A Categorical Equivalence Motivated by Kalman's Construction

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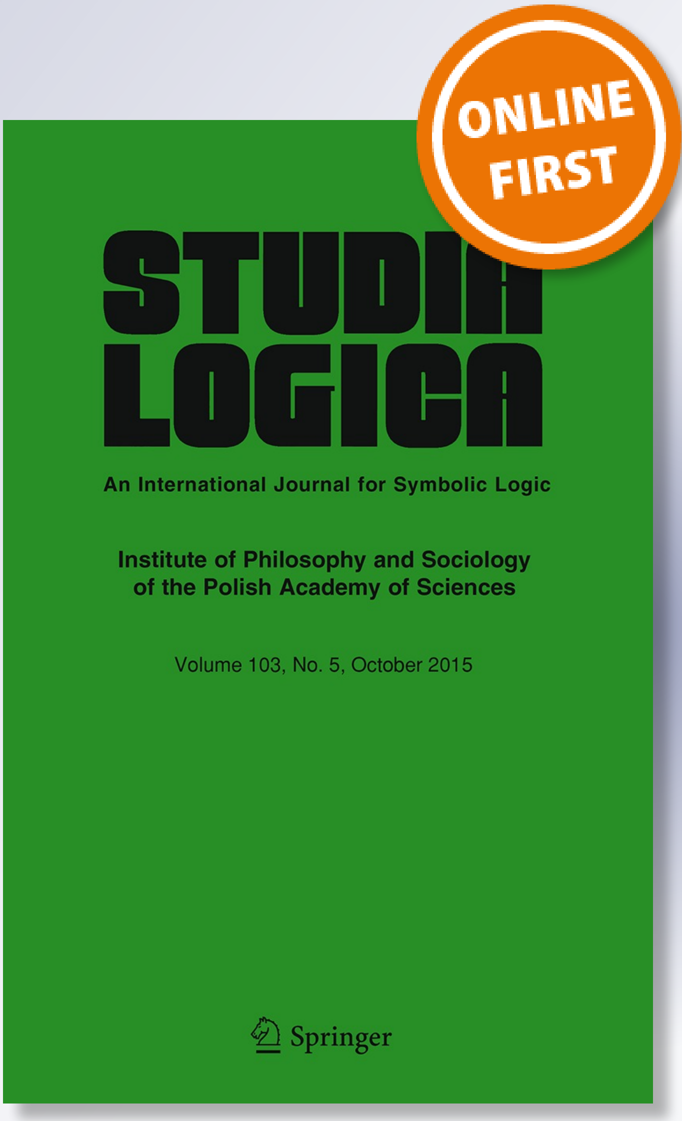
Studia Logica
An International Journal for Symbolic Logic

ISSN 0039-3215

Stud Logica
DOI 10.1007/s11225-015-9632-1



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Abstract. An equivalence between the category of MV-algebras and the category $\text{MV}^{\bullet}$ is given in Castiglioni et al. (*Studia Logica* 102(1):67–92, 2014). An integral residuated lattice with bottom is an MV-algebra if and only if it satisfies the equations $a = \neg\neg a$, $(a \to b) \vee (b \to a) = 1$ and $a \odot (a \to b) = a \wedge b$. An object of $\text{MV}^{\bullet}$ is a residuated lattice which in particular satisfies some equations which correspond to the previous equations. In this paper we extend the equivalence to the category whose objects are pairs (A, I), where A is an MV-algebra and I is an ideal of A.

Keywords: MV-algebras, Ideals, Adjunction, Categorical equivalence.

1. Introduction

In 1958 J. Kalman proved in [7] that if A is a bounded distributive lattice, then

$$\text{K}(A) = \{(a, b) \in A \times A : a \wedge b = 0\}$$

is a centered Kleene algebra by defining

$$(a, b) \vee (d, e) = (a \vee d, b \wedge e),$$
$$(a, b) \wedge (d, e) = (a \wedge d, b \vee e),$$
$$\sim (a, b) = (b, a),$$

$(0, 1)$ as the bottom, $(1, 0)$ as the top and $(0, 0)$ as the center.

Later, in 1986, R. Cignoli proved in [3] the following facts: (1) K can be extended to a functor from the category of bounded distributive lattices to the category of centered Kleene algebras, (2) there is an equivalence between the category of bounded distributive lattices and the category of centered Kleene algebras whose objects satisfy an additional condition called "interpolation property", (3) the category of Heyting algebras is equivalent

Presented by **Daniele Mundici**; *Received* May 1, 2015

Studia Logica
DOI: 10.1007/s11225-015-9632-1

Published online: 22 September 2015

to the category of centered Nelson algebras. In [1], the previous results were extended giving categorical equivalences for some categories of residuated lattices. An equivalence for the category MV of MV-algebras was developed in [2].

Let $A \in$ MV and let $\cdot$ be the product of A. The set

$$\mathsf{K}^{\bullet}(A) := \{(a, b) \in A \times A : a \cdot b = 0\}$$

plays an important role in the construction of a categorical equivalence for MV [2]. Our main goal in this paper is to extend this equivalence by means of a new construction

$$\mathsf{K}^{\bullet}(A, I) := \{(a, b) \in A \times A : a \cdot b \in I\},$$

where I is an ideal of A. If I is the zero ideal, then $\mathsf{K}^{\bullet}(A, I) = \mathsf{K}^{\bullet}(A)$.

In [8,12], L. Monteiro and I. Viglizzo consider the following structure: given an ideal I and a filter F of a bounded distributive lattice $\langle A, \wedge, \vee, 0, 1\rangle$, let $M(A, I, F) = \{(a, b) \in A \times A : a \wedge b \in I \ \& \ a \vee b \in F\}$. In $M(A, I, F)$ were defined binary operations $\vee$, $\wedge$ and $\sim$ as in the case of $\mathrm{K}(A)$. It was proved that the structure $\langle M(A, I, F), \wedge, \vee, \sim, (0, 1), (1, 0)\rangle$ is a De Morgan algebra. Let $A \in$ MV and I an ideal of A. In particular, I is an ideal of the underlying lattice of A. Since $M(A, I, A) = \{(a, b) \in A \times A : a \wedge b \in I\}$ then the definition of $\mathsf{K}^{\bullet}(A, I)$ is analogous to that of $M(A, I, A)$, but changing the infimum operation by the product operation. For additional motivation see [6,9,11].

In [10] the logic Ł$^{\bullet}$ was defined, whose algebraic models are the objects of a category called MV$^{\bullet}$ [2]. Let A_{MV} be the Lindenbaum algebra of the infinite valued propositional calculus Ł of Łukasiewicz and $A_{\mathrm{MV}^{\bullet}}$ the Lindenbaum algebra of the calculus Ł$^{\bullet}$. In [10, Theorem 3.9] it was proved that there exists an ideal I of A_{MV} such that $\kappa(A_{\mathrm{MV}^{\bullet}}) \cong A_{\mathrm{MV}}/I$, where κ is a unary operation defined on objects of MV$^{\bullet}$. Also, in [1, Corollary 5] it was proved that $\kappa(A_{\mathrm{MV}^{\bullet}}) \cup \{\boldsymbol{c}\}$ generates $A_{\mathrm{MV}^{\bullet}}$, where $\boldsymbol{c}$ is a center, that is: $\sim\boldsymbol{c} = \boldsymbol{c}$. So, we have some link between Ł$^{\bullet}$ and Ł. Let us try to explain what this link means.

It is a known fact that in any MV-algebra there exists a bijection between congruences and ideals, and that ideals are in bijection with the filters by means of the involution of the algebra. Also, the filters of A_{MV} are in bijection with the *theories* of Ł, where a theory is a class of formulas that contains the axioms and is closed by the rule of inference Modus Ponens. Taking into account this bijection, we can say, roughly speaking, that the classes of formulas in Ł$^{\bullet}$ of the form $\kappa(\mathrm{X})$ are in correspondence with the classes of formulas of Ł "modulo" the theory corresponding to the filter

$$\neg I = \{\neg a : a \in I\}.$$

For example, the classes of formulas of the form $x_n \oplus x_{n+1}$, for every propositional variable x_n, belong to that theory.

Let us also remark the following fact: if $A \in$ MV and I is an ideal of A, $\mathsf{K}^\bullet(A/I) = \{(a/\theta_I, b/\theta_I) \in A/I \times A/I : \ a/\theta_I \cdot b/\theta_I = 0/\theta_I\}$, where θ_I is the congruence associated to the ideal I (see ([5, Section 1.2]). We think that it is interesting to study the set $\mathsf{K}^\bullet(A, I) = \{(a, b) \in A \times A : a \cdot b \in I\}$ since

$$(a/\theta_I, b/\theta_I) \in \mathsf{K}^\bullet(A/I) \text{ if and only if } (a, b) \in \mathsf{K}^\bullet(A, I).$$

The paper is organized as follows. In Sect. 2 we give some basic results about the categories considered in [2]. We also introduce and study the category $\mathcal{I}$MV: the objects are pairs (A, I), where $A \in$ MV and I is an ideal of A, and the morphisms $f : (A, I) \to (B, J)$ are morphisms $f : A \to B$ in MV which satisfy the condition $I \subseteq f^{-1}(J)$. In Sect. 3 we build up an adjunction between $\mathcal{I}$MV and a new category whose objects are algebras. In Sect. 4 we obtain an equivalence for the category $\mathcal{I}$MV, which is a generalization of the equivalence given in [2] for the category MV. Finally, in Sect. 5 we make some remarks about properties of the constructions developed throughout this work.

2. Preliminary Definitions and Results

Since we are working on ideas and results of the paper [2], we recommend the reader to have the mentioned paper at hand while reading this work. All the residuated lattices considered in this paper are distributive and commutative, so we shall omit mentioning these two conditions in the sequel, assuming them as given. Recall that a residuated lattice is said to be integral if it is bounded above by the unit of the product. All the categories considered in this paper have an underlying class of algebras, so we shall use the same notation for the category and the class of algebras.

Let $\langle A, \wedge, \vee, \cdot, \to, 0, 1\rangle$ be an object in the category IRL_0 of integral residuated lattices with bottom. We define the set $\mathsf{K}^\bullet(A)$ as in the case of MV-algebras. For $A \in \mathrm{IRL}_0$, we define the operations $\vee$, $\wedge$ and $\sim$ as in the case of $\mathrm{K}(A)$. We also define the following binary operations:

$$(a, b) * (d, e) = (a \cdot d, (a \to e) \wedge (d \to b))$$
$$(a, b) \to (d, e) = ((a \to d) \wedge (e \to b), a \cdot e).$$

An involutive residuated lattice is an algebra $\mathbf{T} = \langle T, \wedge, \vee, *, \rightarrow, \sim, 1\rangle$ such that

1. $\langle T, \wedge, \vee, *, \rightarrow, 1\rangle$ is a residuated lattice.
2. $\sim$ is an involution of the lattice that is a dual automorphism, i.e., $\sim\sim x = x$ for every x.
3. $x * y \leq z$ if and only if $x \leq \ \sim(y * (\sim z))$.

