Assisted inflation in Friedmann-Robertson-Walker and Bianchi spacetimes

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We use exact general solutions for the spatially flat FRW and the anisotropic Bianchi type I cosmologies to show that generically uncoupled scalar fields cooperate to make inflation more probable, while the presence of several interacting fields hinders the occurrence of the phenomenon, in accordance with previous results based on particular power-law solutions. Similar conclusions are reached in the case of Bianchi type VI_0 spacetimes, for power-law solutions which are proved to be attractors.

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I. INTRODUCTION

In many inflationary models the effective potential energy density of a scalar field is responsible for an epoch of accelerated inflationary expansion [1]. Very often one assumes that inflation is driven by a scalar field of the Liouville form, i.e., an exponential potential, because this kind of potential arises in various higher-dimensional supergravity [2] and superstring [3] models [4–7].

Although there are many scalar fields in superstring theories, in the past it was often assumed that typically only one scalar field was responsible for inflation, while those having higher exponents were quickly redshifted away. However, it has been found [8] that so-called *assisted inflation* may occur when several scalar fields are present, even if each individual field is too steep to drive the inflation, provided that the fields are uncoupled and interact only through the geometry. On the other hand, if the fields interact directly with each other, the opposite effect may happen and the presence of cross couplings between fields may hinder inflation [9–11].

Assisted inflation has been mainly studied in power-law solutions $(a \propto t^p)$ for the spatially flat Friedmann-Robertson-Walker (FRW) cosmology, which can be shown [8] to be the late-time attractor for evolution of this kind of model. (Recently, Green and Lidsey [12] discussed in the context of assisted inflation late-time evolution in a general geometry.)

The purpose of this work is to extend previous studies on multi-scalar field cosmologies in two directions: first, we will use general solutions (instead of the special power-law ones) to analyze if the presence of several fields generically helps or impedes inflation, and second, going beyond the aforementioned FRW cosmology, we will consider the anisotropic inhomogeneous generalization given by Bianchi type I models. Thus in Sec. II A we deal with n interacting scalar fields in a FRW spacetime and use the general solution to show that the larger the number of interacting scalar fields, the less likely is inflation. Section II B makes plausible for more general scalar field potentials the results obtained in Sec. II A for exponential potentials. We do not know the solution for n non-interacting scalar fields in a FRW cosmology in the general case, but we use the discussion of the late-time attractor

of Sec. II C to restrict the analysis of uncoupled fields in Sec. II D to the particular case in which all fields are assumed to be equal, for which the general solution can be found. We show that non-interacting fields generically cooperate to assist inflation. The density fluctuations corresponding to the last case are discussed in Sec. II E. General solutions of anisotropic Bianchi type I cosmologies with interacting and uncoupled fields are used in the first part of Secs. III A and III B, respectively, to check that also in those cases interacting fields make inflation more difficult while uncoupled fields assit it. The stability of power-law solutions is discussed in Sec. III C. Finally in Sec. IV we turn to power-law solutions of the Bianchi type VI₀ model, reaching again the same conclusions.

II. *n*-SCALAR FIELD PROBLEM IN A FLAT FRW SPACETIME

In the following we will consider two kinds of problems in flat FRW spacetimes in which there are n homogeneous scalar fields driven by exponential potentials. First of all, we will assume that the scalar fields are interacting through a product of exponential potentials. Then we will consider the case in which the scalar fields are uncoupled because the potential is a sum of potentials involving a single field.

A. Interacting *n*-scalar field problem in flat FRW spacetime

The problem of *n* interacting homogeneous scalar fields, ϕ_i , driven by a product of *n* exponential potentials, $V_i = V_{0i} e^{-k_i \phi_i}$, minimally coupled to gravity in a flat Robertson-Walker spacetime, with metric

$$ds^{2} = -dt^{2} + a^{2}(t)[dx^{2} + dy^{2} + dz^{2}], \qquad (1)$$

is formulated by the system of equations

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V,$$
 (2)

$$\ddot{\phi} + 3H\dot{\phi} - V\vec{k} = 0, \tag{3}$$

where $H = \dot{a}/a$ and the potential

$$V(\vec{\phi}) = V_0 \,\mathrm{e}^{-\vec{k}\cdot\vec{\phi}} \tag{4}$$

allows for interactions between the fields. V_0 is the constant $V_{01}V_{02}\cdots V_{0n}$, and $\vec{k} = (k_1, k_2, \dots, k_n)$ is an *n*-component constant vector with respect to an orthonormal basis in the *n*-dimensional Euclidean internal space to which the vector $\tilde{\phi} = (\phi_1, \phi_2, \dots, \phi_n)$, built with the *n* minimally coupled scalar fields, also belongs. In the remainder of this section k, k^2 , and $\dot{\phi}^2$ stand for $|\vec{k}|$, $\vec{k} \cdot \vec{k}$, and $\dot{\phi} \cdot \dot{\phi}$ respectively. Potentials of this type are of interest because they may be considered just as an approximation to a more complex potential. In fact, in higher-dimensional superstring theories, the scalar field is like one of the matter fields that contribute to the action and effective potential of the theory. Loop expansion [3], or expansion in the number of interacting particles [13], of the action leads to a perturbative expression of the potential which is a summation of exponential terms [5,6]. From Eqs. (2),(3), and discarding the trivial static metric solution H = 0, $\frac{1}{2}\dot{\phi}^2 + V = 0$, we get

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2.$$
 (5)

Using this equation and the system (2),(3) one finds that the Klein-Gordon equation (3) has the first integrals

$$\dot{\vec{\phi}} = H\vec{k} + \frac{\vec{c}}{a^3},\tag{6}$$

where $\vec{c} = (c_1, c_2, \dots, c_n)$ is an arbitrary vector integration constant. As Eq. (6) involves only geometrical quantities the Einstein-Klein-Gordon equations uncouple and the general parametric solution can be obtained [14–16].

