Simultaneous triangularization of switching linear systems: arbitrary eigenvalue assignment and genericity

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***Abstract*—A sufficient condition for the stability of arbitrary switching linear systems (SLSs) without control inputs is that the individual subsystems are stable and their evolution matrices are simultaneously triangularizable (ST). This sufficient condition for stability is known to be extremely restrictive and not robust, and therefore of very limited applicability. The situation can be radically different when control inputs are present. Indeed, previous results have established that, depending on the number of states, inputs and subsystems, the existence of feedback matrices for each subsystem so that the corresponding closed-loop matrices are stable and ST can become a generic property, i.e. a property valid for almost every set of system parameters. This note provides novel contributions along two lines. First, we give sufficient conditions for the genericity of the property of existence of feedback matrices so that the subsystem closed-loop matrices are ST (not necessarily stable). Second, we give conditions for the genericity of the property of existence of feedback matrices that, in addition to achieving ST, enable arbitrary eigenvalue selection for each subsystem's closed-loop matrix. The latter conditions are less stringent than existing ones, and the approach employed in their derivation can be interpreted as an extension to SLSs of specific aspects of the notion of eigenvalue controllability for (non-switching) linear systems.**

***Index Terms*—Hybrid systems, arbitrary switching, stability, Lie algebras, eigenvector assignment, controllability, transverse subspaces.**

I. Introduction

Switched systems are dynamical systems that combine a finite number of subsystems by means of a switching signal [1]. This note focuses on switched systems where each subsystem is linear and also on stability under "arbitrary switching", where stability holds for every possible switching signal [2]–[5]. We refer to the switched systems under consideration as *switching linear systems* (SLSs).

A SLS may either be *autonomous* or have *control inputs*. A sufficient condition for the uniform global exponential stability (where 'uniform' means 'over all switching signals') of an autonomous SLS is that every individual subsystem is stable and their evolution matrices are ST (i.e. generate a solvable Lie algebra). This Lie-algebraic stability condition is simple to check numerically and holds both for discrete-time SLSs [6], [7] and continuous-time SLSs [8], [9]. These Lie-algebraic stability conditions, although mathematically elegant and possibly computationally advantageous (cf. [10], [11]), have had very limited applicability due to their restrictiveness and lack of robustness.

The situation can be radically different for SLSs with control inputs, where feedback may be employed to stabilise the SLS. Indeed, previous results [12] established that the existence of feedback matrices for each subsystem so that the closed-loop SLS satisfies the aforementioned Lie-algebraic stability condition can become a *generic* property, namely, a property that is valid for almost every set of system parameters. According to the sufficient conditions given in [12], genericity holds if each subsystem has a "substantial" number of inputs, although possibly fewer inputs than states. The approach of [12] consisted in analyzing the property of transversality of specific subspaces. When the sufficient condition given in [12] holds, the eigenvalues for the closed-loop matrix of each subsystem can also be arbitrarily selected.

In this note, we address the existence of feedback matrices for each subsystem of a SLS with the properties of (a) the closed-loop subsystem matrices become ST and (b) the eigenvalues of each closed-loop subsystem matrix can be arbitrarily selected. First, we give conditions under which feedback matrices exist so that (a) holds (without necessarily ensuring stability) and then show that these conditions are generic in the space of system parameters (i.e. hold for almost every set of system matrices of the given dimensions). Second, we provide conditions under which feedback matrices exist that simultaneously ensure (a) and (b), and also analyze the genericity of these conditions. The conditions derived are less stringent than those given in [12]. The approach that we employ is substantially different from that in [12] and provides a connection with the notion of eigenvalue controllability for (non-switched) linear systems through the generalization of a type of Popov-Belevitch-Hautus (PBH) test.

Notation. The index set $\{1, 2, \dots, N\}$ is denoted $\underline{N}$. The kernel (null space) of a matrix or linear map $A : \mathcal{X} \to \mathcal{Y}$ is denoted $\ker A$ and its image (range), $\operatorname{img} A$. Given a subspace $\mathcal{B} \subset \mathcal{Y}$, the subspace $\{v \in \mathcal{X} : Av \in \mathcal{B}\}$ is denoted $(A)^{-1}\mathcal{B}$. For $x \in \mathbb{C}^{n\times m}$, its transpose is denoted x', its conjugate transpose x^* and its Moore-Penrose generalized inverse $x^\dagger$. If $\mathcal{S}$ is a vector space, then $\mathrm{d}(\mathcal{S})$ denotes the dimension of $\mathcal{S}$, and $\mathcal{S}^\perp$ its orthogonal complement. If I is a finite set, then $\#I$ denotes the number of elements in I. A complex number $\lambda \in \mathbb{C}$ will be called stable if $|\lambda| < 1$ for discrete time or if $\mathbb{R}\mathrm{e}\{\lambda\} < 0$ for continuous-time. A square matrix is stable if all of its eigenvalues are stable. A property is said to hold generically in some space of parameters if it holds at every point except at points belonging to a proper algebraic variety (see [13, §0.16]).