In this case we have that $\sim(y * (\sim z)) = y \rightarrow z$ and $\sim x = x \rightarrow 0$.

An involutive residuated lattice is said to be centered if it has a distinguished element, called a center, that is, a fixed point for the involution. A $\boldsymbol{c}$*–differential residuated lattice* is an integral involutive residuated lattice with bottom and center $\boldsymbol{c}$, satisfying the following Leibniz condition [1, Definition 7.2]: For any $x, y \in T$, $(x * y) \wedge \boldsymbol{c} = ((x \wedge \boldsymbol{c}) * y) \vee ((y \wedge \boldsymbol{c}) * x)$. We denote the category of $\boldsymbol{c}$–differential residuated lattices by DRL.

The algebra $\langle \mathsf{K}^\bullet(A), \wedge, \vee, *, \rightarrow, \sim, (0,1), (1,0), (0,0)\rangle$ is an object of DRL, where $(0,1)$ is the bottom, $(1,0)$ is the top and $\boldsymbol{c} = (0,0)$. The assignment $A \mapsto \mathsf{K}^\bullet(A)$ extends to a functor $\mathsf{K}^\bullet : \mathrm{IRL}_0 \rightarrow \mathrm{DRL}$. For any $T \in \mathrm{DRL}$, consider $\mathsf{C}(T) := \{x \in T \ : \ x \geq \boldsymbol{c}\}$. It is defined a functor $\mathsf{C} : \mathrm{DRL} \rightarrow \mathrm{IRL}_0$, which is left adjoint to $\mathsf{K}^\bullet$ [1, Theorem 7.6].

The adjunction $\mathsf{C} \dashv \mathsf{K}^\bullet : \mathrm{IRL}_0 \rightarrow \mathrm{DRL}$ restricts to an equivalence $\mathsf{C} \dashv \mathsf{K}^\bullet : \mathrm{IRL}_0 \rightarrow \mathrm{DRL}'$ ([1, Corollary 7.8]), where DRL' is the full subcategory of DRL whose objects T satisfy the following condition:

($\mathsf{CK}^\bullet$) *For every pair of elements $z, w \in T$ such that $z, w \geq \boldsymbol{c}$ and $z * w \leq \boldsymbol{c}$, there exists $x \in T$ such that $x \vee \boldsymbol{c} = z$ and $\sim x \vee \boldsymbol{c} = w$.*

If T is an algebra of DRL', there exists a map $\kappa : T \rightarrow T$ that satisfies the following two conditions:

(k1) $\kappa x \wedge \boldsymbol{c} = x \wedge \boldsymbol{c}$,

(k2) $\kappa x \vee \boldsymbol{c} = \boldsymbol{c} \rightarrow x$.

Conversely, if T is an algebra of DRL in which there exists an operator κ that satisfies (**k1**) and (**k2**), then ($\mathsf{CK}^\bullet$) holds on T [2, Theorem 1]. In what follows we denote by DRL' the category whose objects have a unary operator κ in its signature, and verify the corresponding equations. In every integral residuated lattice $\langle A, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$, we define $\neg a = a \rightarrow 0$ for every $a \in A$. If we consider an algebra A of IRL_0, then $\kappa : \mathsf{K}^\bullet(A) \rightarrow \mathsf{K}^\bullet(A)$ is given by $\kappa(a, b) = (\neg b, b)$.

The category MDRL is the full subcategory of DRL' whose objects satisfy the equation

(Inv$^\bullet$) $\quad \sim\kappa x = \kappa\sim\kappa x.$

Let iIRL_0 be the full subcategory of IRL_0 whose objects satisfy the equation

(Inv) $\quad \neg\neg a = a.$

For $T \in \mathrm{DRL}'$ we have that $\kappa(T) = \{x \in T : \kappa x = x\}$. If $T \in \mathrm{MDRL}$, then $\kappa(T) \in \mathrm{iIRL}_0$. If $g : T \to S$ is a morphism in MDRL, then $\kappa(g) : \kappa(T) \to \kappa(S)$ is the morphism in iIRL_0 given by the restriction of g to $\kappa(T)$. On the other hand, if $A \in \mathrm{iIRL}_0$ then $\mathsf{K}^\bullet(A) \in \mathrm{MDRL}$. If $f : A \to B$ is a morphism in iIRL_0, then we define $\mathsf{K}^\bullet(f) : \mathsf{K}^\bullet(A) \to \mathsf{K}^\bullet(B)$ as $(\mathsf{K}^\bullet(f))(a, b) = (f(a), f(b))$, which is a morphism in MDRL. For every $A \in \mathrm{iIRL}_0$ we have the isomorphism $\alpha : A \to \kappa(\mathsf{K}^\bullet(A))$ given by $\alpha(a) = (a, \neg a)$, and for every $T \in \mathrm{MDRL}$ we have the isomorphism $\beta_T : T \to \mathsf{K}^\bullet(\kappa(T))$ given by $\beta_T(x) = (\lambda x, \lambda\sim x)$, where $\lambda x = \sim\kappa\sim x = \kappa(x * \boldsymbol{c})$ (see [2, Lemma 7]). There is a categorical equivalence $\kappa \dashv \mathsf{K}^\bullet : \mathrm{iIRL}_0 \to \mathrm{MDRL}$ [2, Theorem 11].

Recall that an MV-algebra is term equivalent to an integral residuated lattice with bottom $\langle A, \vee, \wedge, \cdot, \to, \neg, 0, 1\rangle$ that satisfies (Inv) and the following equations:

(Lin) $\ (a \to b) \vee (b \to a) = 1,$

(QHey) $\ a \cdot (a \to b) = a \wedge b.$

The category $\mathrm{MV}^\bullet$ is the full subcategory of MDRL whose objects satisfy the equations

(Lin$^\bullet$) $\ (x \to y) \vee (y \to x) \geq \boldsymbol{c},$

(QHey$^\bullet$) $\ \boldsymbol{c} * x * (x \to (y \vee \boldsymbol{c})) = \boldsymbol{c} * (x \wedge y).$

There is a categorical equivalence $\kappa \dashv \mathsf{K}^\bullet : \mathrm{MV} \to \mathrm{MV}^\bullet$ [2, Corollary 15].

Consider the following equation in MDRL:

(P$^\bullet$) $\qquad (\kappa x \to \kappa y) \vee (\kappa y \to \kappa x) = 1.$

Remark 1. In every algebra of MDRL we have that the conditions (P$^\bullet$) and (Lin$^\bullet$) are equivalent. In order to prove it, suppose that we have the condition (Lin$^\bullet$). Hence, $(\kappa x \to \kappa y) \vee (\kappa y \to \kappa x) \geq \boldsymbol{c}$. Thus, by [2, Corollary 12] we obtain that $1 = \kappa((\kappa x \to \kappa y) \vee (\kappa y \to \kappa x)) = (\kappa x \to \kappa y) \vee (\kappa y \to \kappa x)$. Conversely, suppose that we have the condition (P$^\bullet$). It follows from [2, Corollary 12] that in every algebra of DRL' we have the equation $x \leq \kappa x$, so $(x \to \kappa y) \vee (y \to \kappa x) = 1$. By the categorical equivalence between DRL' and IRL_0 it is possible to prove the equation $\boldsymbol{c} \wedge (x \to \kappa y) = \boldsymbol{c} \wedge (x \to y)$.

Hence, $\boldsymbol{c} = (\boldsymbol{c} \wedge (x \to \kappa y)) \vee (\boldsymbol{c} \wedge (y \to \kappa x)) = \boldsymbol{c} \wedge ((x \to y) \vee (y \to x))$, i.e., $(x \to y) \vee (y \to x) \geq \boldsymbol{c}$.

In the following diagram we have the relationship among the above mentioned categories, where inc denotes the inclusion functor:

$$\begin{array}{ccccc} \mathrm{MV} & \xrightarrow{\mathrm{inc}} & \mathrm{iIRL}_0 & \xrightarrow{\mathrm{inc}} & \mathrm{IRL}_0 \\ {\scriptstyle \kappa}\uparrow\downarrow{\scriptstyle \mathsf{K}^\bullet} & & {\scriptstyle \kappa}\uparrow\downarrow{\scriptstyle \mathsf{K}^\bullet} & & {\scriptstyle \kappa}\uparrow\downarrow{\scriptstyle \mathsf{K}^\bullet} \\ \mathrm{MV}^\bullet & \xrightarrow{\mathrm{inc}} & \mathrm{MDRL} & \xrightarrow{\mathrm{inc}} & \mathrm{DRL}'. \end{array}$$

Remark 2. Let $A \in \mathrm{MV}$ and let I, J be ideals of A. Then

(a) $\mathsf{K}^\bullet(A) = \mathsf{K}^\bullet(A, I)$ if and only if $I = \{0\}$.

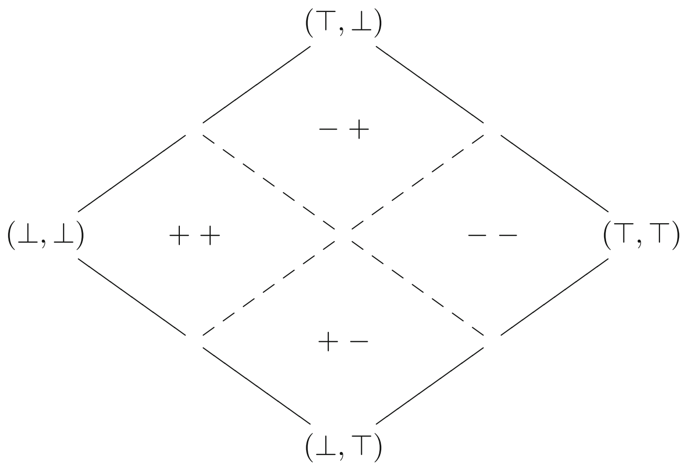
(b) If $I \subseteq J$, then $\mathsf{K}^\bullet(A, I) \subseteq \mathsf{K}^\bullet(A, J)$. In particular, $\mathsf{K}^\bullet(A) \subseteq \mathsf{K}^\bullet(A, I) \subseteq \mathsf{K}^\bullet(A, A) = A \times A$.

(c) $\mathsf{K}^\bullet(A, I)$ is closed under the operations $\vee$, $\wedge$, $\sim$, $*$, $\to$, $(0,1)$, $(1,0)$ and $(0,0)$ given in $\mathsf{K}^\bullet(A)$. There exists in $\mathsf{K}^\bullet(A, I)$ an unique unary operation κ which satisfies (**k1**) and (**k2**): this map takes the form $\kappa(a, b) = (\neg b, b)$, as in the case of $\mathsf{K}^\bullet(A)$. Therefore, we obtain that $\langle \mathsf{K}^\bullet(A, I), \wedge, \vee, *, \to, \sim, \kappa, (0,1), (1,0), (0,0)\rangle$ is an algebra.