One can easily verify that in this kind of model the Einstein-Klein-Gordon equations have power-law solutions

$$a \propto t^{2/k^2},\tag{7}$$

so that they inflate at all times when $k^2 < 2$. We will show below that the general solution has the same kind of behavior when the scale factor *a* is large enough.

Inserting Eq. (6) into Eq. (5) we obtain the following second-order equation for the scale factor a(t):

$$\ddot{s} + s^{m}\dot{s} + \frac{1}{4\cos^{2}\sigma}s^{2m+1} = 0 \quad (\sigma \neq \pi/2),$$
 (8)

where

$$\cos\sigma = \frac{\vec{c} \cdot \vec{k}}{ck},\tag{9}$$

 $m = -6/k^2 < 0$, $c = |\vec{c}|$, *a*, and *t* have been replaced by the new variables *s* and τ defined by

and the overdots stand for derivation with respect to τ . Once $s(\tau)$ is known one can compute, in principle, the scale factor $a(\tau)$ from Eq. (10), and the fields $\vec{\phi}(t)$ from Eq. (6).

Equation (8) is a particular case of the second order nonlinear ordinary differential equation

$$\ddot{s} + \alpha f(s)\dot{s} + \beta f(s)\int f(s)ds + \gamma f(s) = 0, \qquad (11)$$

where f(s) is some real function and α , β , and γ are constant parameters. Depending on the values of k^2 our equation (8) corresponds to the next two cases:

(1) $k^2 \neq 6$ and

$$f(s) = s^{m}, \quad \alpha = 1, \quad \beta = \frac{m+1}{4\cos^{2}\sigma}, \quad \gamma = 0.$$
 (12)

(2) $k^2 = 6$ and

$$f(s) = \frac{1}{s}, \quad \alpha = 1, \quad \beta = 0, \quad \gamma = \frac{1}{4\cos^2\sigma}.$$
 (13)

Inserting the first integrals (6) in the Einstein equation (2), we get a quadratic equation in the expansion rate H which has real solutions only when its discriminant is non-negative. This condition leads to

$$\beta \le \frac{1}{4\left[1 + \frac{2a^{6}V}{c^{2}}\right]} < \frac{1}{4}, \tag{14}$$

discarding the oscillatory solutions of Eq. (11), which would be obtained for a negative potential or $\beta > 1/4$. It follows that, if $k^2 \neq 6$, real solutions exist always for

$$k^2 < k_0^2 = \frac{6}{\sin^2 \sigma}.$$
 (15)

This sets a restriction on the integration constants and the exponent of the exponential potential.

The general solution of Eq. (11) can be obtained making the nonlocal transformation of variables [17]

$$z = \int f(s)ds, \quad \eta = \int f(s)d\tau.$$
(16)

Under this transformation Eq. (8) becomes a linear inhomogeneous ordinary differential equation with constant coefficients

$$z'' + \alpha z' + \beta z + \gamma = 0, \tag{17}$$

which for $\alpha > 0$ and $\beta > 0$ is the equation of a damped harmonic oscillator in a constant external field. Here the prime indicates differentiation with respect to η . In our case the nonlocal transformation of variables (16) is

$$a = s^{-m/3}, \quad \tau = ckt \cos \sigma,$$
 (10)

$$z = \frac{s^{m+1}}{m+1}, \quad \eta = \int s^m d\tau, \tag{18}$$

for $k^2 \neq 6$, and

$$z = \ln s, \quad \eta = \int \frac{d\tau}{s}, \tag{19}$$

for $k^2 = 6$.

Taking $\alpha = 1$ in Eq. (17) we get

$$z = \begin{cases} b_1 e^{\lambda_+ \eta} + b_2 e^{\lambda_- \eta} & \text{for } k^2 \neq 6, \\ b_1 + b_2 e^{-\eta} - \gamma \eta & \text{for } k^2 = 6, \end{cases}$$
(20)

where

$$\lambda_{\pm} = \frac{-1 \pm \sqrt{1 - 4\beta}}{2},\tag{21}$$

and b_1 , b_2 are real integration constants.

We see from Eqs. (10) and (18) that

$$a \propto z^{-m/3(m+1)}$$
, (22)

so that, while for $k^2 > 6$ the scale factor goes to 0 as $z \rightarrow 0$, one has that for $k^2 < 6$ the scale factor goes to infinity when $z \rightarrow 0$. On account of Eq. (20) one may assume that the latter occurs as $\eta \rightarrow \eta_0$, for an appropriate value of η_0 . Let us now write $\eta = \eta_0 + \delta \eta$ and expand Eqs. (20) and (22) about z= 0 and $\eta = \eta_0$, keeping only terms linear in $\delta \eta$, so that $\delta z = \delta \eta$ and

$$a = \delta z^{-m/3(m+1)} = \delta \eta^{-m/3(m+1)}.$$
 (23)

Now, we see from Eqs. (10) and (18) that

$$\delta \eta = a^{-3} \delta \tau \propto a^{-3} \delta t. \tag{24}$$

If we use this in Eq. (23), we get, successively,

$$a \propto a^{m/(m+1)} \delta t^{-m/3(m+1)}$$
 (25)

and

$$a \propto \delta t^{-m/3} = \delta t^{2/k^2}.$$
 (26)

