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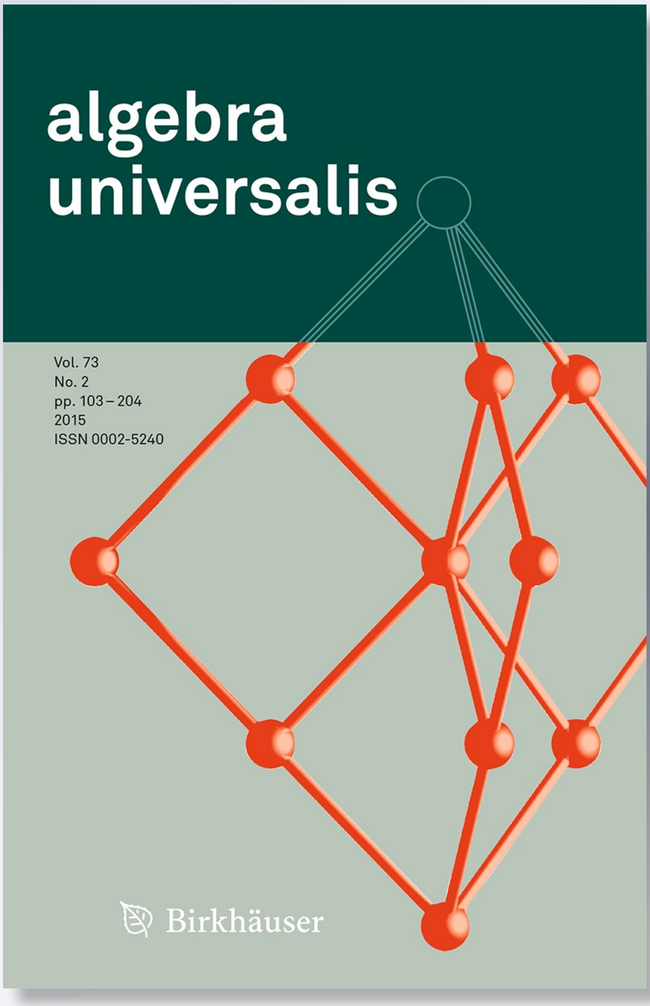

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Compatible operations on commutative weak residuated lattices

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Abstract. Compatibility of functions is a classical topic in Universal Algebra related to the notion of affine completeness. In algebraic logic, it is concerned with the possibility of implicitly defining new connectives.

In this paper, we give characterizations of compatible operations in a variety of algebras that properly includes commutative residuated lattices and some generalizations of Heyting algebras. The wider variety considered is obtained by weakening the main characters of residuated lattices $(A, \wedge, \vee, \cdot, \rightarrow, e)$ but retaining most of their algebraic consequences, and their algebras have a commutative monoidal structure. The order-extension principle $a \leq b$ if and only if $a \rightarrow b \geq e$ is replaced by the condition: if $a \leq b$, then $a \rightarrow b \geq e$. The residuation property $c \leq a \rightarrow b$ if and only if $a \cdot c \leq b$ is replaced by the conditions: if $c \leq a \rightarrow b$, then $a \cdot c \leq b$, and if $a \cdot c \leq b$, then $e \rightarrow c \leq a \rightarrow b$. Some further algebraic conditions of commutative residuated lattices are required.

1. Introduction

The problem of adding connectives to extend a logic in a "natural" way has been broadly studied. For intuitionistic calculus, the paper [4] of X. Caicedo and R. Cignoli emphasizes the algebraic aspect of the problem through the notion of *compatible function*, which translates to the notion of compatible connective in intuitionistic logic. Results in [4] are extended to algebraizable logics by X. Caicedo in [2] (see also [3]).

In [5], compatible functions were studied in commutative residuated lattices, following basically the characterization of compatible functions by means of the relationship between congruences and convex subalgebras ([12]). In [16], compatible functions were studied in the weak Heyting algebra $(A, \wedge, \vee, \rightarrow, 0, 1)$, which satisfy the inequality $a \wedge (a \rightarrow b) \leq b$, using essentially the description of compatible functions by means of the relationship between congruences and open filters [7]. In the present work, we study compatible functions in a new variety that includes the previous ones, providing a common framework to the results given in [5, 16].

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Let us explain the original motivation for the construction of the new algebras. In a weak Heyting algebra $(A, \wedge, \vee, \rightarrow, 0, 1)$, the inequality $a \wedge (a \rightarrow b) \leq b$ is equivalent to saying that for every $a, b, c \in A$, if $b \leq a \rightarrow c$, then $a \wedge b \leq c$ [7, Proposition 4.22]. We generalize commutative residuated lattices, introducing algebras $(A, \wedge, \vee, \cdot, \rightarrow, e)$ that satisfy the inequality $a \cdot (a \rightarrow b) \leq b$ and some additional conditions, which in particular provide that the inequality $a \cdot (a \rightarrow b) \leq b$ is equivalent to the following condition: for every $a, b, c \in A$, if $b \leq a \rightarrow c$, then $a \cdot b \leq c$.