In what follows we will give an example of $\mathsf{K}^\bullet(A, I)$ for an algebra $A \in \mathrm{MV}$ and an ideal I of A.

Example 1. In 1958 Chang introduced the MV-algebra $\mathbf{C}$ [4,5], defined by $\mathbf{C} = \Gamma(\mathbb{Z} \otimes \mathbb{Z}, (1,0))$, where $\mathbb{Z}$ is the set of integer numbers, $\mathbb{Z} \otimes \mathbb{Z}$ is the lexicographic product and Γ is the categorical equivalence between ℓ–groups with strong unit, and the category MV.

Let I be the following ideal of $\mathbf{C}$: $I = \{\perp\} \cup \{m : m > 0\}$. We define $+\,- \; = \{(x, y) \in \mathbf{C} \times \mathbf{C} : (x > 0 \;\&\; y < 0)$ or $(x = \perp \;\&\; y < 0)$ or $(x > 0 \;\&\; y = \top)\}$. The sets $-\,+$, $+\,+$ and $-\,-$ are defined in a similar way. We shall prove that $\mathsf{K}^\bullet(\mathbf{C}, I)$ is the union of the quadrants $+\,-$, $-\,+$ and $+\,+$ that we show in the following graphic of $\mathbf{C} \times \mathbf{C}$. In fact, if $(x, y) \in \mathbf{C} \times \mathbf{C}$ is in $+\,-$, the product $x.y$ is positive or $\perp$ and the same is true for $(x, y) \in \; -\,+$. If $(x, y) \in \; +\,+$, then $x \cdot y$ is $\perp$.



Inspired by the construction of $\mathsf{K}^{\bullet}(A, I)$ we give the following

Definition 1. Let $\mathcal{I}$MV be the category whose objects are pairs (A, I), where $A \in$ MV and I is an ideal of A, and whose morphisms $f : (A, I) \to (B, J)$ are morphisms $f : A \to B$ in MV that satisfy the condition $I \subseteq f^{-1}(J)$.

A Connection Between the Categories MV and $\mathcal{I}$MV

Let $A \in$ MV, and let I be an ideal of A. The relation given by $(a, b) \in \theta_I$ if and only if $d(a, b) \in I$ is a congruence relation in A, where $d(a, b) = (a \ominus b) \oplus (b \ominus a)$. Moreover, the correspondence $I \mapsto \theta_I$ is a bijection from the set of ideals of A onto the set of congruences on A [4, Proposition 1.2.6.]. If θ is a congruence of A and $a \in A$, we write a/I in place of a/θ_I.

Proposition 1. *There is an adjunction* $Q \dashv E : \text{MV} \to \mathcal{I}\text{MV}$.

Proof. Define the functors $E : \text{MV} \to \mathcal{I}\text{MV}$ and $Q : \mathcal{I}\text{MV} \to \text{MV}$. If $A \in$ MV, then $E(A) = (A, \{0\})$. If $f : A \to B$ is a morphism in MV, then

$E(f) = f$. If $(A, I) \in \mathcal{I}\mathrm{MV}$ then $Q(A, I) = A/I$. If $g : (A, I) \to (B, J)$ is a morphism in $\mathcal{I}\mathrm{MV}$, then the condition $I \subseteq g^{-1}(J)$ allows us define a morphism $Q(g) : A/I \to B/J$ in MV as $Q(g)(a/I) = g(a)/J$. Note that for $A \in \mathrm{MV}$ we have that $(Q \circ E)(A) = A/\{0\}$, and for $(A, I) \in \mathcal{I}\mathrm{MV}$ we have that $(E \circ Q)(A, I) = (A/I, \{0\})$.

For every $(A, I) \in \mathcal{I}\mathrm{MV}$ we define the morphism $\eta_{(A,I)} : (A, I) \to (A/I, \{0\})$ in the obvious way, that is, $\eta_{(A,I)}(a) = a/I$. An easy moment's reflection shows that for every morphism $g : (A, I) \to (B, J)$ in $\mathcal{I}\mathrm{MV}$ the following diagram commutes:

$$\begin{array}{ccc}
(A, I) & \xrightarrow{\eta_{(A,I)}} & (A/I, \{0\}) \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle (E\circ Q)(g)} \\
(B, J) & \xrightarrow{\eta_{(B,J)}} & (B/J, \{0\}).
\end{array}$$

Hence, $\eta : 1_{\mathcal{I}\mathrm{MV}} \to E \circ Q$ is a natural transformation. Let $g : (A, I) \to (B, \{0\})$ be a morphism in $\mathcal{I}\mathrm{MV}$. The inclusion $I \subseteq g^{-1}(\{0\})$ allows us to define a morphism $f : A/I \to B$ in MV as $f(a/I) = g(a)$.

It is immediate that the following diagram commutes:

$$\begin{array}{ccc}
(A, I) & \xrightarrow{\eta_{(A,I)}} & (A/I, \{0\}) \\
 & \searrow^{g} & \downarrow{\scriptstyle E(f)} \\
 & & (B, \{0\}).
\end{array}$$

We have that f is the unique morphism in $\mathcal{I}\mathrm{MV}$ with the above mentioned property. Therefore, we have that $Q \dashv E$. ■

3. The Adjunction

In this section we consider a new category we shall call $\mathcal{I}\mathrm{MV}^\bullet$, and we build up functors $\mathsf{K}^\bullet : \mathcal{I}\mathrm{MV} \to \mathcal{I}\mathrm{MV}^\bullet$ and $\kappa : \mathcal{I}\mathrm{MV}^\bullet \to \mathcal{I}\mathrm{MV}$. We also prove that there exists an adjunction $\kappa \dashv \mathsf{K}^\bullet : \mathcal{I}\mathrm{MV} \to \mathcal{I}\mathrm{MV}^\bullet$. Let $A \in \mathrm{MV}$. In what follows we introduce a category whose definition is motivated by properties satisfied for the algebra $\langle \mathsf{K}^\bullet(A, I), \wedge, \vee, *, \to, \sim, \kappa, (0, 1), (1, 0), (0, 0)\rangle$:

Definition 2. We define the category $\mathcal{I}\mathrm{MV}^\bullet$ as the category whose objects $\mathbf{T} = \langle T, \wedge, \vee, *, \to, \sim, \kappa, 0, 1, \boldsymbol{c}\rangle$ are algebras of type $(2, 2, 2, 2, 1, 1, 0, 0, 0)$ which satisfies the following conditions:

1. $\langle T, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice, $*$ is an associative and commutative operation, and $x * y \leq z$ if and only if $x \leq y \to z$ if and only if $x \leq \sim(y * (\sim z))$, for any x, y, z.
2. $\sim$ is an involution of the lattice that is a dual automorphism, i.e., $\sim\sim x = x$ for every x.
3. $\boldsymbol{c} = \sim\boldsymbol{c} = \boldsymbol{c} \to 0$ and $(x * \sim x) \wedge \boldsymbol{c} = 0$.
4. For any x, y, $(x * y) \wedge \boldsymbol{c} = ((x \wedge \boldsymbol{c}) * y) \vee ((y \wedge \boldsymbol{c}) * x)$.
5. κ is defined by the equations (**k1**) and (**k2**).
6. T satisfies the conditions $(\mathbf{Inv})^{\bullet}$, $(\mathbf{P})^{\bullet}$ and $(\mathbf{QHey})^{\bullet}$.

REMARK 3. (a) Let (A, I) be an object in $\mathcal{I}$MV. Then we have that the algebra $\langle \mathsf{K}^{\bullet}(A, I), \wedge, \vee, *, \to, \sim, \kappa, (0, 1), (1, 0), (0, 0)\rangle$ is an object in $\mathcal{I}\mathrm{MV}^{\bullet}$. If $f : (A, I) \to (B, J)$ is a morphism in $\mathcal{I}$MV, then the map $\mathsf{K}^{\bullet}(f) : \mathsf{K}^{\bullet}(A, I) \to \mathsf{K}^{\bullet}(A, J)$ given by $\mathsf{K}^{\bullet}(f)(a, b) = (f(a), f(b))$ is a morphism in $\mathcal{I}\mathrm{MV}^{\bullet}$. Therefore, we have defined a functor $\mathsf{K}^{\bullet}$ from $\mathcal{I}$MV to $\mathcal{I}\mathrm{MV}^{\bullet}$.

(b) $\mathrm{MV}^{\bullet}$ is a full subcategory of $\mathcal{I}\mathrm{MV}^{\bullet}$. Moreover, $\mathrm{MV}^{\bullet} \subsetneq \mathcal{I}\mathrm{MV}^{\bullet}$. In order to prove it, consider the real interval $[0, 1]$ and $A = I = [0, 1]$ endowed with its structure of MV-algebra. Then $\mathsf{K}^{\bullet}([0, 1], [0, 1]) = [0, 1] \times [0, 1]$ and $((\frac{1}{2}, \frac{1}{2}) \to (1, 1)) \vee ((1, 1) \to (\frac{1}{2}, \frac{1}{2})) = (\frac{1}{2}, \frac{1}{2}) \not\geq \boldsymbol{c}$, so $\mathsf{K}^{\bullet}([0, 1], [0, 1]) \notin \mathrm{MV}^{\bullet}$.

(c) There are many differences between $\mathrm{MV}^{\bullet}$ and $\mathcal{I}\mathrm{MV}^{\bullet}$. Let $(A, I) \in \mathcal{I}$MV. The assertion $(a, b) \in \mathsf{K}^{\bullet}(A)$ is equivalent to any of the following assertions: $(a, b) \to (0, 1) = \sim(a, b)$, $(a, b) \leq \kappa(a, b)$ or $(a, b) = ((a, b) \vee (0, 0)) \wedge \kappa(a, b)$. Then, κ does not have necessarily the same behavior in $\mathsf{K}^{\bullet}(A)$ and $\mathsf{K}^{\bullet}(A, I)$. In general, in $\mathcal{I}\mathrm{MV}^{\bullet}$ the equation $x * 1 = x$ is not true: more precisely, an object of $\mathcal{I}\mathrm{MV}^{\bullet}$ is an object of $\mathrm{MV}^{\bullet}$ if and only if it satisfies the equation $x * 1 = x$. Therefore, the objects of $\mathcal{I}\mathrm{MV}^{\bullet}$ do not necessarily have residuated lattices as reducts.

(d) The condition $x * y \leq z$ if and only if $x \leq y \to z$ for any x, y, z is equivalent to the equations $x * (y \vee z) = (x * y) \vee (x * z)$, $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, $x * (x \to y) \leq y$ and $y \leq x \to (x * y)$. Besides the condition $x * y \leq z$ if and only if $x \leq \sim(y * (\sim z))$ for any x, y, z is equivalent to the condition $x \to y = \sim(x * \sim y)$ for any x, y. Therefore, $\mathcal{I}\mathrm{MV}^{\bullet}$ is a variety. Also note that $\boldsymbol{c} * x \leq \boldsymbol{c}$, $\boldsymbol{c} * \boldsymbol{c} = 0$ and $(x * y) \to z = x \to (y \to z)$.

