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Strong cliques and equistability of EPT graphs[☆]

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ABSTRACT

In this paper, we characterize the equistable graphs within the class of EPT graphs, the edge-intersection graphs of paths in a tree. This result generalizes a previously known characterization of equistable line graphs. Our approach is based on the combinatorial features of triangle graphs and general partition graphs. We also show that, in EPT graphs, testing whether a given clique is strong is co-NP-complete. We obtain this hardness result by first showing hardness of the problem of determining whether a given graph has a maximal matching disjoint from a given edge cut. As a positive result, we prove that the problem of testing whether a given clique is strong is polynomial in the class of local EPT graphs, which are defined as the edge intersection graphs of paths in a star and are known to coincide with the line graphs of multigraphs.

1. Introduction

We consider finite simple undirected graphs. A graph $G = (V, E)$ is *threshold* if there exists a weight function $w : V \to \mathbb{N}$ and a threshold $t \in \mathbb{N}$, such that a subset $S \subseteq V$ is a stable set of G if and only if $\sum_{v\in S} w(v) \le t$ [8] (a stable set in a graph is a subset of pairwise non-adjacent vertices). In 1980, Payan introduced equistable graphs as a generalization of threshold graphs: A graph $G = (V, E)$ is called *equistable* if and only if there exists a positive integer t and an *equistable weight function*, that is, a weight function $w : V \to \mathbb{N}$ on the vertices of G such that a subset $S \subseteq V$ is an (inclusion-wise) maximal stable set of G if and only if $\sum_{v\in S} w(v) = t$ [34].

Let us illustrate the definition with two examples. Consider the 4-vertex path, P_4, with vertex set $V = \{v_1, v_2, v_3, v_4\}$ and edge set $\{v_iv_{i+1} \mid 1 \le i \le 3\}$. Since the sets $\{v_1, v_4\}$, $\{v_1, v_3\}$, and $\{v_2, v_4\}$ are all maximal stable sets of the P_4, any equistable weight function $w : V \to \mathbb{N}$ of the P_4 has to satisfy $w(v_1) + w(v_4) = w(v_1) + w(v_3) = w(v_2) + w(v_4) = t$ (for some threshold t), which implies the equation $w(v_2) + w(v_3) = t$, meaning that the non-stable set $\{v_2, v_3\}$ of the P_4 has the same weight as the maximal stable sets. It follows that the P_4 is not equistable. On the other hand, extending the P_4 to the

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bull graph, that is, adding a new vertex, say v_5, and making it adjacent to precisely the two midpoints of the P_4, results in an equistable graph. An equistable weight function of the bull graph is given for example by $w(v_1) = 2$, $w(v_2) = 3$, $w(v_3) = 5$, $w(v_4) = 4$, $w(v_5) = 1$ (the corresponding threshold is $t = 7$).

Equistable graphs were studied in a series of papers [7,20,22–25,27,29–31,34,35]. However, while threshold graphs are well understood, admit several characterizations, efficient recognition algorithms, and efficient algorithms for many optimization problems [26], this is not the case for equistable graphs:

- The set of equistable graphs is not closed under vertex deletion (as shown by the above example).
- Verifying whether a given weight function on the vertices of a graph G is an equistable weight function of G is co-NP-complete [30].
- Finding the maximum cardinality of a stable set or the minimum cardinality of a maximal stable set in an equistable graph is APX-hard [30].
- The complexity status of recognizing equistable graphs is open.
- No characterization of equistable graphs in terms of their combinatorial properties is known.

The only positive known algorithmic result related to the recognition of general equistable graphs is the result of Levit et al. [25] stating that it can be verified in polynomial time if a given graph admits an equistable weight function with weights bounded by a fixed positive integer. (This was recently improved to an FPT algorithm [20].) An application of equistable graphs in parallel computing was given by Korach et al. [23].

In [27], Mahadev et al. introduced a subclass of equistable graphs, the so-called strongly equistable graphs. For a graph G, we denote by $\mathcal{S}(G)$ the set of all maximal stable sets of G, and by $\mathcal{T}(G)$ the set of all other nonempty subsets of $V(G)$. A graph $G = (V, E)$ is said to be *strongly equistable* if for each $T \in \mathcal{T}(G)$ and each $\gamma \leq 1$ there exists a weight function $w : V \to \mathbb{R}_+$ such that $w(S) = 1 \text{forall} S \in \mathcal{S}(G)$, and $w(T) \neq \gamma$. Mahadev et al. showed that every strongly equistable graph is equistable, and conjectured that the converse assertion is valid. The conjecture is known to hold for a class of graphs containing all perfect graphs [27], for series–parallel graphs [22], for line graphs [24], for AT-free graphs [29], and for various product graphs [29]. However, a counterexample to the conjecture of Mahadev et al. was recently found in [31] in the class of the complements of the line graphs of triangle-free graphs (see Fig. 1).

Although no characterizations of equistable graphs in terms of their combinatorial properties are known, there are some necessary and some sufficient conditions of combinatorial flavor for a graph to be equistable. Following [33], we say that a graph is a *triangle graph* if it satisfies the following.

Triangle condition. For every maximal stable set S in $G = (V, E)$ and every edge uv in $G - S$ there is a vertex $s \in S$ such that $\{u, v, s\}$ induces a triangle in G.

The triangle condition was introduced by McAvaney et al. in [28], who proved that all general partition graphs (see below) satisfy the condition. In the equistable graphs literature, a condition equivalent to the triangle condition was also used, expressed in terms of induced 4-vertex paths and maximal stable sets. We say that an induced 4-vertex path $P_4(a, b, c, d)$ in a graph G is *bad* if there exists a maximal stable set S in G containing a and d such that no vertex from S is adjacent both to b and c. Mahadev et al. [27] proved that if G is equistable, then G contains no bad P_4. Equivalently, every equistable graph is a triangle graph [29]. The converse inclusion does not hold; not every triangle graph is equistable. Examples of triangle non-equistable graphs can be found in [6,29]; see also Fig. 1.

Another combinatorially defined graph class related to equistable graphs is the class of general partition graphs. A graph $G = (V, E)$ is a *general partition graph* if there exists a set U and an assignment of non-empty subsets $U_x \subseteq U$ to the vertices of G such that two vertices x and y are adjacent if and only if $U_x \cap U_y \neq \emptyset$, and for every maximal stable set S of G, the set $\{U_x : x \in S\}$ is a partition of U. General partition graphs arise in the geometric setting of lattice polygon triangulations [12] and were studied in a series of papers [2,9–11,21,41].

A *strong clique* in a graph G is a clique (that is, a set of pairwise adjacent vertices) that intersects all maximal stable sets. McAvaney, Robertson and DeTemple proved in [28] that a graph G is general partition if and only if every edge of G is contained in a strong clique. This result together with results from [27,29] implies that every general partition graph is strongly equistable.

The above mentioned graph classes are related as follows:

$$\textit{general partition graphs} \subset \textit{strongly equistable graphs} \subset \textit{equistable graphs} \subset \textit{triangle graphs}. \tag{1}$$

Some examples of graphs showing that the three inclusions are strict are depicted in Fig. 1.

