

# STABLE INFLATIONARY DISSIPATIVE COSMOLOGIES

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The stability of the de Sitter era of cosmic expansion in spatially curved homogeneous isotropic universes is studied. The source of the gravitational field is an imperfect fluid such that the parameters that characterize it may change with time. In this way we extend our previous analysis for spatially-flat spaces as well as the work of Barrow.

## 1. Introduction

Inflationary cosmic expansions driven by a dissipative fluid has attracted some attention in the past as a mechanism able to solve the problems usually attached to the standard hot big-bang scenario (i.e. flatness, horizons, and monopoles abundance) different from those based on one (or more) scalar fields. The main dynamic effect of either the scalar field(s) or the dissipative fluid is to produce a state of cosmic accelerated expansion through the violation of the strong energy condition. If this accelerated state is sufficiently long, and if the Universe is able to exit it, the mentioned problems may find a satisfactory solution. Very often this analysis has been restricted to spatially-flat Friedmann-Lemaître-Robertson-Walker (FLRW for short) space-times, if only because the dependence of the scale factor on the spatial curvature  $k$  becomes negligible shortly after inflation has set in. However, one should be mindful that this curvature may have some impact on the stability of the inflationary solutions. Given the potential importance of such a scenario, it is advisable to establish its generality with respect to the effect of spatial curvature. In <sup>1</sup> we studied the stability of inflationary expansions caused by a dissipative fluid governed by a causal transport equation (i.e. one that respects both relativistic causality and hydrodynamic stability of the fluid <sup>2, 3</sup>), assuming  $k = 0$ . Before going any further, a word of caution seems in order. The stability of the dissipative fluid, in the sense this expression is used here, depends on the physical properties of the fluid under consideration, i.e. on its state and transport equations <sup>4, 5</sup>. Accordingly it should not be confused with the stability of the cosmic expansion -either inflationary or not. The stability of inflationary expansions driven by dissipative fluids in curved FLRW space-times was briefly discussed by Barrow <sup>6, 7</sup>. However, his study was confined to fluids governed by transport equations of Eckart's type <sup>8</sup> which, as is well-known, entails the aforesaid problems of acausality and instability.

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The aim of this paper is to investigate the combined effects of viscosity and curvature extending the analysis of <sup>1</sup> to spatially curved FLRW space-times and to study scenarios not considered previously. This bears some interest because observationally it is doubtful whether the spatial curvature of the Universe is positive, null or negative. On the one hand the average matter density is well below the critical value ( $\sim 0.3$  in critical units), but on the other hand the Universe seems to have entered an accelerated phase nowadays, which strongly hints to a cosmological constant of about 0.7 in the same units -however its value is limited by gravitational lensing <sup>9</sup>. Further, although the location of the first acoustic peak in the CBR spectrum seems to suggest a flat Universe, its exact position is still uncertain <sup>10, 11</sup>. In extending the results of <sup>1</sup> we also extend the work of Barrow's since the transport equation we consider is of causal type and therefore more general than the one used by him -for a brief illustration of the impact of causal transport equations in FLRW cosmology see <sup>12 13 14 15</sup> and <sup>16</sup>.

To analyse the asymptotic stability of the inflationary expansions we will resort again to the second method of Lyapunov <sup>17</sup>, succinctly stated in the appendix of <sup>1</sup>. In addition, some of the scenarios considered in this paper will be similar to those of <sup>1</sup>, whereby we shall avoid unnecessary repetition of details. As it turns out, the set of de Sitter solutions is drastically restricted because of a constraint that links some of the fluid parameters with the rate of cosmic expansion. Our choice of units is  $c = 8\pi G = k_B = 1$ .

