COSMOLOGICAL MODELS IN SEMI-CLASSICAL AND HIGHER ORDER GRAVITATIONAL THEORIES

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Abstract. We have studied semiclassical models with a classical scalar field, giving exact solutions in the cases of a $\lambda\ \varphi^4$ and an exponential self-interaction potential, in the last case we have also studied the influence of the vacuum polarization terms on the stability of the power-law solutions. We have also found cosmological exact solutions to the higher order gravitational equations derived from a Lagrangian with a $R\ \Box^k R$ structure, and investigated the stability of the de Sitter and Minkowski space-time in the sixth order approximation of this theory.

1. Introduction

Higher order gravitational field equations are either introduced by quantum corrections to the theory ((Brout *et al.*, 1979), 1980; (Vilenkin, 1985); (Mijic *et al.*, 1986)), or postulated in a generalised Lagrangian.

In the first context, known as the semiclassical viewpoint, the effects due to the interaction of quantum free matter fields with a classical gravitational field can be calculated in closed form in the one-loop approximation. The back-reaction equations are ((Hartle and Horowitz, 1981); (Birrel and Davies, 1982))

$$G_{ik} + \Lambda\ g_{ik} = -8\pi\ G\left[T^{cl}_{ik} + \left\langle T^{q}_{ik}\right\rangle\right]\ , \tag{1}$$

where G_{ik} is the Einstein tensor, Λ is the cosmological constant, T^{cl}_{ik} is the stress-energy tensor of the classical sources, and $\left\langle T^{q}_{ik}\right\rangle$ is the renormalised vacuum expectation value of the stress-energy tensor operator for the quantum fields. For conformally invariant free quantum fields in conformally flat spacetimes, if the quantum state is the conformal vacuum based on the Minkowski space, it was found that ((Davies, 1977); (Bunch and Davies, 1977))

$$\left\langle T^{q}_{ik}\right\rangle = \frac{1}{6}\bar{\alpha}\ {}^{(1)}H_{ik} + \bar{\beta}\ {}^{(3)}H_{ik} \tag{2}$$

where ${}^{(1)}H_{ik}$ and ${}^{(3)}H_{ik}$ consist only of local terms, and the constants $\bar{\alpha}$ and $\bar{\beta}$ are calculated by the gravitational conformal anomaly ((Duff, 1977); (Brown and

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Cassidy, 1977)) and depend on the number and types of fields present, as well as on the regularisation scheme. In the simplest case of the self-consistent evolution of a Robertson-Walker background interacting with a number of massless conformally invariant free quantum fields, were exactly solved in several cases, and the stability of the de Sitter and Minkowski solutions was investigated ((Starobinsky, 1980); (Barrow and Ottewill, 1983); (Chimento, 1988), 1990; (Chimento and Zuccalá, 1993), 1997). Also, the strong energy condition is frecuently violated in these models, driving the universe into an inflationary period ((Chimento, 1988); (Spindel, 1988)).

The other approach to higher order gravitational theories consist in considering effective Lagrangians with a finite or infinite number of derivatives of the Riemann tensor. In fact, these theories appear in a natural way if gravity emerges from some more fundamental finite quantum theory with a non-local effective Lagrangian having some characteristic scale l_0. For scales much larger than l_0, the expansion of the Lagrangian in terms of the Riemann tensor and its derivatives produces a series that breaks somewhere near the fundamental scale. Some higher order theories of this kind were shown to be conformally equivalent to the second order theory with many scalar fields ((Schmidt, 1990)), given rise to a multi-inflation scheme. This is the case of the family of Lagrangians of the form

$$L = \sqrt{-g}\; F\left(R,\ \Box R,\ \ldots, \Box^k R\right)\ , \tag{3}$$

where, R is the curvature scalar, $g = -\left|\det g_{ij}\right|$, $\Box$ denotes the D'Alembertian and F is a differentiable function of its arguments. The exclusive dependence of F on the Ricci scalar R (and not on the other invariants) is necessary for the theory to contain only scalar particles in addition to the usual massless tensor gravitons. In addition, Lagrangians of the form (3) can be considered as a starting point in the construction of a renormalizable theory of gravity, where the existence of corrective terms for renormalization could be connected to the existence of a Noether symmetry for the Lagrangian ((Capozziello and Lambiase, to appear)).