For some graph classes $\mathcal{C}$ some (or all) of the above inclusions become equalities. Typically, such an equivalency is proved by showing that every triangle graph in $\mathcal{C}$ is a general partition graph. This approach exploits the combinatorial features of triangle graphs and of general partition graphs (instead of the more algebraically-flavored defining properties of equistable and strongly equistable graphs). Clearly, whenever the triangle graphs coincide with the general partition graphs within a class $\mathcal{C}$, we obtain as a corollary also the equality of strongly equistable and equistable graphs within $\mathcal{C}$. This is the case when $\mathcal{C}$ is any of the following graph classes: chordal graphs [35], series–parallel graphs [22], line graphs [24], simplicial graphs [24], very well-covered graphs [24], AT-free graphs [29], nontrivial Cartesian products of graphs [29], nontrivial deleted lexicographic products of graphs where the base is a triangle-free graph [29], and complements of line graphs of forests [6]. Moreover, the first two inclusions in (1) are equalities for complements of line graphs of bipartite graphs [6], and the last inclusion is an equality for distance-hereditary graphs [23].

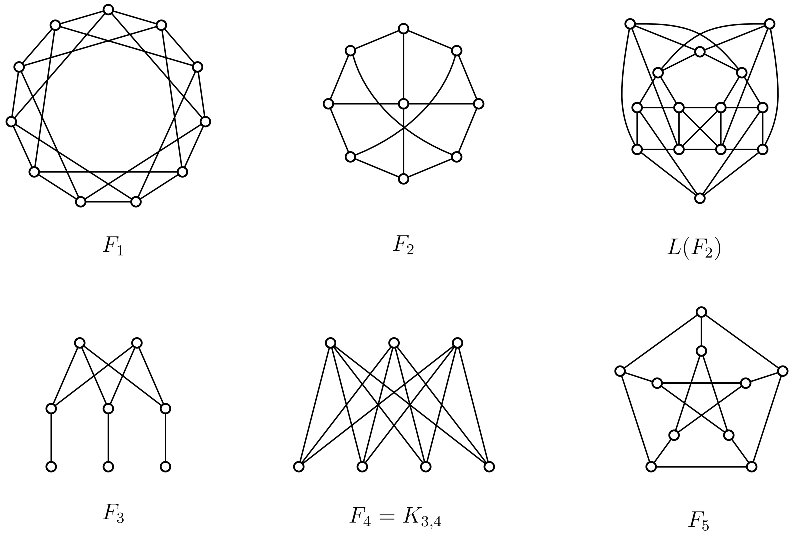

Fig. 1. F_1: A 11-vertex, 22-edge graph such that the complement of its line graph is strongly equistable but not general partition (see [31], where the example is also generalized to an infinite family). F_2: A 9-vertex, 14-edge graph such that the complement of its line graph is equistable but not strongly equistable (see [31]); its line graph is also shown. F_3, F_4, F_5: three graphs such that the complements of their line graphs are triangle but not equistable (see [6,29], where F_3 and F_4 are also generalized to infinite families).

Since interval graphs are chordal, the results of [35] imply that within the class of interval graphs, all the three inclusions in (1) are equalities. Furthermore, the results of [24] imply the same conclusion for the class of line graphs of multigraphs. This motivates the study of the properties in the inclusion chain (1) for the class of *EPT graphs*, a common generalization of interval graphs and of line graphs of multigraphs. EPT graphs were introduced by Golumbic and Jamison [15] as the edge intersection graphs of undirected paths in an undirected tree. Note that interval graphs are exactly the edge intersection graphs of subpaths of a path. Moreover, line graphs of multigraphs were proved to exactly coincide with the so-called *local EPT graphs*, the edge intersection graphs of undirected paths in a star. (See Section 4.2 for further details, including the definition of line graphs of multigraphs.) Golumbic and Jamison showed that recognizing EPT graphs is NP-complete [16]. EPT graphs were studied in a series of papers [1,3–5,17,18,32,37,38].

Our results can be summarized as follows:

In Section 2, we show two simple properties of cliques in EPT graphs.

In Section 3, we show that within EPT graphs, all the four classes of graphs appearing in chain of inclusions (1) coincide. This implies several equivalent characterizations of equistable EPT graphs.

In Section 4, we discuss some algorithmic and complexity issues related to the problem of recognizing EPT equistable graphs. Using the results from Section 3, the problem is reduced to testing whether a given clique in an EPT graph is strong. We prove that this problem is co-NP-complete, thus leaving open the complexity status of determining if a given EPT graph is equistable. As a positive result, we show that the problem of testing whether a given clique is strong is polynomial time solvable in the subclass of local EPT graphs. As a corollary, the problem of testing whether a given clique is strong is also polynomial in the class of line graphs of multigraphs.

As a byproduct, we prove that the problem of deciding if a given graph has a maximal matching such that each of its edges has either both or none of its endpoints in a given subset of vertices, is NP-complete even for bipartite graphs of maximum degree at most 5.

We use standard graph theoretic terminology, see, e.g., [13].

2. Properties of cliques in EPT graphs

In this section we derive two simple properties of cliques in EPT graphs which will be needed later.

We start with some formal definitions related to EPT graphs. Let $\mathcal{P}$ be a multiset of nontrivial simple paths in a host tree T. The *edge intersection graph* $\mathrm{EPT}(T, \mathcal{P})$ is the graph whose vertex set is $\mathcal{P}$, and two paths are adjacent in $\mathrm{EPT}(T, \mathcal{P})$ if and only if they share at least one common edge. An undirected graph G is called an *edge intersection graph of paths in a tree,* or an *EPT graph*, if $G \cong \mathrm{EPT}(T, \mathcal{P})$ for some pair $(T, \mathcal{P})$, where $\cong$ denotes the graph isomorphism relation. In this case we also refer to $(T, \mathcal{P})$ as an *EPT representation* of G. In what follows we denote the vertices of G also by lowercase letters u, v, u_1, v_1, etc.; and, given a vertex v, denote by P_v the path in $\mathcal{P}$ representing v. Let $G = \mathrm{EPT}(T, \mathcal{P})$. For an edge $e \in E(T)$, let K_e be the subset of $V(G)$ defined by

$$K_e = \{v \in V(G) : e \in E(P_v)\};$$

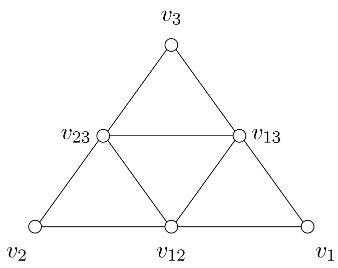

Fig. 2. The 3-sun S_3.

and for a *claw* Y in T (that is, a subgraph of T induced by a vertex and 3 distinct neighbors of it), let K_Y be the subset of $V(G)$ defined by

$$K_Y = \{v \in V(G) : |E(P_v) \cap E(Y)| = 2\}.$$

The vertex y of T incident with all edges in $E(Y)$ is also called the *center* of Y. Clearly, both K_e and K_Y induce cliques of G. The following result is [16, Theorem 1.1]:

Theorem 1 (*Golumbic and Jamison [16]*). *Let $G = \mathrm{EPT}(T, \mathcal{P})$, and K be a clique in G. Then $K \subseteq K_e$ for some edge $e \in E(T)$, or $K \subseteq K_Y$ for some claw Y in T.*

The clique K in the above theorem is called an *edge clique* if $K \subseteq K_e$ for some $e \in E(T)$, and a *claw clique* if this is not the case. Notice that, for a claw Y in T, K_Y is a claw clique if and only if every pair of edges of Y is covered by some path P_v, $v \in K_Y$.

