furthermore, every intersection graph of directed paths of a directed tree is DV [16]. Therefore, H is DV. Furthermore, every family of subtrees of a tree is Helly [10]. Thus $\mathcal{F}$ is Helly.

Now we show that $K(L(\mathcal{F}))$ is isomorphic to G. As $\mathcal{F}$ is Helly and separating, we can apply Lemma 3.8 to obtain that $\mathcal{C}(L(\mathcal{F})) = D\mathcal{F}$. It is simple to prove that two different vertices u and v are adjacent in G if and only if D_u and D_v are adjacent in $K(L(\mathcal{F}))$, that is, the function $f : V(G) \to V(K(L(\mathcal{F})))$ such that $f(v) = D_v$ for all $v \in V(G)$ is a graph isomorphism between G and $K(L(\mathcal{F}))$.

For every $F \in \mathcal{F}$, consider the member $\mathcal{C}_F$ of $D\mathcal{C}(L(\mathcal{F}))$. Then, $\mathcal{C}_F = \{C \in \mathcal{C}(L(\mathcal{F})) : F \in C\} = \{D_v \in D\mathcal{F} : v \in F\}$. Since $F \in \mathcal{X}(G)$, it follows from the isomorphism between G and $K(L(\mathcal{F}))$ that $\{D_v \in D\mathcal{F} : v \in F\} \in \mathcal{X}(K(L(\mathcal{F})))$. Consequently, $D\mathcal{C}(L(\mathcal{F})) \subseteq \mathcal{X}(K(L(\mathcal{F})))$. Therefore, every DV-compatible tree of $K(L(\mathcal{F}))$ is a DV-clique tree of $L(\mathcal{F})$, which completes the proof. □

Consider the graph G of Fig. 7. This graph has three cliques, namely, $\{a, b, c, e\}$, $\{a, b, c, d\}$ and $\{c, d, f\}$. It is dually chordal because the tree T also appearing in the figure is compatible with it. What is more, it is dually RDV because the directed path *eabcdf* is an RDV-compatible tree. By Theorem 4.4, there exists a separating family $\mathcal{F}$ whose members are in $\mathcal{SDC}(G) \cap \mathcal{X}(G)$, the two section graph $\mathcal{S}(\mathcal{F})$ equals G, the intersection graph $L(\mathcal{F})$ is basic DV and $K(L(\mathcal{F})) = G$.

Let T' be any DV-compatible tree of G. Since $\{a, b, c, e\} \cap \{a, b, c, d\} = \{a, b, c\}$ and $\{a, b, c, d\} \cap \{c, d, f\} = \{c, d\}$, we have that $\{a, b, c, d\}$, $\{a, b, c\}$ and $\{c, d\}$ induce directed paths in T'. From this we also deduce that $\{a, b\}$ induces a directed path. Therefore $\{a, b\} \in \mathcal{X}(G)$.

Let $\mathcal{F}' = \mathcal{F} \cup \{\{a, b\}\}$. Thus, by Proposition 4.6, $K(L(\mathcal{F}'))$ equals G and every DV-compatible tree of G is a DV-clique tree of $L(\mathcal{F}')$. However, $\{a, b\} \notin \mathcal{SDC}(G)$, as the compatible tree T in the figure reveals. We infer from Theorem 1.7 that $L(\mathcal{F}')$ is not basic chordal.

As a conclusion from this example, we derive that the equality between the DV-clique trees of a graph and the DV-compatible trees of its clique graph does not ensure that the graph is basic chordal. That is why, when we define basic DV graphs with the goal of making it a subclass of basic chordal graphs, we cannot only focus on the correspondence between the DV-clique trees and the DV-compatible trees of the clique graph, and instead we explicitly say that the graph must be basic chordal. There is a similar conclusion about basic RDV graphs.

To finish this paper, we establish a relationship between hereditary basic chordal graphs and basic DV and basic RDV graphs.

We know how to construct all the clique trees of every hereditary basic chordal graph. Particularly, we can find an RDV-clique tree.

