Journal of Applied Logic 15 (2016) 1–15

Contents lists available at ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

The left adjoint of Spec from a category of lattice-ordered groups

José Luis Castiglioni *, Hernán Javier San Martín

ARTICLE INFO

Article history:
Received 29 June 2015
Received in revised form 14 September 2015
Accepted 19 October 2015
Available online 4 November 2015

Keywords:
Lattice ordered abelian groups
Categorical adjunction
Local order units

ABSTRACT

Let us write $\ell\mathcal{G}_u^f$ for the category whose objects are lattice-ordered abelian groups (l-groups for short) with a strong unit and finite prime spectrum endowed with a collection of Archimedean elements, one for each prime l-ideal, which satisfy certain properties, and whose arrows are l-homomorphisms with additional structure. In this paper we show that a functor which assigns to each object $(A, \hat{u}) \in \ell\mathcal{G}_u^f$ the prime spectrum of A, and to each arrow $f : (A, \hat{u}) \to (B, \hat{v}) \in \ell\mathcal{G}_u^f$ the naturally induced p-morphism, has a left adjoint.

1. Introduction

The algebraic counterpart of the infinite valued Łukasiewicz sentential calculus is the class of MV-algebras. This is probably the first historical motivation for the study of this class of algebras.

Furthermore, there is a close connection between the category of l-groups with a strong unit and the category of MV-algebras. In [8] Mundici established a categorical equivalence Γ between them. This equivalence enables us to translate properties from one class to another. For example, for any l-group A with a strong unit u, the prime spectrum of A is order isomorphic to the prime spectrum of the MV-algebra $\Gamma(A, u)$.

The correspondence that assigns to each (abelian) l-group A with a strong unit its prime spectrum $\mathrm{Spec}(A)$ can be extended to a functor from the category of (abelian) l-groups with a strong unit to the category of spectral root systems [1,3]. Since each categorical equivalence must reflect isomorphisms and $\mathrm{Spec}(\mathbb{Z}, 1) = \mathrm{Spec}(\mathbb{R}, 1)$ although $(\mathbb{Z}, 1)$ is not isomorphic to $(\mathbb{R}, 1)$, it follows that Spec may be not part of a categorical equivalence. Hence, one may naturally wonder whether Spec (of a variant thereof) might yield an adjunction pair. In this paper a left adjoint of Spec is obtained by restricting it to a suitable category of (always abelian) l-groups with a distinguished strong unit.

In [7] the authors study a problem closely related to the one investigated here. The relationship between the results in [7] and the ones in the present paper will be explored mainly in last section.

* Corresponding author.
E-mail addresses: jlc@mate.unlp.edu.ar (J.L. Castiglioni), hsanmartin@mate.unlp.edu.ar (H.J. San Martín).

http://dx.doi.org/10.1016/j.jal.2015.11.001

In what follows we will recall some basic definitions and properties used in this work, and notation will be fixed.

A *partially ordered abelian group* [2,6] is an abelian group $(A, +, -, 0)$ endowed with a partial order relation $\leq$ which is *compatible* with the addition operation: thus, for all $x, y, z \in A$ we have $x + z \leq y + z$ whenever $x \leq y$. When the order relation is total, A is said to be a *totally ordered abelian group*, or *o-group* for short. When the order of A defines a lattice structure, A is called a *lattice-ordered abelian group*, or *l-group*, for short. In any l-group we have $z + (x \vee y) = (z + x) \vee (z + y)$ and $z + (x \wedge y) = (z + x) \wedge (z + y)$. For each element $x \in A$, we define $|x| = x \vee -x = x^+ + x^-$, where $x^+ = x \vee 0$ and $x^- = -x \vee 0$. A *strong (order) unit* u of A is an Archimedean element of A, i.e., an element $0 \leq u \in A$ such that for each $x \in A$ there exists a natural number n such that $|x| \leq nu$, where $nu := u + u + \ldots + u$ (n times). An *l-ideal* (or *convex subgroup*) of an l-group A is a subgroup J of A such that if $a, b \in J$ and $a \leq c \leq b$ then $c \in A$. An l-ideal J of an l-group A is said to be *prime* if and only if J is proper (i.e., $J \neq A$) and the quotient l-group A/J is totally ordered. We define the *(prime) spectrum* of A, $\mathrm{Spec}(A)$, as the set of prime l-ideals of A. Finally, if W is a subset of A we write $\langle W \rangle$ to indicate the l-ideal generated by W.

The following remark is part of the folklore of l-groups [2,6]:

Remark 1. Let A be an l-group with a strong unit u.

(a) Let $W \subseteq A$. Then $b \in \langle W \cup \{a\} \rangle$ if and only if there exist a natural number n and $c \in W$ such that $|b| \leq n|a| + |c|$.
(b) Let $P \in \mathrm{Spec}(A)$. Then $u \notin P$, if $x \geq 0$ and $x \in P$ then $x \wedge u \in P$, if $x \in P$ then $|x| \in P$, and if $|x| \in P$ then $x \in P$.
(c) Let P be a convex subgroup of A. Then $P \in \mathrm{Spec}(A)$ if and only if for every $x, y \in A$, if $x \wedge y = 0$ then $x \in P$ or $y \in P$.
(d) Every proper l-ideal is an intersection of prime l-ideals.

Other elementary properties of l-groups can be found in [6].

A *root system* is a poset X such that for every $x \in X$ the set

$$[x) := \{y \in X : y \geq x\}$$

is a totally ordered subset of X. A *p-morphism* is a morphism of posets $f : X \to Y$ with the following property: given $x \in X$ and $y \in Y$ such that $f(x) \leq y$ there exists $z \in X$ such that $x \leq z$ and $f(z) = y$. Inspired by the known fact that if A is an l-group then $(\mathrm{Spec}(A), \subseteq)$ is a spectral root system, and taking into account some considerations done in [3], in this paper we focus on the link between the category of l-groups with a strong unit, and the category of root systems with p-morphisms as arrows.

Let us fix notation for some of the categories that appear in this paper:

$\ell\mathcal{G}$ = Category of l-groups with a strong unit,
$\ell\mathcal{G}^f$ = Category of l-groups with a strong unit and finite spectrum,
$\mathcal{MV}$ = Category of MV-algebras,
$\mathcal{RS}$ = Category of root systems with p-morphisms as arrows,
$\mathcal{FRS}$ = Category of finite root systems with p-morphisms as arrows.

The rest of the paper is organized as follows. In Section 2 we build up a functor from the category $\mathcal{FRS}$ to the category $\ell\mathcal{G}^f$. In Section 3 we show that it is possible to define a functor from the category $\ell\mathcal{G}$ to the category $\mathcal{RS}$. In Section 4 we define the category of l-groups with local order units. The objects of this category are objects of $\ell\mathcal{G}^f$ together with a family of constants (a strong unit for each prime l-ideal) which satisfy particular properties. Its morphisms are l-homomorphism preserving the strong units which also satisfy additional conditions involving the local units. We prove that there exists a functor from this new category to the category $\mathcal{FRS}$, and conversely. In Section 5 we prove that these functors form an adjoint pair. In the last section we explore the relationship between our work and [7].

We believe it is interesting to note that all the results we obtain in this paper are proved using only basic algebraic facts coming from the theory of l-groups and the theory of posets.

2. Getting an l-group from a finite root system

In this section we build up a functor from $\mathcal{FRS}$ to $\ell\mathcal{G}^f$. We start with some definitions and preliminary results.

Let F be a finite root system and $\mathbb{Z}$ the set of integer numbers. We write $\mathbb{Z}^F$ for the set of functions from F to $\mathbb{Z}$, F^M for the set of maximal elements of F and $\mathbb{Z}^n_{lex} = \mathbb{Z} \otimes \ldots \otimes \mathbb{Z}$ (n times) is $\mathbb{Z} \times \ldots \times \mathbb{Z}$ (n times) with the lexicographic order.

Let $h, k \in \mathbb{Z}^F$. For every $x \in F$ we define $(h + k)(x) = h(x) + k(x)$ and $0(x) = 0$. Let n be the cardinal of $[x)$. Then there exist $x_1, \ldots, x_n$ such that $[x) = \{x_n, x_{n-1}, \ldots, x_1\}$, with $x = x_n < x_{n-1} < \ldots < x_1$. We also define the following binary operations:

$$(h \wedge k)(x) \text{ is the } n\text{th coordinate of } (h(x_1), \ldots, h(x_n)) \wedge (k(x_1), \ldots, k(x_n)) \text{ in } \mathbb{Z}^n_{lex},$$
$$(h \vee k)(x) \text{ is the } n\text{th coordinate of } (h(x_1), \ldots, h(x_n)) \wedge (k(x_1), \ldots, k(x_n)) \text{ in } \mathbb{Z}^n_{lex}.$$

Finally we define the map $e_F : F \to \mathbb{Z}$ by $e_F(x) = 1$ for every $x \in F$.

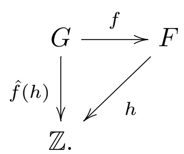
Lemma 1. *$(\mathbb{Z}^F, e_F)$ is an object of $\ell\mathcal{G}^f$.*

It is worth mentioning that our construction, which is a sort of generalization of the lexicographic product, is essentially the classical one used, for instance, in the Conrad–Harvey–Holland Theorem for abelian ℓ-groups [4,5].