Given an EPT graph G and a clique K in G, one may ask whether there exists an EPT representation of G in which K is an edge clique. In this context, we prove the following proposition. The *3-sun* is the graph S_3 depicted in Fig. 2.

Proposition 1. *Let G be an EPT graph, let H be an induced subgraph of G isomorphic to S_3, and let U be the set of all vertices of H of degree 4 in H. Then, U is a claw clique in every EPT representation of G.*

Proof. Assume for a contradiction that $G \cong \mathrm{EPT}(T, \mathcal{P})$ for some T and $\mathcal{P}$ such that $U \subseteq K_e$ for some $e \in E(T)$. Let us write

$$U = \{v_{12}, v_{13}, v_{23}\} \quad \text{and} \quad V(H) \backslash U = \{v_1, v_2, v_3\},$$

where $v_{ij} \in U$ is adjacent to exactly v_i and v_j from $V(H) \backslash U$ (see Fig. 2).

None of the paths P_{v_i} passes through edge e. Indeed, if we had, say, $e \in E(P_{v_1})$, then e would be a common edge of P_{v_1} and $P_{v_{23}}$, contrary to the fact that v_1 is not adjacent to v_{23} in G. It follows that at least two of the paths P_{v_i} must belong to the same connected component of the graph $T - e$. Without loss of generality we may assume that P_{v_1} and P_{v_2} belong to the same component. Now, let x_1 be the vertex of $V(P_{v_1}) \cap V(P_{v_{12}})$ closest to e, and x_2 be the vertex of $V(P_{v_2}) \cap V(P_{v_{12}})$ closest to e. Using the fact that P_{v_1} and P_{v_2} are edge-disjoint, and $P_{v_{12}} \in K_e$, we see that $x_1 \neq x_2$. Without loss of generality we may assume that x_1 is closer to e than x_2. Since $e \in E(P_{v_{23}})$ and the path $P_{v_{23}}$ has an edge in common with P_{v_2}, $P_{v_{23}}$ contains the subpath of T connecting x_2 and e. Consequently, $P_{v_{23}}$ contains the (non-trivial) common subpath of P_{v_1} and $P_{v_{12}}$, which implies that v_{23} is adjacent to v_1 in G, a contradiction. □

Now, we turn to (inclusion-wise) maximal cliques. It can be easily seen that every claw clique is maximal, but this is not always true for edge cliques, not even for the ones of the form K_e. This can happen for two reasons: either $K_e \subset K_f$ for some $f \in E(T) \backslash \{e\}$ or $K_e \subset K_Y$ for some claw Y in T. The first possibility leads to the following definition.

Definition 1. A clique K_e of a graph $\mathrm{EPT}(T, \mathcal{P})$ is *edge-maximal* if for every $e' \in E(T)$, $K_e \subseteq K_{e'}$ implies that $e = e'$.

Notice that the definition of edge-maximal clique depends on the given EPT representation of the graph. For example, if G is the complete graph K_3, then G can be represented either:

- with the host tree T being a 3-vertex path with vertices x, y, and z, and with $\mathcal{P}$ consisting of three paths each containing the three vertices x, y, and z, or
- with the host tree T being the 4-vertex star with center w and leaves x, y, and z, and $\mathcal{P}$ consisting of three paths (x, w, y), (y, w, z), and (z, w, x).

In the former EPT representation, the two cliques K_{xy} and K_{yz} coincide, hence neither of them is edge-maximal; they are both maximal in G. In the latter one, there are three cliques of the form K_e, namely K_{wx}, K_{wy}, and K_{wz}. Each of them is edge-maximal, however none of them is maximal in G.

As the above example shows, there exist EPT representations with no edge-maximal cliques. However, any EPT graph admits an EPT representation in which every clique K_e is edge-maximal (for instance, any representation using a host tree with minimum number of edges). The above example also shows that there exist EPT representations in which every clique K_e is edge-maximal but not maximal. However, as the next proposition shows, every edge-maximal clique that is not maximal is (properly) contained in a claw clique (cf. the proof of Theorem 2).

Proposition 2. *Let $G = \mathrm{EPT}(T, \mathcal{P})$ and $e \in E(T)$ such that K_e is edge-maximal but not maximal. Then $K_e \subset K_Y$ for a claw clique K_Y such that $e \in E(Y)$.*

Proof. Let K be a maximal clique with $K_e \subset K$. By Theorem 1, either $K = K_{e'}$ for some edge $e' \in E(T)$ or K is a claw clique. The first case is excluded since K_e is edge-maximal. Let Y be the claw in T such that the claw clique $K = K_Y$. We must show that $e \in E(Y)$. Let us write $E(Y) = \{e_1, e_2, e_3\}$, and let y be the center of Y. Every path in K_e contains e, but also, being a path of K_Y, includes y and hence also the whole path that goes from y to e (and including e as its last edge); we denote this path by P^*. Since y is not an internal vertex of P^*, we see that P^* contains exactly one of the edges in $\{e_1, e_2, e_3\}$, say e_1. We conclude that $K_e \subseteq K_{e_1}$, and since K_e is edge-maximal, this implies that $e = e_1 \in E(Y)$. □

3. Triangle EPT graphs are general partition graphs

In this section we prove our main result, which gives several equivalent characterizations of equistable EPT graphs.

Theorem 2. *For every EPT graph G, the following conditions are equivalent.*

(i) *Every edge of G is contained in a strong clique.*
(ii) *G is general partition.*
(iii) *G is strongly equistable.*
(iv) *G is equistable.*
(v) *G is triangle.*

Note that the equivalence (i) $\Leftrightarrow$ (ii) as well as the chain of implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) holds for all graphs (not necessarily EPT). This follows from results in [27–29] (see the Introduction for more details). To show Theorem 2 it thus suffices to prove that EPT graphs satisfy the implication (v) $\Rightarrow$ (i). The proof of Theorem 2 will appear at the end of this section and will be based on one lemma about triangle graphs, and three lemmas about triangle EPT graphs, to be proved below. Our first lemma can be, in fact, deduced from the proof of [2, Theorem 1], but for completeness we add a proof here.

Lemma 1. *Let G be a triangle graph, K a maximal clique of G that is not strong, and let S be a maximal stable set S such that $K \cap S = \emptyset$. Then G contains an induced subgraph H isomorphic to the 3-sun S_3 such that the set U of all vertices of H of degree 4 in H satisfies $U \subseteq K$ and $V(H) \backslash U \subseteq S$.*

Proof. Let $v_1 \in S$ be a vertex maximizing $|N(v_1) \cap K|$. By the triangle condition, this maximum is at least two. Since K is maximal, there is a vertex in K that is not adjacent to v_1. Let v_{23} denote such a vertex. The triangle condition implies that for every edge $uv_{23} \in E(G)$ with $u \in K \cap N(v_1)$, there is a vertex $v_2 \in S$ adjacent to both u and v_{23}. Choose a pair (u, v_2) as above so that $|N(v_1) \cap N(v_2) \cap K|$ is maximized. By the choice of v_1 and since $v_{23} \in K \cap (N(v_2) \backslash N(v_1))$, there exists a vertex $v_{13} \in K \cap (N(v_1) \backslash N(v_2))$. By the triangle condition, there exists a vertex $v_3 \in S$ adjacent to both v_{13} and v_{23}.