In Section 2 of the present paper we present several semiclassical models with a classical scalar field, considering the particular cases of a $\lambda\ \varphi^4$ and an exponential self-interaction potential. In both cases we present exact solutions an analyse its features and conditions of existence. Section 3 is devoted to cosmological solutions to the higher order gravitational equations derived from a Lagrangian with a $R\,\Box^k R$ structure. We also investigate the stability of the de Sitter and Minkowski spacetime in the sixth order approximation of this theory. In Section 4 the conclusions are stated.

2. Semiclassical Models

We consider cosmological solutions of the back-reaction equations (1,2) in the spatially flat Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\ \left(dx^2 + dy^2 + dz^2\right)\ , \tag{4}$$

where we have assumed $c = 1$. The 00 component of (1, 2) becomes:

$$\alpha\ \left(2H\ddot{H} - \dot{H}^2 + 6H^2\dot{H}\right) + \beta\ H^4 = H^2 - \frac{1}{2}\,\dot{\varphi}^2 - V(\varphi)\ , \tag{5}$$

where we have defined a new set of constants $\alpha = \frac{1}{2}\,l_p^2\,\bar{\alpha}$ and $\beta = \frac{1}{2}\,l_p^2\,\bar{\beta}$, being $l_p = (16\pi G)^{1/2}$ the Planck length. Also, $H = \dot{a}/a$ is the Hubble expansion rate, and the last two terms in the r.h.s. of (5) represent the energy density of a self-interacting classical scalar field φ, minimally coupled to the metric, satisfying the Klein-Gordon equation

$$\ddot{\varphi} + 3\ H\ \dot{\varphi} + \frac{\partial V}{\partial \varphi} = 0\ . \tag{6}$$

Below we obtain exact solutions of this problem.

2.1. $\lambda\,\varphi^4$- POTENTIAL

We consider the semiclassical equation (5) in the special case $\alpha = 0$, which, as was pointed out in the introduction, is not the case of a generalised lagrangian theory. By computing its first temporal derivative and using (6) we obtain

$$\beta\ H^2\dot{H} = \frac{1}{2}\ \left(\dot{H} + \frac{3}{2}\dot{\varphi}^2\right)\ .$$

We seek for solutions that can be parametrised in terms of the field variable φ, so that $H = H(\varphi)$ and $\dot{\varphi} = \dot{\varphi}(\varphi)$. Then, the last equation becomes

$$\beta\ H^2 H' = \frac{1}{2}\ \left(H' + \frac{3}{2}\ \dot{\varphi}\right)\ . \tag{7}$$

Proposing a linear relation $H = c_1\ \varphi + c_2$ for some constant parameters c_1, c_2, we obtain from (7)

$$\dot{\varphi} = \frac{2}{3}\,c_1\ \left[2\ \beta\ \ (c_1\ \varphi + c_2)^2 - 1\right]\ , \tag{8}$$

and replacing this expression into (5) we obtain a three parametric family of potentials of the form,

$$V(\varphi) = \lambda_4\ \varphi^4 + \lambda_3\ \varphi^3 + \lambda_2\ \varphi^2 + \lambda_1\ \varphi + \lambda_0\ , \tag{9}$$

where λ_i, $i = 0, .., 4$ are functions of the parameters c_1, c_2 and β; then, for a given value of the regularisation parameter β, the choice of the arbitrary constants c_1 and c_2 determines the particular form of the potential (9). The interesting case is that of the potentials implementing spontaneous symmetry breaking. The conditions for the potential to have the SSB form are

$$\begin{aligned} \lambda_4 &> 0\,, \\ -\frac{1}{12}\,\lambda_3^2\,\lambda_2^2 + \frac{1}{4}\,\lambda_3^3\,\lambda_1 + \frac{8}{27}\,\lambda_2^3\,\lambda_4 + \lambda_1^2\,\lambda_4^2 - \lambda_1\,\lambda_2\,\lambda_3\,\lambda_4 &< 0\,, \\ 3\,\lambda_3^2 - 8\,\lambda_2\,\lambda_4 &> 0\,. \end{aligned}$$

This conditions result in rather complicated behaviour of the SSB feature in terms of the original parameters β, c_1 and c_2. The integration of (8) produce the folowing families of solutions to the semiclassical problem (5) with $\alpha = 0$ and a potential of the form (9).