II. Problem Formulation

Consider the discrete- or continuous-time SLS

$$x^+ \;=\; A_{\sigma(t)}x(t) + B_{\sigma(t)}u(\sigma(t), t), \tag{1}$$

where x^+ denotes $x(t+1)$ or $\dot{x}(t)$ and the switching function $\sigma(\cdot)$ takes values in $\underline{N}$. We address the general complex case $x \in \mathbb{C}^n$, $u(i,t) \in \mathbb{C}^{\tilde{m}_i}$, $A_i \in \mathbb{C}^{n\times n}$ and $B_i \in \mathbb{C}^{n\times \tilde{m}_i}$ for all $i \in \underline{N}$. Consider state-feedback control of the form $u(\sigma(t), t) = K_{\sigma(t)}x(t)$, giving rise to the closed-loop system

$$x^+ = A^{\mathrm{CL}}_{\sigma(t)}x(t), \quad \text{where } A^{\mathrm{CL}}_i = A_i + B_iK_i, \quad \text{for } i \in \underline{N}. \tag{2}$$

Since we focus on stability under abitrary switching, we do not need to specify the switching function σ, and we identify the SLS (1) with the set $\mathcal{Z} = \{(A_i, B_i) : i \in \underline{N}\}$. We will refer to $\mathcal{Z}$ as the SLS and to n as its (state) dimension. We will address the existence of feedback matrices for the SLS (1) that make the closed-loop subsystem matrices A^{CL}_i in (2) ST (i.e. generate a solvable Lie algebra). We thus will employ the following definition.

Definition 1: An SLS $\mathcal{Z} = \{(A_i, B_i) : i \in \underline{N}\}$ is said to be

STF (Simultaneously Triangularizable by Feedback):
if K_i exist such that A^{CL}_i as in (2) are ST.

STSF (ST with Stability by Feedback):
if K_i exist such that A^{CL}_i are ST and individually stable.

STFAE (STF with Arbitrary Eigenvalues):
if K_i exist such that A^{CL}_i are ST and their eigenvalues can be individually arbitrarily selected.

SDF (Simultaneously Diagonalizable by Feedback):
if K_i, T exist such that $T^{-1}A^{\mathrm{CL}}_iT$ are diagonal. □

Note that $\mathcal{Z}$ STFAE $\Rightarrow$ $\mathcal{Z}$ STSF[1] $\Rightarrow$ $\mathcal{Z}$ STF $\Leftarrow$ $\mathcal{Z}$ SDF.

In this note, we will derive sufficient conditions for the STF and STFAE properties, and for them to hold for almost every set of system parameters. We also provide a simple necessary and sufficient condition for SDF.

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[1] In the previous publications [12], [14]–[16], the STSF property was named SLASF (Solvable Lie Algebra with Stability by Feedback).

III. PREVIOUS RESULTS

A. Iterative simultaneous triangularization by feedback

Control design that causes the closed-loop subsystem matrices to be ST can be performed by (i) seeking feedback matrices that assign a common eigenvector, (ii) obtaining a reduced-dimension SLS, and (iii) repeating (i) and (ii) for the reduced-dimension SLS, and iterating. This observation is the basis for the iterative algorithms of [12], [14], [16] which, in addition, require stable eigenvalues. This is summarized in the following lemma.

Lemma 1: Let $A_i \in \mathbb{C}^{n\times n}$, $B_i \in \mathbb{C}^{n\times \tilde{m}_i}$, for $i \in \underline{N}$. Suppose that there exist $F_i \in \mathbb{C}^{\tilde{m}_i\times n}$, $\lambda_i \in \mathbb{C}$, and $\tilde{v} \in \mathbb{C}^n$, such that

$$(A_i + B_iF_i)\tilde{v} = \lambda_i\tilde{v}, \quad \text{for all } i \in \underline{N}, \quad \text{with } \tilde{v} \neq 0. \tag{3}$$