The next goal is to define a functor from $\mathcal{I}\mathrm{MV}^{\bullet}$ to $\mathcal{I}$MV.

The proof of the following technical lemma follows from similar ideas to that given in [2].

Lemma 1. *Let $T \in \mathcal{I}\mathrm{MV}^{\bullet}$. Then:*

1. $\kappa(x \wedge y) = \kappa x \wedge \kappa y$.
2. $x \geq \boldsymbol{c}$ *if and only if* $\kappa x = 1$, *and* $\kappa x = 0$ *if and only if* $x \wedge \boldsymbol{c} = 0$.
3. $\kappa(x \wedge \boldsymbol{c}) = \kappa x$.
4. $\kappa x = \kappa y$ *if and only if* $x \wedge \boldsymbol{c} = y \wedge \boldsymbol{c}$.
5. $\kappa\kappa x = \kappa x$.
6. $\kappa(T) = \{x \in T : \kappa x = x\}$.
7. $x \vee \boldsymbol{c} = \kappa{\sim}x \to \boldsymbol{c}$ *and* $\kappa(T)$ *is closed under* $\sim$.
8. *For every* $x \leq \boldsymbol{c}$, $\boldsymbol{c} * \kappa x = \boldsymbol{c} \wedge \kappa x = x$.
9. *For every* $x, y \leq \boldsymbol{c}$, $\kappa x * y = \kappa y * x$.
10. *For every* x, $\sim \kappa x = \kappa((\sim x) * \boldsymbol{c})$.
11. *For every* $x, y \leq \boldsymbol{c}$, $\kappa x * \kappa y = \kappa(\kappa x * y)$, *so* $\kappa x * \kappa y = \kappa(\kappa x * \kappa y)$.
12. *For every* $x \geq \boldsymbol{c}$, $x = \kappa(x * \boldsymbol{c}) \vee \boldsymbol{c}$.
13. *For* $x \geq \boldsymbol{c}$, $y \geq \boldsymbol{c}$, $x = y$ *if and only if* $x * \boldsymbol{c} = y * \boldsymbol{c}$.
14. *For every* x, y, $\kappa(x + y) = \kappa(x) + \kappa(y)$, *where* $x + y = {\sim}({\sim}x * {\sim}y)$.

Corollary 2. *Let $T \in \mathcal{I}\mathrm{MV}^{\bullet}$ and $x, y \in T$. Then:*

(i) $\boldsymbol{c} * (\boldsymbol{c} \to x) = x \wedge \boldsymbol{c}$.

(ii) $\kappa(x \vee y) = \kappa x \vee \kappa y$.

(iii) $\boldsymbol{c} * \kappa x = \boldsymbol{c} \wedge x = \boldsymbol{c} \wedge \kappa x$.

(iv) $\boldsymbol{c} * \kappa(x * y) = \boldsymbol{c} * ((\lambda x * \kappa y) \vee (\lambda y * \kappa x))$, *where* $\lambda x = {\sim}\kappa{\sim}x$.

(v) $\boldsymbol{c} * \kappa x = \boldsymbol{c} * \kappa y$ *if and only if* $\kappa x = \kappa y$.

(vi) $\kappa(x * y) = (\lambda x * \kappa y) \vee (\lambda y * \kappa x)$.

(vii) $\boldsymbol{c} \to \kappa x = \boldsymbol{c} \to x$.

(viii) $\boldsymbol{c} \wedge (x \to \kappa y) = \boldsymbol{c} \wedge (x \to y)$.

Proof. Let us prove (i) by using the item 10. of Lemma 1:

$$\begin{aligned}
\boldsymbol{c} * (\boldsymbol{c} \to x) &= \boldsymbol{c} * \sim (\boldsymbol{c} * \sim x) \\
&= \sim (\boldsymbol{c} \to (\boldsymbol{c} * \sim x)) \\
&= \sim (\kappa(\boldsymbol{c} * \sim x) \vee \boldsymbol{c}) \\
&= \sim (\sim \kappa x \vee \boldsymbol{c}) \\
&= \kappa x \wedge \boldsymbol{c} \\
&= x \wedge \boldsymbol{c}.
\end{aligned}$$

Now we prove (ii). We have, in first place, $(\kappa x \vee \kappa y) \wedge \boldsymbol{c} = (x \wedge \boldsymbol{c}) \vee (y \wedge \boldsymbol{c}) = (x \vee y) \wedge \boldsymbol{c}$. We also prove that $(\kappa x \vee \kappa y) \vee \boldsymbol{c} = \boldsymbol{c} \to (x \vee y)$. In order to show it, note that $\boldsymbol{c} \leq \boldsymbol{c} \to x, \boldsymbol{c} \to y$. Then by item 13. of Lemma 1, the problem is reduced to show that $((\boldsymbol{c} \to x) \vee (\boldsymbol{c} \to y)) * \boldsymbol{c} = (\boldsymbol{c} \to (x \vee y)) * \boldsymbol{c}$.

From (i):

$$\begin{aligned}((\boldsymbol{c} \to x) \vee (\boldsymbol{c} \to y)) * \boldsymbol{c} &= (x \wedge \boldsymbol{c}) \vee (y \wedge \boldsymbol{c}) \\ &= (x \vee y) \wedge \boldsymbol{c} \\ &= (\boldsymbol{c} \to (x \vee y)) * \boldsymbol{c}.\end{aligned}$$

The condition (iii) follows from the condition (i) because $\boldsymbol{c} * \kappa x = \boldsymbol{c} * (\boldsymbol{c} \vee \kappa x) = \boldsymbol{c} * (\boldsymbol{c} \to x) = \boldsymbol{c} \wedge x$.

Now we prove the condition (iv). First note that it follows from condition (i) that

$$\begin{aligned}\boldsymbol{c} * \kappa(x * y) &= \boldsymbol{c} * (\boldsymbol{c} \vee \kappa(x * y)) \\ &= \boldsymbol{c} * (\boldsymbol{c} \to (x * y)) \\ &= (x * y) \wedge \boldsymbol{c} \\ &= ((x \wedge \boldsymbol{c}) * y) \vee ((y \wedge \boldsymbol{c}) * x).\end{aligned}$$

Besides by item 10. of Lemma 1 and the condition (iii) we have that

$$\begin{aligned}\boldsymbol{c} * ((\lambda x * \kappa y) \vee (\lambda y * \kappa x)) &= (\boldsymbol{c} * \lambda x * \kappa y) \vee (\boldsymbol{c} * \lambda y * \kappa x) \\ &= (\boldsymbol{c} * \kappa(x * \boldsymbol{c}) * \kappa y) \vee (\boldsymbol{c} * \kappa(y * \boldsymbol{c}) * \kappa x) \\ &= (x * \boldsymbol{c} * \kappa y) \vee (y * \boldsymbol{c} * \kappa x) \\ &= ((x * (\boldsymbol{c} \wedge y)) \vee ((y * (\boldsymbol{c} \wedge x)).\end{aligned}$$

Thus, we obtain $\boldsymbol{c} * \kappa(x * y) = \boldsymbol{c} * ((\lambda x * \kappa y) \vee (\lambda y * \kappa x))$.

In order to prove the condition (v), suppose that $\boldsymbol{c} * \kappa x = \boldsymbol{c} * \kappa y$. Taking into account (iii) we obtain $x \wedge \boldsymbol{c} = y \wedge \boldsymbol{c}$. Hence, by item 4. of Lemma 1 we have that $\kappa x = \kappa y$.

The item (vi) is consequence of the items (iv) and (v), and the item (vii) follows from Item 5 of Lemma 1.

Finally we prove the item (viii). By items (i) and (vii) we obtain

$$\begin{aligned}\boldsymbol{c} \wedge (x \to \kappa y) &= \boldsymbol{c} * (\boldsymbol{c} \to (x \to \kappa y)) \\ &= \boldsymbol{c} * (x \to (\boldsymbol{c} \to \kappa y)) \\ &= \boldsymbol{c} * (x \to (\boldsymbol{c} \to y)) \\ &= \boldsymbol{c} * (\boldsymbol{c} \to (x \to y)) \\ &= \boldsymbol{c} \wedge (x \to y).\end{aligned}$$

■

Lemma 3. *Let $T \in \mathcal{I}\mathrm{MV}^{\bullet}$. Then $\kappa(T)$ is closed under $\wedge$, $\vee$, 0, 1, $\sim$, $*$ and $\to$. Moreover, $1 * \kappa x = \kappa x$ for every $x \in T$.*

Proof. Straightforward computations based on Lemma 1 and Corollary 2 prove that $\kappa(T)$ is closed under $\wedge$, $\vee$, 0, 1, $\sim$, $*$ and $\rightarrow$. It follows from Lemma 1 and Corollary 2 that if $x \in T$, then

$$\begin{aligned} 1 * \kappa x &= \kappa(\boldsymbol{c}) * \kappa(x \wedge \boldsymbol{c}) \\ &= \kappa(\kappa(x \wedge \boldsymbol{c}) * \boldsymbol{c}) \\ &= \kappa(x \wedge \boldsymbol{c}) \\ &= \kappa x. \end{aligned}$$

■

Corollary 4. *Let $T \in \mathcal{I}\mathrm{MV}^{\bullet}$. Then $\kappa(T) \in \mathrm{MV}$.*

Proof. It follows from Lemma 3 that $\sim x = \neg x$ for every $x \in \kappa(T)$. Hence, $\langle \kappa(T), \wedge, \vee, *, \rightarrow, 0, 1\rangle \in \mathrm{iIRL}_0$. Let $x, y \in \kappa(T)$. Taking into account that T satisfies (QHey$^{\bullet}$) we have that

$$\begin{aligned} \boldsymbol{c} * (x \wedge y) &= \boldsymbol{c} * x * (x \rightarrow (y \vee \boldsymbol{c})) \\ &= \boldsymbol{c} * x * (\boldsymbol{c} \rightarrow (x \rightarrow y)) \\ &= \boldsymbol{c} * x * (x \rightarrow y). \end{aligned}$$

Hence, by (v) of Corollary 2 we have that $\kappa(T)$ satisfies (QHey). It is immediate that the condition (P$^{\bullet}$) on T implies the condition (Lin) on $\kappa(T)$. ■

If $T \in \mathcal{I}\mathrm{MV}^{\bullet}$, we define the following sets:

1. $I_T = \{y \in T : y \wedge \boldsymbol{c} = 0\}$.
2. $J_T = \lambda(I_T)$.

Note that if $(A, I) \in \mathcal{I}\mathrm{MV}$, then $\kappa(\mathsf{K}^{\bullet}(A, I)) = \kappa(\mathsf{K}^{\bullet}(A))$.