$\beta > 0$:

$$\varphi(t) = -\frac{c_2}{c_1} - \frac{1}{c_1\sqrt{2\,\beta}}\;\tanh\left(\nu\;\Delta t\right)\;, \tag{10}$$

$$a(t) = a_0\;\left[\cosh\left(\nu\;\Delta t\right)\right]^{-\frac{3}{4\,c_1^2\,\beta}}\;, \tag{11}$$

and

$$\varphi(t) = -\frac{c_2}{c_1} - \frac{1}{c_1\sqrt{2\,\beta}}\;\coth\left(\nu\;\Delta t\right)\;, \tag{12}$$

$$a(t) = a_0\;\left[\sinh\left(\nu\;\Delta t\right)\right]^{-\frac{3}{4\,c_1^2\,\beta}}\;, \tag{13}$$

$\beta < 0$:

$$\varphi(t) = -\frac{c_2}{c_1} + \frac{1}{c_1\sqrt{2\;|\beta|}}\;\tan\left(\nu\;\Delta t\right)\;, \tag{14}$$

$$a(t) = a_0\;\left[\cos\left(\nu\;\Delta t\right)\right]^{-\frac{3}{4\,c_1^2\,\beta}}\;, \tag{15}$$

where we have defined $\nu \equiv \sqrt{8\,|\beta|}c_1^2\;/\;3$. The solution (10, 11) represent a non-singular transition between two asymptotic stages of different constant φ values corresponding to $H = 1/\sqrt{2\,\beta}$ in the remote past and $H = -1/\sqrt{2\,\beta}$ in the remote future. Solution (12, 13) represents a universe contracting from an initial singular state with an infinite φ value (also an infinite energy density as seen in (8)), to a corresponding final stage approaching a constant φ value with $H = -1/\sqrt{2\,\beta}$. The universe represented by solution (14, 15) has a finite time span, undergoing a symmetric bounce between two infinite values of the scale factor, while the scalar field ranges from a negative infinite to a positive infinite value.

2.2. Exponential potential

It is known that if the self-interaction potential V for the inflaton field φ assumes exponential form, the model give rise to power-law inflation ((Lucchin and Matarrese, 1985); (Halliwell, 1987); (Burd and Barrow, 1988); (Salopek and Bond, 1990)). We shall consider here the effect of the quantum contributions on the inflationary behaviour of this models restricting ourselves to the $\alpha \neq 0$ case with no other assumptions on specific values or signs. It is convenient to write down equation (5) in the form

$$\begin{aligned} &2\,h\,\ddot{h} - \dot{h}^2 + 6\,h^2\dot{h} + \frac{\beta}{\alpha}h^4 = \\ &= \operatorname{sgn}\alpha\,\left(h^2 - \frac{|\alpha|}{12}l_p^4\,\bar{\rho}_{cl}\right)\,, \end{aligned} \tag{16}$$

being $h = \dot{a}/a$, where now, and for the rest of this section, the dot means differentiation with respect to the dimensionless time $\tau = (\frac{1}{2}|\alpha|l_p^2)^{-1/2}\,t$. The classical energy density reads

$$\bar{\rho}_{cl} = \left(|\alpha|\,l_p^2\right)^{-1}\,\dot{\varphi}^2 + V(\varphi)\,, \tag{17}$$

and the Klein-Gordon equation becomes

$$\ddot{\varphi} + 3\,h\,\dot{\varphi} + \frac{1}{2}|\alpha|\,l_p^2\,\frac{\partial V}{\partial\varphi} = 0\,. \tag{18}$$

Let us consider the exponential potential

$$V(\varphi) = V_0\,e^{-A\varphi}\,, \tag{19}$$

for some constant parameter A. By analogy with the classical problem, we search for solutions of the form

$$\begin{aligned} a(\tau) &\sim \tau^n\,, \\ \varphi &= b\,\ln\tau\,, \end{aligned} \tag{20}$$

which gives $h = n/\tau$. Replacing (20) into Klein-Gordon equation (18), we obtain that the parameters b and n are given by

$$A^2 = \frac{4}{b^2} = \frac{4\,(3n-1)}{|\alpha|\,l_p^2\,V_0}\,. \tag{21}$$

Also, the classical part of (16)

$$h^2 - \frac{l_p^2}{12}\left(\dot{\varphi}^2 + |\alpha|\,l_p^2\,V(\varphi)\right) = 0\,, \tag{22}$$

is satisfied by solution (20) if the classical constraint for the exponent

$$n = \frac{l_p^2}{A^2} \,. \tag{23}$$

is assumed ((Lucchin and Matarrese, 1985)). This power-law solution can also satisfy the semiclassical part of (16) if

$$\frac{\beta}{\alpha}\, n^2 - 6\, n + 3 = 0 \,,$$

which gives a second constraint for the exponent

$$n = \frac{3\alpha}{\beta}\, \left[1 \pm \sqrt{1 - \beta/3\alpha}\right] \,. \tag{24}$$

Then, if the conditions (23, 24) are fulfilled, there exist power-law exact solutions of the form (20, 21) to the semiclassical problem (16–19) which are also solutions of the classical Einstein Equation (22). The existence of inflationary power-law solutions in the classical case, is preserved in the semiclassical problem when $0 < \beta/\alpha < 3$, as can be seen from (24).