Let $F, G \in \mathcal{FRS}$, and $f : G \to F$ a p-morphism. We define the map $\hat{f} : (\mathbb{Z}^F, e_F) \to (\mathbb{Z}^G, e_G)$ by

$$\hat{f}(h) := h \circ f.$$

This definition amounts to asking that the following diagram commutes:



Straightforward computations show that $\hat{f}$ preserves $+$, $-$ and that $\hat{f}(e_F) = e_G$.

Lemma 2. *Let $h, k \in \mathbb{Z}^F$. Then*

(a) $\hat{f}(h \wedge k) = \hat{f}(h) \wedge \hat{f}(k)$,
(b) $\hat{f}(h \vee k) = \hat{f}(h) \vee \hat{f}(k)$.

Proof. Let $h, k \in \mathbb{Z}^F$ and $x \in G$. Let m be the maximum element of $[x)$, so $f(m)$ is the maximum element of $[f(x))$. In order to prove it, let $y \geq f(x)$. Since f is a p-morphism then there exists $z \geq x$ such that

$f(z) = y$. But m is the maximum of the chain $[x)$, so $m \geq z$. Thus, $f(m) \geq f(z) = y$, i.e., $f(m)$ is the maximum of $[f(x))$.

First case: suppose $[\hat{f}(h)](m) < [\hat{f}(k)](m)$, i.e., $h(f(m)) < k(f(m))$. Since $f(m)$ is the maximum in $[f(x))$ then $(h \wedge k)(f(x)) = h(f(x))$, i.e.,

$$[\hat{f}(h \wedge k)](x) = [\hat{f}(h)](x). \tag{1}$$

Since m is the maximum of $[x)$,

$$[\hat{f}(h) \wedge \hat{f}(k)](x) = [\hat{f}(h)](x). \tag{2}$$

It follows from equations (1) and (2) that $[\hat{f}(h \wedge k)](x) = [\hat{f}(h) \wedge \hat{f}(k)](x)$.

Second case: suppose $h(f(m)) = k(f(m))$.

If for every $y \geq x$ we have $h(f(y)) = k(f(y))$ then for every $z \geq f(x)$ we have $h(z) = k(z)$. In order to prove it, let $z \geq f(x)$. Since f is a p-morphism then there exists $y \geq x$ such that $f(y) = z$. Then $h(z) = h(f(y)) = k(f(y)) = k(z)$, which was our aim. Then $[\hat{f}(h \wedge k)](x) = [\hat{f}(h) \wedge \hat{f}(k)](x)$.

Finally let $f(n)$ be the first natural number such that $n \geq x$ and $h(f(n)) \neq k(f(n))$. Since f is a p-morphism then $f(n)$ is the first natural number such that $f(n) \geq f(x)$ and $h(z) \neq k(z)$. Therefore, $[\hat{f}(h \wedge k)](x) = [\hat{f}(h) \wedge \hat{f}(k)](x)$. □

Then we have the following

Proposition 1. *There exists a functor from $\mathcal{FRS}$ to $\ell\mathcal{G}^f$.*

In what follows we will prove that if $F \in \mathcal{FRS}$, then there exists an order isomorphism between F and $\mathrm{Spec}(\mathbb{Z}^F)$.

Proposition 2. *Let $F \in \mathcal{FRS}$. The map $\eta_F : F \to \mathrm{Spec}(\mathbb{Z}^F)$ given by*

$$\eta_F(x) := \{h \in \mathbb{Z}^F : h(y) = 0 \text{ for every } y \geq x\}$$

is an order isomorphism. In particular, η_F is a p-morphism.

Proof. When F is clear from the context, we write η in place of η_F. First we will prove that η is a well defined map. Let $x \in F$. We have to prove that $\eta(x)$ is a prime l-ideal of $\mathbb{Z}^F$. It is immediate that $\eta(x)$ is closed under $-$, 0, $\wedge$ and $\vee$. In order to show that $\eta(x)$ is a convex subset of $\mathbb{Z}^F$, let $h \leq k \leq j$ with $h, j \in \eta(x)$ and $k \in \mathbb{Z}^F$. Consider $k \notin \eta(x)$, so there exists $y \geq x$ such that $k(y) \neq 0$. If $k(y) > 0$ then $k \not\leq j$, which is a contradiction. If $k(y) < 0$ then $h \not\leq k$, which is a contradiction again. Thus, $\eta(x)$ is a convex subset of $\mathbb{Z}^F$. Hence, $\eta(x)$ is an l-ideal of $\mathbb{Z}^F$ which is proper because $e_F \notin \eta(x)$. Let $h, k \in \mathbb{Z}^F$ such that $h \wedge k = 0$. We consider $h, k \notin \eta(x)$. Let y_h be the maximum element of $[x)$ such that $h(y_h) \neq 0$ (in a similar way we define y_k). Since $[x)$ is a chain, we can assume $y_k \geq y_h$. If $y_k = y_h$ then $(h \wedge k)(y_k) = h(y_k) \wedge k(y_k) \neq 0$, which is a contradiction. Now suppose $y_k > y_h$. If $k(y_k) < 0$ then $(h \wedge k)(y_k) = k(y_k) \neq 0$, which is an absurd. If $k(y_k) > 0$ then $(h \wedge k)(y) = h(y)$ for every $y \geq x$. Hence, $(h \wedge k)(y_h) = h(y_h) \neq 0$, which is impossible. Hence, $h \in \eta(x)$ or $k \in \eta(x)$. Thus, $\eta(x) \in \mathrm{Spec}(\mathbb{Z}^F)$.

In what follows, we will prove that η is an injective map. For every $x \in F$ we define the map

$$u_x(z) = \begin{cases} 1 & \text{if } x \leq z \\ 0 & \text{if } x \not\leq z. \end{cases}$$

Take $x, y \in F$ and assume $x \neq y$; for example, consider $x \nleq y$. Then $u_x \in \eta(x)$ and $u_x \notin \eta(y)$, so $\eta(x) \neq \eta(y)$. Hence, η is an injective map.

Now we will show that η is a bijective map. It was proved in [3] that there exists an order isomorphism between F and $\mathrm{Spec}(\mathbb{R}^F)$, where the order in the l-group $\mathbb{R}^F$ agrees with the order given by us in the l-group $\mathbb{Z}^F$. Moreover, we can change $\mathrm{Spec}(\mathbb{R}^F)$ by $\mathrm{Spec}(\mathbb{Z}^F)$ in order to obtain an order isomorphism between F and $\mathrm{Spec}(\mathbb{Z}^F)$. Since η is an injective map and F is finite then F and $\mathrm{Spec}(\mathbb{Z}^F)$ have the same cardinal. Hence, η is a bijective map.

Finally we will prove that η is an order isomorphism, i.e., $x \leq y$ if and only if $\eta(x) \subseteq \eta(y)$ for every $x, y \in F$. Let $x \leq y$, $h \in \eta(x)$ and $z \geq y$. Since $x \leq y$ then $z \leq x$. Hence, $h(z) = 0$, i.e. $h \in \eta(y)$. Then we have $\eta(x) \subseteq \eta(y)$. Conversely, let $\eta(x) \subseteq \eta(y)$. If $x \nleq y$ then $u_x \in \eta(x)$ and $u_x \notin \eta(y)$. Thus $\eta(x) \nsubseteq \eta(y)$, which is a contradiction. Therefore $x \leq y$. □

3. Getting a root system from an *l*-group

In this section we build up a functor from the category $\ell\mathcal{G}$ to the category $\mathcal{RS}$.

The following result is part of the folklore of the subject. Although it follows as a consequence of [7, Lemma 13] (see Remark 2, below) through a suitable reframing, we opt to give an elementary and self contained proof of it.

Lemma 3. *Let $f : (A, u) \to (B, v)$ be a morphism in $\ell\mathcal{G}$. Then the map $\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ given by $\mathrm{Spec}(f)(P) = f^{-1}(P)$ is a p-morphism.*

Proof. Let $P \in \mathrm{Spec}(B)$ and $Q \in \mathrm{Spec}(A)$ be such that $f^{-1}(P) \subseteq Q$. Define the set

$$\Sigma := \{Z \in \mathrm{Spec}(B) : P \subseteq Z \text{ and } f^{-1}(Z) \subseteq Q\}.$$

A routine application of Zorn's Lemma shows that the set Σ has a maximal element M. Moreover, M is the maximum element of Σ. In particular, $M \in \mathrm{Spec}(B)$, $P \subseteq M$ and $f^{-1}(M) \subseteq Q$. In what follows we will see that $f^{-1}(M) = Q$. In order to prove it, let $a \in Q$ and define $I = \langle M \cup \{f(a)\}\rangle$. We will show that $f^{-1}(I) \subseteq Q$. Let $b \in f^{-1}(I)$, so there exist a natural number n and $c \in M$ such that $|f(b)| \leq (n\,|f(a)|) + |c|$. Thus, $f(|b|) \leq f(n\,|a|) + |c|$. Then $f(|b| - n\,|a|) \leq |c|$. Thus we have $0 \leq f(((|b| - n\,|a|) \wedge u) \vee 0) \leq |c| \wedge v$. Since $0, |c| \wedge v \in M$ and we have the inclusion $f^{-1}(M) \subseteq Q$, it follows that $((|b| - n\,|a|) \wedge u) \vee 0 \in M$. Also

$$\begin{aligned} ((|b| - n\,|a|) \wedge u) \vee 0 &= ((|b| - n\,|a|) \vee 0) \wedge (u \vee 0) \\ &= ((|b| - n\,|a|) \vee 0) \wedge u. \end{aligned}$$