Proposition 2. *If $(A, I) \in \mathcal{I}\mathrm{MV}$, then $J_{\mathsf{K}^{\bullet}(A,I)} = \{(a, \neg a) : a \in I\}$. In particular, the map $\alpha_{(A,I)} : (A, I) \rightarrow (\kappa(\mathsf{K}^{\bullet}(A, I)), J_{\mathsf{K}^{\bullet}(A,I)})$ given by $\alpha_{(A,I)}(a, b) = (a, \neg a)$ is an isomorphism in $\mathcal{I}\mathrm{MV}$.*

Proof. It follows from straightforward computations (see [2, Theorem 11]). ■

Proposition 3. *If $T \in \mathcal{I}\mathrm{MV}^{\bullet}$, then $(\kappa(T), J_T) \in \mathcal{I}\mathrm{MV}$.*

Proof. We will prove that J_T is an ideal of $\kappa(T)$. It is immediate that $J_T \subseteq \kappa(T)$ and $0 \in \kappa(T)$. Now let us show that $\lambda x * \lambda y \in J_T$ for $x, y \in T$ such that $x \wedge \boldsymbol{c} = y \wedge \boldsymbol{c} = 0$. Using item 14. of Lemma 1 we obtain $\lambda(x * y) = \lambda x * \lambda y$. Besides $(x * y) \wedge \boldsymbol{c} = ((x \wedge c) * y) \vee ((y \wedge c) * x) = 0$. Finally, let $z \in \kappa(T)$ and $x \in T$ such that $z \leq \lambda x$ and $x \wedge \boldsymbol{c} = 0$. We prove that there is $y \in T$ such that $z = \lambda y$ and $y \wedge \boldsymbol{c} = 0$. First note that $z = \lambda z$. Let $y = x \wedge z$. Hence, $y \wedge \boldsymbol{c} = 0$.

Since κ preserves finite meets and joins then $\lambda(x \wedge z) == \lambda x \wedge z = z$, so $z \in J_T$. ∎

Remark 4. If $T \in \mathrm{MV}^\bullet$, then $J_T = \{0\}$.

Motivated by the proof of [2, Theorem 11], we show the next

Theorem 5. *Let $T \in \mathcal{I}\mathrm{MV}^\bullet$. Then the map $\beta_T : T \to \mathsf{K}^\bullet(\kappa(T), J_T)$ given by $\beta_T(x) = (\lambda x, \lambda{\sim}x)$ is a morphism in $\mathcal{I}\mathrm{MV}^\bullet$ which is an injective map.*

Proof. First note that $(z, w) \in \mathsf{K}^\bullet(\kappa(T), J_T)$ if and only if $(z, w) \in \kappa(T) \times \kappa(T)$, and $z * w = \lambda y$ for some $y \in T$ such that $y \wedge \boldsymbol{c} = 0$. The map β_T is well defined. In order to prove it, let $x \in T$. Then $(\lambda x, \lambda{\sim}x) \in \kappa(T) \times \kappa(T)$, $\lambda x * \lambda{\sim}x = \lambda(x * {\sim}x)$ and $(x * {\sim}x) \wedge \boldsymbol{c} = 0$. Thus, $(\lambda x, \lambda{\sim}x) \in \mathsf{K}^\bullet(\kappa(T), J_T)$. Now we prove that β_T is an injective function. Let $x, y \in T$ such that $\beta_T(x) = \beta_T(y)$. Then, $\kappa x = \kappa y$ and $\kappa{\sim}x = \kappa{\sim}y$. Hence, $x \wedge \boldsymbol{c} = y \wedge \boldsymbol{c}$ and $x \vee \boldsymbol{c} = y \vee \boldsymbol{c}$, so $x = y$. Thus, β_T is an injective function. Properties of κ (see Lemma 1 and Corollary 2) show that β_T preserves the operations. ∎

Lemma 6. *Let $g : T \to U$ be a morphism in $\mathcal{I}\mathrm{MV}^\bullet$. Then the map $\kappa(g) : (\kappa(T), J_T) \to (\kappa(U), J_U)$ given by $\kappa(g)(t) = g(t)$ is a morphism in $\mathcal{I}\mathrm{MV}$. Moreover, the assignment $T \mapsto (\kappa(T), J_T)$ extends to a functor $\kappa : \mathcal{I}\mathrm{MV}^\bullet \to \mathcal{I}\mathrm{MV}$.*

Let $g : T \to U$ be a morphism in $\mathcal{I}\mathrm{MV}^\bullet$. Direct computations show that the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{\beta_T} & \mathsf{K}^\bullet(\kappa(T), J_T) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathsf{K}^\bullet(\kappa(g))} \\ U & \xrightarrow{\beta_U} & \mathsf{K}^\bullet(\kappa(U), J_U). \end{array}$$

Hence, we have that $\beta : 1_{\mathcal{I}\mathrm{MV}^\bullet} \to \mathsf{K}^\bullet \circ \kappa$ is a natural transformation, where $1_{\mathcal{I}\mathrm{MV}^\bullet} : \mathcal{I}\mathrm{MV}^\bullet \to \mathcal{I}\mathrm{MV}^\bullet$ is the identity functor. Now consider the morphism $\pi_1 : (\kappa(\mathsf{K}^\bullet(A, I)), J_{\mathsf{K}^\bullet(A,I)}) \to (A, I)$ in $\mathcal{I}\mathrm{MV}$ given by $\pi_1(a, \neg a) = a$. Let $T \in \mathcal{I}\mathrm{MV}^\bullet$ and $g : T \to \mathsf{K}^\bullet(A, I)$ a morphism in $\mathcal{I}\mathrm{MV}^\bullet$. We define the morphism $f : (\kappa(T), J_T) \to (A, I)$ as $f = \pi_1 \circ \kappa(g)$. We will show that the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{\beta_T} & \mathsf{K}^\bullet(\kappa(T), J_T) \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle \mathsf{K}^\bullet(f)} \\ & & \mathsf{K}^\bullet(A, I). \end{array} \qquad (1)$$

Consider the map $\pi_2 : \mathsf{K}^\bullet(A, I) \to A$ given by $\pi_2(a, b) = b$. As $\lambda(a, b) = (a, \neg a)$ in $\mathsf{K}^\bullet(A, I)$, for every $t \in T$ we obtain that

$$\begin{aligned}(\mathsf{K}^\bullet(f) \circ \beta_T)(t) &= \mathsf{K}^\bullet(f)(\lambda t, \lambda{\sim}t)\\ &= (f(\lambda t), f(\lambda{\sim}t))\\ &= ((\pi_1(g(\lambda t)), (\pi_1(g(\lambda{\sim}t)))\\ &= (\pi_1(\lambda g(t)), \pi_1({\sim}\lambda g(t)))\\ &= (\pi_1(g(t)), \pi_2(g(t)))\\ &= g(t).\end{aligned}$$

Thus, the morphism f is such that the diagram (1) commutes. It follows from the definition of κ that f is the unique morphism in $\mathcal{I}$MV with the above mentioned property. Therefore we conclude the following

Theorem 7. *There is an adjunction* $\kappa \dashv \mathsf{K}^\bullet : \mathcal{I}\mathrm{MV} \to \mathcal{I}\mathrm{MV}^\bullet$.

4. The Categorical Equivalence

In this section we restrict the category $\mathcal{I}\mathrm{MV}^\bullet$ in order to obtain a categorical equivalence between the restricted category and $\mathcal{I}$MV. We find a condition (ICK$^\bullet$) and we prove that it is equivalent to the existence of a partial function S which is defined by equations.

Moreover, we prove that the existence of S is equivalent to the existence of a new operation $\boxplus$ defined by equations in $\kappa(T) \times I_T$. We have that $Sx = \kappa t \boxplus y$, where $x = t \vee y$, $t = x \wedge \boldsymbol{c} \in (\boldsymbol{c}]$, $y \in I_T$, where $(\boldsymbol{c}]$ is the set of the elements that are least or equal to $\boldsymbol{c}$ and I_T is defined in Sect. 3.

But the condition (ICK$^\bullet$) is also equivalent to the existence of $\boxplus$ in $T \times T$, defined by equations, as we show in Proposition 4.

The algebra $\langle T, \boxplus, 0\rangle$ is a commutative monoid (Proposition 6).

Let $T \in \mathcal{I}\mathrm{MV}^\bullet$. We consider the following condition:

(ICK$^\bullet$) For any $z, w \geq \boldsymbol{c}$ such that $z * w \leq y \vee \boldsymbol{c}$ for some $y \in I_T$, there exists x such that $x \vee \boldsymbol{c} = z$ and ${\sim}x \vee \boldsymbol{c} = w$.

Remark 5. (a) If $T \in \mathrm{DRL}'$, then the conditions (CK$^\bullet$) and (ICK$^\bullet$) are equivalent.

(b) The condition (ICK$^\bullet$) is equivalent to the following one: for any $z, w \geq \boldsymbol{c}$ such that $(z * w) \vee \boldsymbol{c} = y \vee \boldsymbol{c}$ and $y \wedge \boldsymbol{c} = 0$ for some y, there is x such that $x \vee \boldsymbol{c} = z$ and ${\sim}x \vee \boldsymbol{c} = w$.

(c) If $(A, I) \in \mathcal{I}\mathrm{MV}$, then $\mathsf{K}^\bullet(A, I)$ satisfies (ICK$^\bullet$).

Let $T \in \mathcal{I}\mathrm{MV}^{\bullet}$. We define the following set:

$$M_T = \{x \in T : x \vee \boldsymbol{c} = y \vee \boldsymbol{c} \text{ for some } y \in I_T\}.$$

Remark 6. Let $x \in M_T$. Then, there exists $y \in I_T$ such that $x \vee \boldsymbol{c} = y \vee \boldsymbol{c}$. Let $t = x \wedge \boldsymbol{c}$. It is immediate that $x \vee \boldsymbol{c} = (y \vee t) \vee \boldsymbol{c}$, $x \wedge \boldsymbol{c} = (y \vee t) \wedge \boldsymbol{c}$, from where $x = y \vee t$. Conversely, let $x = y \vee t$, with $y \in I_T$ and $t \leq \boldsymbol{c}$. Then we have that $x \in M_T$ because $x \vee \boldsymbol{c} = y \vee \boldsymbol{c}$. Thus, we obtain that

$$M_T = \{x \in T : x = t \vee y, \text{ for some } t \leq \boldsymbol{c} \text{ and } y \in I_T\}.$$

Moreover, the previous decomposition is unique. Let $x = y \vee t = y' \vee t'$, with $y, y' \in I_T$ and $t, t' \leq \boldsymbol{c}$. Then, $x \wedge \boldsymbol{c} = (y \vee t) \wedge \boldsymbol{c} = t \wedge \boldsymbol{c} = t$. In the same way, $x \wedge \boldsymbol{c} = t'$. So, $t = t'$. Moreover, $y \vee \boldsymbol{c} = y' \vee \boldsymbol{c} = x \vee \boldsymbol{c}$ and $y \wedge \boldsymbol{c} = y' \wedge \boldsymbol{c} = 0$, from where $y = y'$.