## 2. Asymptotic Stability

We consider a FLRW universe filled with a imperfect fluid whose dissipative bulk pressure obeys the causal transport equation <sup>2, 3, 14, 15</sup>

$$\pi + \tau\dot{\pi} = -3\zeta H \quad (1)$$

where  $\tau(> 0)$  is the relaxation time associated at the dissipative process,  $\zeta(> 0)$  the phenomenological coefficient of bulk viscosity,  $H \equiv \dot{a}(t)/a(t)$  denotes the Hubble function, and  $a(t)$  the scale factor of the FLRW metric. The corresponding Einstein field equations are

$$H^2 = \frac{1}{3}\rho - \frac{k}{a^2} \quad (k = 0, \pm 1) \quad (2)$$

$$3(\dot{H} + H^2) = -\frac{1}{2}[\rho + 3(p + \pi)] \quad (3)$$

with  $\rho$  and  $p$  the energy density and hydrostatic pressure, respectively. These latter two quantities are assumed to be linked by the equation of state  $p = (\gamma - 1)\rho$ , where  $\gamma$  denotes the adiabatic index of the cosmic fluid. This, in general, depends on time -though oftenly, for simplicity, it is assumed constant.

The above set of equations yields

$$\ddot{H} + 3\gamma H\dot{H} + \tau^{-1} \left[ \dot{H} + \frac{3}{2}(\gamma + \tau\dot{\gamma}) H^2 - \frac{3}{2} \zeta H \right]$$

$$+\frac{k}{a^2} \left[ \left(1 - \frac{3}{2}\gamma\right) (2H - \tau^{-1}) + \frac{3}{2}\dot{\gamma} \right] = 0 \quad (4)$$

and the latter can be recast as

$$\frac{d}{dt} \left[ \frac{1}{2} \dot{H}^2 + V(H) \right] = D(H, \dot{H}). \quad (5)$$

Here, the left hand side is the time derivative of a Lyapunov function (see appendix in <sup>1</sup>) with

$$\begin{aligned} V(H) = & \frac{1}{2} (\gamma + \tau\dot{\gamma}) \tau^{-1} H^3 + \left[ \frac{k}{a^2} \left(1 - \frac{3}{2}\gamma\right) - \frac{3}{4}\zeta\tau^{-1} \right] H^2 \\ & + \frac{k}{a^2} \left[ \frac{3}{2}\dot{\gamma} - \left(1 - \frac{3}{2}\gamma\right) \tau^{-1} \right] H \end{aligned} \quad (6)$$

and

$$\begin{aligned} D(H, \dot{H}) = & - (3\gamma H + \tau^{-1}) \dot{H}^2 + \frac{1}{2} \tau^{-2} (\tau\dot{\gamma} - \dot{\tau}\gamma + \tau^2\ddot{\gamma}) H^3 \\ & + \frac{3}{4} \tau^{-2} (\zeta\dot{\tau} - \dot{\zeta}\tau) H^2 - \frac{2k}{a^2} \left(1 - \frac{3}{2}\gamma\right) H^3 - \frac{3k}{2a^2} \dot{\gamma} H^2 \\ & - \frac{2k}{a^2} \left[ \frac{3}{2}\dot{\gamma} - \left(1 - \frac{3}{2}\gamma\right) \tau^{-1} \right] H^2 + \frac{k}{a^2} \left[ \frac{3}{2}\ddot{\gamma} + \frac{3}{2}\dot{\gamma}\tau^{-1} + \left(1 - \frac{3}{2}\gamma\right) \tau^{-2}\dot{\tau} \right] H. \end{aligned} \quad (7)$$

As shown in figure 1 the potential (6) has two extrema, a minimum at  $H^+$  and a maximum at  $H^-$ . Their asymptotic behaviours for an expanding evolution at late time are

$$H^+ = \frac{\zeta}{\gamma + \tau\dot{\gamma}} + \mathcal{O}(a^{-2}) \quad (8)$$

and

$$H^- = \frac{k}{9\zeta a^2} \left[ \frac{3}{2}\dot{\gamma} - \left(1 - \frac{3}{2}\gamma\right) \tau^{-1} \right] + \mathcal{O}(a^{-4}). \quad (9)$$