We consider now the stability of the solutions to the problem (16-19). Let (h_0, φ_0) be an exact solution to this problem, by considering the perturbed solution $(h = h_0 + \varepsilon,\ \varphi = \varphi_0 + \delta)$, replacing it into (16–19) and retaining only the first order terms in ε and δ, we obtain

$$h_0\, \ddot{\varepsilon} + \left(\dot{h}_0 + 3\, h_0^2\right)\, \dot{\varepsilon} + \left(\ddot{h}_0 + h_0\, \dot{h}_0 + 2\frac{\beta}{\alpha}\, h_0^3 - \operatorname{sgn}\alpha\, h_0\right)\, \varepsilon =$$

$$= \frac{l_p^2}{12} \operatorname{sgn}\alpha \left(-\dot{\varphi}_0\, \dot{\delta} + A\ \ |\alpha|\, \frac{l_p^2}{2}\, V_0\, e^{-A\varphi_0}\, \delta\right)\,, \tag{25}$$

$$\ddot{\delta} + 3\, h_0\, \dot{\delta} + 3\, \dot{\varphi}_0\, \varepsilon + A^2\ \ |\alpha|\, \frac{l_p^2}{2}\, V_0\, e^{-A\varphi_0}\, \delta = 0 \,. \tag{26}$$

It is possible to obtain ε from (26) as a functional of δ, and replace it into (25) to obtain a fourth-order linear differential equation for the field perturbation δ. For the special case of the power-law solutions (20) this equation is of the form

$$\delta^{(4)} + \frac{(6n+1)}{\tau}\, \delta^{(3)} + \frac{\left(9n^2 + 17n - 7\right)}{\tau^2}\, \ddot{\delta} + \frac{\left(51n^2 - 36n + 6\right)}{\tau^3}\, \dot{\delta} +$$

$$+ \frac{(3n-1)\ (16n-2)}{\tau^4}\, \delta = \operatorname{sgn}\alpha\, \left[\ddot{\delta} + \frac{(3n+1)}{\tau}\, \dot{\delta} + \frac{(3n-1)}{\tau^2}\, \delta\right] \,. \tag{27}$$

The the bracket on the r.h.s. of equation (27) is a quantity whose annulation gives the classical equation for the field perturbation δ, having a general solution of the form $\delta = c_1/\tau + c_2/\tau^{3n-1}$. By this reason, the classical power-law evolutions of the form (20) are stable for $n > 1/3$ ((Halliwell, 1987)). In the present case, however, the vacuum polarization terms appearing in the l.h.s. of (27) could altere this feature. As we are interested in the asymptotic stability of (20) for large τ values, we assume the existence of a special family of solutions of (27) whose asymptotic behaviour in the $\tau \gg 1$ region satisfies $\max\left\{\left|\delta^{(3)}/\delta^{(4)}\right|, \left|\delta^{(3)}/\delta^{(2)}\right|\right\} \ll \tau$, $\max\left\{\left|\dot{\delta}/\delta^{(4)}\right|, \left|\dot{\delta}/\delta^{(2)}\right|\right\} \ll \tau$, and $\max\left\{\left|\delta/\delta^{(4)}\right|, \left|\delta/\delta^{(2)}\right|\right\} \ll \tau$. For a family of solutions defined in such way, Equation (27) asymptotically reduces to $\delta^{(4)} - \operatorname{sgn}\alpha\,\ddot{\delta} = 0$, whose general solution is $\delta(\tau) = C_1\, e^{-\tau} + C_2\, e^{\tau} + C_3\, \tau + C_4$, for $\alpha > 0$, and $\delta(\tau) = C_1\, \cos\Delta\tau + C_2\, \tau + C_3$, for $\alpha < 0$. Both cases are consistent with the defining conditions of the family, and represent the two possible asymptotic bahaviours of $\delta(\tau)$ inside the family, showing the unstable behaviour of the power law solutions (20) with respect to perturbations satisfying such conditions.