Let $v = \tilde{v}/\|\tilde{v}\|$ and select $U \in \mathbb{C}^{n\times n-1}$ so that $W := [v\ U]$ satisfies $W^*W = \mathrm{I}$. Define $A_i^r := U^*(A_i + B_iF_i)U$, $B_i^r := U^*B_i$, $\mathcal{Z} := \{(A_i, B_i) : i \in \underline{N}\}$, $\mathcal{Z}^r := \{(A_i^r, B_i^r) : i \in \underline{N}\}$. Then,

a) $\mathcal{Z}$ is STF if and only if $\mathcal{Z}^r$ is STF.

b) If λ_i are stable, then $\mathcal{Z}$ is STSF if and only if $\mathcal{Z}^r$ is STSF.

Lemma 1 shows that once a common feedback-assignable eigenvector $\tilde{v}$ is found, the question of whether the given n-dimensional SLS $\mathcal{Z}$ is STF or STSF can be reduced to analyzing whether the $n-1$-dimensional SLS $\mathcal{Z}^r$ is so. The determination of whether a given SLS $\mathcal{Z}$ is STF or STSF can thus be tackled by seeking a common feedback-assignable eigenvector (with corresponding stable eigenvalues), obtaining the reduced-dimension SLS $\mathcal{Z}^r$, and repeating the procedure on the reduced-dimension SLS $\mathcal{Z}^r$. This can be done until dimension 1 is reached, in which case the obtained 1-dimensional system would trivially satisfy the triangularizability (solvable Lie algebra) condition. It can also be shown that if this iterative procedure is successful, then feedback matrices exist not only with the above properties but also ensuring that the closed-loop eigenvalues are precisely those appearing in (3) at every iteration. This procedure is the basis for the iterative triangularization by feedback algorithm in [12], [15], [16].

B. Common Eigenvector Assignment by Feedback

This section recalls the structural condition introduced in [16] which, when satisfied, ensures that a feedback-assignable common eigenvector exists and provides a method for its computation.

Define $m_i := \mathrm{rank}(B_i) = \mathrm{d}(\mathrm{img}\, B_i)$, and factor $B_i = b_ir_i$, where $r_i : \mathbb{C}^{\tilde{m}_i} \to \mathbb{C}^{m_i}$ has full row rank and $b_i : \mathbb{C}^{m_i} \to \mathbb{C}^n$ has full column rank. Note that $\mathrm{img}\, B_i = \mathrm{img}\, b_i$. Let

$$\Lambda := [\lambda_1, \lambda_2, \ldots, \lambda_N]', \quad \text{and define} \tag{4}$$

$$Q(\Lambda) := [R(\Lambda), -B], \quad \text{where} \tag{5}$$

$$R(\Lambda) := \begin{bmatrix} \lambda_1 I - A_1 \\ \vdots \\ \lambda_N I - A_N \end{bmatrix}, \quad \text{and } B := \mathrm{blkdiag}\big[b_1, \ldots, b_N\big],$$

where blkdiag denotes block diagonal concatenation.

Lemma 2 (Structural condition [16]): Let

$$p := n + \sum_{i=1}^{N} m_i - Nn. \tag{6}$$

Then,

(a) A vector that can be assigned by feedback as a common eigenvector with corresponding eigenvalues λ_i for $i \in \underline{N}$ exists if and only if $\mathrm{d}(\ker Q(\Lambda)) > 0$.

(b) If $Q(\Lambda)w = 0$ with $w \neq 0$ partitioned as

$$w := [\tilde{v}', u_1', \ldots, u_N']', \tag{7}$$

$$\text{then } \tilde{v} \neq 0, \text{ and } (A_i + B_iF_i)\tilde{v} = \lambda_i\tilde{v} \tag{8}$$

for every F_i satisfying $r_iF_i\tilde{v} = u_i$ and every $i \in \underline{N}$. For each $i \in \underline{N}$ one such F_i is $F_i = (r_i)^\dagger u_i\tilde{v}^\dagger$.

(c) $\mathrm{d}(\ker Q(\Lambda)) \geq p$ for every $\Lambda \in \mathbb{C}^N$ [recall (4)]. Consequently, if $p > 0$, then a feedback-assignable common eigenvector exists for every choice of corresponding eigenvalues.