The solution (9) is unstable, while the stability of the solution (8) is determined by the sign of the leading term of  $D$  in a neighbourhood of the point  $(H_0, 0)$  of the phase plane  $(H, \dot{H})$ , where  $H_0$  is the leading term of  $H^+$

$$D \approx - (3\gamma H + \tau^{-1}) \dot{H}^2 + \frac{1}{3} H^2 \left( H - \frac{3}{2} H^+ \right) \frac{d}{dt} (3\gamma H + \tau^{-1}) + \mathcal{O}(a^{-2}). \quad (10)$$

Here we have assumed that  $H^+$  is a quasi-de Sitter solution. A sufficient condition for this solution to be asymptotically stable is that  $3\gamma H + \tau^{-1}$  increases for large

time. This is equivalent to the condition that  $\zeta/\tau$  increases with time. Then using the relationship <sup>14</sup>

$$\frac{\zeta}{\tau} = v^2 \gamma \rho \quad (11)$$

where  $v$  is speed of the dissipative signal associated to  $\pi$ , this condition is also equivalent that  $\gamma v^2$  is an increasing function of time.

It can be seen that  $H_0$  is an exact de Sitter solution when the constraint

$$\dot{\gamma} + [\tau^{-1} - 2H_0] \gamma = \frac{2}{3} [\tau^{-1} - 2H_0] \quad (12)$$

is satisfied. It gives for  $\gamma \neq 2/3$

$$H_0 = \frac{-1 \pm \sqrt{1 + 6(3\gamma - 2)\tau\zeta}}{2(3\gamma - 2)\tau} \quad (13)$$

and

$$H_0 = \frac{3}{2}\zeta \quad (14)$$

when  $\gamma = 2/3$ . Also the static solution  $H = 0$  exists for  $k = 1$ , when the constraint

$$\tau\dot{\gamma} + \gamma = \frac{2}{3} \quad (15)$$

holds. To analyse its stability we linearize equation (4) about it,

$$\ddot{H} + \tau^{-1}\dot{H} + 3 \left[ \frac{\dot{\gamma}\tau}{a^2} - \frac{1}{2}\zeta\tau^{-1} \right] H = 0. \quad (16)$$

The general solution of (16) grows for any decreasing adiabatic index showing that the static solution is unstable and can be identified with the limiting case of the unstable solution  $H_-$ . This result could be different if a non-standard increasing adiabatic index in the limit  $t \rightarrow \infty$  were admitted. However, by fine tuning the initial condition a one-parameter family of solutions that approaches a static space-time at large times can be obtained. Besides, there exists a one-parameter family of solutions that starts from a static spacetime in the far past and evolves towards a stable de Sitter solution.

### 3. Inflationary Solutions

Accelerated expansion requires a sufficiently negative pressure, hence we will consider evolutions that approach asymptotically to a negatively constant viscous pressure. To this end we will investigate three models, two of them have the viscous pressure coefficient proportional to Hubble time, and in the last one it is determined by causality and stability of scalar perturbations. These three cases are characterized by the specific instances: (i) when  $\dot{\gamma} = 0$  and  $\zeta \propto H^{-1}$ , (ii) when  $\tau$  and  $\zeta$  vary as  $H^{-1}$ , and (iii) when  $\tau^{-1} \propto H$  and  $\zeta = v^2 \gamma \rho \tau$ . We will see them in turn.

### 3.1. Constant adiabatic coefficient model

When  $\zeta = \text{constant}$  (4) can be rewritten as

$$\begin{aligned} & \tau \frac{d}{dt} \left[ \dot{H} + \frac{3}{2}\gamma H^2 - \frac{k}{a^2} \left( 1 - \frac{3}{2}\gamma \right) \right] \\ & + \left[ \dot{H} + \frac{3}{2}\gamma H^2 - \frac{k}{a^2} \left( 1 - \frac{3}{2}\gamma \right) - \frac{3}{2}\zeta H \right] = 0 \end{aligned} \quad (17)$$

accordingly, the bulk viscosity coefficient must be given by

$$\zeta = \frac{2c}{3H} \quad (18)$$

with  $c$  a positive constant, and

$$\dot{H} + \frac{3}{2}\gamma H^2 - \frac{k}{a^2} \left( 1 - \frac{3}{2}\gamma \right) = c \quad (19)$$