Since $u \notin Q$ then $(|b| - n\,|a|) \vee 0 \in Q$. In addition, $n\,|a| \in Q$ because $a \in Q$. Hence, $((|b| - n\,|a|) \vee 0) + n\,|a| \in Q$. Furthermore,

$$\begin{aligned} ((|b| - n\,|a|) \vee 0) + n\,|a| &= (|b| - n\,|a| + n\,|a|) \vee (0 + n\,|a|) \\ &= |b| \vee n\,|a|\,. \end{aligned}$$

Since $0 \in Q$, $|b| \vee n\,|a| \in Q$ and $0 \leq |b| \leq |b| \vee n\,|a|$ then $|b| \in Q$. So $b \in Q$ and then $f^{-1}(I) \subseteq Q$. Hence $I = M$ because $I \in \Sigma$ and $M \subseteq I$. Since $f(a) \in I$, $a \in f^{-1}(I) = f^{-1}(M)$. Thus, $Q \subseteq f^{-1}(M)$. Therefore, $f^{-1}(M) = Q$, which was our aim. □

Remark 2. It is possible to give another proof of Lemma 3 by considering [7, Lemma 13]. In order to show it, let $f : (A, u) \to (B, v)$ be a morphism in $\ell\mathcal{G}$ and $\Gamma : \ell\mathcal{G} \to \mathcal{MV}$ Mundici's functor [2]. In particular, $\Gamma(f) : \Gamma(a, u) \to \Gamma(B, v)$ is an homomorphism in $\mathcal{MV}$. Moreover, from [7, Lemma 13] it follows that

Spec : $\mathrm{Spec}(\Gamma(A,u)) \to \mathrm{Spec}(\Gamma(B,v))$ is a p-morphism (here we also write Spec for the functor from $\mathcal{MV}$ to $\mathcal{RS}$ defined in the usual way [2,7]). Moreover, the correspondence $\phi_A : P \mapsto \{x \in A : |x| \wedge u \in P\}$ defines an isomorphism from the poset $(\mathrm{Spec}(\Gamma(A,u)), \subseteq)$ onto the poset $(\mathrm{Spec}(A), \subseteq)$ [3, Corollary 1.3]. The inverse isomorphism is given by the correspondence $\psi_A : Q \mapsto Q \cap [0,u]$, where $[0,u] = \{x \in A : 0 \leq x \leq u\}$. We also have that ϕ_A and ψ_A are p-morphisms. Straightforward computations show the commutativity of the following diagram:

$$\begin{array}{ccc} \mathrm{Spec}(\Gamma(B,v)) & \xrightarrow{\mathrm{Spec}(\Gamma(f))} & \mathrm{Spec}(\Gamma(A,u)) \\ {\scriptstyle \phi_B}\downarrow & & \downarrow{\scriptstyle \phi_A} \\ \mathrm{Spec}(B) & \xrightarrow{\mathrm{Spec}(f)} & \mathrm{Spec}(A). \end{array}$$

Therefore, $\mathrm{Spec}(f) = \phi_A \circ \mathrm{Spec}(\Gamma(f)) \circ \psi_B$, and as a consequence, $\mathrm{Spec}(f)$ is a p-morphism.

As a straightforward consequence of Lemma 3, we have the following result.

Proposition 3. *There exists a functor from $\ell\mathcal{G}$ to $\mathcal{RS}$.*

4. On *l*-groups with local order units

In this section we define a category whose objects are objects of $\ell\mathcal{G}^f$ together with local order units (a strong unit for each prime l-ideal) satisfying specific properties, and whose arrows are l-homomorphisms satisfying additional conditions. We prove that there exists a functor from this category to the category $\mathcal{FRS}$, and conversely.

Let A be an l-group and $P \in \mathrm{Spec}(A)$. We define the set

$$I_P := \{Q \in \mathrm{Spec}(A) : P \nsubseteq Q\}.$$

If $g : A \to B$ is a morphism in $\ell\mathcal{G}$, we also define the set

$$C_{(P,g)} := \{Q \in \mathrm{Spec}(B) : g^{-1}(Q) = P\}.$$

We write $\mathrm{Max}(A)$ for the set of maximal l-ideals of A. If $P \notin \mathrm{Max}(A)$, we write $S(P)$ for the successor of P. Let $P \in \mathrm{Spec}(A)$, so in particular P is an l-group. If u_P is a strong unit of P we define

$$\delta_P = \begin{cases} u - u_P & \text{if } P \in \mathrm{Max}(A) \\ u_{S(P)} - u_P & \text{if } P \notin \mathrm{Max}(A). \end{cases}$$

Definition 1. The category $\ell\mathcal{G}^f_u$ of l-groups with local order units is defined as follows:

Objects: Structures $(A, \hat{u})$ with the following properties:

1. A is an l-group with finite spectrum.
2. $\hat{u} = \{u\} \cup \{u_P\}_{P \in \mathrm{Spec}(A)}$, where u is a strong unit of A and u_P is a strong unit of P for each $P \in \mathrm{Spec}(A)$.
3. $u = \sum_{P \in \mathrm{Spec}(A)} \delta_P$, and $\delta_P \geq 0$ for each $P \in \mathrm{Spec}(A)$.
4. $u_P = \begin{cases} \sum_{P \nsubseteq Q} \delta_Q & \text{if } I_P \neq \emptyset \\ 0 & \text{if } I_P = \emptyset. \end{cases}$

Arrows: $g : (A, \hat{u}) \to (B, \hat{v})$ is a morphism if it satisfies the following conditions:

5. $g : (A, u) \to (B, v)$ is a morphism in $\ell\mathcal{G}$.
6. $g(\delta_P) = \begin{cases} \sum_{Q \in C_{(P,g)}} \delta_Q & \text{if } C_{(P,g)} \neq \emptyset \\ 0 & \text{if } C_{(P,g)} = \emptyset. \end{cases}$

If $(A, \hat{u}) \in \ell\mathcal{G}_u^f$, then $\mathrm{Spec}(A) \in \mathcal{FRS}$. If $g : (A, \hat{u}) \to (B, \hat{v})$ is a morphism in $\ell\mathcal{G}_u^f$, then, from Lemma 3, it follows that $\mathrm{Spec}(g) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a morphism in $\mathcal{FRS}$. Hence, we have

Proposition 4. Spec *is a functor from* $\ell\mathcal{G}_u^f$ *to* $\mathcal{FRS}$.

In what follows we will see that there exists a functor from $\mathcal{FRS}$ to $\ell\mathcal{G}_u^f$.

Let $F \in \mathcal{FRS}$. For every $P \in \mathrm{Spec}(\mathbb{Z}^F)$ there exists a unique $x_P \in F$ such that $\eta_F(x_P) = P$. We define maps $u_P : F \to \mathbb{Z}$ by

$$u_P(y) = \begin{cases} 0 & \text{if } x_P \leq y \\ 1 & \text{if } x_P \nleq y. \end{cases}$$

Lemma 4. *Let* $F \in \mathcal{FRS}$. *If* F *has* n *elements then* $\{\delta_P\}_{P \in \mathrm{Spec}(\mathbb{Z}^F)}$ *is the canonical base of* $\mathbb{Z}^n$, *where* δ_P *is defined as in Definition 1. Moreover, the structure* $(\mathbb{Z}^F, \hat{u}_F)$ *is an object of* $\ell\mathcal{G}_u^f$, *where* $\hat{u}_F = \{e_F\} \cup \{u_P\}_{P \in \mathrm{Spec}(\mathbb{Z}^F)}$.

Proof. Let P be a maximal l-ideal in $\mathbb{Z}^F$, so $\delta_P = e_F - u_P$. Let $y \in F$. If $x_P = y$ (i.e., $x_P \leq y$) then $\delta_P(y) = e_F(y) - u_P(y) = 1 - 0 = 1$. If $x_P \neq y$ (i.e. $x_P \nleq y$) then $\delta_P(y) = e_F(y) - u_P(y) = 1 - 1 = 0$. Let $P \in \mathrm{Spec}(\mathbb{Z}^F)$ be such that P is not maximal. Hence, $\delta_P = u_{S(P)} - u_P$. Let $y \in F$ and consider $y \neq x_P$. If $x_P < y$ then $x_{S(P)} \leq y$, so $\delta_P(y) = u_{S(P)}(y) - u_P(y) = 0 - 0 = 0$. If $x_P = y$ then $x_P \leq y$ and $x_{S(P)} \nleq y$, so $\delta_P(y) = u_{S(P)}(y) - u_P(y) = 1 - 0 = 1$. Let now $x_P \nleq y$. Hence, $x_{S(P)} \nleq y$ and as a consequence, $\delta_P(y) = u_{S(P)}(y) - u_P(y) = 1 - 1 = 0$. Then, $\delta_P(y) = 1$ if $y = x_P$ and $\delta_P(y) = 0$ if $y \neq x_P$. Finally, take $P, Q \in \mathrm{Spec}(\mathbb{Z}^F)$ such that $P \neq Q$. In particular, $x_P \neq x_Q$. Thus, $\delta_P(x_P) = 1$ and $\delta_Q(x_P) = 0$, so $\delta_P \neq \delta_Q$. Thus, $\{\delta_P\}_{P \in \mathrm{Spec}(\mathbb{Z}^F)}$ is the canonical base of $\mathbb{Z}^n$.