Lemma 2 gives a structural condition, namely $p > 0$, for a feedback-assignable common eigenvector $\tilde{v}$ to exist for each choice of corresponding eigenvalues λ_i. This condition is *structural* because the quantities involved in the computation of p are only matrix ranks and dimensions. Note that p is the difference between the number of columns and the number of rows of $Q(\Lambda)$ in (5). If the structural condition $p > 0$ is satisfied, a feedback-assignable common eigenvector $\tilde{v}$ and its corresponding feedback matrices F_i can be computed as follows:

1) Select the corresponding (stable) closed-loop eigenvalues λ_i for each subsystem $i \in \underline{N}$ and build Λ as in (4);
2) Find a vector $w \neq 0$ with components partitioned as in (7) so that $Q(\Lambda)w = 0$ (namely, so that $w \in \ker Q(\Lambda)$);
3) Construct $\tilde{v}$, and $u_1, \ldots, u_N$ from the components of w in (7).
4) $F_i = (r_i)^\dagger u_i\tilde{v}^\dagger$ for every $i \in \underline{N}$.

C. The Structural Condition

If the structural condition $p > 0$, given by Lemma 2, holds for the given SLS $\mathcal{Z}$, then a feedback-assignable common eigenvector and the corresponding feedback matrices can be computed for every choice of corresponding closed-loop eigenvalues. We may then proceed along the lines of Lemma 1 and simplify the problem to the reduced-dimension system $\mathcal{Z}^r$. It would thus be useful to know whether the structural condition holds for $\mathcal{Z}^r$. From Lemma 1, it follows that the relationship between $m_i = \mathrm{rank}\, B_i$ and $m_i^r := \mathrm{rank}\, B_i^r$ depends on the common feedback-assignable eigenvector $\tilde{v}$ as follows:

$$m_i^r = \begin{cases} m_i & \text{if } \tilde{v} \notin \mathcal{B}_i, \\ m_i - 1 & \text{if } \tilde{v} \in \mathcal{B}_i, \end{cases} \qquad \text{with } \mathcal{B}_i := \mathrm{img}\, B_i. \tag{9}$$

Let $n_r := n - 1$ denote the dimension of $\mathcal{Z}^r$, and let p_r denote the quantity p in (6) when computed for the reduced-dimension SLS $\mathcal{Z}^r$:

$$p_r = n_r + \sum_{i=1}^{N} m_i^r - Nn_r.$$

Employing (9), we can straightforwardly arrive at

$$p_r \geq p - 1, \tag{10}$$

with equality if and only if

$$\tilde{v} \in \mathcal{B}, \qquad \text{with } \mathcal{B} := \bigcap_{i\in\underline{N}} \mathcal{B}_i. \tag{11}$$

From (10), we see that the situation where the structural condition holds for $\mathcal{Z}$ but not for $\mathcal{Z}^r$, i.e. $p > 0$ and $p_r \not> 0$, can only happen if $p = 1$ and (11) simultaneously hold.

D. Sufficient condition for genericity of STSF

Theorem 3 of [12] gives a sufficient condition for the genericity of the STSF property, i.e. the existence of feedback matrices that render the closed-loop SLS (2) stable by ensuring that the matrices A_i^{CL} generate a solvable Lie algebra. This sufficient condition is the following.

Theorem 1 (Adapted from Theorem 3 of [12]): If the state dimension n, the number of subsystems N, and the number of control inputs $\tilde{m}_i \leq n$ for each $i \in \underline{N}$, are such that

$$n + \sum_{i\in\underline{N}} \max\{0, 2\tilde{m}_i - n\} - Nn \geq 0, \tag{12}$$

then the STSF property holds for almost every set of system parameters $A_i \in \mathbb{C}^{n\times n}$ and $B_i \in \mathbb{C}^{n\times \tilde{m}_i}$ for all $i \in \underline{N}$. □

In [12], the matrices B_i are assumed to have full column rank, and hence no distinction needs to be made between $\tilde{m}_i$ and m_i because a matrix $B_i \in \mathbb{C}^{n\times \tilde{m}_i}$ with $\tilde{m}_i \leq n$ generically satisfies $m_i = \operatorname{rank} B_i = \tilde{m}_i$.

When the SLS dimensions are nontrivial, i.e. when $1 \leq \tilde{m}_i \leq n-1$, $n \geq 2$, and $N \geq 2$, we showed in [12] that for (12) to hold it is necessary that $2\tilde{m}_i > n$ and hence (12) becomes

$$n + \sum_{i=1}^{N}(2\tilde{m}_i - n) - Nn = 2\left(n + \sum_{i=1}^{N}\tilde{m}_i - Nn\right) - n \geq 0$$

and using (6) and the fact that $\tilde{m}_i = m_i$ generically when $\tilde{m}_i \leq n$,

$$2p \geq n. \tag{13}$$

Genericity of the STSF property is ensured in [12] by deriving conditions under which the satisfaction of the structural condition $p > 0$ for the SLS $\mathcal{Z}$ also implies satisfaction of the structural condition $p_r > 0$ for the reduced-dimension SLS $\mathcal{Z}^r$ (according to Lemma 1).