Fot $T \in \mathcal{I}\mathrm{MV}^{\bullet}$ we define (if it is possible) a function $S : M_T \to T$ through the following equalities:

(S1) $Sx \wedge \boldsymbol{c} = x \wedge \boldsymbol{c}$.

(S2) $Sx \vee \boldsymbol{c} = (\boldsymbol{c} \wedge {\sim} x) \to x$.

Let $A \in \mathrm{MV}$. In $\mathsf{K}^{\bullet}(A)$, the map κ plays a crucial role. If we restrict to the elements least or equal $\boldsymbol{c}$, we have the inequality $x \leq \kappa x$. In the pairs, $(0, b) \leq (b \to 0, b)$. We can generalize this situation to $\mathsf{K}^{\bullet}(A, I)$, for $(A, I) \in \mathcal{I}\mathrm{MV}^{\bullet}$, by considering the inequality $(a, b) \leq (b \to a, b)$ with $a \in I$. This idea motivates the definition of the function S because $M_{\mathsf{K}^{\bullet}(A,I)} = \{(a, b) : a \in I\}$ and $S : M_{\mathsf{K}^{\bullet}(A,I)} \to \mathsf{K}^{\bullet}(A, I)$ is given by $S(a, b) = (b \to a, b)$. We are going to see that S also has an important role in $\mathsf{K}^{\bullet}(A, I)$.

We remark that when T has the form $\mathsf{K}^{\bullet}(A, I)$, the domain M_T of S is in bijection to the set $(\boldsymbol{c}] \times I_T$, where I_T is in this case equal to $I \times \{1\}$. In the particular case $I = \{0\}$, I_T is reduced to $\{(0, 1)\}$ and then the domain of S is $(\boldsymbol{c}] \times \{(0, 1)\}$, which is isomorphic to $(\boldsymbol{c}]$.

Thus, if we restrict to the case $I = \{0\}$, $(\mathrm{IC}\mathsf{K}^{\bullet})$ becomes $(\mathrm{C}\mathsf{K}^{\bullet})$ and S becomes κ. In fact, the operator κ is determined by its values in the set $(\boldsymbol{c}]$.

Lemma 8. *Let $T \in \mathcal{I}\mathrm{MV}^{\bullet}$. The following conditions are equivalent:*

1. *T satisfies* $(\mathrm{IC}\mathsf{K}^{\bullet})$.
2. *There exists a function $S : M_T \to T$ which satisfies* **(S1)** *and* **(S2)**.

Proof. 1. $\Rightarrow$ 2. Let $x \in M_T$. Then there is y such that $x \vee \boldsymbol{c} = y \vee \boldsymbol{c}$ and $y \wedge \boldsymbol{c} = 0$. We define $w = {\sim} x \vee \boldsymbol{c}$ and $z = {\sim} x \to (x \vee \boldsymbol{c})$. We have that $z * w \leq y \vee \boldsymbol{c}$. Besides $w \geq \boldsymbol{c}$, and $z \geq \boldsymbol{c}$ because ${\sim} x * (\boldsymbol{c} \wedge {\sim} x) \leq \boldsymbol{c}$.

Thus, there is t such that $t \vee \boldsymbol{c} = z$ and $\sim t \vee \boldsymbol{c} = w$. Put $Sx = t$. Hence, $Sx \wedge \boldsymbol{c} = \sim w = x \wedge \boldsymbol{c}$ and $Sx \vee \boldsymbol{c} = z = (\boldsymbol{c} \wedge \sim x) \to x$.

2. $\Rightarrow$ 1. Suppose that there exists S. Let $z, w \geq \boldsymbol{c}$ be such that $z * w \leq y \vee \boldsymbol{c}$ and $y \wedge \boldsymbol{c} = 0$ for some y. Since $(\sim w \vee y) \vee \boldsymbol{c} = y \vee \boldsymbol{c}$ and $y \wedge \boldsymbol{c} = 0$ then $\sim w \vee y \in M_T$. Then we define the element $x = z \wedge S(\sim w \vee y)$. We will prove that $x \vee \boldsymbol{c} = z$ and $\sim x \vee \boldsymbol{c} = w$.

The condition $z * (\sim y \wedge w) \leq z * w \leq y \vee \boldsymbol{c}$ implies the following inequality

$$z \leq (w \wedge \sim y) \to (y \vee \boldsymbol{c}). \tag{2}$$

Then using (2) we have that

$$\begin{aligned} x \vee \boldsymbol{c} &= (S(\sim w \vee y) \vee \boldsymbol{c}) \wedge (z \vee \boldsymbol{c}) \\ &= ((\boldsymbol{c} \wedge \sim y \wedge w) \to (y \vee \sim w)) \wedge z \\ &= ((w \wedge \sim y) \to (y \vee \boldsymbol{c})) \wedge z \\ &= z. \end{aligned}$$

Finally we have that

$$\begin{aligned} \sim x \vee \boldsymbol{c} &= \sim S(\sim w \vee y) \vee \boldsymbol{c} \\ &= \sim (S(\sim w \vee y) \wedge \boldsymbol{c}) \\ &= \sim ((\sim w \vee y) \wedge \boldsymbol{c}) \\ &= w. \end{aligned}$$

■

Remark 7. If $(A, I) \in \mathcal{I}\mathrm{MV}$ we define a map $\boxplus : \kappa(\mathsf{K}^\bullet(A, I)) \times I_{\mathsf{K}^\bullet(A,I)} \to \mathsf{K}^\bullet(A, I)$ by $\boxplus((\neg b, b), (d, 1)) = (\neg b \oplus d, b)$. The soundness of the definition follows from that $(\neg b \oplus d) \cdot b = (b \to d) \cdot b \leq d \in I$. Then, $(\neg b \oplus d, b) \in \mathsf{K}^\bullet(A, I)$.

Let $T \in \mathcal{I}\mathrm{MV}^\bullet$. Recall that for every $x, y \in T$ the operation $+$ is defined by $x + y = \sim(\sim x * \sim y)$.

Motivated by the Remark 7, for $T \in \mathcal{I}\mathrm{MV}^\bullet$ we define (if it is possible) a function $\boxplus : \kappa(T) \times I_T \to T$ through the following equalities (for $x \in T$ and $y \in I_T$):

($\boxplus\kappa$ **1**) $(\kappa x \boxplus y) \wedge \boldsymbol{c} = (\kappa x + y) \wedge \boldsymbol{c}$,

($\boxplus\kappa$ **2**) $(\kappa x \boxplus y) \vee \boldsymbol{c} = (\boldsymbol{c} * \kappa x) + (y \vee \boldsymbol{c})$.

Remark 8. Let $(A, I) \in \mathcal{I}\mathrm{MV}$. Then $(a, b) \in M_{\mathsf{K}^\bullet(A,I)}$ if and only if $a \in I$, and $y \in I_{\mathsf{K}^\bullet(A,I)}$ if and only if there exists $a \in I$ such that $y = (a, 1)$.

For every $(a, b) \in M_{\mathsf{K}^\bullet(A,I)}$ there exist $x \leq \boldsymbol{c}$ and $y \in I_{\mathsf{K}^\bullet(A,I)}$ such that $S(a, b) = \kappa x \boxplus y$. We can consider $x = (0, b)$ and $y = (a, 1)$.

Lemma 9. *Let $T \in \mathcal{I}\mathrm{MV}^\bullet$, $x \in T$ and $y \in I_T$. Then $\boldsymbol{c} * \sim y = \boldsymbol{c}$, $1 \to y = 0$, $y + \kappa x = \kappa x$ and $\kappa y = 0$.*

PROOF. Let $x \in T, y \in I_T$. Then $\boldsymbol{c} * \sim y = \boldsymbol{c} * (\sim y \vee \boldsymbol{c}) = \boldsymbol{c} * 1 = \boldsymbol{c}$, so $\boldsymbol{c} * \sim y = \boldsymbol{c}$. As $y \in I_T$ we obtain $(1 * \sim y) \vee \boldsymbol{c} = 1$ and $(1 * \sim y) \wedge \boldsymbol{c} = ((1 \wedge \boldsymbol{c}) * \sim y) \vee ((\sim y \wedge \boldsymbol{c}) * 1) = \boldsymbol{c} \vee ((\sim y \wedge \boldsymbol{c}) * 1) = \boldsymbol{c}$, so $1 * \sim y = \sim y$, i.e., $1 \to y = 0$. Besides, $y + \kappa x = \sim(\sim y * \sim \kappa x) = \sim(\sim y * (1 * \sim \kappa x)) = \sim((1 * \sim y) * \sim \kappa x) = \sim(1 * \sim \kappa x) = \kappa x$. Finally we have that $0 = \kappa(y \wedge \boldsymbol{c}) = \kappa y \wedge \kappa \boldsymbol{c} = \kappa y$. ∎

Note that

$$(\kappa x + y) \wedge \boldsymbol{c} = \kappa x \wedge \boldsymbol{c} = x \wedge \boldsymbol{c},$$
$$(\boldsymbol{c} * \kappa x) + (y \vee \boldsymbol{c}) = (\boldsymbol{c} \wedge x) + (y \vee \boldsymbol{c}).$$

Using the previous lemma, the following can be proved

LEMMA 10. *Let $T \in \mathcal{I}\mathrm{MV}^{\bullet}$. There exists S which satisfies* (**S1**) *and* (**S2**) *if and only if there exists $\boxplus$ which satisfies* ($\boxplus\kappa$**1**) *and* ($\boxplus\kappa$**2**).

PROOF. Suppose that there exists S. In first place, let $x \in T$, $x \leq \boldsymbol{c}$ and $y \in I_T$. We define $\kappa x \boxplus y = S(x \vee y)$. Then we have that

$$\begin{aligned}(\kappa x \boxplus y) \wedge \boldsymbol{c} &= (x \vee y) \wedge \boldsymbol{c} \\ &= x \qquad\qquad .\end{aligned}$$

Besides we have that

$$\begin{aligned}(\kappa x \boxplus y) \vee \boldsymbol{c} &= (\boldsymbol{c} \wedge \sim(x \vee y)) \to (x \vee y) \\ &= (\boldsymbol{c} \wedge \sim y) \to (x \vee y) \\ &= (\boldsymbol{c} * \kappa{\sim} y) \to (x \vee y) \\ &= \kappa{\sim} y \to (\boldsymbol{c} \to (x \vee y)) \\ &= \kappa{\sim} y \to (\boldsymbol{c} \vee \kappa(x \vee y)) \\ &= \kappa{\sim} y \to (\boldsymbol{c} \vee \kappa x) \\ &= \kappa{\sim} y \to (\boldsymbol{c} \to x) \\ &= (\boldsymbol{c} * \kappa{\sim} y) \to x.\end{aligned}$$

On the other hand,

$$\begin{aligned}x + (y \vee \boldsymbol{c}) &= \sim(\sim x * (\sim y \wedge \boldsymbol{c})) \\ &= \sim(\sim x * \kappa{\sim} y * \boldsymbol{c}) \\ &= (\boldsymbol{c} * \kappa{\sim} y) \to x \qquad .\end{aligned}$$

So, $(\kappa x \boxplus y) \vee \boldsymbol{c} = x + (y \vee \boldsymbol{c})$.