It is immediate that $(\mathbb{Z}^F, \hat{u}_F)$ satisfies conditions 1, 2 and 3 of Definition 1. In what follows we will see that condition 4 is also verified. Let $P \in \mathrm{Spec}(\mathbb{Z}^F)$ such that $I_P \neq \emptyset$. For $y \in F$, we have

$$u_P(y) = 0 \text{ if and only if } \left(\sum_{P \nsubseteq Q} \delta_Q \right)(y) = 0. \tag{3}$$

In order to prove it, consider $u_P(y) = 0$, i.e., $x_P \leq y$. Let $Z \in \mathrm{Spec}(\mathbb{Z}^F)$ such that $P \nsubseteq Z$. Thus, $x_P \nleq x_Z$, so $y \neq x_Z$. Hence, $\delta_Z(y) = 0$ and as a consequence, $(\sum_{P \nsubseteq Q} \delta_Q)(y) = 0$. Conversely, let $(\sum_{P \nsubseteq Q} \delta_Q)(y) = 0$, i.e., for every $Q \in \mathrm{Spec}(\mathbb{Z}^F)$ such that $P \nsubseteq Q$ we have $\delta_Q(y) = 0$ (i.e., $x_Q \neq y$). Since $x_{\eta(y)} = y$ then $P \subseteq \eta(y)$, i.e., $x_P \leq y$. Thus, $u_P(y) = 0$. Then we have proved condition (3). For every $y \in F$ we have $u_P(y) \in \{0, 1\}$ and $(\sum_{P \nsubseteq Q} \delta_Q)(y) \in \{0, 1\}$, so if $I_P \neq \emptyset$ then $u_P = \sum_{P \nsubseteq Q} \delta_Q$. Finally consider $I_P = \emptyset$, and suppose that there exists $y \in F$ such that $u_P(y) = 1$. Thus, $x_P \nleq y$. So $P \nsubseteq \eta_f(y)$, which is a contradiction. Then, $u_P = 0$ whenever $I_P = \emptyset$. Therefore, we have proved the condition 4 of Definition 1. □

Let $\eta_F : F \to \mathrm{Spec}(\mathbb{Z}^F)$ and $\eta_G : G \to \mathrm{Spec}(\mathbb{Z}^G)$ be the isomorphisms given in Proposition 2, where F and G are objects in $\mathcal{FRS}$. Let $f : F \to G$ be a morphism in $\mathcal{FRS}$. We will prove the commutativity of the following diagram:

$$\begin{array}{ccc} F & \xrightarrow{\eta_F} & \mathrm{Spec}(\mathbb{Z}^F) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{Spec}(\hat{f})} \\ G & \xrightarrow{\eta_G} & \mathrm{Spec}(\mathbb{Z}^G). \end{array}$$

Note that previous diagram commutes if and only if for every $x \in F$,

$$(\mathrm{Spec}(\hat{f}))(\eta_F(x)) = \eta_G(f(x)).$$

For $x \in F$ we have

$$\begin{aligned} (\mathrm{Spec}(\hat{f}))(\eta_F(x)) &= (\hat{f}^{-1})(\eta_F(x)) \\ &= \{h \in \mathbb{Z}^G : \hat{f}(h) \in \eta_F(x)\} \\ &= \{h \in \mathbb{Z}^G : h(f(y)) = 0 \text{ for every } y \geq x\} \end{aligned}$$

and

$$\eta_G(f(x)) = \{h \in \mathbb{Z}^G : h(z) = 0 \text{ for every } z \geq f(x)\}.$$

In order to prove that $(\mathrm{Spec}(\hat{f}))(\eta_F(x)) = \eta_G(f(x))$, let $h \in (\mathrm{Spec}(\hat{f}))(\eta_F(x))$, i.e., $h(f(y)) = 0$ for every $y \geq x$. Assume $z \geq f(x)$. Since f is a p-morphism then there exists $y \geq x$ such that $f(y) = z$. Since $y \geq x$, $h(f(y)) = 0$, i.e., $h(z) = 0$. Thus, $h \in \eta_G(f(x))$.

Conversely, let $h \in \eta_G(f(x))$, i.e., $h(z) = 0$ for every $z \geq x$. Let $y \geq x$, so in particular $f(y) \geq f(x)$. From the assumption we have that $h(f(y)) = 0$. Hence, $h \in (\mathrm{Spec}(\hat{f}))(\eta_F(x))$. Then, $(\mathrm{Spec}(\hat{f}))(\eta_F(x)) = \eta_G(f(x))$, which was our aim. Therefore, we obtain the following

Lemma 5. *Let F, G be objects in $\mathcal{FRS}$ and $f : F \to G$ a morphism in $\mathcal{FRS}$. Then $(\mathrm{Spec}(\hat{f}))(\eta_F(x)) = \eta_G(f(x))$ for every $x \in F$.*

Lemma 6. *Let $f : F \to G$ be a morphism in $\mathcal{FRS}$. Then $\hat{f} : (\mathbb{Z}^G, \hat{u}_G) \to (\mathbb{Z}^F, \hat{u}_F)$ is a morphism in $\ell\mathcal{G}_u^f$.*

Proof. Let $P \in \mathrm{Spec}(\mathbb{Z}^F)$. We will argue by cases.

First case: If $C_{(P,\hat{f})} = \emptyset$ and there exists $x \in F$ such that $[\hat{f}(\delta_P)](x) = \delta_P(f(x)) \neq 0$ then $\delta_P(f(x)) = 1$. Hence, $\eta_G(f(x)) = P$. By Lemma 5 $(\mathrm{Spec}(\hat{f}))(\eta_F(x)) = P$. Hence $\eta_F(x) \in C_{(P,\hat{f})}$, which is a contradiction because $C_{(P,\hat{f})} = \emptyset$. Thus, if $C_{(P,\hat{f})} = \emptyset$ then $[\hat{f}(\delta_P)](x) = 0$ for every $x \in F$.

Second case: Consider $C_{(P,\hat{f})} \neq \emptyset$, and let $x \in F$. First we will prove that

$$[\hat{f}(\delta_P)](x) = 1 \text{ if and only if } \left(\sum_{Q \in C_{(P,\hat{f})}} \delta_Q\right)(x) = 1. \tag{4}$$

Suppose that $[\hat{f}(\delta_P)](x) = \delta_P(f(x)) = 1$. From Lemma 5, we have $P = \eta_G(f(x)) = (\mathrm{Spec}(\hat{f}))(\eta(x)) = (\hat{f})^{-1}(\eta_F(x))$ and as a consequence, $\eta_F(x) \in C_{(P,\hat{f})}$. Then, $\left(\sum_{Q \in C_{(P,\hat{f})}} \delta_Q\right)(x) = 1$. Conversely, suppose that $\left(\sum_{Q \in C_{(P,\hat{f})}} \delta_Q\right)(x) = 1$. Hence, there exists $Q \in C_{(P,\hat{f})}$ such that $\hat{f}^{-1}(Q) = P$ and $\delta_Q(x) = 1$, i.e., $\eta_F(x) = Q$. From Lemma 5, it follows that $P = \hat{f}^{-1}(Q) = \hat{f}^{-1}(\eta_F(x)) = \eta_G(f(x))$. Thus, $[\hat{f}(\delta_P)](x) = \delta_P(f(x)) = 1$. Hence, we have proved (4). Since $[\hat{f}(\delta_P)](x) \in \{0, 1\}$ and $\left(\sum_{Q \in C_{(P,\hat{f})}} \delta_Q\right)(x) \in \{0, 1\}$, we have that $C_{(P,\hat{f})} \neq \emptyset$ implies $[\hat{f}(\delta_P)](x) = \left(\sum_{Q \in C_{(P,\hat{f})}} \delta_Q\right)(x)$ for every $x \in F$. $\square$

If $F \in \mathcal{FRS}$, we define $\Lambda(X) = \mathbb{Z}^F$. If $f : F \to G$ is a morphism in $\mathcal{FRS}$, we define $\Lambda(f) : (\mathbb{Z}^G, \hat{u}_G) \to (\mathbb{Z}^F, \hat{u}_F)$ as $\Lambda(f) = \hat{f}$. Then, from Lemmata 4 and 6 we have

Corollary 7. *Λ is a functor from $\mathcal{FRS}$ to $\ell\mathcal{G}_u^f$.*

5. An adjunction

We prepare the necessary material for the proof that the functor $\Lambda : \mathcal{FRS}^{op} \to \ell\mathcal{G}_u^f$ is left adjoint to Spec.