In Section IV, we first will provide a sufficient condition for the STF property and its genericity to hold, and then will relax the sufficient condition (13) by analyzing properties of the matrix $Q(\Lambda)$ in (5) and the relationship with eigenvalue controllability for linear time-invariant systems.

E. Transversality of subspaces

The genericity results of [12] are based on the property of transversality of subspaces. In this note, we also will employ the latter property but will apply it to a different set of subspaces.

Definition 2 (Transverse): Two subspaces $\mathcal{S}, \mathcal{T}$ of an ambient space $\mathcal{X}$ are said to be transverse when the dimension of their intersection is minimal, given the dimensions of $\mathcal{S}$ and $\mathcal{T}$, i.e. when

$$\mathrm{d}(\mathcal{S}\cap\mathcal{T}) = \max\{0, \mathrm{d}(\mathcal{S}) + \mathrm{d}(\mathcal{T}) - \mathrm{d}(\mathcal{X})\}. \tag{14}$$

Equivalently, $\mathcal{S}$ and $\mathcal{T}$ are transverse when the dimension of their sum is maximal. A set $S = \{\mathcal{S}_1, \ldots, \mathcal{S}_N\}$ of subspaces of an ambient space $\mathcal{X}$ is transverse when both the intersection of the subspaces in every subset of S has minimal dimension *and* the sum of the subspaces in every subset of S has maximal dimension[2] (see [12]).

Transversality of a set of subspaces according to Definition 2 is a generic property and hence almost every set containing a finite number of subspaces taken "randomly" among all subspaces of $\mathcal{X}$ will be transverse. We require the following properties related to transversality (see [12] for proofs).

Lemma 3: Let $S = \{\mathcal{S}_1, \ldots, \mathcal{S}_N\}$ be a set of subspaces of the ambient space $\mathcal{X}$, and define

$$q := \mathrm{d}(\mathcal{X}) + \sum_{i\in\underline{N}}\mathrm{d}(\mathcal{S}_i) - N\mathrm{d}(\mathcal{X}).$$

Then,

(a) $\mathrm{d}(\mathcal{S}_i\cap\mathcal{S}_j) = \mathrm{d}(\mathcal{S}_i) + \mathrm{d}(\mathcal{S}_j) - \mathrm{d}(\mathcal{S}_i + \mathcal{S}_j)$ for all $i, j \in \underline{N}$.

(b) If S is transverse, then $\mathrm{d}\left(\bigcap_{i\in\underline{N}}\mathcal{S}_i\right) = \max\{0, q\}$.

(c) If S is transverse and $q \geq 0$, then $\mathrm{d}(\mathcal{S}_i + \mathcal{S}_j) = \mathrm{d}(\mathcal{X})$ for all $i, j \in \underline{N}$ with $i \neq j$.

(d) Let $J = I \cup \{j\}$, with $J \subset \underline{N}$ and $\#J = \#I + 1$. Suppose that $q \geq 0$ and that $\{\mathcal{S}_i : i \in I\}$ is transverse. Then, $\{\mathcal{S}_i : i \in J\}$ is transverse if and only if $\bigcap_{i\in I}\mathcal{S}_i + \mathcal{S}_j = \mathcal{X}$.

[2] Repeated elements in S have to be considered so that, e.g., $S = \{\mathcal{S}_1, \mathcal{S}_2\}$ with $\mathcal{S}_1 = \mathcal{S}_2$ is not transverse but $S = \{\mathcal{S}_1\}$ is.

Note that p in (6) corresponds to q in Lemma 3 with $\mathcal{X} = \mathbb{C}^n$ and applied to the set $S = \{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$. This is because $\mathcal{B}_i \subset \mathbb{C}^n$ according to (9) and $m_i = \mathrm{d}(\mathcal{B}_i)$ by definition.

IV. MAIN RESULTS

The main results of this note are provided in Sections IV-A, IV-B and IV-C. Section IV-A gives sufficient conditions for a SLS to be STF, i.e. simultaneously triangularizable by feedback, and for the STF property to be generic. Section IV-B gives a necessary and sufficient condition for simultaneous diagonalizability by feedback, which is a special case of STF. Section IV-C relaxes the sufficient condition (12) by means of an approach related to the notion of eigenvalue controllability for LTI systems. In Section IV-D, we discuss the results derived and the link with eigenvalue controllability.