We see then that if $k^2 < 2$, the solution does inflate at least along some time interval. Now, since $k^2 = k_1^2 + k_2^2 + \cdots$ $+ k_n^2$, one concludes from our analysis of the general solution that the larger the number of interacting scalar fields, the less likely will be $k^2 < 2$ as well that inflation takes place. On the other hand, for $6 < k^2 < k_0^2$, one has $0 < \beta < 1/4$, which implies that both $\lambda_{\pm} < 0$; thus Eqs. (18) and (20) tell us that the scale factor goes to infinity when $z \rightarrow \infty$ and $\eta \rightarrow -\infty$ (with no loss of generality we are taking both constants $b_1, b_2 > 0$). By keeping only the dominant terms we have for very large negative η that

$$a \propto \mathrm{e}^{-m\lambda_{-}\eta/3(m+1)}.$$
 (27)

This relation and Eq. (24), which is generally valid, yield

$$\frac{\delta a}{a} \propto a^{-3} \,\delta t \tag{28}$$

whose general solution is $a \propto t^{1/3}$. In this case the solution does not inflate and it approaches the free scalar field solution. These results are in accordance with those obtained with particular solutions by other authors [9–11].

It would be surprising if for $k^2=6$ the scale factor had a behavior drastically departing from that suggested by the previous analysis when $k^2 \in (k_0^2, 6) \cup (6, 2)$. In fact, we see from Eqs. (10) and (19) that $z=3 \log a$ so that, as a consequence of Eq. (20), *a* goes to infinity as $\eta \rightarrow -\infty$. From Eq. (20) we see that $\log z \sim -\eta$, when $\eta \rightarrow -\infty$ and, by using Eq. (24),

$$\frac{\delta a}{a\log a} \propto \frac{\delta z}{z} \sim -\delta \eta \propto \frac{\delta t}{a^3},\tag{29}$$

which can be written as

$$\frac{\delta a}{a} \sim a^{-3} \,\delta t \log a \,. \tag{30}$$

In the asymptotic regime in which $a \rightarrow \infty$ we have $a \,\delta a < a^2 \,\delta a / \log a < a^2 \,\delta a$, so that from Eq. (30) we get $a^2 \leq t \leq a^3$ and, finally,

$$t^{1/3} \leq a \leq t^{1/2}$$
, when $a \to \infty$. (31)

We see that the solution does not inflate.

B. More general potentials

The results of the above section show that we can introduce an *n*-dimensional Euclidean internal vector space containing the *n*-component vector $\vec{\phi} = (\phi_1, \phi_2, \dots, \phi_n)$ built with *n* minimally coupled scalar fields. Let us assume now that the potential has the general form

$$V = V(\Phi), \quad \Phi = \vec{k} \cdot \vec{\phi}. \tag{32}$$

In this case the Einstein equation (2) remains unchanged; however, the Klein-Gordon equation (3) becomes

$$\vec{\phi} + 3H\vec{\phi} + V'\vec{k} = 0.$$
 (33)

where the prime indicates derivatives with respect to the variable Φ . Since Eqs. (2), (32) and (33) are invariant under rotations of the orthogonal axes in the Euclidean space mentioned above, we may choose the first axis of this internal space along the vector \vec{k} . Then the Klein-Gordon equation splits into one equation for $\phi_1 = \phi$,

$$\ddot{\phi} + 3H\dot{\phi} + kV' = 0, \tag{34}$$

and an n-1 free field Klein-Gordon equation for $\phi_2 = \cdots = \phi_n = \psi$:

$$\ddot{\psi} + 3H\dot{\psi} = 0. \tag{35}$$

From Eq. (35) we obtain the first integral $\dot{\psi} = c_0/a^3$, where c_0 is an arbitrary integration constant. Hence the original *n*-scalar field problem is equivalent to considering a self-interacting scalar field with stiff matter. In fact, the Einstein equation (2) now reads

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V + \frac{c_{\psi}^2}{2a^6},\tag{36}$$

where c_{ψ}^2 is the sum of n-1 positive defined integration constants.

Taking into account that the scalar field ϕ depends only on *t*, its energy-momentum tensor may be written in the perfect fluid form

$$T_{ik} = (p_{\phi} + \rho_{\phi})u_{i}u_{k} + p_{\phi}g_{ik}, \qquad (37)$$

where

$$\rho_{\phi} = \frac{1}{2}\dot{\phi}^{2} + V(\phi),$$

$$p_{\phi} = \frac{1}{2}\dot{\phi}^{2} - V(\phi).$$
(38)

The fluid interpretation of the scalar field has proved very useful in the study of inflationary and *Q*-matter scenarios [18]. In particular it leads us to consider its equation of state $p_{\phi} = (\gamma_{\phi} - 1)\rho_{\phi}$. On the other hand, the state equation for fluid representing stiff matter is $p_f = (\gamma_f - 1)\rho_f$ with $\gamma_f = 2$. Because of the additivity of the stress-energy tensor, it makes sense to consider an effective perfect fluid description with equation of state $p = (\gamma - 1)\rho$ where $p = p_f + p_{\phi}$, $\rho = \rho_f + \rho_{\phi}$ and

$$\gamma = \frac{\gamma_f \rho_f + \gamma_\phi \rho_\phi}{\rho_f + \rho_\phi} \tag{39}$$

is the overall (i.e., effective) adiabatic index. For this effective perfect fluid the dynamical equations are

$$3\frac{\dot{a}^2}{a^2} = \rho \tag{40}$$

and

$$\frac{\ddot{a}}{a} = -\frac{1}{6} [\rho + 3p], \tag{41}$$

where p and ρ are the density and pressure of the effective perfect fluid. They involve the self-interacting scalar field and the *n*-1 free scalar fields. Inflationary solutions occur when $\ddot{a} > 0$; this means that the expansion is dominated by a gravitationally repulsive stress that violates the strong energy condition $\rho + 3p < 0$ or equivalently $\gamma < 2/3$. When we impose this condition on Eq. (39), we obtain that

$$V(\Phi) > \dot{\phi}^2 + \frac{c_{\psi}^2}{a^6}.$$
 (42)