is a first integral of (17). Assuming that the adiabatic index does not vary with time, this equation can be easily reduced to an equivalent mechanical system by the substitution  $a = s^{2/3\gamma}$  (in the appendix we obtain its parametric general solution), with first integral

$$\frac{\dot{s}^2}{2} + \frac{9\gamma^2 k}{8} s^{2(1-2/3\gamma)} - \frac{3\gamma c}{4} s^2 = \frac{3\gamma E}{2} \quad (20)$$

where  $E$  plays the role of a mechanical energy. This expression shows that the curvature and viscosity terms behave as “conservative forces”. In particular viscosity provides an effective spring with negative elastic constant, and it determines the late time behaviour of this model. Next we will integrate this system for some values of  $\gamma$  that yield simple solutions and were considered in <sup>6</sup> to compare with the results obtained for fluids governed by Eckart’s transport equation:

I) When  $\gamma = 1/3$  we obtain

$$a(t) = c_1 + c_2 \exp(\sqrt{2ct}) + c_3 \exp(-\sqrt{2ct}) \quad (21)$$

provided the constraint  $8cc_2c_3 - k - 2cc_1^2 = 0$  holds. In the limit  $c \rightarrow 0$  the usual perfect fluid solutions are recovered

$$a(t) = a_1 t^2 + a_2 t + a_3 \quad (22)$$

provided the constraint  $4a_1a_3 - a_2^2 - k = 0$  is satisfied. This family of solutions includes several interesting possibilities. Solutions with and without initial singularity, bouncing solutions and solutions with a finite time-span, as well as the Milne solution when  $k = -1$  and  $a_1 = 0$ .

II) When  $\gamma = 2/3$  there are three families of solutions depending on the sign of  $E$ :

1. For  $E > 0$

$$a(t) = \sqrt{\frac{2E}{c}} \sinh(\sqrt{c}\Delta t) \quad (23)$$

i.e. the scale factor initially grows as  $\sqrt{2E}\Delta t$  independently of  $c$ .

2. For  $E < 0$

$$a(t) = \sqrt{-\frac{2E}{c}} \cosh(\sqrt{c}\Delta t) \quad (24)$$

this family of non-singular solutions bounce at  $\Delta t = 0$  when the scale factor attains its minimum value  $a_m = \sqrt{-2E/c}$ .

3. For  $E = 0$

$$a(t) = \sqrt{\frac{1}{c}} \exp(\sqrt{c}\Delta t). \quad (25)$$

The three families of solutions exist for all  $k = 0$  or  $\pm 1$ , and for large time they approach the inflationary de Sitter scenario  $a(t) \approx \exp \sqrt{c}\Delta t$ . These results extend those of §4 in <sup>6</sup> obtained in the framework of the non-causal Eckart's theory, and where the viscosity was assumed to follow a power-law dependence upon the density.

III) When  $\gamma = 4/3$  (i.e. a radiation-dominated universe) one has

$$a^2(t) = \frac{1}{c} \left[ k + b \exp(-\sqrt{2ct}) + \frac{1}{8b} (1 - 4cE) \exp(\sqrt{2ct}) \right] \quad (26)$$

where  $b$  is an integration constant. This family of solutions comprises three types of behaviours depending on the value of  $E$ . When  $E > 0$  singular solutions occur for all  $k$ . However, when  $-1/4c < E < 0$  a singularity arises only if  $k < 0$ . Near the singularity the behaviour of the solution (26) is radiation-like. When  $E < -c$  there is no singularity.

Again the large-time behaviour is qualitatively the same as that found with Eckart's theory. Otherwise the evolution behaves differently.

IV) When  $\gamma > 2/3$  it follows, using the results of the appendix, a family of solutions that near the singularity evolves as  $a \approx t^{2/3\gamma}$ , and at late time it exhibits a de Sitter expansion  $a \approx \exp \sqrt{2c/3\gamma}t$ , irrespective of the spatial curvature.