Definition 1.1. A *generalized commutative residuated lattice*, *GCRL* for short, is an algebra $(A, \wedge, \vee, \cdot, \rightarrow, e)$ that satisfies the following conditions for every $a, b, c \in A$:

1. $(A, \cdot, e)$ is a commutative monoid,
2. $(A, \vee, \wedge)$ is a lattice,
3. $(a \rightarrow b) \wedge (a \rightarrow c) = a \rightarrow (b \wedge c)$,
4. $(a \rightarrow c) \wedge (b \rightarrow c) = (a \vee b) \rightarrow c$,
5. $(a \rightarrow b) \cdot (b \rightarrow c) \leq a \rightarrow c$,
6. $e \leq a \rightarrow a$.

For $a \in A$ and $n \geq 1$, we define inductively $a^0 = e$ and $a^n = a \cdot a^{n-1}$. We also define $\Box^0(a) = a$, $\Box(a) = e \rightarrow a$ and the iterated operator $\Box^n$ in the usual way. The map $\Box$ preserve finite meets, so in particular $\Box$ is monotonic.

Definition 1.2. A *commutative weak residuated lattice*, *CWRL* for short, is a *GCRL* that satisfies the following conditions for every $a, b, c \in A$:

(**R1**) $a \cdot (a \rightarrow b) \leq b$,
(**R2**) $(a \cdot b) \vee (a \cdot c) = a \cdot (b \vee c)$,
(**R3**) $\Box(a) \leq b \rightarrow (b \cdot a)$,
(**R4**) $a \leq (a \rightarrow e) \rightarrow e$,
(**R5**) $\Box(a^2) \leq \Box(a) \cdot \Box(a)$,
(**R6**) $\Box(a) \rightarrow (\Box(a) \rightarrow e) \leq \Box^2(a) \rightarrow e$,
(**R7**) $(\Box(a) \rightarrow e) \cdot (\Box(a) \rightarrow e) \leq \Box(a^2) \rightarrow e$.

We write CWRL for the variety of *CWRLs*. A *commutative residuated lattice* (*CRL* for short) is an ordered algebraic structure $(A, \wedge, \vee, \cdot, \rightarrow, e)$, where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, e)$ is a commutative monoid, and $\rightarrow$ is a binary operation such that for every $a, b, c \in A$, the condition $a \cdot b \leq c$ if and only if $b \leq a \rightarrow c$ is satisfied. The *CRLs* form a variety; we write CRL for this variety. It follows from properties of *CRLs* [12, 13] that CRL is a subvariety of CWRL; in particular, the inequalities (**R5**), (**R6**) and (**R7**) are equalities in CRL. Moreover, if $A \in$ CWRL, then $A \in$ CRL if and only if $\Box(a) = a$ for every $a \in A$ (Remark 2.5).

A *weak Heyting algebra*, or *WH-algebra* [1, 7], is an ordered algebraic structure $(A, \wedge, \vee, \rightarrow, 0, 1)$, where $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\rightarrow: A \times A \rightarrow A$ is a map such that for all $a, b, c \in A$, the following conditions are satisfied:

1. $(a \to b) \wedge (a \to c) = a \to (b \wedge c)$,
2. $(a \to c) \wedge (b \to c) = (a \vee b) \to c$,
3. $(a \to b) \wedge (b \to c) \leq a \to c$,
4. $a \to a = 1$.

A *RWH-algebra* [7] is a WH-algebra that in addition satisfies the following inequality:

(R) $a \wedge (a \to b) \leq b$.

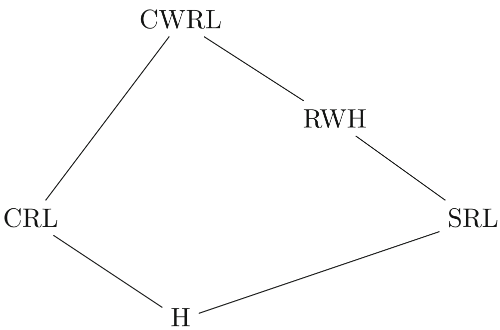
These algebras are a generalization of Heyting algebras. The variety of *RWH*-algebras will be denoted by RWH. This is a subvariety of CWRL in the following sense: if $(A, \wedge, \vee, \to, 0, 1) \in$ RWH, then $(A, \wedge, \vee, \wedge, \to, 1) \in$ CWRL; in particular, the inequalities (**R5**), (**R6**) and (**R7**) are equalities in RWH.

A *subresiduated lattice* is a *RWH*-algebra that in addition satisfies the following inequality:

(T) $a \to b \leq c \to (a \to b)$.

Subresiduated lattices were introduced by G. Epstein and A. Horn in [9], and they were also studied in [7]. The variety of subresiduated lattices will be denoted by SRL. We write H to indicate the variety of Heyting algebras.

We obtain the following diagram:



The aims of this work are the following:

(i) to study compatible functions in CWRL;
(ii) to extend results about compatible functions in CRL [5];
(iii) to extend results about compatible functions in RWH and in SRL [16].

Note that CWRL generalizes CRL as RWH generalizes H: the conceptual step is the same. The choice of the conditions (**R2**)–(**R7**) is needed to characterize the compatible functions as a natural generalization of the case of the variety CRL. This will become more clear in the development of this work.

In Section 2, we give some basic results about the variety CWRL. In Section 3, we study the structure of the congruence lattice of any algebra of CWRL, which we shall need later. Then in Section 4, we give characterizations for compatible functions, and we prove that the variety CWRL is locally affine complete. Finally, in Section 5, we give a method to build up unary compatible functions.

2. Basic results

The next elemental lemmas will be important for this work.