Proposition 4.7. *Let G be a hereditary basic chordal graph. Then G is an RDV graph.*

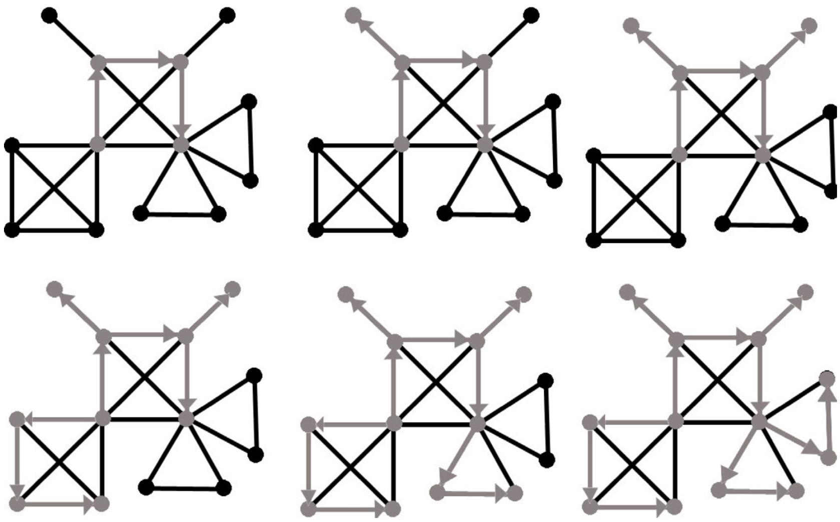

Fig. 8. A graphical representation of the procedure to obtain an *RDV*-compatible tree of $K(G)$.

Proof. We know that $K(G)$ is a block graph. Now we find an *RDV*-compatible tree for $K(G)$.

If $K(G)$ is complete, then every directed hamiltonian path of $K(G)$ is an *RDV*-compatible tree of $K(G)$.

Otherwise, let $D_1, D_2, \ldots, D_k$ be an ordering of $\mathcal{C}(K(G))$ such that $D_i \bigcap (\bigcup_{j=1}^{i-1} D_j) \neq \emptyset$ for $2 \leq i \leq k$. Such an ordering exists because $K(K(G))$ is connected. We construct the *RDV*-compatible tree of $K(G)$ as follows (see Fig. 8 for an example),

For $i = 1$, let P_1 be an oriented hamiltonian path of D_1. Let C_1 be its initial vertex. Set $T_1 = P_1$ and $i = 2$.

For $i = n$, where $n \geq 2$, let $D_n \bigcap (\bigcup_{j=1}^{n-1} D_j) = \{C_n\}$. The intersection has only one element because $K(G)$ is a block graph. Let P_n be a directed hamiltonian path of D_n starting at C_n. Let T_n be the tree formed by uniting T_{n-1} and P_n. While $i < k$, set $i = m + 1$. Otherwise end.

It is not difficult to prove in an inductive way that T_k is rooted at C_1. Furthermore, $T_k[D_i] = P_i$ for $1 \leq i \leq k$. Therefore, T_k is an *RDV*-compatible tree of $K(G)$.

In either case, the *RDV*-compatible tree of $K(G)$ is also an *RDV*-clique tree of G by Proposition 4.2. Hence G is an *RDV*-graph. □

Proposition 4.7 shows that hereditary basic chordal graphs are a subclass of basic *RDV*-graphs. A careful review of the proof of Proposition 4.7 also yields that, for every clique C of a hereditary basic chordal graph G, there exists an *RDV*-clique tree of G rooted at C. We just need to set D_1 equal to a clique of $K(G)$ containing C and to make P_1 start at C. In view of this fact, we characterize the *RDV* graphs with *RDV*-clique trees that can be rooted at any vertex.

Theorem 4.8. *Let G be a graph. Then G is RDV and, for every $C \in \mathcal{C}(G)$, there exists an RDV-clique tree of G rooted at C if and only if G is a gem-free chordal graph.*

Proof. Suppose that G is a gem-free chordal graph. We prove that G is *RDV* and that, for every $C \in \mathcal{C}(G)$, there exists an *RDV*-clique tree of G rooted at C by induction on $|\mathcal{C}(G)|$. It is trivially true if G has one clique, i.e., G is a complete graph.

Suppose that the property is true for every chordal gem-free graph G with $|\mathcal{C}(G)| \leq k$, were $k \geq 1$.

Let G be a chordal gem-free graph with $k + 1$ cliques and let C be any clique of G. Take a clique tree T of G in a way that the degree of C in T is minimum.

If the degree of C in T is one, then C is a simplicial clique. Let C' be the only vertex adjacent to C in T. Consider the chordal gem-free graph G' obtained after removing the simplicial vertices in C from G. It holds that $\mathcal{C}(G') = \mathcal{C}(G) \setminus \{C\}$. By the induction hypothesis, G' is *RDV* and it has an *RDV*-clique tree T_1 rooted at C'. Let T_2 be the tree obtained from T_1 by adding the vertex C and the oriented edge CC'. Thus T_2 is an *RDV*-clique tree of G rooted at C.