### 3.2. Constant interactions number per expansion time model

The expanding universe defines a natural time-scale – the expansion time  $H^{-1}$ . Any particle species will remain in thermal equilibrium with the cosmic fluid so long as the interaction rate is high enough to allow rapid adjustment to the falling temperature. The relaxation time  $\tau$  is the characteristic collisional time of hydrodynamical processes occurring after the quantum era. Then a necessary condition for maintaining thermal equilibrium is

$$\tau < H^{-1} \quad (27)$$

Now  $\tau$  is determined by

$$\tau \simeq \frac{1}{n\sigma v} \quad (28)$$

where  $\sigma$  is the interaction cross-section,  $n$  is the number density of the target particles with which the given species is interacting, and  $v$  is the mean relative speed of interacting particles. So we have that  $\nu = (\tau H)^{-1}$  is the number of interactions in an expansion time. Now we will consider the case when  $\nu$  is a constant larger than one.

To find the solutions of equation (4) that satisfy this property we insert the ansatz

$$\gamma = \frac{2}{3}(1 + \epsilon) \quad (29)$$

along with (18) in the equation (4). A set of solutions can be found for a variable adiabatic index. Expression (4) splits in two equations,

$$\ddot{H} + (2 + \nu)H\dot{H} + \nu H^3 - \nu cH = 0 \quad (30)$$

plus a linear equation in  $\epsilon$ ,

$$\left[ H^2 + \frac{k}{a^2} \right] \dot{\epsilon} + \left[ 2H \left( \dot{H} - \frac{k}{a^2} \right) + \tau^{-1} \left( H^2 + \frac{k}{a^2} \right) \right] \epsilon = 0 \quad (31)$$

provided  $\tau^{-1} = \nu H$  with  $\nu$  a constant. Solving (31), we get

$$\gamma = \frac{2}{3} \left( 1 + b \frac{a^{2-\nu}}{k + \dot{a}^2} \right) \quad (32)$$

where  $b$  is a free parameter. To make explicit the  $t$  dependence of the adiabatic index we need solutions of (30) that satisfy  $0 \leq \gamma \leq 2$ . Equation (30) can be transformed into a second order linear differential equation whose general solution, obtained in <sup>24</sup>, reads

$$H^2 = c + \frac{c_1}{a^2} + \frac{c_2}{a^\nu} \quad (33)$$

where  $c_1$  and  $c_2$  are arbitrary integration constants. There it was shown that simple explicit solutions of (30) can be obtained when  $\nu = 1$  and when  $\nu = 4$ . In this case the solutions are

$$a(t) = c_1 + c_2 \exp(\sqrt{ct}) + c_3 \exp(-\sqrt{ct}) \quad (34)$$

and

$$a(t) = [c_1 + c_2 \exp(2\sqrt{ct}) + c_3 \exp(-2\sqrt{ct})]^{1/2} \quad (35)$$

respectively. For  $t \gg c^{-1/2}$ , both (34) and (35) describe inflationary expansions regardless the initial conditions.

### 3.3. *Stable causal sound perturbations model*

In this subsection we will investigate the consequences imposed by causality and stability on scalar longitudinal sound perturbations. It can be seen that the relationship between the viscosity coefficient and the dissipative contribution to the sound speed is  $\zeta = v^2 \gamma \rho \tau$ .

In this case equation (4) transforms into

$$h'' + (3\gamma + \nu) h' + 3\gamma (\nu - 3v^2) h = 0 \quad (36)$$

where the variable  $h = H^2 + k/a^2$  is proportional to the energy density of the fluid and the prime indicates derivative with respect to  $\eta = \ln a$ . Except for  $\nu = 3v^2$ , the general solution of this equation leads asymptotically to power-law behaviors<sup>13,18</sup>. When  $\nu = 3v^2$  the solution exhibit de Sitter expansion at late time. Exact solutions of (36) are

$$a(t) = \frac{1}{\sqrt{b}} \sinh(\sqrt{b}\Delta t), \quad (k = -1) \quad (37)$$

$$a(t) = \frac{1}{\sqrt{b}} \cosh(\sqrt{b}\Delta t), \quad (k = 1) \quad (38)$$

where  $b$  is an integration constant. These solutions are attractors and therefore of singular importance as they indicate the leading behaviour for large cosmological times. It is remarkable that the expressions (37) and (38) do not explicitly contain any quantity directly associated to viscosity (or particle creation) even though it is precisely this effect that drives the exponential expansion. In this case the same effect generates the translational invariance of the energy density.