Next, we provide intermediate results (Lemmas 4, 5, and 6) that will be required by the theorems in Sections IV-A and IV-C. The proof of these intermediate results is given in Section V. In the sequel, a superscript ^{r} on a quantity indicates that the corresponding quantity is to be computed for the reduced-dimension $\mathcal{Z}^r$ (as per Lemma 1) instead of for $\mathcal{Z}$; e.g. since $\mathcal{B}_i = \operatorname{img} B_i$, then $\mathcal{B}_i^r := \operatorname{img} B_i^r$.

Lemma 4: Let $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_N\}$ be transverse and $p \geq 0$ [with $\mathcal{B}_i$ as in (9) and p as in (6)]. Then, $\{\mathcal{B}_1^r, \mathcal{B}_2^r, \ldots, \mathcal{B}_N^r\}$ is transverse.

Lemma 5: Let $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_N\}$ be transverse and $p = 0$. Then, there exists $\Lambda \in \mathbb{C}^N$ for which $\operatorname{rank} Q(\Lambda) < Nn$.

Lemma 6: Suppose that $p > 0$ and $\operatorname{rank} Q(\Lambda) = Nn$ for all $\Lambda \in \mathbb{C}^N$. Then, $\operatorname{rank} Q^r(\Lambda) = N(n-1)$ for all $\Lambda \in \mathbb{C}^N$, where Q^r denotes the matrix (5) computed for the reduced-dimension SLS $\mathcal{Z}^r$ of Lemma 1.

Lemma 4 shows that if the quantity p in (6) is nonnegative, then the property of transversality of the input spaces $\mathcal{B}_i$ is preserved for the reduced-dimension SLS $\mathcal{Z}^r$ obtained as per Lemma 1. Lemma 5 identifies a condition under which the matrix $Q(\Lambda)$ in (5), which has Nn rows, cannot have full row rank. Lemma 6 shows that if the structural condition $p > 0$ is satisfied and if the matrix $Q(\Lambda)$ has full row rank at every $\Lambda \in \mathbb{C}^N$, then this latter condition will be preserved for the reduced-dimension SLS.

A. Simultaneous triangularization by feedback

Our first main result is the following.

Theorem 2: Let $p \geq 0$ and suppose that $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_N\}$ is transverse. Then, $\mathcal{Z}$ is STF.

Proof: If $p > 0$, a feedback-assignable common eigenvector $\tilde{v}$ can be found according to Lemma 2 and as explained in Section III-B. Then, we may proceed along the lines of Lemma 1 and obtain the reduced-dimension SLS $\mathcal{Z}^r$. According to (10), then $p_r \geq 0$.

If $p = 0$, $Q(\Lambda)$ is a $Nn \times Nn$ square matrix. By Lemma 5 there exists $\Lambda \in \mathbb{C}^N$ such that $\operatorname{rank} Q(\Lambda) < Nn$, and hence $\mathrm{d}(\ker Q(\Lambda)) > 0$. Lemma 2 then shows that a feedback-assignable common eigenvector $\tilde{v}$ exists for $\mathcal{Z}$, and we may proceed as per Lemma 1. Note that if $p = 0$, then $p_r = p-1$ is not possible because by Lemma 3b) we have $\bigcap_{i\in\underline{N}}\mathcal{B}_i = 0$ and hence (11) cannot hold. Consequently from (10)–(11), $p_r \geq p = 0$.

In either case ($p > 0$ or $p = 0$), Lemma 4 establishes that $\{\mathcal{B}_1^r, \mathcal{B}_2^r, \ldots, \mathcal{B}_N^r\}$ is transverse. We have thus shown that the conditions $p \geq 0$ and $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_N\}$ transverse imply the existence of a feedback-assignable common eigenvector $\tilde{v}$, and that the corresponding reduced-dimension SLS $\mathcal{Z}^r$ satisfies the same conditions, namely $p_r \geq 0$ and $\{\mathcal{B}_1^r, \mathcal{B}_2^r, \ldots, \mathcal{B}_N^r\}$ transverse. We may thus iterate the computation of a feedback-assignable common eigenvector and dimension reduction until dimension 1 is reached, showing that the given SLS $\mathcal{Z}$ is STF by Lemma 1a). ■