Let $(A, \hat{u}) \in \ell\mathcal{G}_u^f$. For every $\widetilde{P} \in \mathrm{Spec}(\mathbb{Z}^{\mathrm{Spec}(A)})$, let $P \in \mathrm{Spec}(A)$ the unique element of $\mathrm{Spec}(A)$ such that $\eta_{\mathrm{Spec}(A)}(P) = \widetilde{P}$. From Lemma 4 we have that the assignment $\delta_{\widetilde{P}} \mapsto \delta_P$ from $\mathbb{Z}^{\mathrm{Spec}(A)}$ to A can be extended to a unique morphism of groups

$$\varepsilon_A : \mathbb{Z}^{\mathrm{Spec}(A)} \to A.$$

We define $\widetilde{u} := e_{\mathrm{Spec}(A)}$ and $\widehat{u} := \{\widetilde{u}\} \cup \{u_{\widetilde{P}}\}_{P \in \mathrm{Spec}(\mathbb{Z}^{\mathrm{Spec}(A)})}$. From Proposition 4 and Corollary 7 we conclude that $(\mathbb{Z}^{\mathrm{Spec}(A)}, \widehat{u}) \in \ell\mathcal{G}_u^f$.

Lemma 8. *If $\widetilde{P} \in \mathrm{Spec}(\mathbb{Z}^{\mathrm{Spec}(A)})$ then $\varepsilon_A(u_{\widetilde{P}}) = u_P$. Moreover, $\varepsilon_A(\widetilde{u}) = u$.*

Proof. Let $\widetilde{P} \in \mathrm{Spec}(\mathbb{Z}^{\mathrm{Spec}(A)})$. Suppose that $I_P \neq \emptyset$. Then $u_{\widetilde{P}} = \sum_{\widetilde{P} \nsubseteq \widetilde{Q}} \delta_{\widetilde{Q}}$. Thus,

$$\varepsilon_A(u_{\widetilde{P}}) = \sum_{\widetilde{P} \nsubseteq \widetilde{Q}} \varepsilon_A(\delta_{\widetilde{Q}}) = \sum_{P \nsubseteq Q} \delta_Q = u_P.$$

If $I_P = \emptyset$ (equivalently, $I_{\widetilde{P}} = \emptyset$) then $u_P = 0$ and $u_{\widetilde{P}} = 0$, so we also have $\varepsilon_A(u_{\widetilde{P}}) = u_P$.

In order to prove that $\varepsilon_A(\widetilde{u}) = u$, note that $\widetilde{u} = \sum \delta_{\widetilde{P}}$. Hence,

$$\varepsilon_A(\widetilde{u}) = \sum \varepsilon_A(\delta_{\widetilde{P}}) = \sum \delta_P = u. \qquad \square$$

In what follows we will give some technical results which we need for this section.

Lemma 9. *Let $(A, \hat{u}) \in \ell\mathcal{G}_u^f$. Then*

- (a) *If $P \in \mathrm{Spec}(A)$ is such that $nu \in P$, then $n = 0$.*
- (b) *If $\alpha_0, \alpha_1, \ldots, \alpha_j \in \mathbb{Z}$, $P_{j+1} \subset P_j \subset \ldots \subset P_1$ are in $\mathrm{Spec}(A)$ and $\alpha_0 u + \alpha_1 u_1 + \ldots + \alpha_j u_j \in P_{j+1}$ then $\alpha_k = 0$ for every $k = 0, \ldots, j$, where $u_j = u_{P_j}$.*
- (c) *Let A be an o-group, and $\{0\} \subset P_n \subset P_{n-1} \subset \ldots \subset P_1$ all the elements of $\mathrm{Spec}(A)$. Let $\alpha_i, \beta_i \in \mathbb{Z}$ for $i = 0, \ldots, n$. Then $\alpha_0 u + \alpha_1 u_1 + \ldots + \alpha_n u_n \leq \beta_0 u + \beta_1 u_1 + \ldots + \beta_n u_n$ if and only if $(\alpha_0, \alpha_1, \ldots, \alpha_n) \leq (\beta_0, \beta_1, \ldots, \beta_n)$ in $\mathbb{Z}^{n+1}_{lex}$.*

Proof. We first settle item (a). If there exists $n > 0$ such that $nu \in P$ then $-nu \leq u \leq nu$. Since $-nu \in P$ and $nu \in P$ then $u \in P$, which is a contradiction. The case $n < 0$ is similar. Hence, $n = 0$.

In order to prove (b), suppose $P_{j+1} \subset P_j \subset \ldots \subset P_1$ and $a = \alpha_0 u + \alpha_1 u_1 + \ldots + \alpha_j u_j \in P_{j+1}$. Since $a \in P_1$ and $\alpha_1 u_1 + \ldots + \alpha_j u_j \in P_1$ then $\alpha_0 u \in P_1$, so $\alpha_0 = 0$ follows from item (a). Since $\alpha_1 u_1 + \ldots + \alpha_j u_j \in P_2$ and $\alpha_2 u_2 + \ldots + \alpha_j u_j \in P_2$, $\alpha_1 u_1 \in P_2$ and hence, $\alpha_1 = 0$. We can repeat this reasoning, and in the final step we obtain $\alpha_j u_j \in P_{j+1}$, so $\alpha_j = 0$.

We finally settle (c). Suppose $(\alpha_0, \alpha_1, \ldots, \alpha_n) \leq (\beta_0, \beta_1, \ldots, \beta_n)$ in $\mathbb{Z}^{n+1}_{lex}$, and suppose that there exists $0 \leq i \leq n-2$ such that $\alpha_0 = \beta_0, \ldots, \alpha_i = \beta_i$ and $\alpha_{i+1} < \beta_{i+1}$. Hence,

$$\alpha_{i+1} + 1 \leq \beta_{i+1}. \tag{5}$$

Further, $(\alpha_{i+2} - \beta_{i+2})u_{i+2} + \ldots + (\alpha_n - \beta_n)u_n \in P_{i+2}$ and

$$(\alpha_{i+2} - \beta_{i+2})u_{i+2} + \ldots + (\alpha_n - \beta_n)u_n < u_{i+1}. \tag{6}$$

In order to prove the inequality (6), suppose $(\alpha_{i+2} - \beta_{i+2})u_{i+2} + \ldots + (\alpha_n - \beta_n)u_n \geq u_{i+1}$. Since $0 \leq u_{i+1} \leq (\alpha_{i+2} - \beta_{i+2})u_{i+2} + \ldots + (\alpha_n - \beta_n)u_n \in P_{i+2}$, then $u_{i+1} \in P_{i+2}$. Then $P_{i+1} \subseteq P_{i+2} \subset P_{i+1}$, which is a contradiction. From equations (5) and (6) it follows that

$$\alpha_{i+1}u_{i+1} + (\alpha_{i+2} - \beta_{i+2})u_{i+2} + \ldots + (\alpha_n - \beta_n)u_n < \alpha_{i+1}u_{i+1} + u_i \leq \beta_{i+1}u_{i+1}.$$

Then $\alpha_{i+1}u_{i+1} + \ldots + \alpha_n u_n < \beta_{i+1}u_{i+1} + \ldots + \beta_n u_n$. Therefore, $\alpha_0 u + \alpha_1 u_1 + \ldots + \alpha_n u_n \leq \beta_0 u + \beta_1 u_1 + \ldots + \beta_n u_n$.

Conversely, suppose $\alpha_0 u + \alpha_1 u_1 + \ldots + \alpha_n u_n \leq \beta_0 u + \beta_1 u_1 + \ldots + \beta_n u_n$, i.e., $\gamma_0 u + \gamma_1 u_1 + \ldots + \gamma_n u_n \leq 0$, where $\gamma_i = \alpha_i - \beta_i$ for $i = 0, \ldots, n$. We have two possible cases.

First case: suppose $\gamma_0 u + \gamma_1 u_1 + \ldots + \gamma_n u_n \notin P$ for any $P \in \mathrm{Spec}(A)$. If $\gamma_i = 0$, for $i = 0, \ldots, n$, then $(\alpha_1, \ldots, \alpha_n) = (\beta_1, \ldots, \beta_n)$. Thus we can assume there exists a maximum number i such that $\gamma_i \neq 0$. We will prove that $\gamma_i < 0$. In order to prove it, suppose $\gamma_i > 0$. Then $0 \leq u_i \leq \gamma_i u_i$. Thus, $\gamma_{i+1}u_{i+1} + \ldots + \gamma_n u_n \leq -\gamma_i u_i \leq 0$ and $\gamma_{i+1}u_{i+1} + \ldots + \gamma_n u_n \in P_{i+1}$ and as a consequence, $\gamma_i u_i \in P_{i+1}$. Hence, from item (a) we get $\gamma_i = 0$, which is a contradiction. Then, $\gamma_i < 0$ and $(\alpha_0, \alpha_1, \ldots, \alpha_n) \leq (\beta_0, \beta_1, \ldots, \beta_n)$.

Second case: suppose there exists $P_i \in \mathrm{Spec}(A)$ such that $\gamma_0 u + \gamma_1 u_1 + \ldots + \gamma_n u_n \in P_i$.