## 4. Additional Inflationary Scenarios

### 4.1. Variable $\tau$

When either  $\gamma$  or  $\zeta$  is a constant the solution (13) is asymptotically stable provided  $\tau$  is a decreasing function. When  $\tau$  is a function that first decreases and in a second stage increases, there is a first period of exponential inflation followed by a graceful exit. On the other hand, when the relationship (11) holds,  $\tau$  presents a minimum provided  $v^2$  has a maximum. In this case the Universe enters an inflationary stage and afterwards exits it. This parallels the cosmic scenario of §2.2 in<sup>1</sup>. There the cosmic fluid was modelled as a mixture of radiation and heavy particles that decayed at a very high rate into (more stable) lighter particles (less massive modes), with high or moderate multiplicity<sup>19</sup>.

### 4.2. Variable $\tau$ and $\gamma$

Another model of dissipatively driven inflation arises when  $\zeta$  is a constant (or nearly a constant) and  $\gamma$  varies as

$$\gamma(t) = \frac{2}{3} \left[ 1 + \frac{c}{\tau(t)} \right] \quad (39)$$



with  $c$  a constant. From (7) we see that  $D$  is negative-definite when  $\dot{\tau} < 0$ , and accordingly the de Sitter expansion results asymptotically stable. Likewise  $\gamma(t)$  increases when  $c > 0$ .

By choosing  $c = \tau(t_1)/2$  and  $\tau(t_1)/\tau(t_2) = 2$  one follows a model in which initially all the energy is in the form of non-relativistic particles,  $\gamma = 1$ , and it is gradually transferred to relativistic ones, so that finally  $\gamma = 4/3$ . This corresponds to the dissipative process of decay of dust particles into radiation. Additional dissipation may arise from the interaction matter-radiation. Once the heavy particles have decayed, the Universe exits inflation. A similar scenario in flat space was reported in <sup>1</sup>, (see equations (17) & (18) there) but in that work  $\tau$  was considered constant instead.

Assuming that the dust particles are primeval mini-black holes of rest mass  $m$  and that the thermodynamic properties of the mixture of black holes and relativistic particles correspond to a Boltzmann gas of vanishing chemical potential, the adiabatic index reads

$$\gamma(z) = 1 + \frac{K_2(z)}{zK_1(z) + 3K_2(z)} \quad (40)$$

where  $K_n$  are modified Bessel functions of the second kind, and  $z \equiv m/T$  the dimensionless inverse temperature. By equating the right hand sides of (39) and (40) we can describe the continuous process of decay of mini-black holes from  $t = t_1$ , when the black hole energy density dominates the Universe, until  $t_2$  when the mini-black holes have completely evaporated away and the Universe becomes radiation-dominated. Here it is understood that all the black holes have the same mass and therefore the same temperature, and that this one equals the temperature of the massless component of the cosmic fluid at the beginning of the evaporation. The temperature behaves as in the  $k = 0$  case <sup>1</sup>

$$\frac{z'}{z} = \frac{12K_2(z) + 3zK_1(z) - Bz^2}{12K_2(z) + 5zK_1(z) + z^2K_0(z)} \quad (41)$$

where  $B = 9H\zeta/A_0$ , with  $A_0$  a constant and  $' \equiv d/Hdt$ . From this equation follows that  $z'$  is negative for large  $z$ , which agrees with the warm inflationary scenario of above.