3. Lagrangian Models

A particular subfamily of (3) corresponds to the linear functional

$$F\left(R,\ \Box R,\ \ldots,\Box^k R\right) = \sum_{A=0}^{k} \gamma_A\ R\ \Box^A R\ . \tag{28}$$

Also, the Einsteinian term R should be included in (28) to yield the correct weak-field limit ((Schmidt, 1994)). As we are interested in cosmological problems, we also include a cosmological constant Λ, to obtain, by means of the variation of L with respect to the metric, the complete $(4k+4)$-order equation

$$\Lambda\ g_{ab} + G_{ab} + \sum_{j=0}^{k} \gamma_j\ E_{ab}^{(j)} = -8\pi G\ T_{ab}\ , \tag{29}$$

where we have included a classical source with a stress-energy tensor T_{ab}, and where $E_{ab}^{(j)}$, $j \geq 1$, is obtained from the variation of the jth-order term $L^{(j)} = \sqrt{-g}\ R\ \Box^j R$, reading ((Schmidt, 1990))

$$\begin{aligned} E_{ab}^{(j)} &= 2\ R_{ab}\Box^j R - 2\ \left(\Box^j R\right)_{;\ ab} + 2\ g_{ab}\left(\Box^{j+1} R - \tfrac{1}{4} R\ \Box^j R\right) + \\ &+ \sum_{A=1}^{j}\left[\tfrac{1}{2} g_{ab}\left(\Box^{j-A} R\ \left(\Box^{A-1} R\right)_{;\ c}\right)^{;\ c} - \left(\Box^{j-A} R\right)_{;\ (a}\left(\Box^{A-1} R\right)_{;\ b)}\right]\ . \end{aligned} \tag{30}$$

We shall consider here cosmological solutions of (29) in the spatially flat Robertson-Walker metric (4).

3.1. EXACT SOLUTIONS

Several vacuum exact solutions to (29, 30) with the metric (4) are easily found for any order. In fact, the de Sitter universe

$$H^2 = \frac{\Lambda}{3}\,, \quad \Lambda > 0\,, \tag{31}$$

where $H = \dot{a}\,/\,a$ is the Hubble variable, satisfies $E_{ab}^{(j)} = 0$ for $j \geq 1$, since $\Box^j R = 0$ for this evolution. Of course in the case $\Lambda = 0$, we recover the Minkowski space-time, a trivial solution to (29). Also, in the particular case $\gamma_0 = \frac{1}{8\Lambda}$, $\Lambda \neq 0$, we have found a one-parametric family of solutions of the form

$$H^2 = \frac{C}{a^4} + \frac{\Lambda}{3}\,, \tag{32}$$

where C is an integration constant. This solution, as well as the de Sitter's one, satisfies (29) for any order, since (32) is a solution of constant scalar curvature $R = -4\,\Lambda$, whose integration lads to

$\Lambda > 0$, $C > 0$,

$$a(t) = a_0\; (\sinh \mu\; \Delta t)^{1/2}\,, \tag{33}$$

$\Lambda > 0$, $C < 0$,

$$a(t) = a_0\; (\cosh \mu\; \Delta t)^{1/2}\,, \tag{34}$$

$\Lambda < 0$, $C > 0$,

$$a(t) = a_0\; (\sin \mu\; \Delta t)^{1/2}\,, \tag{35}$$

where $\mu = \sqrt{4\;|\Lambda|\,/\,3}$ and $a_0 = \sqrt{|3\;C\,/\,\Lambda|}$. Solutions (33) have an initial singularity, while (34) undergo a symmetric bounce, both have final de Sitter stages. The universes represented by (35) have a finite time span between an initial and a final singularity. In the context of inflationary paradigm, where H is slowly varying, it should be assumed $\gamma_0 \geq 10^{10}\; m_p^{-2}$ in order to have sufficiently small adiabatic perturbations at the last stages of inflation ((Starobinsky, 1983)), then, the exact inflationary solutions (33-34) apply when $\Lambda \leq 10^{-10}\; l_p^2$ which is certainly the case, since the observational limit for Λ is many orders of magnitude bellow this value.

In the case when a radiation source is present, a term of the form $\sigma\,/\,a^4$ is added to the l.h.s. of Equation (29), and the radiation dominated Friedman universe

$$a(t) = (4\,\sigma)^{1/4}\; t^{1/2}\,,$$

becomes an exact solution for any order of the theory.