Hence, when there is only one scalar field the solution inflates for $V(\phi) > \dot{\phi}^2$. However, the *n* interacting scalar fields produce a desassisted inflation because the solutions inflate only if Eq. (42) holds. Using the effective perfect fluid description we can generalize this analysis to an arbitrary potential $V(\phi_1, \phi_2, \ldots, \phi_n)$. In this case we obtain that $V(\phi_1, \phi_2, \ldots, \phi_n) > \dot{\phi}_1^2 + \dot{\phi}_2^2 + \cdots + \dot{\phi}_n^2$, so that the interacting *n* scalar fields might make inflation more unlikely in a FRW spacetime.

C. n-scalar field attractor problem in the FRW spacetime

Let us now assume that the *n* homogeneous scalar fields, ϕ_i , in the FRW spacetime are driven by a general potential $V = V(\phi_i)$. In that case the Einstein-Klein-Gordon equations are

$$3H^2 = \frac{1}{2} \sum_{i=1}^{n} \dot{\phi}_i^2 + V, \qquad (43)$$

$$\ddot{\phi}_i + 3H\dot{\phi}_i + V_{,\phi_i} = 0,$$
 (44)

where $V_{,\phi_i}$ stand for $\partial V / \partial \phi_i$. From these equations we get

$$\dot{H} = -\frac{1}{2} \sum_{i=1}^{n} \dot{\phi}_{i}^{2} \,. \tag{45}$$

In order to investigate the stable scalar field configurations, we introduce the quantity

$$\omega = \frac{\sum_{i=1}^{n} \dot{\phi}_i^2}{n \dot{\phi}_{\alpha}^2},\tag{46}$$

which reduces to $\omega = 1$ for the configuration $\phi_1 = \phi_2 = \cdots = \phi_n$. Using Eqs. (43)–(46) we find the differential equation for ω :

$$\dot{\omega} = 2 \frac{n V_{,\phi_{\alpha}} \dot{\phi}_{\alpha} \omega - \dot{V}}{n \dot{\phi}_{\alpha}^2}.$$
(47)

(In this section no summation convention applies to repeated Greek indexes.) If we further assume that the potential satisfies the condition

$$\dot{V} = nV_{,\phi_{\alpha}}\dot{\phi}_{\alpha}, \qquad (48)$$

Equation (47) becomes

$$\dot{\omega} = 2 \frac{V_{,\phi_{\alpha}}}{\dot{\phi}_{\alpha}} (\omega - 1) \tag{49}$$

which has a fixed point solution: $\omega = 1$. Furthermore, the general solution of Eq. (49) can be found using Eq. (44):

$$\omega = 1 + \frac{c}{a^6 \dot{\phi}_{\alpha}^2} \tag{50}$$

where c is an arbitrary integration constant. However, it is useful to express the last solution in terms of geometrical quantities with the aid of Eqs. (45) and (46). The final result is

$$\omega = \left(1 + \frac{nc}{2a^6\dot{H}}\right)^{-1}.$$
 (51)

Evaluating Eq. (51) in the asymptotic regime, it can be easily shown that the particular solution $\omega = 1$ is an attractor for evolutions that behave asymptotically as $a \propto t^{\nu}$ with $\nu > 1/3$.

This result strongly suggests that the special case in which all scalar fields are equal may be the late-time attractor of more general problems. In fact, this has been proved in [9] for potential (4) with $k_1 = \cdots = k_n$ and $V_{01} = \cdots = V_{0n}$, which guarantee assumption (48). We did not use this fact in the previous section because we were able to use the general solution with no additional assumption on k_i , V_{0i} , and ϕ_i , but it will be useful to simplify somewhat the problem analyzed in the following section.

D. Non-interacting *n*-scalar field problem in the FRW spacetime

Let us now assume that the *n* homogeneous scalar fields, ϕ_i , in the spacetime given by Eq. (1) do not interact directly, but are driven by a sum of *n* exponential potentials V_i $= V_{0i} e^{-k_i \phi_i}$. In that case the Einstein-Klein-Gordon equations are

$$3H^{2} = \sum_{i=1}^{n} \left[\frac{1}{2} \dot{\phi}_{i}^{2} + V_{i} \right],$$
 (52)

$$\ddot{\phi}_i + 3H\dot{\phi}_i - k_i V_i = 0. \tag{53}$$

From now on we will consider a simplified problem in which $k_1 = k_2 = \cdots = k_n \equiv k$ and $V_{01} = V_{02} = \cdots = V_{0n} \equiv V_0$. As discussed in the previous section, we can expect in this particular case that in the asymptotic evolution all scalar fields tend to a common limit. That this is actually the case has been proved in [8]. As a consequence, in the remainder of this section we will take $\phi_1 = \phi_2 = \cdots = \phi_n \equiv \phi$, so that $V_1 = V_2 = \cdots = V_n \equiv V = V_0 e^{-k\phi}$ and Eqs. (52)–(53) reduce to

$$3H^2 = \frac{n}{2}\dot{\phi}^2 + nV,$$
 (54)

$$\ddot{\phi} + 3H\dot{\phi} - kV = 0. \tag{55}$$

One can easily get, from Eqs. (54)-(55),

$$\dot{H} = -\frac{n}{2}\dot{\phi}^2 \tag{56}$$

and the first integral of the Klein-Gordon equation (55)

$$\dot{\phi} = \frac{k}{n}H + \frac{c}{a^3},\tag{57}$$

where c is an arbitrary integration constant.