Theorem 2 gives a sufficient condition for the existence of feedback matrices that cause the closed-loop SLS to be ST. Since transversality is a generic property, we immediately have the following result on the genericity of simultaneous triangularization by feedback. Define:

$$\tilde{p} := n + \sum_{i \in \underline{N}} \min\{\tilde{m}_i, n\} - Nn. \tag{15}$$

Theorem 3: If the state dimension n, the number of subsystems N, and the number of control inputs $\tilde{m}_i$ for each subsystem $i \in \underline{N}$, are such that $\tilde{p}$ in (15) satisfies $\tilde{p} \geq 0$, then the property that $\mathcal{Z}$ is STF is generic in the space of parameters of $A_i \in \mathbb{C}^{n\times n}$, $B_i \in \mathbb{C}^{n\times \tilde{m}_i}$, $i \in \underline{N}$.

Proof: Generically, we have $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_N\}$ transverse, $m_i = \operatorname{rank} B_i = \min\{\tilde{m}_i, n\}$, and hence $\tilde{p} = p$. The proof then follows straightforwardly by application of Theorem 2. ■

B. Simultaneous diagonalization by feedback

Proposition 1 below gives a necessary and sufficient condition for the simultaneous diagonalizability by feedback of the given SLS.

Proposition 1: There exist $K_i \in \mathbb{C}^{\tilde{m}_i \times n}$ and an invertible $T \in \mathbb{C}^{n\times n}$ such that $T^{-1} A_i^{\text{CL}} T = \operatorname{diag}((\Lambda^1)_i, \ldots, (\Lambda^n)_i)$ for all $i \in \underline{N}$, with A_i^{CL} as in (2), if and only if there exist n vectors w^j such that $Q(\Lambda^j)w^j = 0$ for $j = 1, 2, \ldots, n$, and such that the vectors formed by the first n components of w^j are linearly independent (l.i.).

Proof: ($\Rightarrow$) Let $\tilde{v}^j$ denote the j-th column of T, $(\Lambda^j)_i$ the i-th component of Λ^j, and define $u_i^j = r_i K_i \tilde{v}^j$, where $B_i = b_i r_i$ as explained in Section III-B. It follows that $B_i K_i \tilde{v}^j = b_i u_i^j$ and hence $[(\Lambda^j)_i \mathrm{I} - A_i]\tilde{v}^j - b_i u_i^j = 0$ for all $j = 1, 2, \ldots, n$ and $i \in \underline{N}$. This is equivalent to $Q(\Lambda^j)w^j = 0$ with $w^j = \operatorname{col}[\tilde{v}^j, u_1^j, \ldots, u_N^j]$, according to (5). The $\tilde{v}^j$ are l.i. because T is invertible.

($\Leftarrow$) Partition the components of each w^j as $w^j = \operatorname{col}[\tilde{v}^j, u_1^j, \ldots, u_N^j]$. Then, $\tilde{v}^j$ are l.i. For each $i \in \underline{N}$, let $K_i \in \mathbb{C}^{\tilde{m}_i \times n}$ be the unique solution to the system of equations $K_i \tilde{v}^j = r_i^\dagger u_i^j$ for $j = 1, 2, \ldots, n$. Note that K_i satisfies $r_i K_i \tilde{v}_j = u_i^j$ for every $j = 1, 2, \ldots, n$. By Lemma 2 b), we have $A_i^{\text{CL}} \tilde{v}^j = (\Lambda^j)_i \tilde{v}^j$ for each $j = 1, 2, \ldots, n$ and $i \in \underline{N}$. Put $T = [\tilde{v}^1, \tilde{v}^2, \ldots, \tilde{v}^n]$. ■

Proposition 1 may become especially useful when the structural condition $p > 0$ holds, since in this case Lemma 2 ensures that for every $\Lambda \in \mathbb{C}^N$, the equation $Q(\Lambda)w = 0$ will have a nonzero solution w.

C. Simultaneous triangularization with eigenvalue assignment

The following result gives a novel sufficient condition for the STFAE property, i.e. the existence of feedback matrices that make the closed-loop SLS be ST and allow arbitrary placement of the closed-loop eigenvalues for each subsystem.

Theorem 4: The SLS $\mathcal{Z}$ is STFAE if

i) $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_N\}$ is transverse, and
ii) $\operatorname{rank} Q(\Lambda) = Nn$ for all $\Lambda \in \mathbb{C}^N$.