3.2. Stability of the de Sitter solutions

The de Sitter space-time was found to be stable for Lagrangians of the form $F = (-R)^u$, while unstable for $F = R\,\Box^k R$ ((Kluske, 1995)). Here, we analyze the stability of the de Sitter solution for the complete gravitational equation (29) up to 6th order ($k = 1$) in the spatially flat RW metric (4)

$$\begin{aligned} -\tfrac{\Lambda}{3} + H^2 - 6\,\alpha\,\left(2\,H\,\ddot{H} - \dot{H}^2 + 6\,H^2\dot{H}\right) - \\ -6\,\gamma\,\left(2\,H\,H^{(4)} - 2\,\dot{H}\,H^{(3)} + 12\,H^2\,H^{(3)} + \ddot{H}^2 + \right. \\ \left. +24\,H\,\dot{H}\,\ddot{H} + 10\,H^3\ddot{H} - 8\,\dot{H}^3 + 32\,H^2\dot{H}^2 - 24\,H^4\dot{H}\right) = 0\,, \end{aligned} \tag{36}$$

with the notation $\alpha \equiv \gamma_0$ and $\gamma \equiv \gamma_1$, which are constants of dimension length squared and length to the fourth power respectively. We shall consider here small perturbations around the exact solution (31) $H_0 \equiv \sqrt{\Lambda\,/\,3}$. By setting $H = H_0 + \delta$, $|\delta| \ll |H_0|$ in Equation (36) and retaining only terms of first order in δ, a linear fourth-order differential equation for the perturbation $\delta(t)$ is obtained. In the case of the de Sitter solution (31), this linear equation reads

$$\begin{aligned} \gamma\,\delta^{(4)} + 6\,\gamma\,H_0\,\delta^{(3)} + \left(\alpha + 5\,\gamma\,H_0^2\right)\ddot{\delta} + \\ +3\,H_0\left(\alpha - 4\,\gamma\,H_0^2\right)\dot{\delta} - \tfrac{1}{6}\delta = 0\,, \end{aligned}$$

whose characteristic roots are given by

$$\lambda_{\pm\mp} = -\frac{3}{2}\,H_0 \pm \frac{H_0}{2}\sqrt{17 - 2\,\xi \mp 2\sqrt{(\xi - 4)^2 + \frac{6}{\gamma\,\Lambda^2}}}\,, \tag{37}$$

where we have defined the non-dimensional parameter $\xi \equiv \frac{3\alpha}{\gamma\Lambda}$. We separate the analysis of the signs of the real parts of the exponents (37) into two limiting cases.

If $|\xi| \ll 1$, the case corresponding to the 'fast' de Sitter solutions, where the Hubble parameter H_0 is much bigger than the characteristic expansion scale given by $\sqrt{|\alpha\,/\,\gamma|}$, the sixth-order term in (36) dominates over the fourth-order one, and we have two subcases. If $\left|\Lambda^2\,\gamma\right| \gg 1$ the exponents behave as $\lambda_{++} \approx H_0$, $\lambda_{+-} \approx -4\,H_0$, $\lambda_{-+} \approx -3\,H_0$, $\lambda_{--} \approx -\frac{\alpha}{6\,\gamma\,H_0}$, being the dimension of the stability subspace equal to 3 if $\mathrm{sgn}(\alpha/\gamma) = 1$ and 2 in the opposite case. If $\left|\Lambda^2\,\gamma\right| \ll 1$, and $\gamma > 0$ we have $\lambda_{\pm+} \approx \pm\,\gamma^{-1/4}$, $\mathrm{Re}\left[\lambda_{\pm-}\right] \approx -\frac{3}{2}\,H_0$, corresponding to a three-dimensional stability subspace, if $\gamma < 0$ instead, we have $\mathrm{Re}\left[\lambda_{++}\right] = \mathrm{Re}\left[\lambda_{--}\right] \approx -|\gamma|^{-1/4}$ and $\mathrm{Re}\left[\lambda_{+-}\right] = \mathrm{Re}\left[\lambda_{-+}\right] \approx |\gamma|^{-1/4}$, giving rise to a two-dimensional stability subspace. As can be spectated from (Kluske, 1995), the 'fast' de Sitter solutions (31) are always unstable.

If $|\xi| \gg 1$, the case corresponding to the 'slow' de Sitter solutions, where the Hubble parameter H_0 is much smaller than $\sqrt{|\alpha\,/\,\gamma|}$, the fourth-order term in (36) dominates over the sixth-order one, giving rise to four limit subcases. If $0 < \gamma \ll \frac{3}{2}\,\alpha^2$, with $\alpha > 0$ we have $\lambda_{\pm+} \approx -\frac{3}{2}\,H_0 \pm\ (6\,\alpha)^{-1/2}$ and $\mathrm{Re}\left[\lambda_{\pm-}\right] \approx -\frac{3}{2}\,H_0$, producing a four-dimensional stability subspace (stable de Sitter solution) if $\Lambda >$