As in the previous section, the Einstein-Klein-Gordon equations have power-law solutions [8]

$$a \propto t^{2n/k^2},\tag{58}$$

so that this type of solutions inflates at all times when $k^2 < 2n$. We will now show that the general solution of the Einstein-Klein-Gordon equations (54) and (55) has the same kind of behavior when the scale factor *a* is large enough.

Inserting Eq. (57) into Eq. (56) we obtain the following second-order equation for the scale factor a(t):

$$\ddot{s} + s^m \dot{s} + \frac{1}{4} s^{2m+1} = 0, \tag{59}$$

where $m = -6n/k^2 < 0$, the overdots mean derivative with respect to τ and we have used, instead of *a* and *t*, the new variables *s* and τ defined by

$$a = s^{-m/3}, \quad \tau = ckt. \tag{60}$$

We have again a particular case of Eq. (11) with $\alpha = 1$ and to find the general exact solution of Eq. (59) one has to consider the following two possibilities:

(1) $k^2 \neq 6n$ and

$$f(s) = s^m, \quad \beta = \frac{m+1}{4}, \quad \gamma = 0.$$
 (61)

(2) $k^2 = 6n$ and

$$f(s) = \frac{1}{s}, \quad \beta = 0, \quad \gamma = \frac{1}{4}.$$
 (62)

The general solution of Eq. (59) can be obtained by performing the nonlocal transformation of variables given in Eqs. (18),(19) with the new value of *m*, which reduces Eq. (59) to the linear inhomogeneous ordinary differential equation with constant coefficients (17).

If one now repeats the analysis of the previous section by systematically taking $\sigma = 0$ and using the new *m*, the same final results for *z* and *a* are obtained: just replace $k^2 = k_1^2 + \cdots + k_n^2$ by k^2/n in the quantities involved in Eqs. (20) and (22). One readily concludes that, when $a \rightarrow \infty$, these models inflate if $k^2 < 2n$, so that the fields cooperate to make inflation more likely in the so-called "assisted inflation," which was first discussed—but only for power-law solutions—in [8].

This result could have been anticipated since the present case is included in the mathematical problem set in Sec. II A

by Eqs. (8) and (9) by just taking $\cos \sigma = \pm 1$ (which corresponds to the simplified problem under consideration) and $m = -6n/k^2$.

E. Density fluctuations

The fact that the contributions of the density fluctuations differ significantly in different inflationary universe models has motivated a detailed study of all the alternatives. In this context, it is interesting to derive the spectral indices for the perturbations that would be created during the periods of inflation described by the solutions given in the last section. It is well known that, for multi-scalar field models, the spectrum of the curvature perturbation reads [19]

$$P_{S} = \left(\frac{H}{2\pi}\right)^{2} \frac{\partial N}{\partial \phi_{i}} \frac{\partial N}{\partial \phi_{j}} \delta_{ij}, \qquad (63)$$

where N is the number of *e*-foldings of inflationary expansion remaining, and there is a summation over *i* and *j*. In the case considered in the previous section, where all the scalar fields are equal, Eq. (63) yields [8]

$$P_{S}(\tilde{k}) = \left(\frac{H}{2\pi}\right)^{2} \frac{1}{n} \frac{H^{2}}{\phi^{2}} \bigg|_{aH = \tilde{k}},$$
(64)

where *H* and $\dot{\phi}$ have to be evaluated at the time when the wave number of interest, \tilde{k} , leaves the horizon during inflation. Also in this case, the spectral index $n_{\tilde{k}}(\tilde{k})$ defined as

$$n_{S}(\tilde{k}) = 1 + \frac{\mathrm{d}\ln P_{S}}{\mathrm{d}\ln \tilde{k}}$$

is given by [19,8]

$$1 - n_S = -2\frac{H}{H^2}.$$
 (65)

The availability of exact solutions allows us to express the relevant quantities as functions of the variable η introduced in Eq. (16). The scale factor reads [see Eq. (22), which is satisfied for uncoupled scalar fields with $m = -6n/k^2$, as one can easily see]

$$a(\eta) = [(m+1) \ z(\eta)]^{-m/3(m+1)}$$
(66)

where $z(\eta)$ is given by Eqs. (20),(21). The spectrum $P_S(\vec{k})$ obtained from Eq. (64) is shown in Fig. 1, for k=3, c=-1, $b_1=-1$, $b_2=0.001$ and different values of n. Here, inflation is possible for $n \ge 5$, and one can see that the peak of the spectral distribution moves towards the high frequency region as n increases. The corresponding spectral index n_S is shown in Fig. 2. From Eq. (65) and the general solutions (20), (21) and (66), it can be shown that the value of $n_S \rightarrow 1$ in the asymptotic region $a \ge 1$ as the number n of present fields increases. This feature is exemplified in Fig. 2, where we can see how the larger the value of n, the closer is the spectrum to the scale invariance [8].

III. *n*-SCALAR FIELD PROBLEM IN BIANCHI TYPE I MODELS

Now we turn to the general Bianchi type I model with n homogeneous scalar fields driven by exponential potentials. As we proceeded in the case of FRW spacetimes, we will first assume that the scalar fields are interacting through a product of exponential potentials, and then we will consider



FIG. 1. Spectrum of the curvature perturbations for k=3, c=-1, $b_1=-1$, and $b_2=0.001$. As *n* increases (inflation is possible when $n \ge 5$) the spectral distribution peak is shifted to high frequencies.



the case in which the scalar fields are uncoupled because the potential is a sum of potentials involving a single field.