In second place, for any $x \in T$, we have that $\kappa x \boxplus y = \kappa(x \wedge \boldsymbol{c}) \boxplus y$. Then it suffices to define $\kappa x \boxplus y = S((x \wedge \boldsymbol{c}) \vee y)$ and the proof follows.

Conversely, let $x \in M_T$. Then there is $y \in I_T$ such that $x \vee \boldsymbol{c} = y \vee \boldsymbol{c}$. We define $Sx = \kappa x \boxplus y$. Then we have that

$$\begin{aligned}Sx \wedge \boldsymbol{c} &= (\kappa x \boxplus y) \wedge \boldsymbol{c} \\ &= x \wedge \boldsymbol{c}\end{aligned}$$

and

$$\begin{aligned} Sx \vee \boldsymbol{c} &= (\kappa x \boxplus y) \vee \boldsymbol{c} \\ &= (\boldsymbol{c} \wedge x) + (y \vee \boldsymbol{c}) \\ &= \sim((c \vee \sim x) * (\sim y \wedge \boldsymbol{c})) \\ &= \sim(\sim x * (\sim y \wedge \boldsymbol{c})) \\ &= (\boldsymbol{c} \wedge \sim y) \to x \\ &= (\boldsymbol{c} \wedge \sim x) \to x. \end{aligned}$$

■

Remark 9. For $(A, I) \in \mathcal{I}\mathrm{MV}$ we define a map $\boxplus : \mathsf{K}^\bullet(A, I) \times \mathsf{K}^\bullet(A, I) \to \mathsf{K}^\bullet(A, I)$ by $\boxplus((a, b), (d, e)) = (a \oplus d, b \cdot e)$. In order to prove the well definition of this map, let $i = a \cdot b$ and $j = d \cdot e$. Hence $k = i \oplus j \in I$, and $a \oplus d \leq (b \to i) \oplus (e \to j) = (\neg b \oplus i) \oplus (\neg e \oplus j) = \neg(b \cdot e) \oplus k = (b \cdot e) \to k$. Thus, $(a \oplus d) \cdot b \cdot e \leq ((b \cdot e) \to k) \cdot b \cdot e \leq k \in I$. Therefore, $(a \oplus d) \cdot b \cdot e \in I$.

Motivated by the Remark 9, we will extend the domain of $\boxplus$ to $T \times T$.

For $T \in \mathcal{I}\mathrm{MV}^\bullet$ we define (if it is possible) a function $\boxplus : T \times T \to T$ through the following equalities:

($\boxplus$**1**) $(x \boxplus y) \wedge \boldsymbol{c} = (x + y) \wedge \boldsymbol{c}$,

($\boxplus$**2**) $(x \boxplus y) \vee \boldsymbol{c} = (\boldsymbol{c} * x) + (y \vee \boldsymbol{c})$.

Remark 10. The second equation above seems to be "asymmetric" in the variables x and y. But

$$(\boldsymbol{c} * x) + (y \vee \boldsymbol{c}) = (\boldsymbol{c} * y) + (x \vee \boldsymbol{c}).$$

Indeed,

$$\begin{aligned} (\boldsymbol{c} * x) + (y \vee \boldsymbol{c}) &= \sim(\sim(\boldsymbol{c} * x) * (\sim y \wedge \boldsymbol{c})) \\ &= \sim((\boldsymbol{c} \to \sim x) * (\kappa{\sim}y * \boldsymbol{c})) \\ &= \sim((\boldsymbol{c} * (\boldsymbol{c} \to \sim x)) * \kappa{\sim}y) \\ &= \sim((\boldsymbol{c} \wedge \sim x) * \kappa{\sim}y) \\ &= \sim((\kappa{\sim}x * \boldsymbol{c}) * \kappa{\sim}y) \\ &= \boldsymbol{c} \to (\lambda x + \lambda y). \end{aligned}$$

As we can see, in the last equation there is not any distinction between x and y.

Then we have the following

Proposition 4. *Let $T \in \mathcal{I}\mathrm{MV}^\bullet$. Then T satisfies the condition* (ICK$^\bullet$) *if and only if there exists the map $\boxplus : T \times T \to T$ which satisfies* ($\boxplus$**1**) *and* ($\boxplus$**2**).

PROOF. Suppose that T satisfies the condition (ICK$^\bullet$). By Corollary 11 we have that the map $\beta_T : T \to \mathsf{K}^\bullet(\kappa(T), J_T)$ is an isomorphism, so by Remark 9 we have that there is $\hat{\boxplus}$ in $\mathsf{K}^\bullet(\kappa(T), J_T)$ which satisfies the conditions ($\boxplus$**1**) and ($\boxplus$**2**). We shall prove that the map $\boxplus : T \times T \to T$ is given by $z \boxplus w = \beta_T^{-1}(x \hat{\boxplus} y)$, where $z = \beta_T^{-1}(x)$ and $w = \beta_T^{-1}(y)$. First note that β_T^{-1} preserves all the operations of the algebra (in particular, it preserves $+$ because this operation is defined using $*$ and $\sim$). Thus we have that

$$\begin{aligned}(z \boxplus w) \wedge \boldsymbol{c} &= (\beta_T^{-1}(x \hat{\boxplus} y)) \wedge \boldsymbol{c}\\ &= \beta_T^{-1}(x \hat{\boxplus} y) \wedge \beta_T^{-1}(\boldsymbol{c})\\ &= \beta_T^{-1}((x \hat{\boxplus} y) \wedge \boldsymbol{c})\\ &= \beta_T^{-1}((x + y) \wedge \boldsymbol{c})\\ &= (\beta_T^{-1}(x) + \beta_T^{-1}(y)) \wedge \beta_T^{-1}(\boldsymbol{c})\\ &= (z + w) \wedge \boldsymbol{c}.\end{aligned}$$

Besides we have that

$$\begin{aligned}(z \boxplus w) \vee \boldsymbol{c} &= (\beta_T^{-1}(x \hat{\boxplus} y)) \vee \boldsymbol{c}\\ &= \beta_T^{-1}(x \hat{\boxplus} y) \vee \beta_T^{-1}(\boldsymbol{c})\\ &= \beta_T^{-1}((x \hat{\boxplus} y) \vee \boldsymbol{c})\\ &= \beta_T^{-1}((x * \boldsymbol{c}) + (y \vee \boldsymbol{c}))\\ &= (\beta_T^{-1}(x) * \beta_T^{-1}(\boldsymbol{c})) + (\beta_T^{-1}(y) \vee \beta_T^{-1}(\boldsymbol{c}))\\ &= (z * \boldsymbol{c}) + (w \vee \boldsymbol{c}).\end{aligned}$$

Conversely, if there exists $\boxplus : T \times T \to T$ which satisfies ($\boxplus$**1**) and ($\boxplus$**2**), then there exists $\boxplus : \kappa(T) \times I_T \to T$ which satisfies ($\boxplus\kappa$**1**) and ($\boxplus\kappa$**2**). Therefore, by Proposition 5 we have that T satisfies the condition (ICK$^\bullet$). ∎

PROPOSITION 5. *Let $T \in \mathcal{IMV}^\bullet$, and let β_T be the morphism given in Theorem 5. The following conditions are equivalent:*

(1) *β_T is a surjective map.*

(2) *For every $z, w \in \kappa(T)$ such that $z * w = \lambda y$ and $y \wedge \boldsymbol{c} = 0$ for some $y \in T$, there is $x \in T$ such that $\lambda x = z$ and $\sim\kappa x = w$.*

(3) *T satisfies* (ICK$^\bullet$).

(4) *There exists a function $S : M_T \to T$ that satisfies* (**S1**) *and* (**S2**).

(5) *There exists a map $\boxplus : \kappa(T) \times I_T \to T$ that satisfies* ($\boxplus\kappa$**1**) *and* ($\boxplus\kappa$**2**).

(6) *There exists a map $\boxplus : T \times T \to T$ that satisfies* ($\boxplus$**1**) *and* ($\boxplus$**2**).

PROOF. The equivalence between (1) and (2) follows from the definition of β_T.

(2) $\Rightarrow$ (3). Let $z, w \geq \boldsymbol{c}$ such that $(z * w) \vee \boldsymbol{c} = y \vee \boldsymbol{c}$ and $y \wedge \boldsymbol{c} = 0$, for some $y \in T$. We define $\overline{z} = \lambda z$ and $\overline{w} = \lambda w$. It is immediate that $\overline{z} * \overline{w} = \lambda y$. Thus, there is $x \in T$ such that $\lambda x = \overline{z}$ and $\sim\kappa x = \overline{w}$. Hence, $x \vee \boldsymbol{c} = z$ and $\sim x \vee \boldsymbol{c} = w$. Thus, we have shown the condition (3).

(3) $\Rightarrow$ (2). Let $z, w \in \kappa(T)$ such that $z * w = \lambda y$ and $y \wedge \boldsymbol{c} = 0$, for some $y \in T$. We define $t = z \vee \boldsymbol{c}$ and $u = w \vee \boldsymbol{c}$. We have that $t, u \geq \boldsymbol{c}$. As $c * (z \vee w) = z \vee w$, we obtain and $t * u = (z * w) \vee \boldsymbol{c} = \lambda y \vee \boldsymbol{c} = y \vee \boldsymbol{c}$. Then, there is $x \in T$ such that $x \vee \boldsymbol{c} = z \vee \boldsymbol{c}$ and $\sim x \vee \boldsymbol{c} = w \vee \boldsymbol{c}$. It follows from properties of κ the fact that $\lambda x = z$ and $\sim\kappa x = w$. Hence, we have proved the condition (2).

The equivalence between (3) and (4) follows from Lemma 8, and the equivalence between (4) and (5) follows from Lemma 4. Finally, the equivalence between (3) and (6) follows from Proposition 4. ■

Therefore we conclude the following

COROLLARY 11. *There is an equivalence between the category* $\mathcal{I}\mathrm{MV}$ *and the the full subcategory of* $\mathcal{I}\mathrm{MV}^\bullet$ *whose objects satisfy the conditions* (2)*,* (3)*,* (4)*,* (5) *or* (6) *of Proposition 5.*

Let $f : T \to U$ be a morphism in the category $\mathcal{I}\mathrm{MV}^\bullet$ where the objects satisfy the condition ($\mathsf{ICK}^\bullet$). Then, by the proposition above, there exist the binary operations $\boxplus$ and $\hat{\boxplus}$ on T and U (respectively). Straightforward computations show that for every $x, y \in T$ we have that $f(x \boxplus y) \wedge \boldsymbol{c} = (f(x) \hat{\boxplus} f(y)) \wedge \boldsymbol{c}$ and $f(x \boxplus y) \vee \boldsymbol{c} = (f(x) \hat{\boxplus} f(y)) \vee \boldsymbol{c}$, so $f(x \boxplus y) = f(x) \hat{\boxplus} f(y)$. Then we obtain the following

COROLLARY 12. *The class of algebras of* $\mathcal{I}\mathrm{MV}^\bullet$ *which satisfy* ($\mathsf{ICK}^\bullet$) *is a variety if we consider* $\boxplus$ *in the signature of the algebras. Moreover, the category associated to the previous variety is equivalent to* $\mathcal{I}\mathrm{MV}$*.*

5. Final Remarks

In this section we are looking for answers for the following questions:

(1) When does an object of $\mathcal{I}\mathrm{MV}^\bullet$ satisfy the equation $x * 1 = x$?