Suppose $i = n$. Since $\gamma_0 u + \gamma_1 u_1 + \ldots + \gamma_n u_n \in P_n$ and $\gamma_n u_n \in P_n$, then $\gamma_0 u + \gamma_1 u_1 + \ldots + \gamma_{n-1}u_{n-1} \in P_n$. From item (b) we get $\gamma_0 = \ldots = \gamma_{n-1} = 0$. Hence, $\gamma_n u_n = \gamma_0 u + \gamma_1 u_1 + \ldots + \gamma_n u_n \leq 0$. We also have $u_n > 0$ because $P_n \neq \{0\}$. If $\gamma_n > 0$ then $u_n \leq \gamma_n u_n \leq 0$, which is a contradiction. Thus, $\gamma_n \leq 0$ and we have $(\alpha_0, \alpha_1, \ldots, \alpha_n) \leq (\beta_0, \beta_1, \ldots, \beta_n)$.

Now, suppose $i \neq n$. Then we can assume there exists a natural number k with the property $\gamma_0 u + \gamma_1 u_1 + \ldots + \gamma_n u_n \in P_k - P_{k+1}$. Let k be the maximum natural number with the previous property. Since $\gamma_k u_k + \ldots + \gamma_n u_n \in P_k$ then $\gamma_0 u + \ldots \gamma_{k-1}u_{k-1} \in P_k$. Hence, from item (b) the equalities $\gamma_0 = \ldots = \gamma_{k-1} = 0$ follow. Since $\gamma_k u_k + \ldots + \gamma_n u_n \notin P_{k+1}$, then $\gamma_k < 0$. In order to show it, assume $\gamma_k \geq 0$. Then we have $\gamma_{k+1}u_{k+1} + \ldots + \gamma_n u_n \leq \gamma_k u_k + \ldots + \gamma_n u_n \leq 0$ and $\gamma_{k+1}u_{k+1} + \ldots + \gamma_n u_n \in P_{k+1}$. Thus, $\gamma_k u_k + \ldots + \gamma_n u_n \in P_{k+1}$, which is a contradiction. Hence, $\gamma_k < 0$. Therefore, $(\alpha_0, \alpha_1, \ldots, \alpha_n) \leq (\beta_0, \beta_1, \ldots, \beta_n)$. □

For later purposes, let us recall that if M and N are lattices, N is totally ordered and $f : N \to M$ is a morphism of posets then f preserves $\wedge$ and $\vee$.

Lemma 10. *Let $(A, \hat{u}) \in \ell\mathcal{G}_u^f$, with A an o-group. The map $\varepsilon_A : \mathbb{Z}^{\mathrm{Spec}(A)} \to A$ is a morphism of l-groups.*

Proof. We will prove that ε_A preserves $\wedge$ and $\vee$. The conclusion of the lemma is immediate if $\mathrm{Spec}(A)$ has only one element, so we can assume that $\mathrm{Spec}(A)$ has more than one element. Let $\{P_1, \ldots, P_{n+1}\}$ be the set of elements of $\mathrm{Spec}(A)$, with P_{n+1} the zero l-ideal and $P_{n+1} \subset P_n \subset \ldots \subset P_1$. Let $h, j \in \mathbb{Z}^{\mathrm{Spec}(A)}$. Thus, $h = \sum_{i=1}^n a_i \delta_{\widetilde{P_i}}$ and $j = \sum_{i=1}^n b_i \delta_{\widetilde{P_i}}$, where $a_i, b_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. In particular, $h = a_1(\widetilde{u} - \widetilde{u}_1) + a_2(\widetilde{u}_1 - \widetilde{u}_2) + \ldots + (a_n - a_{n-1})\widetilde{u}_{n-1} = a_1\widetilde{u} + (a_2 - a_1)\widetilde{u}_1 + \ldots + (a_n - a_{n-1})\widetilde{u}_{n-1}$, where $\widetilde{u}_i = u_{\widetilde{P_i}}$. Hence, $\varepsilon(h) = a_1(u - u_1) + a_2(u_1 - u_2) + \ldots + (a_n - a_{n-1})u_{n-1} = a_1 u + (a_2 - a_1)u_1 + \ldots + (a_n - a_{n-1})u_{n-1}$, where we write u_i in place of u_{P_i}. Put $\varepsilon(h) = a$ and $\varepsilon(j) = b$. In order to prove that ε preserves $\wedge$ and $\vee$, we only need to show that ε_A preserves the order (because $\mathbb{Z}^{\mathrm{Spec}(A)}$ is totally ordered). Suppose that $h \leq j$. From Lemma 9 it follows that $(a_1, a_2 - a_1, \ldots, a_n - a_{n-1}) \leq (b_1, b_2 - b_1, \ldots, b_n - b_{n-1})$ in the lexicographic order. Then, applying Lemma 9 again, we conclude that $a \leq b$, which was our aim. □

Let A be an l-group and $P \in \mathrm{Spec}(A)$. Consider the map $\rho : A \to A/P$ given by $\rho(a) := a/P$.

Remark 3. Let A be an l-group and $P, Q \in \mathrm{Spec}(A)$ such that $P \subseteq Q$. Then $\rho^{-1}(\rho(Q)) = Q$ and $\rho(Q) \in \mathrm{Spec}(A)$.

Let $(A, \hat{u}) \in \ell\mathcal{G}_u^f$ and $P \in \mathrm{Spec}(A)$. We define

$$\hat{u}/P = \{u/P\} \cup \{u_{\rho^{-1}(Z)}/P\}_{Z\in \mathrm{Spec}(A)}.$$

We next prove that

$$u/P := \sum_{Z\in\mathrm{Spec}(A/P)} \delta_{\rho^{-1}(Z)}/P. \tag{7}$$

Since $u = \sum_{Q\in\mathrm{Spec}(A)} \delta_Q$ then $u/P = \sum_{Q\in\mathrm{Spec}(A)} \delta_Q/P$. The proof of (7) amounts to proving

$$\sum_{Q\in\mathrm{Spec}(A)} \delta_Q/P = \sum_{Z\in\mathrm{Spec}(A/P)} \delta_{\rho^{-1}(Z)}/P. \tag{8}$$

In order to show equality (8), we need to prove that $Q \in \mathrm{Spec}(A)$ and $Q \neq \rho^{-1}(Z)$ for every $Z \in \mathrm{Spec}(A/P)$ implies $\delta_Q \in P$ (i.e., $\delta_Q/P = 0$). Let $Q \in \mathrm{Spec}(A)$ and suppose that $Q \neq \rho^{-1}(Z)$ for every $Z \in \mathrm{Spec}(A/P)$. From Remark 3, it follows that $P \nsubseteq Q$. Hence $u_P = \sum_{P\nsubseteq R} \delta_R \geq \delta_Q$. Thus, we have $0 \leq \delta_Q \leq u_P$, whence from $0 \in P$ and $u_P \in P$ we finally get $\delta_Q \in P$.

A routine variant of the proof above now yields,

Lemma 11. *Let $(A, \hat{u}) \in \ell\mathcal{G}_u^f$. Then $(A, \hat{u}/P) \in \ell\mathcal{G}_u^f$. Moreover, the map $\rho : (A, \hat{u}) \to (A, \hat{u}/P)$ is a morphism in $\ell\mathcal{G}_u^f$.*

Lemma 12. *Let $(A, \hat{u}) \in \ell\mathcal{G}_u^f$. Then for every $P \in \mathrm{Spec}(A)$ the map $\varepsilon^P : \mathbb{Z}^{\mathrm{Spec}(A)} \to A/P$ given by $\varepsilon^P(h) = \varepsilon_A(h)/P$ is a morphism of l-groups.*

Proof. Let $(A, \hat{u}) \in \ell\mathcal{G}_u^f$ and $P \in \mathrm{Spec}(A)$. Then, A/P is a chain. Consider the morphism ρ in $\ell\mathcal{G}_u^f$ given in Lemma 11. In particular, $\widehat{\mathrm{Spec}(\rho)}$ is a morphism in $\ell\mathcal{G}^f$. Furthermore, from Lemmata 10 and 11 we have that $\varepsilon_{A/P} : \mathbb{Z}^{\mathrm{Spec}(A/P)} \to A/P$ is a morphism of l-groups. Thus, $\varepsilon_{A/P} \circ \widehat{\mathrm{Spec}(\rho)} : \mathbb{Z}^{\mathrm{Spec}(A)} \to A/P$ is a morphism of l-groups. In what follows, we will prove that $\varepsilon_{A/P} \circ \widehat{\mathrm{Spec}(\rho)} = \varepsilon^P$, which amounts to proving $(\varepsilon_{A/P} \circ \widehat{\mathrm{Spec}(\rho)})(\delta_{\widetilde{Q}}) = \varepsilon^P(\delta_{\widetilde{Q}})$ for every $Q \in \mathrm{Spec}(A)$.