$\frac{4}{3\alpha}$, and a two-dimensional stability subspace in the opposite case. When $\alpha < 0$, the exponents become $\lambda_{\pm+} \approx \pm \sqrt{\frac{|\alpha|}{\gamma}}$ and $\lambda_{\pm-} \approx -\frac{3}{2}\, H_0$, giving rise to a three-dimensional stability subspace. In the limit $\frac{3}{2}\,\alpha^2 \ll \gamma$, the inequality $\Lambda < \frac{4}{3\,(6\,\gamma)^{1/2}}$ is required for consistency, giving $\lambda_{\pm+} \approx -\frac{3}{2}\, H_0 \pm\, (6\,\gamma)^{-1/4}$ and $\mathrm{Re}\left[\lambda_{\pm-}\right] \approx -\frac{3}{2}\, H_0$, corresponding to a three-dimensional stability subspace. If $-\frac{3}{2}\,\alpha^2 \ll \gamma < 0$, with $\alpha > 0$ we have $\lambda_{\pm+} \approx \pm \sqrt{\frac{\alpha}{|\gamma|}}$ and $\lambda_{\pm-} \approx -\frac{3}{2}\, H_0 \pm\, (6\,\alpha)^{-1/2}$, giving a three-dimensional stability subspace if $\Lambda > \frac{2}{9\,\alpha}$, and a two-dimensional stability subspace in the opposite case. With $\alpha < 0$ instead, $\mathrm{Re}\left[\lambda_{\pm\pm}\right] \approx -\frac{3}{2}\, H_0$, giving a stable de Sitter solution. Finally, if $\gamma \ll -\frac{3}{2}\,\alpha^2$, where $\Lambda < \frac{2}{3\,|6\,\gamma|^{1/2}}$ is required for consistency, the exponents are $\mathrm{Re}\left[\lambda_{+\pm}\right] \approx -\frac{3}{2}\, H_0 + \frac{1}{\sqrt{2}}\,|6\,\gamma|^{-1/4}$ and $\mathrm{Re}\left[\lambda_{-\pm}\right] \approx -\frac{3}{2}\, H_0 - \frac{1}{\sqrt{2}}\,|6\,\gamma|^{-1/4}$, producing a two-dimensional stability subspace.

3.3. Stability of the Minkowski space-time

We are also interested in considering the problem of Minkowski space-time stability in the kth-order gravitational theory described by (3, 28), with the constraint given by the metric (4). In this case $H_0 = 0$, and the linear equation for the perturbation vanishes, so we have to consider quadratic terms. This makes necessary to obtain the expansion of the kth-order term (30) up to second order in the Hubble variable H and its derivatives, for which suffices to consider the expressions of the curvature scalar R, and the relevant components of the Ricci tensor R_{ab}, to first order in H, which are given by $R \approx -6\,\dot{H}$, $R_{00} \approx -3\,\dot{H}$. Already being the D'Alembertian an operator of first order, $\Box = \Box_0 + \Box_1$ with $\Box_0 \equiv \partial_t^2$, and $\Box_1 \equiv 3\,H\,\partial_t$, it is not difficult to compute the first-order expression of $\Box^k$, which is found to be

$$\Box^k \approx \Box_0^k + \sum_{A=0}^{k-1} \Box_0^{k-A-1}\,\Box_1\,\Box_0^k\,. \tag{38}$$

With these approximations it can be found, after some algebra, the kth-order term $E^{(k)}$ of the field equation quadratic in H. Let us consider the field equation up to 8th-order (with the terms $k = 1, 2$)

$$\begin{aligned} &H^2 - \gamma_0\,\left(2\,H\,\ddot{H} - \dot{H}^2\right) - \gamma_1\,\left(2\,H\,H^{(4)} - 2\,\dot{H}\,H^{(3)} + \ddot{H}^2\right) - \\ &-\gamma_2\,\left(2\,H\,H^{(6)} - 2\,\dot{H}\,H^{(5)} + 2\,\ddot{H}\,H^{(4)} - \left(H^{(3)}\right)^2\right) = 0\,. \end{aligned} \tag{39}$$