Suppose now that the degree of C in T is larger than one. Let $T_1, T_2, \ldots, T_j$ be the connected components of $T - C$ and, for $1 \leq i \leq k$, let T'_i be the tree obtained from T_i by adding the vertex C and the edge of T incident on C and T_i. Let $G_{T'_i}$ be the subgraph of G induced by the vertices in the cliques of G appearing in T'_i. It holds that $\mathcal{C}(G_{T'_i}) = V(T'_i)$, for $1 \leq i \leq j$.

By the induction hypothesis, $G_{T'_i}$ is an *RDV* graph and it has an *RDV*-clique tree $\overline{T_i}$ rooted at C. Let $\overline{T}$ be the tree such that $V(\overline{T}) = \mathcal{C}(G)$ and $E(\overline{T}) = \bigcup_{i=1}^{j} E(\overline{T_i})$. We now prove that $\overline{T}$ is an *RDV*-clique tree of G rooted at C.

It is clear that $\overline{T}$ is rooted at C. Let v be any vertex of G. If $\mathcal{C}_v$ is contained in the vertices of T'_l, for some, $1 \leq l \leq k$, then $T[\mathcal{C}_v] = T'_l[\mathcal{C}_v]$, which is a directed path.

Now we show that it is impossible for $\mathcal{C}_v$ not to be contained in the vertices of T'_l, for some $1 \leq l \leq k$. Suppose to the contrary that there exist two different numbers m and n between 1 and k such that $\mathcal{C}_v$ contains one clique C_m in $V(T_m)$ and one clique C_n in $V(T_n)$. Since $T[\mathcal{C}_v]$ is a subtree, we can assume without loss of generality that both C_m and C_n are adjacent to C in T. As $v \in C_m \cap C_n$ and G is gem-free, we infer from Proposition 2.2 that C_mC_n is the edge of some clique tree of G. Apply Theorem 1.4 to conclude that $T + C_mC_n - CC_n$ is a clique tree or $T + C_mC_n - CC_m$ is a clique tree, thus contradicting our choice of T.

Combine the last two paragraphs to infer that $\overline{T}$ is an *RDV*-clique tree of G rooted at C, as we desired.

Suppose now that G is a chordal graph with vertices v_1, v_2, v_3, v_4, v_5 inducing a gem like the one of Fig. 3. Let C_1 be a clique containing $\{v_1, v_2, v_3\}$, let C_2 be a clique containing $\{v_1, v_2, v_4\}$ and C_3 be a clique containing $\{v_1, v_4, v_5\}$.

We now prove that G does not have any *RDV*-clique tree rooted at C_2. Let T be any *RDV*-clique tree of G. By considering the directed paths induced by $\mathcal{C}_{v_1}, \mathcal{C}_{v_2}$ and $\mathcal{C}_{v_4}$, we conclude that $T[C_1, C_3]$ is a directed path containing C_2. The latter implies that C_2 cannot be a root of T.

Therefore, no *RDV*-clique tree of G is rooted at C_2. This completes the proof. □

As a consequence, the class of *RDV* graphs that can be rooted everywhere form a superclass of hereditary basic chordal graphs. We now add one more condition to obtain a new characterization of basic chordal graphs.

Theorem 4.9. *Let G be a graph. Then G is hereditary basic chordal if and only if G is RDV and, for every two different cliques C and C' with nonempty intersection, there exists an RDV-clique tree of G rooted at C and with the edge CC'.*

Proof. Suppose that G is hereditary basic chordal graph. We can construct an *RDV*-clique tree for G that is rooted at C and with the edge CC' by following the same procedure as in the proof of Proposition 4.7 and by requiring that D_1 is the clique of $K(G)$ containing the edge CC' and that CC' is the initial edge of P_1.

Conversely, suppose that G is *RDV* and, for every two different cliques C and C' with nonempty intersection, there exists an *RDV*-clique tree of G rooted at C and with the edge CC'. By Theorem 4.8, G is gem-free. Next we prove that G is dart-free.

Suppose to the contrary that v_1, v_2, v_3, v_4, v_5 are vertices of G inducing a dart in G like the one of Fig. 3. Let C_1 be a clique containing $\{v_1, v_2, v_3\}$, let C_2 be a clique containing $\{v_1, v_2, v_4\}$ and C_3 be a clique containing $\{v_1, v_5\}$.