Lemma 2.1. *Let A be a GCRL that satisfies* (**R2**), *and let $a, b, c \in A$.*

(a) *If $a \leq b$, then $a \cdot c \leq b \cdot c$.*
(b) *If $b \leq e$, then $a \cdot b \leq b$.*

Proof. (a): This is a direct consequence of the condition (**R2**).
(b): This follows from the item (a). □

Lemma 2.2. *Let A be a GCRL. For every $a, b, c \in A$, if $a \leq b$, then we have $c \to a \leq c \to b$, $b \to c \leq a \to c$, and $e \leq a \to b$. Moreover, if A satisfies* (**R1**), *then $e \to e = e$.*

Proof. Let $a, b, c \in A$ and $a \leq b$. Then from the proof of [7, Proposition 3.22], $b \to c \leq a \to c$ and $c \to a \leq c \to b$. Finally, $a \to b \geq b \to b \geq e$. □

Remark 2.3. Let A be a *GCRL* with top 1 and $e = 1$. Then, $a \to a = 1$ and $a \to 1 = 1$ for every $a \in A$. Moreover, A satisfies the inequalities (**R4**), (**R6**), and (**R7**).

Lemma 2.4. *Let A be a GCRL.*

(a) *Suppose that A satisfies the equation* (**R2**). *Then A satisfies the inequality* (**R1**) *if and only if for every $a, b, c \in A$, if $a \leq b \to c$, then $b \cdot a \leq c$.*
(b) *A satisfies the inequality* (**R3**) *if and only if for every $a, b, c \in A$, if $b \cdot a \leq c$, then $\Box(a) \leq b \to c$.*

Proof. (a): Suppose that A satisfies the equation (**R1**) and let $a, b, c \in A$ be such that $a \leq b \to c$. It follows from Lemma 2.1 that $b \cdot a \leq b \cdot (b \to c) \leq c$, so $b \cdot a \leq c$. Conversely, suppose that for every $a, b, c \in A$, if $a \leq b \to c$, then $b \cdot a \leq c$. For $a, b \in A$, we have that $a \to b \leq a \to b$. Hence, $a \cdot (a \to b) \leq b$.

(b): Suppose that A satisfies the inequality (**R3**). Let $a, b, c \in A$ be such that $b \cdot a \leq c$. Then $\Box(a) \leq b \to (b \cdot a) \leq b \to c$. Hence, $\Box(a) \leq b \to c$. Conversely, suppose that for every $a, b, c \in A$, if $b \cdot a \leq c$, then $\Box(a) \leq b \to c$. Let $a, b \in A$. Taking into account that $b \cdot a \leq b \cdot a$, we conclude that $\Box(a) \leq b \to (b \cdot a)$. □

The following remark comes from the previous lemma.

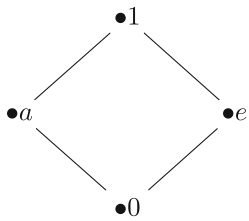
Remark 2.5. Let $A \in$ CWRL.

(i) $\Box(a) \leq a$ for every $a \in A$. Moreover, $\Box^n(a) \leq \Box^m(a)$ when $n \geq m$.
(ii) $A \in$ CRL if and only if $\Box(a) = a$ for every $a \in A$.

Finally, we consider two examples of *CWRLs*.

Example 2.6. Let $\mathbb{R}$ be the real numbers and let $(0, 1] = \{a \in \mathbb{R} : 0 < a \leq 1\}$. For every $k \geq 1$, we define the structure $A = ((0, 1], \wedge, \vee, \cdot, \to, 1)$, where $\cdot$ is the usual product of real numbers and $a \to b = (\frac{b^k}{a^k}) \wedge 1$. Straightforward computations show that $A \in$ CWRL. For $k > 1$, we have that $A \notin$ CRL because for $a \neq 1$, we obtain $\Box(a) = a^k < a$. For $k = 1$, $A \in$ CRL.

Example 2.7. Let H be the following poset:



Consider the following binary operations:

$\cdot$	0	a	e	1
0	0	0	0	0
a	0	a	a	a
e	0	a	e	1
1	0	a	1	1

$\to$	0	a	e	1
0	1	1	1	1
a	0	1	0	1
e	0	0	e	1
1	0	0	0	1

We have that $A = (H, \wedge, \vee, \cdot, \to, e) \in$ CWRL. Moreover, $\Box(a) = 0$. Therefore $A \notin$ CRL.

Remark 2.8. The varieties CRL and RWH are proper subvarieties of CWRL.

3. Convex subalgebras

In this section, we study the structure of the congruence lattice of any *CWRL*, which we shall need later in order to give characterizations for compatible functions. Since we are building on ideas of the paper [12], we recommend the reader to have that paper at hand while reading this section.

We will refer to a subset H of a commutative weak residuated lattice A as being a *subalgebra* of A provided H is closed with respect to the operations defined on A. Let $A \in$ CWRL and let H be a *convex subalgebra* of A, that is, a subalgebra such that if $a, b \in H$ and $a \leq c \leq b$, then $c \in H$. We write $\mathrm{Sub_C}(A)$ for the set of convex subalgebras of A. We write $\mathrm{Con}(A)$ for the set of congruences of A. If $\theta \in \mathrm{Con}(A)$ and $a \in A$, we write a/θ for the equivalence class of A. We will see that there is an order isomorphism between $\mathrm{Con}(A)$ and $\mathrm{Sub_C}(A)$.