A. Interacting *n*-scalar field problem in the anisotropic Bianchi type I model

The general Bianchi type I model is the anisotropic generalization of the spatially flat FRW universe expanding differently in the x, y, and z directions. In the usual synchronous form its line element is given by

$$ds^{2} = -dT^{2} + a_{1}^{2}(T)dx^{2} + a_{2}^{2}(T)dy^{2} + a_{3}^{2}(T)dz^{2}.$$
 (67)

For convenience we use the semiconformal coordinates

$$dt = \frac{dT}{a_3}, \quad e^f = a_3^2, \quad G = a_1 a_2, \quad e^p = \frac{a_1}{a_2}, \quad (68)$$

to cast the metric (67) into the form

$$ds^{2} = e^{f(t)}(-dt^{2} + dz^{2}) + G(t)(e^{p(t)}dx^{2} + e^{-p(t)}dy^{2}).$$
(69)

We first consider, as in Sec. II A, *n* scalar fields ϕ_i interacting directly through the exponential potential (4). The problem of *n* interacting homogeneous scalar fields, ϕ_i , driven by a product of *n* exponential potentials V_i $= V_{0i}e^{-k_i\phi_i}$, minimally coupled to gravity in the Bianchi I spacetime (69), is formulated by the following system of Einstein-Klein-Gordon equations:

$$\dot{p} = \frac{a}{G},\tag{70}$$

$$e^{f} = \frac{\ddot{G}}{2VG},\tag{71}$$

$$\frac{\ddot{G}}{G} - \frac{1}{2} \left(\frac{\dot{G}}{G}\right)^2 - \frac{\dot{G}}{G} \dot{f} + \frac{1}{2} \dot{p}^2 = -\dot{\phi}^2,$$
(72)

FIG. 2. The spectral index $n_S(\tilde{k})$ for the general solution approaches 1 in the asymptotic region $a \ge 1$ as the number of fields increases.

$$\ddot{\vec{\phi}} + \frac{\dot{G}}{G}\dot{\vec{\phi}} - e^f V \vec{k} = 0, \qquad (73)$$

where a is an arbitrary integration constant. It can be easily seen that the vector

$$\dot{\vec{\phi}} = \frac{\dot{G}}{G}\frac{\ddot{k}}{2} + \frac{\ddot{m}}{G} \tag{74}$$

(where \vec{m} is a *n*-dimensional vector whose components are integration constants) is a first integral of the Klein-Gordon equation set (73). Inserting Eq. (70) into Eq. (74) the general solution of the Klein-Gordon equations is found:

$$\vec{\phi} = \vec{\phi}_0 + p \, \frac{\vec{m}}{a} + \frac{\vec{k}}{2} \ln G,$$
 (75)

where $\vec{\phi}_0$ is an arbitrary constant vector. Equations (70)–(72) along with Eq. (75) uncouple and their solutions can be obtained if one is able to solve the following third-order equation for *G*:

$$G\ddot{G}^{2} - \ddot{G}\dot{G}G + \left(\frac{1}{2} - \frac{k^{2}}{4}\right)\ddot{G}\dot{G}^{2} + \left(m^{2} + \frac{a^{2}}{2}\right)\ddot{G} = 0.$$
 (76)

Once G(t) is known, in principle one can compute p(t) and $\vec{\phi}(t)$ from Eqs. (70) and (75), respectively; f(t) is then obtained from Eq. (71).

The Einstein-Klein-Gordon equations admit power-law solutions, $G = t^{\alpha}$, but they happen to be isotropic and, thus, equal to the ones discussed in Sec. II A. Thus we will analyze, instead, the general solution of Eq. (76).

Equation (76) has the first integral

$$G\frac{\ddot{G}}{\dot{G}} + (K-1)\dot{G} + \frac{M^2}{\dot{G}} = C,$$
 (77)

where C is an arbitrary constant and

$$K \equiv \frac{k^2}{4} - \frac{1}{2}, \quad M^2 \equiv m^2 + \frac{a^2}{2}.$$
 (78)

If instead of t and G we use the new variables z and τ defined, for $C \neq 0$, in

$$G = z^{1/K}, \quad t = -\frac{\tau}{C}, \tag{79}$$

then, Eq. (77) becomes

$$z'' + z^{-1/K} z' + \frac{KM^2}{C^2} z^{1-2/K} = 0,$$
(80)

where a prime denotes the derivative with respect to τ . This equation is, once more, a particular case of Eq. (11) and can be linearized by using the non-local transformation (16), which in this case is

$$y \equiv \int z^{-1/K} dz = K \frac{z^{1-1/K}}{K-1}, \quad \eta \equiv \int z^{-1/K} d\tau = -\frac{C}{a} p$$
(81)

for $K \neq 1$ and

$$y \equiv \int z^{-1} dz = \ln z, \quad \eta \equiv \int z^{-1} d\tau = -\frac{C}{a} p \qquad (82)$$

for K = 1. If we take

$$\beta \equiv (K-1)\frac{M^2}{C^2}, \quad \gamma = 0 \tag{83}$$

for $K \neq 1$ and

$$\beta = 0, \quad \gamma = \frac{M^2}{C^2} \tag{84}$$

for K = 1, Eq. (80) reduces to two particular cases of Eq. (17) for $\alpha = 1$. The trivial solution of this equation gives the implicit general solution of Eq. (76) which can be written, for arbitrary a, M and non-vanishing C, as

$$G = [e^{-\eta/2} (C_1 e^{\lambda \eta} + C_2 e^{-\lambda \eta})]^{1/(K-1)}$$
(85)

for $K \neq 1$ and as

$$G = C_1 e^{-\gamma \eta + C_2 e^{-\eta}} \tag{86}$$

for K=1. Here C_1 and C_2 are integration constants and $\lambda = \sqrt{1-4\beta}/2$.