This equation has the fine property of being linearizable, in fact, computing the time derivative of (39) and dividing by $2\,H$ ($H \neq 0$ is assumed), we obtain

$$\gamma_2\,H^{(7)} + \gamma_1\,H^{(5)} + \gamma_0\,H^{(3)} - \dot{H} = 0\,, \tag{40}$$

which permits to study the stability of the Minkowski solution $H_0 = 0$. Here, we are going to study the stability of Minkowski space-time in the sixth-order theory obtained from (28) with $k = 1$, which is conformally equivalent to the Einstein theory with two interacting scalar fields, and is related to cosmological models with double inflation ((Gottlober *et al.*, 1990)). The general solution of equation (40) when $\gamma_2 = 0$ is

$$H(t) = A + B\, e^{\lambda_+ t} + C\, e^{-\lambda_+ t} + D\, e^{\lambda_- t} + E\, e^{-\lambda_- t} \,, \tag{41}$$

where A, B, C, D and E are integration constants and

$$\lambda_\pm^2 = \frac{-\alpha}{2\,\gamma} \mp \sqrt{\frac{\alpha^2}{4\,\gamma^2} + \frac{1}{6\,\gamma}} \,, \tag{42}$$

with the notation $\alpha \equiv \gamma_0$ and $\gamma \equiv \gamma_1$. By replacing (41) in (39) the following constraint among the coefficients is obtained

$$A^2 = 8\,\left[B\,C\,\left(1 - 3\,\alpha\,\lambda_+^2\right) + D\,E\,\left(1 - 3\,\alpha\,\lambda_-^2\right)\right] \,. \tag{43}$$

It is immediately seen from (41) that Minkowski space-time can not be strictly stable because of the opposite signs of the exponents. To analyze its stability in detail, we consider two cases.

If $\alpha^2 + \frac{2}{3}\,\gamma > 0$, we have $\lambda_\pm^2 \in \mathbf{R}$, and, defining $\Delta \equiv \sqrt{\alpha^2 + \frac{2}{3}\,\gamma}$, two subcases are considered: The first, corresponding to $\gamma < 0$ with $\Delta < |\alpha|$, obtaining $\mathrm{Re}\left[\lambda_\pm\right] = \left(\frac{-(\alpha\mp\Delta)}{2\,\gamma}\right)^{1/2}$, for $\alpha > 0$, giving rise to a two-dimensional stability subspace, or $\mathrm{Re}\left[\lambda_\pm\right] = 0$, for $\alpha < 0$, which produces an oscillatory scale factor whose frecuencies are

$$\mathrm{Im}\left[\lambda_\pm\right] = \sqrt{\frac{(\alpha \mp \Delta)}{2\,\gamma}} \,. \tag{44}$$

We note that this frecuencies coincide with the masses of the interacting scalar fields found in (Gottlober *et al.*, 1990) for the conformally equivalent problem. The second subcase, corresponding to $\gamma > 0$ with $|\alpha| < \Delta$, obtaining $\mathrm{Re}\left[\lambda_+\right] = \mathrm{Im}\left[\lambda_-\right] = \sqrt{\frac{(\Delta-\alpha)}{2\,\gamma}}$ and $\mathrm{Re}\left[\lambda_-\right] = \mathrm{Im}\left[\lambda_+\right] = 0$, produces a one-dimensional stability subspace.

If $\alpha^2 + \frac{2}{3}\gamma < 0$, $\lambda_\pm^2$ have complex values and we obtain

$$\mathrm{Re}\left[\lambda_+\right] = \mathrm{Re}\left[\lambda_-\right] = \frac{1}{2}\sqrt{\frac{\sqrt{\alpha^2 + \Delta^2} + \alpha}{|\gamma|}} \,,$$

producing a two-dimensional stability subspace.

4. Conclusions

We have investigated higher order gravitational field equations, being them either introduced by quantum corrections to the theory, or postulated in a generalised Lagrangian. In the first case we have studied several homogeneous isotropic semi-classical cosmological models with the presence of a self interacting classical scalar field. Considering a $\lambda\varphi^4$ self interaction potential, we found exact solutions to this problem with a vacuum polarization term. In the case of exponential potentials we have shown the existence of inflationary power-law solutions when the range of the regularisation parameters is $0 < \beta/\alpha < 3$. We have also shown that, unlike the classical case, the vacuum polarisation terms make the power-law solutions unstable for every value of the exponent n. In the second case of the Lagrangian models, we study the stability of the de Sitter space-time for the complete 6^{th} order theory of the kind (29). We have shown the detailed dependence of the stability of the de Sitter space-time upon the characteristic expansion scale (or the equivalent vacuum energy) given by $\sqrt{|\alpha / \gamma|}$, which is an upper bound for the stability of the de Sitter universe. Also, in our study of stability of Minkowski space-time (under isotropic and homogeneous perturbations) we have recovered the masses of the scalar fields appearing in the conformal picture given in (Gottlober *et al.*, 1990).

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