Let $Q \in \mathrm{Spec}(A)$. If there exists $Z \in \mathrm{Spec}(A/P)$ such that $Q = \rho^{-1}(Z)$ (this Z is necessarily unique) then straightforward computations show that $\delta_{\widetilde{Q}} \circ \mathrm{Spec}(\rho) = \delta_{Q/P}$, so $(\varepsilon_{A/P} \circ \widehat{\mathrm{Spec}(\rho)})(\delta_{\widetilde{Q}}) = \varepsilon^P(\delta_{\widetilde{Q}})$. If for every $Z \in \mathrm{Spec}(A/P)$ we have $Q \neq \rho^{-1}(Z)$ then, from Remark 3, it follows $P \nsubseteq Q$. In particular, $\delta_Q \in P$. Then $(\varepsilon_{A/P} \circ \widehat{\mathrm{Spec}(\rho)})(\delta_{\widetilde{Q}}) = \varepsilon^P(\delta_{\widetilde{Q}}) = 0$. Hence, we have $(\varepsilon_{A/P} \circ \widehat{\mathrm{Spec}(\rho)})(\delta_{\widetilde{Q}}) = \varepsilon^P(\delta_{\widetilde{Q}})$ for every $\widetilde{Q} \in \mathrm{Spec}(\mathbb{Z}^{\mathrm{Spec}(A)})$. Therefore, ε^P is a morphism of l-groups. □

Lemma 13. *Let $(A, \hat{u}) \in \ell\mathcal{G}_u^f$. If $P, Q \in \mathrm{Spec}(A)$ are such that $\varepsilon_A^{-1}(Q) = \widetilde{P}$, then $P = Q$. Moreover, for every $P \in \mathrm{Spec}(A)$ we have $\varepsilon_A^{-1}(P) = \widetilde{P}$.*

Proof. Let $P, Q \in \mathrm{Spec}(A)$ such that $\varepsilon_A^{-1}(Q) = \widetilde{P}$. We have $u_{\widetilde{P}} \in \widetilde{P}$ and by Lemma 8, $\varepsilon_A(u_{\widetilde{P}}) = u_P \in Q$. Hence, $P \subseteq Q$. On the other hand, $\varepsilon_A(u_{\widetilde{Q}}) = u_Q \in Q$. Thus, $u_{\widetilde{Q}} \in \varepsilon_A^{-1}(Q) = \widetilde{P}$. So, $u_{\widetilde{Q}} \in \widetilde{P}$. Then we have $\widetilde{Q} \subseteq \widetilde{P}$, i.e., $Q \subseteq P$. Therefore $P = Q$.

Finally, we will see that $\varepsilon_A^{-1}(P) = \widetilde{P}$ for every $P \in \mathrm{Spec}(A)$. We have $\varepsilon_A(u_{\widetilde{P}}) = u_P \in P$, so $u_{\widetilde{P}} \in \varepsilon_A^{-1}(P)$. Then $\widetilde{P} \subseteq \varepsilon_A^{-1}(P)$. On the other hand, let $u_{\widetilde{Q}}$ be a strong unit of $\varepsilon_A^{-1}(P)$. Hence, $\varepsilon_A(u_{\widetilde{Q}}) = u_Q \in P$. Since

$u_Q \in Q$ then $Q \subseteq P$ and $\widetilde{Q} \subseteq \widetilde{P}$. However $u_{\widetilde{Q}}$ is a strong unit of $\varepsilon_A^{-1}(P)$ and $\widetilde{Q}$. Thus, $\varepsilon_A^{-1}(P) = \widetilde{Q} \subseteq \widetilde{P}$. Then we have $\varepsilon_A^{-1}(P) = \widetilde{P}$. $\square$

Now, we give the first main result of this section.

Proposition 5. *Let $(A, \hat{u}) \in \ell\mathcal{G}_u^f$. Then, $\varepsilon_A : (\mathbb{Z}^{\mathrm{Spec}(A)}, \widehat{u}) \to (A, \hat{u})$ is a morphism in $\ell\mathcal{G}_u^f$.*

Proof. Let $(A, \hat{u})$ be an object of $\ell\mathcal{G}_u^f$. By Lemma 12, we have $\varepsilon_A(h \wedge j)/P = (\varepsilon_A(h)/P) \wedge (\varepsilon_A(j)/P)$ and $\varepsilon_A(h \vee j)/P = (\varepsilon_A(h)/P) \vee (\varepsilon_A(j)/P)$, for every $P \in \mathrm{Spec}(A)$. Since the intersection of all prime l-ideals of A is the zero l-ideal then $\varepsilon_A(h \wedge j) = \varepsilon_A(h) \wedge \varepsilon_A(j)$ and $\varepsilon_A(h \vee j) = \varepsilon_A(h) \vee \varepsilon_A(j)$. From Lemma 8, it follows that ε_A is a morphism in $\ell\mathcal{G}$. By Lemma 13, we have $C_{(P,\varepsilon_A)} \neq \emptyset$ and

$$\varepsilon_A(\delta_{\widetilde{P}}) = \delta_P = \sum_{Q \in C_{(P,\varepsilon_A)}} \delta_Q.$$

Therefore, ε_A is a morphism in $\ell\mathcal{G}_u^f$. $\square$

Remark 4. Let $g : (A, \hat{u}) \to (B, \hat{v})$ be a morphism in $\ell\mathcal{G}_u^f$, and $P \in \mathrm{Spec}(A)$. Then

$$\widehat{\mathrm{Spec}(g)}(\delta_{\widetilde{P}}) = \begin{cases} \sum_{Q \in C_{(P,g)}} \delta_{\widetilde{Q}} & \text{if } C_{(P,g)} \neq \emptyset \\ 0 & \text{if } C_{(P,g)} = \emptyset. \end{cases}$$

Let $\varepsilon_A : (\mathbb{Z}^{\mathrm{Spec}(A)}, \widehat{u}) \to (A, \hat{u})$ and $\varepsilon_B : (\mathbb{Z}^{\mathrm{Spec}(B)}, \widehat{v}) \to (B, v)$ be the morphisms in $\ell\mathcal{G}_u^f$ we have defined above. Let $g : (A, \hat{u}) \to (B, \hat{v})$ be a morphism in $\ell\mathcal{G}_u^f$. We will prove the commutativity of the following diagram:

$$\begin{array}{ccc} (\mathbb{Z}^{\mathrm{Spec}(A)}, \widehat{u}) & \xrightarrow{\ \varepsilon_A\ } & (A, \hat{u}) \\ {\scriptstyle \widehat{\mathrm{Spec}(g)}}\downarrow & & \downarrow{\scriptstyle g} \\ (\mathbb{Z}^{\mathrm{Spec}(B)}, \widehat{v}) & \xrightarrow{\ \varepsilon_B\ } & (B, \widehat{v}). \end{array}$$

The commutativity of the previous diagram is equivalent to proving

$$(g \circ \varepsilon_A)(\delta_{\widetilde{P}}) = (\varepsilon_B \circ (\widehat{\mathrm{Spec}(g)}))(\delta_{\widetilde{P}}),$$

for every $\widetilde{P} \in \mathrm{Spec}(\mathbb{Z}^{\mathrm{Spec}(A)})$.

Let us first note that

$$(g \circ \varepsilon_A)(\delta_{\widetilde{P}}) = g(\varepsilon_A(\delta_{\widetilde{P}})) = g(\delta_P).$$

If $C_{(P,g)} \neq \emptyset$, then, by Remark 4,

$$(\varepsilon_B \circ (\widehat{\mathrm{Spec}(g)}))(\delta_{\widetilde{P}}) = \varepsilon_B \left(\sum_{Q \in C_{(P,g)}} \delta_{\widetilde{Q}} \right) = \sum_{Q \in C_{(P,g)}} \delta_Q.$$

Since g is a morphism in $\ell\mathcal{G}_u^f$ then $(g \circ \varepsilon_A)(\delta_{\widetilde{P}}) = (\varepsilon_B \circ (\widehat{\mathrm{Spec}(g)}))(\delta_{\widetilde{P}})$.

Finally, if $C_{(P,g)} = \emptyset$, then $g(\delta_P) = (\varepsilon_B \circ (\widehat{\mathrm{Spec}(g)}))(\delta_{\widetilde{P}}) = 0$.

Therefore, we have proved that $(g \circ \varepsilon_A)(\delta_{\widetilde{P}}) = (\varepsilon_B \circ (\widehat{\mathrm{Spec}(g)}))(\delta_{\widetilde{P}})$.

The second main result of this section is the following one.

Theorem 14. *The functor* $\Lambda : \mathcal{FRS}^{op} \to \ell\mathcal{G}_u^f$ *is left adjoint to* Spec*.*

6. Final remarks

In [7] the authors study a problem closely related to the one investigated here. In what follows, we study some connections between the results in [7] and those in the present paper.

Let F be a finite root system. Since F is finite, the set of minimal elements of F, say $\min(F)$, is also finite. We associate to F another root system F^+ such that any root in F^+ is a chain. If $\min(F) = \{m_1, \ldots, m_n\}$, we can take

$$F^+ := \coprod_{m \in \min(F)} [m).$$

Moreover, there is a unique surjective p-morphism $\kappa : F^+ \to F$ making, for every $m \in \min(F)$, the following diagram commute,

$$\begin{array}{ccc} F^+ & \overset{\kappa}{\twoheadrightarrow} & F \\ {\scriptstyle i_m}\nwarrow & & \nearrow{\scriptstyle j_m} \\ & [m). & \end{array} \tag{9}$$

Here, i_m and j_m are the natural inclusions of $[m)$ into F^+ and F respectively. We call such a morphism $\kappa : F^+ \to F$ a *covering of* F.