Let T be any *RDV*-clique tree of G rooted at C_1. Thus $\mathcal{C}_{v_1}$ induces a directed path in T containing C_1, C_2 and C_3. We have three possibilities in this path, namely, $C_1 \in T[C_2, C_3]$, $C_2 \in T[C_1, C_3]$ and $C_3 \in T[C_1, C_2]$. The first possibility cannot hold because T is rooted at C_1. Furthermore. $\mathcal{C}_{v_2}$ induces a directed path in T containing C_1 and C_2 but not C_3. Therefore $C_2 \in T[C_1, C_3]$. Consequently C_1 is not adjacent to C_3 in T.

Thus, there is no *RDV*-clique tree of G that is rooted at C_1 and has the edge C_1C_3, which is a contradiction.

It necessarily follows that G is dart-free. Therefore, G is hereditary basic chordal. □

5. Concluding remarks

This is our second work on basic chordal graphs. The first one [6] approached them in a more general way, with a special focus on the relation between the clique graphs of a chordal graph and the compatible trees of its clique graph, which was the question that originally motivated us to define the class.

In this occasion we thought that trying to find some subclasses of basic chordal graphs and some of its principal characteristics represented a step forward in our study of the class.

As a result, we defined hereditary basic chordal, basic *DV* and basic *RDV* graphs and found several characterizations for them and for its clique graphs.

It is particularly pleasant for us the fact that hereditary basic chordal graphs are quite versatile and have so many characterizations, in a way that they transcend the study of basic chordal graphs, especially because of the different ways in which the class presented itself to many researchers. We do not discard that there are even many more characterizations to be discovered for hereditary basic chordal graphs.

Our study of basic *DV* and basic *RDV* graphs did not yield many characterizations compared to the case of hereditary basic chordal graphs, but it was enough to show the interesting way in which basic chordal graphs behave in relation to specific types of clique trees. Trying to find additional characterizations for these two classes could be the subject for future research.

References

[1] H.J. Bandelt, E. Prisner, Clique graphs and Helly graphs, J. Combin. Theory Ser. B 51 (1991) 34–45.
[2] A. Brandstädt, F. Dragan, V. Chepoi, V. Voloshin, Dually chordal graphs, SIAM J. Discrete Math. 11 (1998) 437–455.
[3] A. Brandstädt, V.B. Le, Simplicial powers of graphs, in: Proceedings of COCOA 2008, in: LNCS, vol. 5165, 2008, pp. 160-170 (Extended abstract), Full version electronically available in Theor. Computer Science.
[4] A. Brandstädt, P. Wagner, Characterising (k,l)-leaf powers, Discrete Appl. Math. 158 (2010) 110–122.
[5] J. Cáceres, A. Márquez, O.R. Oellermann, M.L. Puertas, Rebuilding convex sets in graphs, Discrete Math. 293 (2005) 26–37.
[6] P. De Caria, M. Gutierrez, On the correspondence between tree representations of chordal and dually chordal graphs, Discrete Appl. Math. 164 (2014) 500–511.
[7] G.A. Dirac, On rigid circuit graphs, Abh. Math. Sem. Univ. Hamburg 25 (1961) 71–76.
[8] M. Farber, Characterizations of strongly chordal graphs, Discrete Math. 43 (1983) 173–189.
[9] F. Gavril, The intersection graphs of subtrees in trees are exactly the chordal graphs, J. Combin. Theory Ser. B 116 (1974) 47–56.
[10] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, second ed., in: Annals of Discrete Mathematics, vol. 57, Elsevier, 2004.
[11] M. Gutierrez, Tree-Clique Graphs, Estudos e comunicações do Instituto de Matemática. Vol 63, Universidade Federal do Rio de Janeiro, 1996, pp. 7–26.

[12] M. Gutierrez, J. Meidanis, Algebraic theory for the clique operator, J. Braz. Comp. Soc. 7 (2001) 53–64.
[13] H. Habib, J. Stacho, Reduced clique graphs of chordal graphs, European J. Combin. 33 (2012) 712–735.
[14] S.T. Hedetniemi, P.J Slater, Line graphs of triangleless graphs and iterated clique graphs, Lect. Notes Math. 303 (1972) 139–147.
[15] W. Keneddy, Strictly Cordal Graphs and Phylogenetic Roots (Master thesis), University of Alberta, 2005.
[16] C.L. Monma, V.K. Wei, Intersection graphs of paths in a tree, J. Combin. Theory Ser. B 41 (1986) 141–181.
[17] E. Prisner, J. Szwarcfiter, Recognizing clique graphs of directed and rooted path graphs, Discrete Appl. Math. 94 (1999) 321–328.