Lemma 3.1. *Let $A \in$ CWRL. If $\theta \in \mathrm{Con}(A)$, then $e/\theta \in \mathrm{Sub_C}(A)$.*

Proof. The proof is similar to that of the case of commutative residuated lattices (see [12, Lemma 2.1]). $\Box$

Let $A \in \mathrm{CWRL}$ and $H \in \mathrm{Sub_C}(A)$. We define the following sets:

$$\theta_H = \{(a,b) \in A \times A : a \cdot h \leq b \text{ and } b \cdot h \leq a \text{ for some } h \in H\},$$
$$K_1 = \{(a,b) \in A \times A : (a \to b) \wedge e \in H \text{ and } (b \to a) \wedge e \in H\},$$
$$K_2 = \{(a,b) \in A \times A : h \leq a \to b \text{ and } h \leq b \to a \text{ for some } h \in H\}.$$

Lemma 3.2. *Let* $A \in \mathrm{CWRL}$ *and* $H \in \mathrm{Sub_C}(A)$. *Then* $\theta_H = K_1 = K_2$.

Proof. Straightforward computations based on Lemma 2.4 show $\theta_H = K_2$. It is immediate that $K_1 \subseteq K_2$. To see that $K_2 \subseteq K_1$, let $(a,b) \in K_2$. Thus, there exists $h \in H$ such that $h \leq a \to b$ and $h \leq b \to a$. Hence, $h \wedge e \leq (a \to b) \wedge e \leq e$ and $h \wedge e \leq (b \to a) \wedge e \leq e$. As $h \wedge e \in H$, by the convexity of H we have that $(a \to b) \wedge e \in H$ and $(b \to a) \wedge e \in H$. Therefore, $K_2 \subseteq K_1$. □

Lemma 3.3. *Let* $A \in \mathrm{CWRL}$. *If* $H \in \mathrm{Sub_C}(A)$, *then* $\theta_H \in \mathrm{Con}(A)$.

Proof. It is routine to prove that θ_H is an equivalence relation.

Let $(a,b), (c,d) \in \theta_H$. Easy computations prove $(a \vee c, b \vee d), (a \cdot c, b \cdot d) \in \theta_H$ (see the proof of [12, Lemma 2.2]). Using Lemma 3.2, we will prove that $(a \wedge c, b \wedge d) \in \theta_H$ and $(a \to c, b \to d) \in \theta_H$. By definition of θ_H, there exist $h, j \in H$ such that $h \leq a \to b$, $h \leq b \to a$, $j \leq c \to d$, and $j \leq d \to c$. Clearly, we can replace h and j by $k = h \wedge j$. Thus, we have that

$$(a \wedge c) \to (b \wedge d) = ((a \wedge c) \to b) \wedge ((a \wedge c) \to d) \geq (a \to b) \wedge (c \to d) \geq k.$$

Similarly, we can prove that $(b \wedge d) \to (a \wedge c) \geq k$. Thus, $(a \wedge c, b \wedge d) \in \theta_H$.

On the other hand, $k \cdot (a \to c) \cdot k \leq (b \to a) \cdot (a \to c) \cdot (c \to d) \leq b \to d$. Hence, $(a \to c) \cdot k^2 \leq b \to d$. The inequality $(b \to d) \cdot k^2 \leq a \to c$ follows in a like manner. Therefore, we obtain $(a \to c, b \to d) \in \theta_H$. □

Lemma 3.4. *Let* $A \in \mathrm{CWRL}$ *and* $a, h \in A$ *such that* $a \cdot h \leq e$. *Then* $a \leq \Box(h) \to e$.

Proof. Let $a \cdot h \leq e$. By Lemma 2.4, we have that $\Box(h) \leq a \to e$, so by (**R4**) and Lemma 2.2, we obtain that $a \leq (a \to e) \to e \leq \Box(h) \to e$. □

Theorem 3.5. *If* A *is a member of* CWRL, *then* $\mathrm{Con}(A)$ *is order isomorphic to* $\mathrm{Sub_C}(A)$. *The isomorphism is established via the assignments* $\theta \mapsto e/\theta$ *and* $H \mapsto \theta_H$.

Proof. The stated correspondences follow by lemmas 3.1 and 3.3. Let $\theta \in \mathrm{Con}(A)$ and $H \in \mathrm{Sub_C}(A)$. The fact that $\theta = \theta_{e/\theta}$ and $H \subseteq e/\theta_H$ is proved as in the proof of [12, Theorem 2.3]. In order to prove that $e/\theta_H \subseteq H$, let $a \in e/\theta_H$, that is, $(a,e) \in \theta_H$. Thus, there exists $h \in H$ such that $a \cdot h \leq e$ and $h \leq a$. It follows from Lemma 3.4 that $h \leq a \leq \Box(h) \to e$. Hence, $a \in H$ by convexity. Straightforward computations prove that the stated bijections are order-preserving. □

Remark 3.6. (1) Let $A \in$ CRL. Theorem 3.5 generalizes [12, Theorem 2.3], which establishes an order isomorphism between the lattice of congruences of A and the lattice of convex subalgebras of A.