(2) What properties are satisfied by the binary operation $\boxplus$?

We start with the following

LEMMA 13. *Let* $T \in \mathcal{I}\mathrm{MV}^\bullet$ *and* $x \in T$*. Then* $x \leq x * 1$*.*

Proof. Let $x \in T$. Thus, by Corollary 2 we have that

$$\begin{aligned}(x * 1) \wedge \boldsymbol{c} &= ((x \wedge \boldsymbol{c}) * 1) \vee ((1 \wedge \boldsymbol{c}) * x)\\ &= (\boldsymbol{c} * (\boldsymbol{c} \to x) * 1) \vee (\boldsymbol{c} * x)\\ &= (\boldsymbol{c} * (\boldsymbol{c} \to x)) \vee (\boldsymbol{c} * x)\\ &= (\boldsymbol{c} \wedge x) \vee (\boldsymbol{c} * x) \geq c \wedge x.\end{aligned}$$

Besides we have that $((x * 1) \vee \boldsymbol{c}) * \boldsymbol{c} = (x * 1) * \boldsymbol{c} = x * \boldsymbol{c} = (x \vee \boldsymbol{c}) * \boldsymbol{c}$. By distributivity we conclude that $x \leq x * 1$. ∎

Lemma 14. *Let $T \in \mathcal{I}\mathrm{MV}^\bullet$ and $x \in T$. The following conditions are equivalent:*

1. $\boldsymbol{c} * x \leq x$.
2. $\boldsymbol{c} * x \leq c \wedge x$.
3. $x \to x = 1$, *i.e.*, $x * 1 \leq x$.

Proof. 1. ⇒ 2. We have that $\boldsymbol{c} * x \leq x$ and $\boldsymbol{c} * x \leq \boldsymbol{c}$, i.e., $c * x \leq \boldsymbol{c} \wedge x$.

2. ⇒ 1. It is immediate.

1. ⇒ 3. We have that $\boldsymbol{c} \vee (x \to x) = x \to x$, i.e., $0 = c \wedge (x * {\sim} x) = x * {\sim} x$. Hence, $x \to x = 1$.

3. ⇒ 1. It is immediate. ∎

For $T \in \mathcal{I}\mathrm{MV}^\bullet$ and $x \in T$, we have that $x * 1 \leq x$ if and only if $x * 1 = x$. Then we have the following

Corollary 15. *Let $T \in \mathcal{I}\mathrm{MV}^\bullet$. Then $T \in \mathrm{MV}^\bullet$ if and only if for any x it holds the condition (1), (2) or (3) of Lemma 14.*

We end this paper doing some final considerations about the operation $\boxplus : T \times T \to T$.

Lemma 16. *Let $T \in \mathcal{I}\mathrm{MV}^\bullet$. Then for every $x, y \in T$ we have the following conditions:*

1. $(\boldsymbol{c} * x) + (y \vee \boldsymbol{c}) = \boldsymbol{c} \to (\lambda x + \lambda y) = \boldsymbol{c} \vee (\lambda x + \lambda y)$.
2. *If there exists the map $\boxplus$ then $\lambda(x \boxplus y) = \lambda x \boxplus \lambda y$.*

Proof. 1. Let $x, y \in T$. It follows from Remark 10 that $(\boldsymbol{c} * x) + (y \vee \boldsymbol{c}) = \boldsymbol{c} \to (\lambda x + \lambda y)$. Besides

$$\begin{aligned}\boldsymbol{c} \to (\lambda x + \lambda y) &= \boldsymbol{c} \vee \kappa(\lambda x + \lambda y)\\ &= \boldsymbol{c} \vee (\kappa\lambda x + \kappa\lambda y)\\ &= \boldsymbol{c} \vee (\lambda x + \lambda y).\end{aligned}$$

2. Let $x, y \in T$. We have that

$$\begin{aligned}\lambda(x \boxplus y) \wedge \boldsymbol{c} &= \sim(\boldsymbol{c} \vee \kappa{\sim}(x \boxplus y)) \\ &= \sim(\boldsymbol{c} \to \sim(x \boxplus y)) \\ &= \boldsymbol{c} * (x \boxplus y) \\ &= \boldsymbol{c} * (\boldsymbol{c} \vee (x \boxplus y)) \\ &= \boldsymbol{c} * (\boldsymbol{c} \to (\lambda x + \lambda y)) \\ &= \boldsymbol{c} \wedge (\lambda x + \lambda y) \\ &= \boldsymbol{c} \wedge (\lambda x \boxplus \lambda y),\end{aligned}$$

and

$$\begin{aligned}\lambda(x \boxplus y) \vee \boldsymbol{c} &= \sim(\boldsymbol{c} \wedge \kappa{\sim}(x \boxplus y)) \\ &= \sim(\boldsymbol{c} \wedge \sim(x \boxplus y)) \\ &= \boldsymbol{c} \vee (x \boxplus y) \\ &= \boldsymbol{c} \to (\lambda x + \lambda y) \\ &= \boldsymbol{c} \to (\lambda\lambda x + \lambda\lambda y) \\ &= (\lambda x \boxplus \lambda y) \vee \boldsymbol{c}.\end{aligned}$$

Hence, $\lambda(x \boxplus y) = \lambda x \boxplus \lambda y$. ■

PROPOSITION 6. *Let $T \in \mathcal{I}\mathrm{MV}^{\bullet}$ such that there exists $\boxplus$. Then $\langle T, \boxplus, 0\rangle$ is a commutative monoid.*

PROOF. It is immediate that $\boxplus$ is commutative.

Let $x, y, z \in T$. We will prove that $\boxplus$ is associative.

First we have that

$$\begin{aligned}(x \boxplus (y \boxplus z)) \wedge \boldsymbol{c} &= (x + (y \boxplus z)) \wedge \boldsymbol{c} \\ &= \boldsymbol{c} * (c \to (x + (y \boxplus z))) \\ &= \boldsymbol{c} * \sim(\boldsymbol{c} * \sim x * \sim(y \boxplus z)) \\ &= \boldsymbol{c} * \sim(\boldsymbol{c} * \sim x * (\boldsymbol{c} \vee \sim(y \boxplus z))) \\ &= \boldsymbol{c} * \sim(\boldsymbol{c} * \sim x * \sim(\boldsymbol{c} \wedge (y \boxplus z))) \\ &= \boldsymbol{c} * \sim(\boldsymbol{c} * \sim x * \sim(\boldsymbol{c} \wedge (y + z))) \\ &= \boldsymbol{c} * \sim(\boldsymbol{c} * \sim x * (\boldsymbol{c} \vee (\sim y * \sim z))) \\ &= \boldsymbol{c} * \sim(\boldsymbol{c} * \sim x * (\sim y * \sim z)).\end{aligned}$$

By this reason we also have that

$$\begin{aligned}((x \boxplus y) \boxplus z) \wedge \boldsymbol{c} &= (z \boxplus (x \boxplus y)) \wedge \boldsymbol{c} \\ &= \boldsymbol{c} * \sim(\boldsymbol{c} * \sim z * (\sim x * \sim y)).\end{aligned}$$

Thus, using the associativity and commutativity of $*$ we obtain

$$(x \boxplus (y \boxplus z)) \wedge \boldsymbol{c} = ((x \boxplus y) \boxplus z) \wedge \boldsymbol{c}. \tag{3}$$

Second, we have that

$$\begin{aligned}(x \boxplus (y \boxplus z)) \vee \boldsymbol{c} = \boldsymbol{c} &\to (\lambda x + \lambda(y \boxplus z)) \\ &= \sim(\boldsymbol{c} * \sim\lambda x * \sim\lambda(y \boxplus z)) \\ &= \sim(\boldsymbol{c} * \sim\lambda x * (\boldsymbol{c} \vee \sim\lambda(y \boxplus z))) \\ &= \sim(\boldsymbol{c} * \sim\lambda x * \sim(\boldsymbol{c} \wedge \lambda(y \boxplus z))) \\ &= \sim(\boldsymbol{c} * \sim\lambda x * \sim(\boldsymbol{c} \wedge (\lambda y \boxplus \lambda z))) \\ &= \sim(\boldsymbol{c} * \sim\lambda x * \sim(\boldsymbol{c} \wedge (\lambda y + \lambda z))) \\ &= \sim(\boldsymbol{c} * \sim\lambda x * (\sim\lambda y * \sim\lambda z)).\end{aligned}$$

Besides we have that

$$\begin{aligned}((x \boxplus y) \boxplus z) \vee \boldsymbol{c} &= (z \boxplus (x \boxplus y)) \vee \boldsymbol{c} \\ &= \sim(\boldsymbol{c} * \sim\lambda z * (\sim\lambda x * \sim\lambda y)).\end{aligned}$$

Hence, using again the associativity and commutativity of $*$ we obtain

$$(x \boxplus (y \boxplus z)) \vee \boldsymbol{c} = ((x \boxplus y) \boxplus z) \vee \boldsymbol{c}. \tag{4}$$

Then by Eqs. (3) and (4) we obtain $x \boxplus (y \boxplus z) = (x \boxplus y) \boxplus z$.
Finally we will prove that $x \boxplus 0 = x$ for every $x \in T$. We have that

$$\begin{aligned}(x \boxplus 0) \wedge \boldsymbol{c} &= (x + 0) \wedge \boldsymbol{c} \\ &= \boldsymbol{c} * (\boldsymbol{c} \to (x + 0)) \\ &= \boldsymbol{c} * \sim(\boldsymbol{c} * (\sim x) * 1) \\ &= \boldsymbol{c} * \sim(\boldsymbol{c} * \sim x) \\ &= \boldsymbol{c} * (\boldsymbol{c} \to x) \\ &= x \wedge \boldsymbol{c},\end{aligned}$$

and

$$\begin{aligned}(x \boxplus 0) \vee \boldsymbol{c} = \boldsymbol{c} &\to (\lambda x + 0) \\ &= \boldsymbol{c} \to \lambda x \\ &= \sim(\boldsymbol{c} * \kappa\sim x) \\ &= \sim(\boldsymbol{c} \wedge \sim x) \\ &= x \vee \boldsymbol{c}.\end{aligned}$$

Therefore, we obtain that $x \boxplus 0 = x$. ■

Acknowledgements. This work was supported by Consejo Nacional de Investigaciones Científicas y Tecnológicas, Argentina [PIP 112-201101-00636] and Universidad Nacional de La Plata [11/X667].

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