The time dependence of this temperature near  $t_1$  and  $t_2$  for a generic  $\tau(t)$  can be made explicit by expanding (39) about  $t = t_1$  and (40) for  $z \rightarrow \infty$ . Thus one follows  $T \propto t - t_1$ , while in the opposite limit (i.e.  $t \rightarrow t_2$  and  $z \rightarrow 0$ ) it yields  $T \propto (t_2 - t)^{-1/2}$ . This reproduces the results of <sup>1</sup> but this time with a generic  $\tau(t)$ . Again the production of relativistic particles at the final stage of the black holes evaporation is accompanied by a huge increase of the temperature of the cosmic fluid -which also agrees with a previous study of this process <sup>20</sup>.

The entropy production per unit volume in the radiation fluid is given by the well-known expression <sup>2, 3, 14</sup>,  $\dot{S} = \pi^2/(\zeta T)$ . It has the limiting behaviours

$$\dot{S} \simeq -\frac{3\tau(t_1)\pi^2(t_1)}{m\zeta\dot{\tau}(t_1)(t - t_1)}, \quad t \rightarrow t_1 \quad (42)$$

$$\dot{S} \simeq 2 \frac{\pi^2(t_2) \sqrt{6\dot{\tau}(t_2)(t-t_2)}}{m\zeta}, \quad t \rightarrow t_2 \quad (43)$$

where

$$\begin{aligned} \pi^2(t) = \frac{1}{\tau^2(t)} \left\{ \frac{1}{2\tau^2(t_1)} \left( 1 + \frac{2\tau(t)}{\tau(t_1)} \right)^2 \left[ 3\tau(t_1)\zeta \left( 2 + \frac{3}{2}\tau(t_1)\zeta \right) + \right. \right. \\ \left. \left. 1 - (1 + 3\tau(t_1)\zeta) \sqrt{1 + 6\tau(t_1)\zeta} + 1 \right] + \right. \\ \left. \frac{k}{a^2(t)} \left( 1 + \frac{2\tau(t)}{\tau(t_1)} \right) \left( 1 + 3\tau(t_1)\zeta - \sqrt{1 + 6\tau(t_1)\zeta} \right) + \frac{\tau^2(t_1)k^2}{a^4(t)} \right\} \quad (44) \end{aligned}$$

and

$$a(t) = a_0 \exp \left\{ \left[ \frac{\sqrt{1 + 6\zeta\tau(t_1)} - 1}{2\tau(t_1)} \right] t \right\}. \quad (45)$$

The entropy production rate in the radiation fluid happens to be very high at the beginning of the evaporation, but decreases sharply about the end of this process. As in <sup>1</sup> the net rate of radiation particle production per mini-black hole and unit of volume varies roughly as  $(\rho + p)^{-1}$ , where in this case  $\rho$  and  $p$  refer to the radiation fluid only <sup>21</sup>.

## 5. General case

In this section we consider the de Sitter solution (13) and assume that  $\gamma, \tau$  and  $\zeta$  vary with time. Again this more general situation may occur during the decay of massive particles into lighter ones and also during the decay of four-dimensional fundamental strings into massive and massless particles -admittedly this second possibility is more speculative. In this case the algebraic relationship holds

$$\gamma(t) = \frac{2}{3} + \frac{3\zeta(t) - 2H_0}{6H_0^2\tau(t)}. \quad (46)$$

By adequately choosing the behaviour of  $\zeta(t)$  and  $\tau(t)$  the decay into radiation, reaching relativistic gas state when  $\gamma = 4/3$ , can be described. After the decay a condensation phase back into non-relativistic matter may occur. It ends when  $\gamma$  returns to 1. A scenario compatible with the latter phase is the quantum tunneling of radiation into black holes <sup>22</sup>. This may arise very naturally because of the instability of the hot radiation against spontaneous condensation <sup>23</sup>. (This is altogether different from the whole disappearance of the radiation by black hole accretion). During this period both the viscosity coefficient and the relaxation time may be chosen as monotonic decreasing functions, provided  $\zeta/\tau$  grows with time.

Again we may interpret this behaviour in terms of a two-fluid model, where the viscosity coefficient arises because of the particle production process from the decay of massive non-relativistic particles into light ones. Shortly after the beginning of

the decay the particle production rate is large and the energy density of the fluid becomes dominated by the light component. Later on, as the decay rate slows down, the effect of adiabatic dilution by the fast exponential expansion of the Universe turns out to be more important. Accordingly the non-relativistic energy density takes over again, since it goes down as  $a^{-3}$ , while the relativistic energy density goes down at the faster rate of  $a^{-4}$ .