To check whether a model inflates, we will look at the sign of the deceleration parameter $q = -\theta^{-2}(3\dot{\theta} + \theta^2)$, where $\theta = u_{;a}^a$ is the expansion and $\dot{\theta} = \theta_{,a}u^a$, u^a being the four-velocity of the cosmic fluid. Since in this case we are dealing with comoving coordinates, $u^a = (e^{-f/2}, 0, 0, 0)$, one can see that, apart from a positive factor, the deceleration is

$$q \propto 9K\dot{G}^2 - 3(2km + C) + (km - C)^2 + 9M.$$
 (87)

We see from Eq. (85) that when K < 1 (i.e., when $k^2 < 6$) *G* blows up for some value $\eta = \eta_0 = (1/2\lambda) \log (-C_2/C_1)$, provided that $C_1C_2 < 0$. If we expand Eq. (85) around this value ($\eta = \eta_0 + \delta \eta$) we get

$$G \propto \delta \eta^{1/(K-1)},\tag{88}$$

and from Eqs. (79) and (81)

$$d\eta = \frac{d\tau}{G} = -\frac{C}{G}dt, \qquad (89)$$

so that

$$\dot{G} = -\frac{C}{G}G' \propto \frac{1}{K-1}\delta\eta^{-1}.$$
(90)

As a consequence, when $G \rightarrow \infty$ the deceleration parameter (87) is

$$q \propto 9K \frac{C^2}{(K-1)^2} \frac{1}{\delta \eta^2}$$
 (when $\delta \eta \rightarrow 0$), (91)

and there is inflation if K < 0, i.e., if $k^2 = k_1^2 + k_2^2 + \cdots + k_n^2$ <2. We conclude that in these anisotropic universes also a greater number of interacting scalar fields makes inflation less likely.

B. Non-interacting *n*-scalar field problem in the Bianchi type I model

We will now assume that the *n* homogeneous scalar fields ϕ_i in the metric (67) do not interact directly, but are driven by a sum of *n* exponential potentials, $V_i = V_{0i} e^{-k_i \phi_i}$. To simplify the task of finding exact solutions of the Einstein-Klein-Gordon equations, we will further assume that $k_1 = k_2 = \cdots$ $= k_n \equiv k$, $\phi_1 = \phi_2 = \cdots = \phi_n \equiv \phi$ and $V_1 = V_2 = \cdots = V_n \equiv V$ $= V_0 e^{-k\phi}$, so that aforementioned equations can be written as

$$\dot{p} = \frac{a}{G},\tag{92}$$

$$e^f = \frac{\ddot{G}}{2nVG},\tag{93}$$

$$\frac{\ddot{G}}{G} - \frac{1}{2} \left(\frac{\dot{G}}{G}\right)^2 - \frac{\dot{G}}{G} \dot{f} + \frac{1}{2} \dot{p}^2 = -n \, \dot{\phi}^2, \tag{94}$$

$$\ddot{\phi} + \frac{\dot{G}}{G}\dot{\phi} - k \,\mathrm{e}^{f}V = 0, \tag{95}$$

where a is an arbitrary integration constant. It is easy to check that

$$\dot{\phi} = \frac{k}{2n}\frac{\dot{G}}{G} + \frac{m}{G} \tag{96}$$

is a first integral of the Klein-Gordon equation (95), in terms of the new integration constant m. Inserting Eq. (92) into Eq. (96) the general solution of the Klein-Gordon equations is found:

$$\phi = \phi_0 + p \frac{m}{a} - \frac{k}{2n} \ln G, \qquad (97)$$

where ϕ_0 is an arbitrary constant. Equations (92)–(94) along with Eq. (97) uncouple and their solutions can be obtained if one is able to solve the following third-order equation for *G*:

$$G\ddot{G}^{2} - \ddot{G}\dot{G}G + \left(\frac{1}{2} - \frac{k^{2}}{4n}\right)\ddot{G}\dot{G}^{2} + \left(m^{2} + \frac{a^{2}}{2}\right)\ddot{G} = 0.$$
(98)

Since this equation is the same as Eq. (76), once one replaces $k^2 = k_1^2 + k_2^2 + \cdots + k_n^2$ by k^2/n , we may repeat the calculations of the previous section to reach the opposite conclusion: if several non-interacting scalar fields are present, they will cooperate to "assist" the inflation, which will be more likely and occurs for $k^2 < 2n$.

C. Stability of power-law solutions in Bianchi type I model

For many purposes it is interesting to investigate the stability of the solutions of Eq. (76). In particular, we hope that the solution representing an accelerated expansion of the universe and the solutions that correspond to the assisted inflation are stable. To this end we introduce the variable

$$\Omega = \frac{\dot{h}}{h^2},\tag{99}$$

where $h = \dot{G}/G$, in Eq. (76):

$$\dot{\Omega} + \left[\Omega + K - \frac{M^2}{h^2 G^2}\right] (\Omega + 1)h = 0.$$
 (100)

This equation has the fixed point solution $\Omega = -1$. Note that Eq. (100) has also the fixed point solution $\Omega = -K$ if $\dot{G} \rightarrow \infty$ asymptotically. The corresponding asymptotic limits of these solutions can be obtained by solving Eq. (99) for them. The final result is $G \propto t$ and $G \propto t^{1/K}$ respectively. Let us investigate the stability of these solutions when *G* blows up. From Eq. (100) it is easy to see that $\Omega = -1$ is unstable because expanding the solutions about it, $\Omega = -1 + \epsilon$ with $\epsilon \ll 1$, the sign of $\dot{\epsilon}$ depends on the slope of potential and the initial conditions. In fact, the corresponding solution $G \propto t$ does not satisfies Einstein equations [cf. Eq. (93)] and was introduced when multiplying Eq. (94) by $G^2\ddot{G}$ to obtain Eq. (98). On the other hand, the asymptotic solution $\Omega = -K$ is stable because the dynamical equation for the perturbation ϵ ,

$$\dot{\boldsymbol{\epsilon}} = -\frac{1-K}{Kt}\boldsymbol{\epsilon} \tag{101}$$

near the attractor indicates that ϵ decreases for K < 1. In particular, the inflationary solutions, which occur for K < 0, are stable.