Since $v_{13} \in N(v_1) \cap N(v_3) \cap K$ but v_{13} is not adjacent to v_2, the choice of v_2 implies that there exists a vertex $v_{12} \in N(v_1) \cap N(v_2) \cap K$ that is not adjacent to v_3. But now, the vertex set $\{v_1, v_2, v_3, v_{12}, v_{13}, v_{23}\}$ induces a desired copy of S_3 in G. □

Lemma 2. *Let $G = \mathrm{EPT}(T, \mathcal{P})$ be a triangle graph, and let $e = xy \in E(T)$ such that K_e is edge-maximal, and $K_e \subset K_Y$, where Y is a claw in T with center y, and K_Y is a claw clique that is not strong. Then:*

(i) *There exist vertices $v_1, v_2 \in K_e$ such that $E(P_{v_1}) \cap E(P_{v_2}) = \{e\}$.*
(ii) *$K_e = K_X \cap K_Y$ for some claw X in T with center x.*
(iii) *K_X is a claw clique.*

Proof. Let $E_x \subseteq E(T)$ be the set of edges incident with x but not with y, and let r denote the number of edges in E_x which are covered by some path P_v, $v \in K_e$. Let $\{e_1, \ldots, e_r\}$ be the set of edges in E_x covered by some path P_v, $v \in K_e$, and let f_1 and f_2 be the two edges of Y different from e.

Proof of (i). Property (i) holds even without the assumption that G satisfies the triangle property. As proved in [16, Theorem 1], a finite collection of closed intervals on a line has strong Helly number 2. This means that, if we denote by I the intersection of all intervals in the collection, then there exist two intervals in the collection, say I_1 and I_2, such that $I_1 \cap I_2 = I$. We can apply this observation as follows. Since $K_e \subset K_Y$, each path of K_e contains exactly one of f_1 and f_2. If $r = 0$, then we can take any two vertices $v_1, v_2 \in K_e$ such that $f_i \in E(P_{v_i})$ for $i \in \{1, 2\}$. Notice that such vertices do exist because K_Y is a claw clique. Suppose now that $r \geq 1$. Let P_0 be the subpath of T formed by the edges e_1, e and f_1. Consider the subpaths $P_0 \cap P_v$ where v runs over the set K_e. This results in a collection of subpaths of P_0, which can thus be seen as a collection of intervals. Clearly, each of these subpaths covers e. Moreover, it can be shown using the edge-maximality of K_e that the intersection of all intervals corresponds to the subpath induced by e.

By the above discussion, there exist distinct $v_1, v_2 \in K_e$ such that $E(P_{v_1} \cap P_0) \cap E(P_{v_2} \cap P_0) = \{e\}$. We claim that we may also assume that f_1 is covered by exactly one of the paths P_{v_1} and P_{v_2}.

Indeed, if this is not the case, we have $E(P_{v_1}) \cap E(P_0) = \{e\}$ and $E(P_{v_2}) \cap E(P_0) \in \{\{e\}, \{e, e_1\}\}$. Since E_Y is a claw clique and $K_e \subset K_Y$, then there exists an element $v \in K_e$ such that $f_1 \in E(P_v)$. Thus, we just replace v_2 with v.

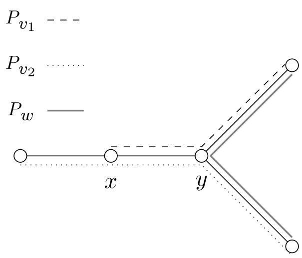

Fig. 3. The claw Y with center y and paths P_{v_1}, P_{v_2}, P_w.

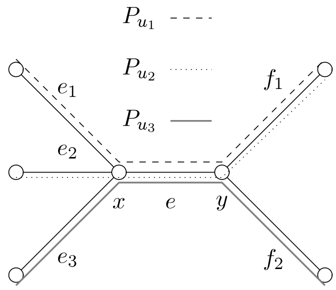

Fig. 4. The claw Y with center y and paths $P_{u_1}, P_{u_2}, P_{u_3}$.

Finally, we will show that we may also assume that e_1 is covered by exactly one of the paths P_{v_1} and P_{v_2}.

Indeed, if this is not the case, we have $E(P_{v_1}) \cap E(P_0) = \{e\}$ and $E(P_{v_2}) \cap E(P_0) = \{e, f_1\}$. Since $e_1 \in E_x$, there exists an element $w \in K_e$ such that $e_1 \in E(P_w)$. And, since $K_e \subset K_Y$, we have that $f_1 \in E(P_w)$ or $f_2 \in E(P_w)$. In the former case, we replace v_2 with w; and in the latter, we replace v_1 with w. This completes the proof of (i).

Before proving (ii), we first show that

$$E(P_v) \cap E_x \neq \emptyset \quad \text{for every } v \in K_e. \tag{2}$$

If not, then there is a vertex $v_1 \in K_e$ such that P_{v_1} ends at vertex x. Since $K_e \subset K_Y$, $v_1 \in K_Y$ also holds. Furthermore, since K_Y is a claw clique, there exists a vertex $v_2 \in K_e$ such that $E(P_{v_1}) \cap E(Y) \neq E(P_{v_2}) \cap E(Y)$. Part of the paths P_{v_1} and P_{v_2} are depicted in Fig. 3, where P_{v_2} possibly ends at vertex x.

Since K_Y is not strong, there is a maximal stable set S such that $K_Y \cap S = \emptyset$. Applying the triangle condition to v_1v_2 and S, we find a vertex $w \in S$ which is adjacent to both v_1 and v_2. It follows that $e \in E(P_w)$ or $E(Y)\backslash\{e\} \subseteq E(P_w)$ (in the latter case P_w is shown in the picture). In either case we find that $w \in K_Y$ (in the first case we use that $K_e \subseteq K_Y$), contradicting that $K_Y \cap S = \emptyset$.

Proof of (ii). Recall that r denotes the number of edges in E_x which are covered by some path P_v, $v \in K_e$. It follows from (2) that in order to finish the proof of part (ii) of the lemma it is sufficient to show that $r = 2$. Observe that $r \neq 1$ because of (2) and the fact that K_e is edge-maximal.

Suppose that $r \geq 3$. Consider the bipartite graph B with parts $\{e_1, \ldots, e_r\}$ and $\{f_1, f_2\}$ in which e_i is adjacent to f_j if and only if there exists a path P_v with $v \in K_e$ and $\{e_i, f_j\} \subseteq E(P_v)$. Since $K_e \subset K_Y$, every vertex e_i is of degree at least 1 in B. By (2), every vertex in $\{f_1, f_2\}$ is of degree at least 1 in B. It follows that B has a matching M of size 2. By adding to M one edge incident with a vertex in $\{e_1, \ldots, e_r\}$ not covered by the matching, we see that we may assume that $\{e_1f_1, e_2f_1, e_3f_2\} \subseteq E(B)$ (by renaming the vertices of B if necessary). That is, $e_i \in E(P_{u_i})$ for some $u_i \in K_e \subset K_Y$, where P_{u_1} and P_{u_2} cover the same two edges of Y (namely e and f_1) whereas P_{u_3} bifurcates in y, see Fig. 4.

Now, apply the triangle condition to u_1u_3 and S. This results in a vertex $w_{13} \in S$, adjacent to both u_1 and u_3. It follows that $P_{w_{13}}$ cannot share any edge with Y, and therefore, $\{e_1, e_3\} \subseteq E(P_{w_{13}})$ (see Fig. 4). By the same reason S has a vertex w_{23} which is adjacent to both u_2 and u_3, and also $\{e_2, e_3\} \subseteq E(P_{w_{23}})$. These give, however, that $e_3 \in E(P_{w_{13}}) \cap E(P_{w_{23}})$, which, since $w_{13} \neq w_{23}$, implies that w_{13} and w_{23} are adjacent, contradicting that both are from S.