(2) Let $A \in$ RWH. A filter H of A is said to be an *open filter* if for every $a \in H$, $\Box(a) \in H$. If A is seen as an algebra of CWRL, then the open filters are exactly the convex subalgebras. Theorem 3.5 also generalizes [7, Theorem 6.12], which establishes an order isomorphism between the lattice of open filters of A and the lattice of congruences of A.

Lemma 3.7. *Let $A \in$ CWRL. Then for every $n \geq 1$, the following inequalities hold:*

(a) $\Box^n(a^2) \leq \Box^n(a) \cdot \Box^n(a)$,
(b) $\Box^n(a) \to (\Box^n(a) \to e) \leq \Box^{n+1}(a) \to e$,
(c) $(\Box^n(a) \to e) \cdot (\Box^n(a) \to e) \leq (\Box^n(a^2) \to e)$.

Proof. This follows from an induction based on the inequalities **(R5)**, **(R6)**, and **(R7)**. □

For any $A \in$ CWRL, we will write A^- for the *negative cone* of A, that is, $A^- = \{x \in A : x \leq e\}$. For any $S \subseteq A$, we will let $C[S]$ denote the smallest convex subalgebra containing S and will let $C[a] = C[\{a\}]$. In what follows, we will let $\langle S \rangle$ denote the *submonoid* of $(A, \cdot, e)$ generated by S. We will write $\mathbb{N}$ for the set of natural numbers.

The following result is analogous to [12, Lemma 2.7].

Lemma 3.8. *Let $A \in$ CWRL and $S \subseteq A^-$. Then*

$$C[S] = \{x \in A : \Box^n(h) \leq x \leq \Box^n(h) \to e, \ \textit{for some } h \in \langle S \rangle \ \textit{ and } n \in \mathbb{N}\}.$$

Proof. If $a \in A^-$, then $\Box(a) \in A^-$. Let $S \subseteq A^-$. Then, $\langle S \rangle \subseteq A^-$. Put

$$K = \{x \in A : \Box^n(h) \leq x \leq \Box^n(h) \to e, \ \text{for some } h \in \langle S \rangle \ \text{ and } n \in \mathbb{N}\}.$$

It is clear that $S \subseteq K \subseteq C[S]$. It will suffice to show that K is a convex subalgebra of A. Let $a, b \in K$. Thus, there are $h_a, h_b \in \langle S \rangle$ and $n, m \in \mathbb{N}$ such that $\Box^n(h_a) \leq a \leq \Box^n(h_a) \to e$ and $\Box^m(h_b) \leq b \leq \Box^m(h_b) \to e$. We may replace h_a and h_b by $h = h_a \cdot h_b$, and n, m by $k = \max\{n, m\}$. The set K is convex because if $\Box^k(h) \leq a \leq x \leq b \leq \Box^k(h) \to e$, then $x \in K$. On the other hand, $\Box^k(h) \leq a \wedge b \leq a \vee b \leq \Box^k(h) \to e$, so K is closed under meets and joins. Now, by Lemma 3.7, we obtain

$$\Box^k(h^2) \leq \Box^k(h) \cdot \Box^k(h) \leq a \cdot b \leq (\Box^k(h) \to e) \cdot (\Box^k(h) \to e) \leq \Box^k(h^2) \to e.$$

Thus, K is also closed under product.

Finally, we show that K is closed under arrow (for it, we use Lemma 3.7 again). First, observe that $a \cdot \Box^k(h) \leq e$ and $\Box^k(h) \leq b$. Then we obtain $a \cdot \Box^k(h^2) \leq a \cdot \Box^k(h) \cdot \Box^k(h) \leq e \cdot b = b$. Thus, by Lemma 2.4, we have that

$$\Box^{k+1}(h^2) \leq a \to b. \tag{3.1}$$

Taking into account that $\Box^k(h) \leq a$ and $b \leq \Box^k(h) \to e$, we obtain that $a \to b \leq \Box^k(h) \to b \leq \Box^k(h) \to (\Box^k(h) \to e)$, so

$$a \to b \leq \Box^k(h) \to (\Box^k(h) \to e) \leq \Box^{k+1}(h) \to e \leq \Box^{k+1}(h^2) \to e. \tag{3.2}$$

Therefore, by inequalities (3.1) and (3.2), we conclude that $a \to b \in K$. $\Box$

The good description of $C[S]$ given in the previous lemma justifies the choice of the inequalities **(R5)**, **(R6)**, and **(R7)**.

Corollary 3.9. *Let $A \in$ CWRL and $a \in A^-$. Then*

$$C[a] = \{x \in A : \Box^n(a^m) \leq x \leq \Box^n(a^m) \to e, \text{ for some } n, m \in \mathbb{N}\}.$$

Moreover, if $x \in A^-$, then $x \in C[a]$ if and only if there are n, m such that $\Box^n(a^m) \leq x$.

4. Compatible functions

In this section, we characterize the compatible functions in the variety of commutative weak residuated lattices, and we use this result to prove that the variety CWRL is locally affine complete. We start with the following.

Definition 4.1. Let A be an algebra and let $f\colon A^k \to A$ be a function.

1. We say that f is *compatible with a congruence* θ *of* A if $(a_i, b_i) \in \theta$ for $i = 1, \ldots, k$ implies $(f(a_1, \ldots, a_k), f(b_1, \ldots, b_k)) \in \theta$.
2. We say that f is a *compatible function* of A provided it is compatible with all the congruences of A.