The adiabatic index (39) can be used to estimate the dissipative contribution to the speed of sound  $v^2$  in (11)

$$v^2 = \frac{2H_0^2\zeta}{(4\tau + 3\zeta - 2H_0)(H_0^2 - \frac{k}{a^2})}. \quad (47)$$

In the limit  $t \rightarrow \infty$ ,  $v^2$  is a monotonic function of  $\zeta$  and  $\tau$  within the region  $2/3 < \gamma$ . It vanishes when  $\zeta = 0$  and reaches a maximum value  $v^2 = 2/3$  when  $\zeta/H_0 \gg 1$ .

## 6. Concluding Remarks

We have applied the second method of Lyapunov to analyse the stability of cosmic inflationary expansions, driven by a dissipative fluid governed by a transport equation that allows for relaxation (i.e. of causal type), for non spatially flat FLRW metrics. The parameters characterizing the fluid (adiabatic index, viscosity coefficient and relaxation time) may vary as the expansion proceeds. This is interesting as repeatedly stressed, a dissipative fluid can phenomenologically mimic a perfect one where particle production takes place <sup>25, 26, 6, 15</sup>, either from the quantum vacuum, or by the decay of pre-existing heavy particles, or by the decay of massive modes of fundamental strings into massless modes.

Essentially the curvature does not modify the results found in our previous paper <sup>1</sup> concerning the stability of the de Sitter solution. This result shows the insensibility of the viscosity-driven inflationary scenario with respect to initial conditions even when spatial curvature is present. However, it severely restricts the set of de Sitter solutions encompassed by this model because a constraint (12), linking the adiabatic index with the relaxation time, must be satisfied. Nevertheless, since the de Sitter solution is asymptotically stable for a wide set of behaviours of the fluid parameters, inflation appears rather natural. The static solution,  $H = 0$ , is found to be unstable; however this result may be altered if a non-standard increasing adiabatic index could enter the play. This work generalizes own previous study in spatially-flat FLRW universes <sup>1</sup> as well as those of Barrow <sup>6, 7</sup>. Further, new solutions describing the effects of dissipation have also been found. They either go over a stable de Sitter scenario from an initial power-law singularity or asymptotically approach a static universe if a fine tuning of the initial condition is made.

Lastly, an expression for the dissipative contribution to the sound speed have been obtained (47). In general the latter goes down as the Universe expands, something rather natural as the impact of dissipative effects is thought to diminish with expansion.

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## Appendix

Equation (19) with the change of variable  $a = s^{2/3\gamma}$  becomes

$$\ddot{s} + F(s) = 0, \quad F(s) = \frac{3\gamma}{2} \left( \frac{3\gamma}{2} - 1 \right) k s^{1-4/3\gamma} - \frac{3\gamma c}{2} s. \quad (48)$$

Then, by the nonlocal transformation

$$z = \int F(s) ds, \quad \eta = \int F(s) dt \quad (49)$$

equation (48) turns into a linear second order differential equation with constant coefficient

$$z'' + 1 = 0 \quad (50)$$

where the prime indicates derivative with respect to  $\eta$ . Solving it the general solution of (48) in parametric form follows. This can be achieved by inserting this solution in the second equation of (49)

$$-\frac{\eta^2}{2} + c_1\eta + c_2 = \frac{9\gamma^2}{8}ks^{2(1-2/3\gamma)} - \frac{3\gamma c}{4}s^2 \quad (51)$$

The transformation law between  $\eta$  and  $t$  follows from solving (51) for  $s$  and inserting the resulting expression in the second equation of (49)

$$t - t_0 = \int \frac{d\eta}{F(s(\eta))}. \quad (52)$$

## Figure Caption

**Figure 1** Graph of the potential  $V(H)$ , given by equation (6), for each value of the spatial curvature  $k$ .