IV. n-SCALAR FIELD PROBLEM IN A BIANCHI TYPE VI₀ MODEL

The Bianchi type VI_0 model can be written as follows:

$$ds^{2} = e^{f(t)}(-dt^{2} + dz^{2}) + G(t)(e^{z}dx^{2} + e^{-z}dy^{2}).$$
(102)

We first consider, as in previous sections, *n* scalar fields ϕ_i interacting through the exponential potential (4). The corresponding Einstein-Klein-Gordon equations are

$$e^f = \frac{\ddot{G}}{2VG},$$
 (103)

$$\frac{\ddot{G}}{G} - \frac{1}{2} \left(\frac{\dot{G}}{G} \right)^2 - \frac{\dot{G}}{G} \dot{f} + \frac{1}{2} = -\dot{\phi}^2,$$
(104)

$$\ddot{\vec{\phi}} + \frac{\dot{G}}{G}\dot{\vec{\phi}} - e^f V \vec{k} = 0.$$
(105)

As formerly, one can check that the vector

$$\dot{\vec{\phi}} = \frac{\dot{G}}{G}\frac{\vec{k}}{2} + \frac{\vec{m}}{G},\tag{106}$$

where \vec{m} is a *n*-dimensional arbitrary constant vector, is a first integral of the Klein-Gordon equation set (105). By using this result and the value of \hat{f} one obtains from Eq. (103), we get

$$G\ddot{G}^{2} - \ddot{G}\dot{G}G + \left(\frac{1}{2} - \frac{k^{2}}{4}\right)\ddot{G}\dot{G}^{2} + \frac{1}{2}\ddot{G}G^{2} + m^{2}\ddot{G} = 0.$$
(107)

The solutions of this equation are not known. However, investigating the stability of its fixed points, the asymptotic behavior of the general solution can be obtained in a simple way. In order to see whether assisted inflation works in Bianchi type VI₀ metrics it is sufficient to analyze the special case $m^2 = 0$. In terms of the variable Ω defined by Eq. (99), Eq. (107) becomes

$$\Omega + \left[\Omega + K - \frac{1}{2h^2} \right] (\Omega + 1)h = 0.$$
 (108)

This equation has three fixed points: $\Omega_1 = -1$, $\Omega_2 = -K$ if $h \rightarrow \infty$ asymptotically and $\Omega_3 = 0$, which correspond to $G \propto t$, $G \propto t^{1/K}$, and $G \propto e^{t/\sqrt{2K}}$ respectively. Now, we investigate the stability of these solutions when *G* blows up. Expanding the solution about fixed points, that is, making $\Omega = \Omega_{1,2,3} + \epsilon$ with $\epsilon \ll 1$, we get

$$\dot{\epsilon} = \frac{1}{2h}\epsilon$$
 (109)

for Ω_1 , which shows it is unstable, as in the case discussed in the previous section. On the other hand, one obtains Eq. (101) for the linear approximation around Ω_2 , and

$$\dot{\epsilon} = -\sqrt{\frac{1}{2K}}\epsilon$$
 (110)

in the case of Ω_3 . We conclude that the asymptotic solution Ω_2 is stable for K < 0, which means $k^2 < 2$ (for this set of potential slopes we have inflation), and Ω_3 is stable for K > 0. Note that $G \propto t^{1/K}$ is only an asymptotic solution of Eq. (107), with m = 0; however, it acts as an attractor for all solutions that are close to it.

In this special case in which $m^2 = 0$, one can readily see that the deceleration parameter for the solutions $G \propto t^{1/K}$ is (after recovering the implicit absolute value around t)

$$q \propto \frac{K}{1-K} |t|^{-2-1/K},$$
 (111)

which is negative for $-1/2 \le K \le 0$. Again, the more interacting scalar fields, the less likely is inflation, which ensues when $k^2 = k_1^2 + k_2^2 + \cdots + k_n^2 \le 2$.

If one assumes now that the scalar fields do not interact directly and one sets all the fields equal, as in Secs. II D and III B, it is easily seen that the attracting power-law solutions inflate when $k^2 < 2n$, so that the scalar fields cooperate to assist inflation.

V. CONCLUSIONS

We have studied the effects of the appearance of more than one scalar field both in FRW spacetimes and in anisotropic Bianchi type I cosmologies. Instead of using the important but particular power-law solutions, we have taken advantage of the general solution to analyze the generic behavior in FRW cosmology. In Bianchi type I cosmologies the power-law solutions are isotropic and, thus, very particular, so that the use of the general solution is even more illustrative. In all cases we have found, in agreement with calculations made by other authors with power-law solutions in FRW cosmology, that the existence of more than one scalar field assists inflation provided that they are uncoupled and interact only through expansion. Also, in this case, the spectrum of density perturbations becomes closer to scale invariance as the number of fields increases. If, on the contrary, the fields interact directly with each other, inflation is less likely to occur. The same behavior has been obtained in Bianchi type VI₀ universes, but (not having available the general solution) only for power-law solutions, which have been shown to be attractors when inflation arises. These results reinforce our belief that the presence of several uncoupled scalar fields in more general cosmologies fosters inflation, but that, in contradistinction, mutually interacting scalar fields tend to hinder the inflationary process.

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