If $A \in$ CWRL and $a, b \in A$, denote by $\theta(a, b)$ the smallest congruence that contains the element (a, b). We also define $a \leftrightarrow b = (a \to b) \wedge (b \to a)$, $d(a, b) = (a \leftrightarrow b) \wedge e$, and $p(a, b) = ((a \to b) \wedge e) \cdot ((b \to a) \wedge e)$.

The following lemma is useful in order to give a description of compatible functions.

Lemma 4.2. *Let $A \in$ CWRL and $a, b \in A$.*

(a) *If $\theta \in \mathrm{Con}(A)$, then $(a, b) \in \theta$ if and only if $d(a, b) \in e/\theta$.*
(b) *If $\theta \in \mathrm{Con}(A)$, then $(a, b) \in \theta$ if and only if $p(a, b) \in e/\theta$.*
(c) *$e/\theta(a, b) = C[d(a, b)] = C[p(a, b)]$.*

Proof. (a): The proof is similar to the proof of [13, Lemma 3.1].

(b): The proof is similar to the proof of [5, Lemma 5].

(c): This follows from the previous items and Theorem 3.5:

$$e/\theta(a, b) = \bigcap_{(a,b)\in\theta} e/\theta = \bigcap_{d(a,b)\in e/\theta} e/\theta = C[d(a, b)].$$

In a similar way, we can prove that $e/\theta(a, b) = C[p(a, b)]$. $\Box$

Let H be an algebra and let $f: A \to A$ be a function. Recall the following convenient remark:

f is a compatible function if and only if $(f(a), f(b)) \in \theta(a, b)$ for every a, b.

Proposition 4.3. *Let A be a* CWRL *and let $f: A \to A$ be a function. The following conditions are equivalent:*

1. *f is compatible.*
2. *For every $a, b \in A$ there exist $n, m \in \mathbb{N}$ with $\Box^n(d(a, b)^m) \leq d(f(a), f(b))$.*
3. *For every $a, b \in A$ there exist $n, m \in \mathbb{N}$ with $\Box^n(p(a, b)^m) \leq p(f(a), f(b))$.*

Proof. This follows from Lemma 4.2 and Corollary 3.9. □

Let A be an algebra and $f: A^k \to A$ a function. For every $b_i \in A$ (where $i = 1, \ldots, k$), we define $\bar{b} = (b_1, \ldots, b_k)$ and $\bar{b}(i) = (b_1, \ldots, b_{i-1}, b_{i+1} \ldots, b_k)$. Then we define the functions $f_{\bar{b}(i)}: A \to A$ by

$$f_{\bar{b}(i)}(a) = f(b_1, \ldots, b_{i-1}, a, b_{i+1}, \ldots, b_k).$$

The following remark is a consequence of the definition of k-ary compatible function on an algebra.

Remark 4.4. Let A be an algebra and let $f: A^k \to A$ be a function. The function f is compatible if and only if for every $\bar{b} \in A^k$, the functions $f_{\bar{b}(i)}$ are compatible.

Corollary 4.5. *Let $A \in$* CWRL *and let $f: A^k \to A$ be a function. The following conditions are equivalent:*

1. *f is compatible.*
2. *For every $\overline{a} = (a_1, \ldots, a_k) \in A^k$ and $\bar{b} = (b_1, \ldots, b_k) \in A^k$ there are $n, m \in \mathbb{N}$ such that*

$$\Box^n(d(a_1, b_1)^m) \cdot \Box^n(d(a_2, b_2)^m) \cdot \cdots \cdot \Box^n(d(a_k, b_k)^m) \leq d(f(\overline{a}), f(\bar{b})). \quad (4.1)$$

3. *For every $\overline{a} = (a_1, \ldots, a_k) \in A^k$ and $\bar{b} = (b_1, \ldots, b_k) \in A^k$ there are $n, m \in \mathbb{N}$ such that*

$$\Box^n(p(a_1, b_1)^m) \cdot \Box^n(p(a_2, b_2)^m) \cdot \cdots \cdot \Box^n(p(a_k, b_k)^m) \leq p(f(\overline{a}), f(\bar{b})). \quad (4.2)$$

Proof. Suppose that f is a compatible function, and let $\overline{a}, \bar{b} \in A^k$. By Proposition 4.3 there are $n_1, \ldots, n_k, m_1, \ldots, m_k \in \mathbb{N}$ such that

$$\begin{aligned}
&\Box^{n_1}(d(a_1, b_1)^{m_1}) \leq d(f(\overline{a}), f(b_1, a_2, a_3, \ldots, a_k)),\\
&\Box^{n_2}(d(a_2, b_2)^{m_2}) \leq d(f(b_1, a_2, a_3, \ldots, a_k), f(b_1, b_2, a_3, \ldots, a_k)),\\
&\qquad\vdots\\
&\Box^{n_k}(d(a_k, b_k)^{m_k}) \leq d(f(b_1, b_2, \ldots, b_{k-1}, a_k), f(\bar{b})).
\end{aligned}$$

Easy computations show that if $x, y, z \in A$, then $d(x, y) \cdot d(y, z) \leq d(x, z)$. So, it is easily seen that

$$\Box^n(d(a_1, b_1)^m) \cdot \Box^n(d(a_2, b_2)^m) \cdot \cdots \cdot \Box^n(d(a_k, b_k)^m) \leq d(f(\overline{a}), f(\bar{b})),$$

where $n = \max\{n_i : i = 1, \ldots, k\}$ and $m = \max\{m_i : i = 1, \ldots, k\}$. Therefore, we deduce the inequality (4.1).