Given a covering $\kappa : F^+ \to F$, the functor $\Lambda : \mathcal{FRS}^{op} \to \ell\mathcal{G}_u^f$ defines a monomorphism $\Lambda(\kappa) : \Lambda(F) \to \Lambda(F^+)$. Applying Λ to diagram 9, we get the following commutative diagram,

$$\begin{array}{ccccc} \Lambda(F) & \overset{\Lambda(\kappa)}{\rightarrowtail} & \Lambda(F^+) & \rightarrowtail\!\!\twoheadrightarrow & \prod_m \Lambda([m)) \\ & {\scriptstyle \Lambda(j_m)}\searrow & & \swarrow{\scriptstyle pr_m} & \\ & & \Lambda([m)). & & \end{array} \tag{10}$$

Here, $p_m : \prod_m \Lambda([m)) \to \Lambda([m))$ is the corresponding projection map.

Note that for every $h \in \Lambda([m))$, there exists $h' \in \Lambda(F)$ such that $(pr_m \circ \Lambda(\kappa))(h') = h$. Hence, $\Lambda(F)$ is a subdirect product of the totally ordered groups $\Lambda([m))$, for $m \in \min(F)$.

We have seen that Λ is part of an adjunction, more precisely, $\Lambda \dashv \mathrm{Spec}$. If we restrict the codomain of Λ to its image, we get a dual equivalence

$$\Lambda : \mathcal{FRS}^{op} \rightleftarrows \Lambda(\mathcal{FRS}) : \mathrm{Spec}.$$

Here, $\Lambda(\mathcal{FRS})$ is the full subcategory of $\ell\mathcal{G}_u^f$ whose objects are certain free ordered $\mathbb{Z}$-modules, as described in Lemma 4.

Let F be a finite root system. A *labelling on* F is a function $\lambda : F^+ \to \mathbb{Z}^{>0}$. A pairs (F, λ), with F a finite root system and λ a labelling on F, is called a *labelled root system*. For two labelled root systems (F, λ) and (G, μ), we say that a p-morphism $\varphi : F \to G$ is a *morphism of labelled root systems*. Let us consider the category $l\mathcal{FRS}$, whose objects are labelled root systems and whose morphisms are morphisms of labelled root systems.

Let (F, λ) be a finite labelled root system. We associate to (F, λ) a lattice ordered group with order unit in the following way.

Recall that $\{\delta_x \mid x \in \Lambda([m))\}$ is a basis for $\Lambda([m))$, and their union a basis for $\Lambda(F^+)$. Let us define $\lambda_m : \Lambda([m)) \to \Lambda([m))$ as the unique morphism in $\ell\mathcal{G}_u^f$ given by $\lambda_m(\delta_x) := \lambda(x)\delta_x$. Let us write $A(\lambda_m)$ for the subobject of the ordered group $\mathbb{Z}^{\otimes k_m} \cong \Lambda([m))$, obtained by considering $\mathrm{im}(\lambda_m) \leq \Lambda([m)) \cong \mathbb{Z}^{\otimes k_m}$. The group $A(\lambda_m)$ is totally ordered, with basis $\{\lambda_m(\delta_x) \mid x \in \Lambda([m))\}$. In particular, we have that $A(\lambda_m) \cong \mathbb{Z}^{\otimes k_m}$ in $\ell\mathcal{G}_u^f$.

Let us now consider the inclusion

$$\Lambda(F) \xrightarrow{\Lambda(\kappa)} \Lambda(F^+) \cong \prod_m \Lambda([m)) \cong \prod_m \mathbb{Z}^{\otimes k_m}$$

and the product morphism $\widetilde{\lambda} := \prod\limits_m \lambda_m : \prod\limits_m \Lambda([m)) \to \prod\limits_m \mathbb{Z}^{\otimes k_m}$. We have $\mathrm{im}(\widetilde{\lambda}) \cong \prod\limits_m A(\lambda_m)$. Write $\widehat{\lambda}$ for the composition $\widehat{\lambda} := \widetilde{\lambda} \circ \Lambda(\kappa)$. Since $\widetilde{\lambda}$ and $\Lambda(\kappa)$ are both injective, so is $\widehat{\lambda}$. Then $\mathrm{im}(\widehat{\lambda})$ is a $\ell\mathcal{G}_u^f$ subobject of $\prod_m A(\lambda_m)$, whose spectrum is isomorphic to F. Write $A(\lambda)$ for $\mathrm{im}(\widehat{\lambda})$. Any algebra of the form $A(\lambda)$ for some labelled root system (F, λ) is called a *QF-group*. We have to define now what a morphism between QF-groups is.

Let (F, λ) and (G, μ) be two labelled root systems and $\varphi : F \to G$ a morphism between them. Since $\widehat{\lambda}$ and $\widehat{\mu}$ are isomorphisms, there exists a unique morphism φ^l yielding the following commutative diagram in $\ell\mathcal{G}_u^f$

$$\begin{array}{ccc} \Lambda(F) & \xrightarrow{\widehat{\lambda}} & A(\lambda) \\ \uparrow{\scriptstyle \Lambda(\varphi)} & & \uparrow{\scriptstyle \varphi^l} \\ \Lambda(G) & \xrightarrow[\widehat{\mu}]{} & A(\mu). \end{array}$$

A *morphism of QF-groups* is one of the form φ^l, for some φ between their spectra.

Let us now consider the subcategory QF of $\ell\mathcal{G}_u^f$, whose objects are QF-groups and whose morphism are the QF-groups morphisms defined above. Let us notice that this category resembles the category QFC of [7]. However, since we are taking subdirect products in $\ell\mathcal{G}_u^f$, we do not get a full subcategory of $\ell\mathcal{G}$, contrary to QFC, which is a full subcategory of $\mathcal{MV}$.

A straightforward computation shows that $\Lambda \dashv \mathrm{Spec}$ is a dual equivalence between the categories $l\mathcal{FRS}$ and QF. While this result is not equivalent to the main result of [7], it follows the same lines of thought.

We end this section by giving an alternative elementary proof of [7, Lemma 13] which could be of interest in itself.

Lemma 15. *Let A, B be MV-algebras and $f : A \to B$ a homomorphism, then,* $\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is a p-morphism.*

Proof. Let $P \in \mathrm{Spec}(B)$ and $Q \in \mathrm{Spec}(A)$ such that $f^{-1}(P) \subseteq Q$. Write $\Sigma := \{Z \in \mathrm{Spec}(B) : P \subseteq Z \text{ and } f^{-1}(Z) \subseteq Q\}$. It has a maximal element M. Moreover, $M \in \mathrm{Spec}(B)$, $P \subseteq M$ and $f^{-1}(M) \subseteq Q$.

In order to see that $f^{-1}(M) = Q$, let us take $a \in Q$ and the ideal I of MV-algebras generated by $M \cup \{f(a)\}$. We will show that $f^{-1}(I) \subseteq Q$.

Let $b \in f^{-1}(I)$. There are a natural number n and an element $c \in M$ such that $f(b) \leq nf(a) \oplus c = f(na) \oplus c = \neg(\neg f(na) \odot \neg c) = \neg(f(\neg na) \odot \neg c)$. Thus, $f(b \odot \neg na) = f(b) \odot f(\neg na) \leq c \in M$, and

$f(b \odot \neg na) \in M$. Since $f^{-1}(M) \subseteq Q$, it follows that $b \odot \neg na \in Q$. Hence, there is an element $d \in Q$ such that $b \leq \neg na \to d = na \oplus d \in Q$, which implies that $b \in Q$. Thus, $f^{-1}(I) \subseteq Q$, and we have $I = M$, so $f(a) \in M$, i.e., $a \in f^{-1}(M)$. Hence, $f^{-1}(Q) = M$. □

Acknowledgements

This work was supported by Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina [PIP 112-201101-00636] and Universidad Nacional de La Plata [11/X667]. Part of this research was carried out in Milan by the second author, within the MATOMUVI project (FP7-PEOPLE-2009-IRSES, contact number 247584) of the European Union.

We would specially like to acknowledge Vincenzo Marra, who was the academic host of the second author in Milan, for suggesting us the problem and for several useful comments and remarks about this manuscript.

References

[1] R. Cignoli, F. Glushankof, F. Lucas, Prime spectra of lattice-ordered Abelian groups, J. Pure Appl. Algebra 136 (1999) 217–229.
[2] R. Cignoli, I. D'Ottaviano, D. Mundici, Algebraic Foundations of Many-Valued Reasoning, Trends in Logic, Studia Logica Library, vol. 7, Kluwer Academic Publishers, 2000.
[3] R. Cignoli, A. Torrens, The poset of prime l-ideals of an Abelian l-group with a strong unit, J. Algebra 184 (1996) 604–612.
[4] P. Conrad, Some structure theorems for lattice-ordered groups, Trans. Am. Math. Soc. 99 (1961) 212–240.
[5] P. Conrad, J. Harvey, C. Holland, The Hahn embedding theorem for Abelian lattice-ordered groups, Trans. Am. Math. Soc. 108 (1963) 143–169.
[6] M.R. Darnel, Theory of Lattice-Ordered Groups, Monographs and Textbooks in Pure and Applied Mathematics, vol. 187, Marcel Dekker, Inc., New York, 1995.
[7] A. Di Nola, R. Grigolia, MV-algebras in duality with labelled root systems, Discrete Math. 243 (2002) 79–90.
[8] D. Mundici, Interpretation of AF C^* algebras in Łukasiewicz sentential calculus, J. Funct. Anal. 65 (1986) 15–63.