Proof of (iii). In order to show that K_X is a claw clique we have to find a path $P \in \mathcal{P}$ for which $E(P) \cap E(X) = \{xx_1, xx_2\}$. Recall that S is a maximal stable set such that $S \cap K_Y = \emptyset$. Apply the triangle condition to the edge v_1v_2, where v_1, v_2 are as in (i), and the maximal stable set S. This results in a vertex $w \in S$ adjacent to both v_1 and v_2. Since $w \notin K_Y$ and $E(P_{v_1}) \cap E(P_{v_2}) = \{e\}$, the path P_w will be a good choice for P. The lemma is proved. □

Lemma 3. *Let $G = \mathrm{EPT}(T, \mathcal{P})$ be a triangle graph, and let $e = xy \in E(T)$ such that K_e is edge-maximal, and $K_e = K_X \cap K_Y$, where X and Y are claws in T with center x and y respectively, and K_X and K_Y are both claw cliques. Then K_X or K_Y is strong.*

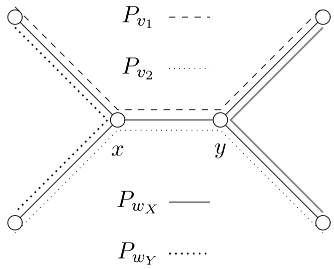

Fig. 5. The claws X, Y, and paths P_{v_1}, P_{v_2}, P_{w_X}, P_{w_Y}.

Proof. Assume, towards a contradiction, that none of K_X and K_Y is strong. In particular, Lemma 2 applies to G. Let v_1 and v_2 be the vertices in K_e described in Lemma 2(i). The claws X, Y together with the parts of the paths P_{v_1} and P_{v_2} are depicted in Fig. 5.

Since K_X and K_Y are not strong, there exist maximal stable sets S_X and S_Y such that $S_X \cap K_X = S_Y \cap K_Y = \emptyset$. Let $S'_X \subseteq S_X$ be the set of those vertices $v \in S_X$ for which P_v lies entirely in the component of $T - e$ containing x. Since $v_1, v_2 \in K_X$, these two vertices are not in S_X, and we may apply the triangle condition to the edge v_1v_2 and S_X. This gives a vertex $w_X \in S_X$, which is adjacent to both v_1 and v_2. It follows that $w_X \in K_Y \backslash K_e$, see Fig. 5. Let us repeat the above argument with the claw clique K_Y. This gives rise to a subset $S'_Y \subseteq S_Y$ consisting of those vertices $v \in S_Y$ for which P_v lies in the component of $T - e$ containing y, and to a vertex $w_Y \in S_Y$ for which $w_Y \in K_X \backslash K_e$, see Fig. 5.

Let S be a maximal stable set of G which contains $S'_X \cup S'_Y$. We claim that S contains a vertex t from K_e. This follows immediately if one of v_1 and v_2 is in S, hence we may assume that $\{v_1, v_2\} \cap S = \emptyset$. Then applying the triangle condition to the edge v_1v_2 and the maximal stable set S, we find a vertex in S which is adjacent to both v_1 and v_2. Let t be such a vertex. Clearly, $t \in K_X \cup K_Y$ (see Fig. 5). Assume that $t \notin K_e$. Since $K_e = K_X \cap K_Y$, we may assume that $t \in K_Y \backslash K_e$ (the other possibility, that is, when $t \in K_X \backslash K_e$, can be treated in the same manner). This means that P_t lies entirely in the component of $T - e$ that contains y. Consequently, the set $(S_Y \backslash S'_Y) \cup \{t\}$ is stable since $S_Y \cap K_Y = \emptyset$. On the other hand, since $S'_Y \subset S$ and $t \in S$, $S'_Y \cup \{t\}$ is also stable, and we conclude that $S_Y \cup \{t\}$ is a stable set. This contradicts the fact that S_Y is a maximal stable set, and therefore $t \in K_e$.

Since $t \in K_e$, $t \in K_Y$. Also, $w_X \in K_Y$, and from this we can conclude that t and w_X are adjacent, and none of them is in S_Y. Applying the triangle condition to the edge tw_X and S_Y, we find a vertex $s \in S_Y$ which is adjacent to both t and w_X. Since $s \in S_Y$, we have $s \notin K_Y$, which implies $s \notin K_e$, and so the path P_s lies entirely in one of the two components of $T - e$. Using also that s is adjacent to $w_X \in K_Y$, it can be seen that this will be the component that contains y, i.e., $s \in S'_Y$. This gives that $s \in S$, which, together with the facts that s is adjacent to t and $t \in S$, is in contradiction with the fact that S is a stable set. The lemma is proved. □

Lemma 4. *Let $G = \mathrm{EPT}(T, \mathcal{P})$ be a triangle graph, and let $e \in E(T)$ such that K_e is edge-maximal. Then one of the following holds.*

(i) *K_e is maximal, and it is strong.*
(ii) *$K_e \subset K_Y$ for some strong claw clique K_Y such that $e \in E(Y)$.*

Proof. Suppose first that K_e is maximal. If K_e is not strong, then it follows from Lemma 1 that G contains an induced subgraph H isomorphic to S_3 such that $U \subseteq K_e$ where U is the set of all vertices of H of degree 4 in H. Thus U is an edge clique in G. This is, however, impossible by Proposition 1. Therefore K_e is strong, and (i) follows.

Suppose next that K_e is not maximal. By Proposition 2, $K_e \subset K_Y$ for a claw clique K_Y such that $e \in E(Y)$. Now, Lemmas 2 and 3 yield that, either K_Y is strong, or $K_e = K_X \cap K_Y$ for a strong claw clique K_X. In either case (ii) holds, and this completes the proof of the lemma. □

Proof of Theorem 2. Recall that the chain of implications (i) $\Rightarrow \cdots \Rightarrow$ (v) holds for all graphs. To see that (v) implies (i), let G be a triangle EPT graph. We choose among all possible EPT representations of G one using a host tree with minimum number of edges, namely $(T, \mathcal{P})$. Thus, it is clear that if $e \in E(T)$, then K_e is edge-maximal. Let $uv \in E(G)$ be an arbitrary edge of G, and $e' \in E(T)$ be an edge common to the two paths P_u and P_v. By the previous remark $K_{e'}$ is edge-maximal, thus Lemma 4 implies that either $K_{e'}$ is a strong clique (if it is maximal), or otherwise $K_{e'}$ is contained in a strong claw clique. In either case, uv is contained in a strong clique of G (recall that u and v belong to $K_{e'}$). Since the choice of edge uv was arbitrary, this proves the implication. □

4. Algorithmic and complexity issues

As stated in the Introduction, the complexity status of recognizing equistable graphs is open. It is therefore an interesting question to determine the complexity status of the problem of determining whether a given EPT graph is equistable. Recognizing EPT graphs was proved to be NP-complete by Golumbic and Jamison [16], therefore we assume that the graph is given by an EPT representation. The result in [30, Theorem 7] shows that the following problem is co-NP-complete: given a graph G and a weight function $w : V(G) \to \mathbb{N}$ on its vertices, is w an equistable weight function of G? The family of graphs appearing in that hardness proof is extremely simple: it consists of the graphs consisting of disjoint edges (that is, nK_2 for $n \geq 1$). Since these graphs are EPT, this immediately implies that the following problem is co-NP-complete: *given an EPT representation of a graph G, and a weight function* $w : V(G) \to \mathbb{N}$ *on its vertices, is* w *an equistable weight function of G?* This result seems to indicate that any potential polynomial recognition algorithm for equistable EPT graphs would have to rely on the structural properties of equistable graphs, as even the 'correctness' of equistable weight functions is hard to verify.