Conversely, suppose that the inequality (4.1) holds. Let $\theta \in \mathrm{Con}(A)$ and $a_i \theta b_i$ for $i = 1, \ldots, k$. By Lemma 4.2, we obtain $d(a_i, b_i)\theta e$, so

$$\Box^n(d(a_1, b_1)^m) \cdot \Box^n(d(a_2, b_2)^m) \cdot \cdots \cdot \Box^n(d(a_k, b_k)^m) \in e/\theta.$$

By inequality (4.1), we have that

$$\Box^n(d(a_1, b_1)^m) \cdot \Box^n(d(a_2, b_2)^m) \cdot \cdots \cdot \Box^n(d(a_k, b_k)^m) \leq d(f(\overline{a}), f(\overline{b})) \leq e.$$

By the convexity of $\theta(e)$, we obtain $d(f(\overline{a}), f(\overline{b})) \in e/\theta$. Taking into account Lemma 4.2, we deduce that $(f(\overline{a}), f(\overline{b})) \in \theta$, i.e., f is compatible.

The equivalence between 1 and 3 can be proved in a similar manner. $\Box$

The question of whether there are compatible functions different from polynomials naturally arises. In the variety of boolean algebras, the answer is no [14], i.e., that variety is *affine complete*. On the other hand, the variety H of Heyting algebras is not affine complete [4]. However, H is *locally* affine complete in the sense that any restriction of a compatible function to a finite subset is a polynomial. Moreover, the variety CRL is locally affine complete [5, Corollary 9], and also the variety RWH [16, Corollary 7].

Remark 4.6. Let $A \in$ CWRL, let $f : A^k \to A$ be a compatible function, and let B be a finite subset of A^k. Let n and m, respectively, be the maximum of the natural numbers associated in item 2. of Corollary 4.5 to all pairs $(\overline{b}, \overline{x})$, where $\overline{x}$ and $\overline{b}$ range over all points of B. The monotonicity of $\Box$ and properties of d imply that

$$\Box^n(d(b_1, x_1)^m) \cdot \Box^n(d(b_2, x_2)^m) \cdot \cdots \cdot \Box^n(d(b_k, x_k)^m) \leq d(f(\overline{b}), f(\overline{x})). \quad (4.3)$$

In the following, we prove the locally affine completeness of the variety CWRL.

Theorem 4.7. *Let $A \in$ CWRL, let $f : A^k \to A$ be a compatible function, let B be a finite subset of A^k, and let $\overline{x} \in B$. Let*

$$T_{\overline{x}} = \{\Box^n(d(b_1, x_1)^m) \cdot \Box^n(d(b_2, x_2)^m) \cdot \cdots \cdot \Box^n(d(b_k, x_k)^m) \cdot f(\overline{b}) : \overline{b} \in B\},$$

where n and m are the natural numbers associated to the pair $(\overline{b}, \overline{x})$ in Remark 4.6. Then, $f(\overline{x}) = \bigvee T_{\overline{x}}$.

Proof. Let $\overline{x} \in B$. For every $\overline{b} \in B$, by (4.3) we have that

$$\Box^n(d(b_1, x_1)^m) \cdot \Box^n(d(b_2, x_2)^m) \cdot \cdots \cdot \Box^n(d(b_k, x_k)^m) \leq f(\overline{b}) \to f(\overline{x}).$$

Hence,

$$\Box^n(d(b_1, x_1)^m) \cdot \Box^n(d(b_2, x_2)^m) \cdot \cdots \cdot \Box^n(d(b_k, x_k)^m).f(\overline{b}) \leq f(\overline{x}).$$

This proves that $f(\overline{x})$ is an upper bound of $T_{\overline{x}}$.

On the other hand, since $\Box^n(d(x_i, x_i)^m) = e$ for every $i = 1, \ldots, k$, we have that $\Box^n(d(x_1, x_1)^m) \cdot \cdots \cdot \Box^n(d(x_k, x_k)^m) \cdot f(\overline{x}) = f(\overline{x})$. Therefore, $f(\overline{x}) = \bigvee T_{\overline{x}}$. $\Box$

Corollary 4.8. *The variety* CWRL *is locally affine complete.*

It follows from the previous corollary that every finite algebra in CWRL is affine complete.

5. The minimum operator

In the following, we use similar ideas to those in [5, 6, 10] in order to study compatible functions in the variety CWRL in terms of the minimum operator.

Definition 5.1. Let A be a poset and let $g\colon A \times A \to A$ be a function. We say that g satisfies the condition ($\mathbf{M}$) if the following condition holds:

$$\text{For all } a, b, c \in A,\ c \geq b \text{ implies } g(a, c) \leq g(a, b). \tag{$\mathbf{M}$}$$

The proof of the following lemma [5, Lemma 10] follows from the fact that if A is a $\vee$-semilattice and g is a function that satisfies the condition ($\mathbf{M}$), then $g(a, g(a, b) \vee b) \leq g(a, b) \vee b$ for every $a, b \in A$.