By Theorem 2, a given EPT graph is equistable if and only if every edge of it is contained in a strong clique. Since EPT graphs have only polynomially many maximal cliques (cf. Theorem 1), this suggests that in order to test whether an EPT graph G given by an EPT representation is equistable, one could enumerate (in polynomial time) all the maximal cliques of G, test for each of them whether it is strong, and check whether the strong cliques cover all edges. Thus, the problem is reduced to testing whether a given clique in an EPT graph is strong. Formally, the decision problem of testing whether a given clique in a graph is strong can be stated as follows:

Strong Clique	
Input:	A graph G and a clique K in G.
Question:	Is K strong?

For general graphs, Zang proved that the Strong Clique problem is co-NP-complete [40]. (Also, we remark that a related result that the problem of testing whether a graph contains a strong clique is NP-hard [19].) We show in this section that, unfortunately, the problem of testing whether a given clique is strong remains co-NP-complete also for EPT graphs, thus leaving open the complexity status of determining if a given EPT graph is equistable (equivalently: general partition/strongly equistable/triangle).

The decision problem of testing whether a clique in an EPT graph is strong can be formally stated as follows:

Strong EPT Clique	
Input:	An EPT representation $(T, \mathcal{P})$ of an EPT graph G, a clique K in G.
Question:	Is K strong?

The Strong Edge Clique problem is the problem obtained from the Strong EPT Clique problem by imposing the additional restriction on the clique K as being an edge clique of G. In Section 4.1 we prove that the Strong Edge Clique problem is NP-complete even if the host tree T is of diameter 3. Consequently, the same is true for the Strong EPT Clique problem. Then, in Section 4.2, we show that this result is sharp, in the sense that if T is of diameter at most 2, then the problem can be solved in polynomial time.

4.1. Testing strong cliques in EPT graphs

Our hardness proof is done in two steps. First, we introduce and show hardness of a new matching problem, the problem of testing whether a given graph has a maximal matching disjoint from a given edge cut. Formally, given a graph G and a subset $X \subseteq V(G)$, we say that a matching M in G is *separated by* X if every edge of M is either contained in X or disjoint from it.

Separated Maximal Matching	
Input:	A graph G and a subset of vertices $X \subseteq V$.
Question:	Does G contain a maximal matching separated by X?

Theorem 3. *The* Separated Maximal Matching *problem is NP-complete, even for bipartite graphs of maximum degree at most* 5.

Proof. The problem is clearly in NP, as testing whether a given matching is separated by X can be done in polynomial time. The hardness reduction is from the 3-SAT problem in which every variable occurs four times, a problem showed NP-hard by Tovey [39].

Assume given a 3-SAT formula F on variables $x_1, \dots, x_n$ and clauses $c_1, \dots, c_m$ in which each variable occurs four times. For $j = 1, \dots, m$, we denote by $\ell_j{}^1$, $\ell_j{}^2$ and $\ell_j{}^3$ the three literals occurring in clause c_j.

We build a bipartite graph $G_F = (A, B; E)$ as follows (see Fig. 6):

- $A = \{x_1, \dots, x_n\} \cup \bigcup_{j=1}^{m}\{\ell_j{}^1, \ell_j{}^2, \ell_j{}^3\}$ (disjoint union),
- $B = \cup_{i=1}^{n}\{x_i{}^T, x_i{}^F\} \cup \{c_1{}^1, \dots, c_m{}^1\} \cup \{c_1{}^2, \dots, c_m{}^2\}$ and

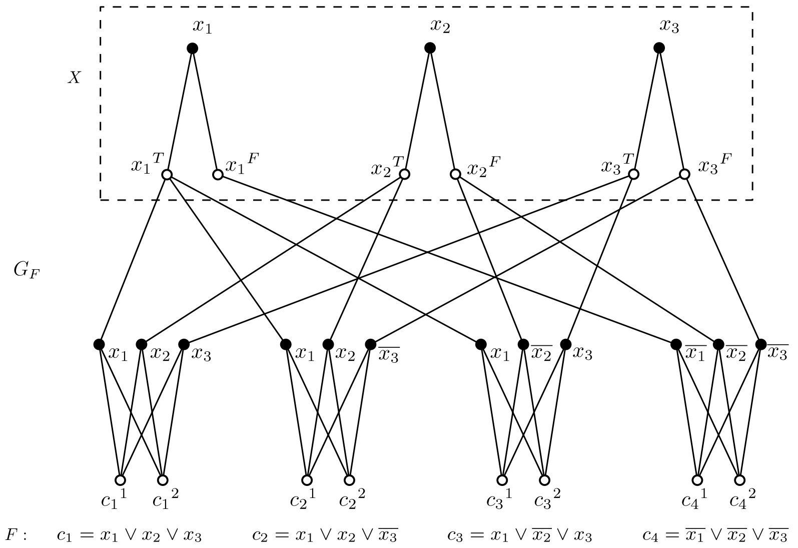

Fig. 6. An example construction of the bipartite graph G_F and its vertex subset X from a formula F. In the example, the formula has variables x_1, x_2, x_3 and clauses $c_1 = x_1 \vee x_2 \vee x_3$, $c_2 = x_1 \vee x_2 \vee \overline{x_3}$, $c_3 = x_1 \vee \overline{x_2} \vee x_3$, and $c_4 = \overline{x_1} \vee \overline{x_2} \vee \overline{x_3}$.

- $E = E_1 \cup E_2$ with

$$E_1 = \bigcup_{i=1}^{n} \{x_i x_i^T, x_i x_i^F\} \cup \bigcup_{j=1}^{m} \{\ell_j^1 c_j^1, \ell_j^2 c_j^1, \ell_j^3 c_j^1, \ell_j^1 c_j^2, \ell_j^2 c_j^2, \ell_j^3 c_j^2\}$$

depending only on m and n whereas the instance specific part E_2 is given by

$$\begin{aligned} E_2 = &\{x_i^T \ell_j^p : 1 \le i \le n,\ 1 \le j \le m,\ \ell_j^p \text{ is the positive literal } x_i\} \\ &\cup \{x_i^F \ell_j^p : 1 \le i \le n,\ 1 \le j \le m,\ \ell_j^p \text{ is the negative literal } \overline{x_i}\}. \end{aligned}$$

Finally, we let $X = \cup_{i=1}^{n} \{x_i, x_i^T, x_i^F\}$.

Since every variable occurs in F four times, G_F is of maximum degree at most 5. Clearly, the above transformation can be carried out in polynomial time. The proof of the theorem will therefore follow from the two lemmas below.

Lemma 5. *If F is satisfiable, then there exists a maximal matching M of G separated by X.*

Proof. Let ϕ be a satisfying truth assignment for F. For every $i = 1, \ldots, n$, put in M the edge $x_i x_i^T$ if $\phi(x_i) = \mathit{true}$ and otherwise the edge $x_i x_i^F$ if $\phi(x_i) = \mathit{false}$. For every $j = 1, \ldots, m$, let ℓ_j^p and ℓ_j^q be two literals of clause c_j such that c_j evaluates to *true* under ϕ on the remaining literal. Then put in M the edges $c_j^1 \ell_j^p$ and $c_j^2 \ell_j^q$. By construction, M is a matching of G separated by X. Also, M is a maximal matching of G since G is bipartite and the only nodes left exposed by M in the A color class of G are all nodes ℓ_j^r corresponding to literals evaluating to *true* under ϕ. These nodes have no exposed neighbor in B since all nodes c_j^1 and c_j^2 are covered, as well as all the nodes of the form x_i^T such that $\phi(x_i) = \mathit{true}$ and all the nodes of the form x_i^F such that $\phi(x_i) = \mathit{false}$. □

Lemma 6. *If there exists a maximal matching M of G separated by X, then F is satisfiable.*

Proof. For every $i = 1, \ldots, n$, $M \cap \{x_i x_i^T, x_i x_i^F\} \neq \emptyset$ by the maximality of M, by the fact that x_i has degree only 2, and by the fact that these two edges incident with x_i are the only edges incident to x_i^T or x_i^F which may possibly belong to M as implied by the fact that every edge of M is either contained in X or disjoint from it. This fact allows us to define a truth assignment ϕ by setting $\phi(x_i) = \mathit{true}$ if $x_i x_i^T \in M$ and $\phi(x_i) = \mathit{false}$ if $x_i x_i^F \in M$. The arguments in the above proof can now be reversed to show that this ϕ is indeed a satisfying truth assignment. Indeed, for every $j = 1, \ldots, m$, let $p \in \{1, 2, 3\}$ be such that neither $\ell_j^p c_j^1$ nor $\ell_j^p c_j^2$ belongs to M. Such a p must exist since M is a matching. Then, since M is maximal, it must be the case that the corresponding literal in F must evaluate to true. □

This completes the proof of Theorem 3. □

Theorem 4. *The* Strong Edge Clique *problem is co-NP-complete, even if T is a tree of diameter* 3.

Proof. The problem is in co-NP, since a maximal stable set S disjoint from a given edge clique K is a short certificate for the fact that K is not strong.

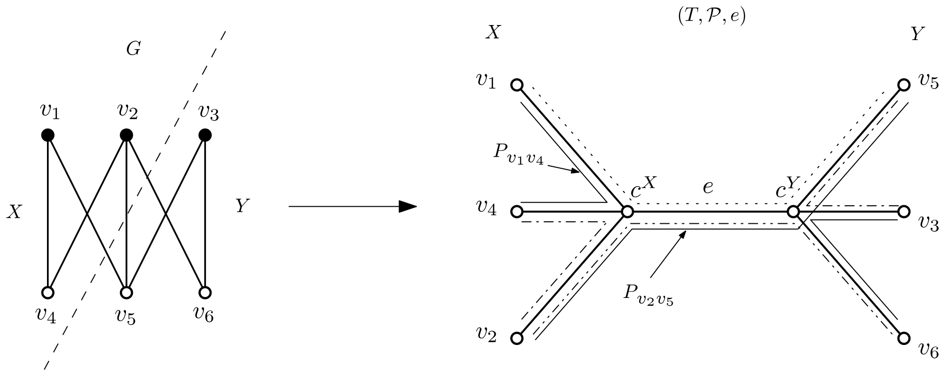

Fig. 7. An example construction of the instance $(T, \mathcal{P}, K_e)$ of the Strong Edge Clique problem from a bipartite graph G with a given vertex subset X. For clarity, only two path names are displayed.

To show hardness, we reduce from the Separated Maximal Matching problem. Let (G, X) be an instance to the Separated Maximal Matching problem where G is a graph, and $X \subseteq V(G)$. We will now show how to construct, in polynomial time, an instance $(T, \mathcal{P}, K_e)$ of the Strong Edge Clique problem such that G contains a maximal matching separated by X if and only if the edge clique K_e is not strong in $H = \text{EPT}(T, \mathcal{P})$.

Let $V = V(G)$ and $Y = V \setminus X$.

- The tree T has $V(G) \cup \{c^X, c^Y\}$ as vertex set, where c^X and c^Y are two new vertices. The edge set of T is $E(T) = \{vc^X \mid v \in X\} \cup \{c^Xc^Y\} \cup \{c^Yv \mid v \in Y\}$.
- The collection $\mathcal{P}$ of subpaths of T consists of the following paths:
  - For every edge $uv \in E(G)$ such that $\{u, v\} \subseteq X$ or $\{u, v\} \subseteq Y$, add to $\mathcal{P}$ the length-two subpath in T connecting u and v, denoted by P_{uv}. (That is, $E(P_{uv}) = \{uc^X, c^Xv\}$ if $\{u, v\} \subseteq X$, and $E(P_{uv}) = \{uc^Y, c^Yv\}$, otherwise.) Such paths will be referred to as *internal paths*, and edges of G generating them *internal edges*.
  - For every edge $uv \in E(G)$ such that $u \in X$ and $v \in Y$, add to $\mathcal{P}$ the length-three subpath in T connecting u and v, denoted again by P_{uv}. (That is, $E(P_{uv}) = \{uc^X, c^Xc^Y, c^Yv\}$.) Such paths will be referred to as *crossing paths*, and edges of G generating them *crossing edges*.
- Set K_e to be the edge clique corresponding to the edge $e = c^Xc^Y$ of T.
- Set $H = \text{EPT}(T, \mathcal{P})$. For simplicity, we will identify the paths P_{uv} with vertices of H in the rest of the proof.

An example construction is shown in Fig. 7.

Clearly, the above transformation can be carried out in polynomial time. Notice that the edge $e = c^Xc^Y$ is dominating in T, and that for every edge $vw \in E(G)$, we have $P_{uv} \in K_e$ if and only if P_{uv} is a crossing path. Moreover, the graph H is isomorphic to the graph obtained from the line graph of G by turning the set of vertices corresponding to the crossing edges into a clique. (For the definition of a line graph, we refer to Section 4.2.)

It remains to show that G contains a maximal matching separated by X if and only if the edge clique K_e is not strong in H.

Suppose first that G contains a maximal matching M separated by X. Let S denote the subcollection of $\mathcal{P}$ consisting of all paths of the form P_{uv} where $uv \in M$. Since every path corresponding to a vertex in K_e is a crossing path, while all paths in S are internal, we have $S \cap K_e = \emptyset$. We claim that S is a maximal stable set in H. Since M is a matching, S is a stable set. Moreover, maximality of M implies maximality of S. Indeed, if S were not maximal, then there would exist an edge $uv \in E(G)$ such that path P_{uv} was edge-disjoint from all paths in S, and no matter whether uv is internal or crossing, it could be added to M to obtain a bigger matching. Thus, S is a maximal stable set in H disjoint from K_e, and hence K_e is not strong.

Conversely, if K_e is not strong, then there exists a collection S of pairwise edge-disjoint internal paths that forms a maximal stable set of H. Reversing the above arguments, it can be seen that the set of edges uv of G such that $P_{uv} \in S$ forms a maximal matching separated by X in G. This completes the proof. □

4.2. Testing strong cliques in local EPT graphs and in line graphs of multigraphs

In this section, we show that the result of Theorem 4 is sharp, in the sense that the Strong EPT Clique problem can be solved in polynomial time on instances such that T is a tree of diameter at most 2. Such instances are easily seen to be equivalent to the special class of EPT graphs known as *local EPT graphs*, defined in [16] as graphs having a *local EPT representation*, that is, an EPT representation $(T, \mathcal{P})$ such that all paths in $\mathcal{P}$ share a common vertex.