Lemma 5.2. *Let A be a $\vee$-semilattice, and let $g\colon A \times A \to A$ be a function that satisfies the condition* ($\mathbf{M}$)*. The following conditions are equivalent:*

(a) *There exists a map $f\colon A \to A$ given by $f(a) = min\ \{b \in A : g(a, b) \leq b\}$.*
(b) *There exists a map $h : A \to A$ that satisfies the following conditions for every $a, b \in A$:*
  (i) $g(a, h(a)) \leq h(a)$,
  (ii) $h(a) \leq g(a, b) \vee b$.

Moreover, in this case we have that $f = h$.

Then we have the following characterization for unary compatible functions.

Proposition 5.3. *Let $A \in$ CWRL and let $f\colon A \to A$ be a function. The following conditions are equivalent:*

1. *f is compatible.*
2. *There exists a function $g\colon A \times A \to A$ that satisfies* ($\mathbf{M}$)*, compatible in the first variable and such that $f(a) = min\ \{b \in A : g(a, b) \leq b\}$ for every $a \in A$.*
3. *There exists a function $\hat{g}\colon A \times A \to A$ that satisfies* ($\mathbf{M}$)*, compatible in the first variable and such that it satisfies the following conditions for every $a, b \in A$:*
   (i) $\hat{g}(a, f(a)) \leq f(a)$,
   (ii) $f(a) \leq \hat{g}(a, b) \vee b$.

Moreover, in this case we have that $g = \hat{g}$.

Proof. Let f be compatible. We define $g\colon A \times A \to A$ by $g(a, b) = f(a)$. Hence, condition 2 is satisfied.

The equivalence between 2 and 3 follows from Lemma 5.2.

In order to prove that condition 3 implies condition 1, let $a, b \in A$. By Proposition 4.3 (and taking into account that g is compatible in the first variable), we have that there exist $n, m \in \mathbb{N}$ such that

$$\Box^n(d(a,b)^m) \leq d(g(a,f(b)), g(b,f(b))) \leq (g(a,f(b)) \to g(b,f(b))) \wedge e. \quad (5.1)$$

Using that $g(b, f(b)) \leq f(b)$, we obtain that

$$(g(a,f(b)) \to g(b,f(b))) \wedge e \leq (g(a,f(b)) \to f(b)) \wedge e. \quad (5.2)$$

It follows from $f(a) \leq g(a, f(b)) \vee f(b)$ that

$$(f(a) \to f(b)) \wedge e \geq ((g(a,f(b)) \vee f(b)) \to f(b)) \wedge e = (g(a,f(b)) \to f(b)) \wedge e.$$

By inequalities (5.1) and (5.2), we have that $\Box^n(d(a,b)^m) \leq (f(a) \to f(b)) \wedge e$. In a similar way, we can prove that $\Box^n(d(a,b)^m) \leq (f(b) \to f(a)) \wedge e$. Thus, $\Box^n(d(a,b)^m) \leq d(f(a), f(b))$. Therefore, it follows from Proposition 4.3 that f is compatible. $\Box$

We end this work with some examples of compatible functions.

Let $A \in$ CWRL. It follows from Proposition 5.3 that for any $n \geq 1$, the binary term $g_n(a,b) = b^n \to a$ induces the compatible function $S_n \colon A \to A$ defined by $S_n(a) = \min\{b \in A : g_n(a,b) \leq b\}$. This function is a generalization of the successor function defined on Heyting algebras [4, 15]. See also [5].

Example 5.4. Let $n \geq 1$. If A is the algebra given in Example 2.6, then the function S_n takes the form $S_n(a) = a^{\frac{k}{nk+1}}$.

Remark 5.5. Let $n \geq 1$. If A is the algebra given in Example 2.7, then S_n does not exist.

In what follows, we will write A to indicate algebras in RWH, and $\neg a$ to indicate $a \to 0$.

Let $n \in \mathbb{N}$. In [16], the compatible function $s_n : A \to A$ was defined through the conditions $a \leq s_n(a)$, $s_n(a) \leq b \vee (\Box^n(b) \to a)$, and $\Box^n(s_n(a)) \to a \leq a$, and it was proved that the existence of this map implies that $s_n(a) = \min\{b \in A : \Box^n(b) \to a \leq b\}$. The function s_0 is the successor function in the sense of [8]. It follows from Proposition 5.3 that if there exists s_0, then there exists S_n, and $S_n = s_0$. The existence of S_1 is equivalent to the existence of S_n for every $n \geq 1$. Moreover, $S_1 = S_n$.

Now we define the function $G \colon A \to A$ through the following conditions: $(G(a) \to a) \wedge \neg\neg a \leq G(a)$ and $G(a) \leq b \vee ((b \to a) \wedge \neg\neg a)$. It follows from Proposition 5.3 that G is a compatible function which can be also defined as $G(a) = \min\{b \in A : (b \to a) \wedge \neg\neg a \leq b\}$. This function is a generalization of Gabbay's function defined on Heyting algebras (see [10, 11]).

Example 5.6. Let H be the poset given in the Example 2.7. Consider the following binary operation given in [16]:

$\rightarrow$	0	a	e	1
0	1	1	1	1
a	e	1	e	1
e	0	0	1	1
1	0	0	e	1

The algebra $(H, \rightarrow)$ is in SRL and it is not a Heyting algebra. We also have that there is no function s_0. Straightforward computations show that $S_1(0) = S_1(a) = e$, $S_1(e) = a$, $S_1(1) = 1$, $G(0) = G(a) = 0$, $G(e) = a$, and $G(1) = 1$.

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