Our proof will rely on the following characterization of local EPT graphs.

Theorem 5 (*Golumbic and Jamison [16]*). *A graph G is a local EPT graph if and only if G is the line graph of a multigraph.*

Recall that given two graphs G and H, we say that G is the *line graph* of H, and write $G = L(H)$, if the vertex set of G is the edge set of H, and two distinct edges of H are adjacent as vertices of G if and only if they have a common endpoint. If this is the case, then graph H is called a *root graph* of G. A graph G is said to be a *line graph* if $G \cong L(H)$ for some graph H. These definitions are extended in the natural way to *line graphs of multigraphs*. In multigraphs, we allow loops and multiple edges (however, we remark that the set of the line graphs of multigraphs does not change if loops are not allowed). For our proof below, we will also need the definitions of the notions of matchings, triangles, and stars for multigraphs. A *matching* in multigraph H is defined as a set of pairwise disjoint edges, but with loops allowed. A *triangle* in H is the set of all non-loop edges in the subgraph of H induced by a set U of three pairwise adjacent vertices. A *star* is the set of all edges incident with some vertex $v \in V(H)$ (all loops at v, if any, are part of the star).

We also note that if G is the line graph of a multigraph H, then the maximal stable sets in G are exactly the maximal matchings in H. Moreover, there are two types of maximal cliques in G: triangles in H, and inclusion-wise maximal stars in H not contained in any triangle.

Theorem 6. *The* Strong EPT Clique *problem is solvable in polynomial time for local EPT graphs given by a local EPT representation.*

Proof. Let G be a local EPT graph given by a local EPT representation $(T, \mathcal{P})$. Let $v \in V(T)$ be a vertex common to all paths in $\mathcal{P}$. We may assume without loss of generality that T is isomorphic to a star centered at v (that is, a graph of the form $K_{1,n}$ in which v is of degree n). Then, G is isomorphic to the line graph of a multigraph H, defined as follows. The vertex set of H is given by the set $V(T)\backslash\{v\}$, each path of length two in $\mathcal{P}$ from a vertex u to a vertex w adds an edge to H joining u and w, and each path $P \in \mathcal{P}$ of length 1 adds a loop to H at the endpoint of P other than v.

Let K be a maximal clique in G. We analyze the two cases, depending on whether K is a triangle or a star.

Suppose first that K is a triangle in H induced by a vertex set U. We claim that K is a strong clique in G if and only if H contains no two disjoint edges each of which is either a loop at a vertex in U or connects a vertex of U with a vertex of $V(H)\backslash U$. On the one hand, if H contains two disjoint edges, say e and f, each of which is either a loop at a vertex in U or connects a vertex of U with a vertex of $V(H)\backslash U$, then any maximal matching M in H with $\{e, f\} \subseteq M$ is a maximal stable set in G disjoint from K, and K is not strong. On the other hand, if K is not strong, then there exists a maximal stable set S in G disjoint from K. Then, S is a matching in H, and by maximality, every edge in K (in the multigraph H) has an endpoint in common with an edge from S. This implies that S covers at least two vertices, say u and u', of U. Since any edge connecting u and u' is in K, vertices u and u' are covered by two distinct (and disjoint) edges in S. Clearly, each of these two edges is either a loop at a vertex in U or connects a vertex of U with a vertex of $V(H)\backslash U$.

Suppose now that K is an inclusion-wise maximal star in H not contained in a triangle, and let $v \in V(H)$ be a vertex incident with all edges in K. In this case, we claim that K is a strong clique in G if and only if one of the following two conditions holds: either (i) H (and thus K) has a loop at v, or (ii) H has no loop at v and no matching of H leaves v uncovered and covers all vertices adjacent to v. Indeed, if K has a loop at v, then $K \subseteq V(G)$ consists of v and all neighbors of v (that is, it is a *simplicial clique*), and in this case K is easily seen to be strong. If H has no loop at v and H has a matching M that leaves v uncovered but covers all vertices adjacent to v, then extending M to any maximal matching yields a maximal stable set in G disjoint from K, so K is not strong in this case. Finally, suppose that K is not strong. Then, H contains a maximal matching M disjoint from K. In particular, this means that H has no loop at v (since otherwise any loop at v could be added to M to obtain a larger matching, contradicting the maximality of M), that v is not covered by M (otherwise M would not be disjoint from K), and that all vertices adjacent to v are covered by M (since if there is a vertex v' adjacent to v and not covered by M, then adding an edge between v and v' to M would yield a larger matching, thus contradicting the maximality of M).

It remains to argue that the above conditions can be tested in polynomial time. Testing the condition verifying whether a given triangle is a strong clique is clearly polynomial. The only remaining nontrivial part is to test whether, given a vertex v of H, there is a matching in H not covering v and covering all vertices adjacent to v, where $v \in V(H)$ is a vertex such that H has no loop at v. Following similar ideas as in [24], this problem can be reduced to solving an instance of the maximum weight matching problem on the graph $H(v)$, where $H(v)$ denotes the subgraph of H induced by all edges not containing v but containing a neighbor of v. Every edge in $H(v)$ connecting two distinct neighbors of v is assigned weight 2, and every other edge in $H(v)$ gets weight 1. Also, $H(v)$ can be transformed to a simple graph $H'(v)$ by replacing any parallel edges connecting two vertices with a single edge, and replacing every loop at a vertex x with an edge connecting x to a new vertex x'. It is easy to see that graph $H'(v)$ has a matching of total weight $|N_H(v)|$ if and only if $H(v)$ contains a matching covering all neighbors of v, which in turn is equivalent to the condition that H has a matching not covering v and covering all vertices adjacent to v. Since the maximum weight matching problem is polynomial (see, e.g., [14]), the proof is complete. □

Some remarks related to Theorem 6 are in order. Roussopoulos [36] proved that given a graph G, testing whether G is a line graph and computing a root H of G (if one exists) can be done in linear time. Using this result, it is not difficult to obtain a polynomial time algorithm to test whether a given graph G is the line graph of a multigraph (and if so, to compute a root of G). Using Theorem 5, these observations have implications for local EPT graphs. Every local EPT graph G is the line graph of a multigraph H, which can be efficiently computed from G. Moreover, from H one can immediately obtain an EPT representation of G. It follows that the Strong EPT Clique problem for local EPT graphs can be solved in polynomial time even if a local EPT representation of the input graph is not given. Summarizing, we obtain:

Theorem 7. *The* Strong Clique *problem is solvable in polynomial time for line graphs of multigraphs (equivalently: for local EPT graphs).*

Recall that line graphs have only polynomially many maximal cliques, which can be efficiently enumerated. Furthermore, a line graph is equistable if and only if every edge of it is contained in a strong clique (this follows from Theorem 2, and was also proved in [24]). Therefore, Theorem 7 gives an argument for the existence of a polynomial time recognition algorithm for equistable line graphs which is, in our opinion, conceptually simpler than that of [24].

We conclude by mentioning some open problems related to the results of this paper.

Question 1. *What is the computational complexity of recognizing general partition/strongly equistable/equistable/triangle graphs?*

The recognition of triangle graphs was conjectured to be co-NP-complete by Kloks et al. [21].

Question 2. *What is the computational complexity of determining whether an EPT graph given by an EPT representation has any (equivalently: all) of the above four properties?*

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