Proof: Since $Q(\Lambda)$ has Nn rows, $n + \sum_{i\in\underline{N}} m_i$ columns, and $\operatorname{rank} Q(\Lambda) = Nn$, then $Q(\Lambda)$ must have at least as many columns as rows and hence $p \geq 0$. Taking i) and Lemma 5 into account, then $p = 0$ contradicts ii), and hence $p > 0$. From Lemma 4 then $\{\mathcal{B}_1^r, \mathcal{B}_2^r, \ldots, \mathcal{B}_N^r\}$ is transverse and from Lemma 6 then $\operatorname{rank} Q^r(\Lambda) = N(n-1)$ for all $\Lambda \in \mathbb{C}^N$. We have thus shown that if conditions i)–ii) hold for the SLS $\mathcal{Z}$, then the same conditions hold for the reduced-dimension SLS $\mathcal{Z}^r$ of Lemma 1. Consequently, we can iterate on the reduced-dimension SLS $\mathcal{Z}^r$ and so on, until a SLS of dimension 1 is reached, showing that $\mathcal{Z}$ is STF. In addition, the fact that $p > 0$ at every iteration allows for the arbitrary selection of the closed-loop eigenvalues, and hence $\mathcal{Z}$ is STFAE. ■

The next result deals with the genericity of the STFAE property.

Theorem 5: If the state dimension n, the number of subsystems N, and the number of control inputs $\tilde{m}_i$ for each subsystem $i \in \underline{N}$, are such that

$$\tilde{m}_i \geq 1 \quad \text{and} \quad \tilde{p} \geq N_D, \tag{16}$$

where N_D denotes the number of subsystems for which $\tilde{m}_i \leq n-1$, then the STFAE property holds generically in the space of system parameters $A_i \in \mathbb{C}^{n\times n}$, $B_i \in \mathbb{C}^{n\times \tilde{m}_i}$, $i \in \underline{N}$. □

The proof of Theorem 5 requires the following result, whose proof is given in Section V.

Proposition 2: Consider the following expressions:

$$\operatorname{rank} Q(\Lambda) = Nn \text{ for all } \Lambda \in \mathbb{C}^N, \tag{17}$$

$$\operatorname{rank} Q(\Lambda) < Nn \text{ for some } \Lambda \in \mathbb{C}^N. \tag{18}$$

In the space of parameters $A_i \in \mathbb{C}^{n\times n}$, $B_i \in \mathbb{C}^{n\times \tilde{m}_i}$, $i \in \underline{N}$,

i) If (16) holds, then (17) holds generically.
ii) If (16) does not hold, then (18) holds generically.

Proof of Theorem 5: Generically, we have $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_N\}$ transverse, $m_i = \operatorname{rank} B_i = \min\{\tilde{m}_i, n\}$, and hence $\tilde{p} = p$. The proof then follows straightforwardly by application of Proposition 2 i) and Theorem 4. ■

D. Discussion

Theorem 2 gives a novel sufficient condition for the STF property (recall Definition 1). Theorem 2 of [12] gives a sufficient condition for the STSF property, which implies STF. The latter condition is much more restrictive than that of Theorem 2 above, as we next explain. The condition in [12, Theorem 2] is the following: the set of subspaces $S = \{\mathcal{B}_i \cap A_i^{-1}\mathcal{B}_i : i \in \underline{N}\}$ is transverse, the quantity q of Lemma 3 applied to the set S satisfies $q \geq 0$, and (A_i, B_i) is controllable for all $i \in \underline{N}$. Note that the quantity p in (6) equals q of Lemma 3 when applied to the set $\{\mathcal{B}_i : i \in \underline{N}\}$. Since $\mathrm{d}(\mathcal{B}_i \cap A_i^{-1}\mathcal{B}_i) \leq \mathrm{d}(\mathcal{B}_i)$, where the inequality is usually strict, we see that demanding $q \geq 0$ for the set S requires many subsystems to have more inputs than if only $p \geq 0$ is needed. The price we pay for the relaxation of the conditions is that only STF (without stability) is ensured.

If $p > 0$ holds for a given SLS, then a feedback-assignable common eigenvector exists for every choice Λ of corresponding eigenvalues for each subsystem, as follows from Lemma 2. Although not explicitly stated in [12], the conditions of [12, Theorem 2] actually ensure the STFAE property, because $p > 0$ is ensured to hold for $\mathcal{Z}$, then for the reduced-dimension $\mathcal{Z}^r$, and then iteratively so until dimension 1 is reached. However, the sufficient conditions for genericity in [12, Theorem 3] which were analyzed in Section III-D are more stringent than those of Theorem 5. This follows by comparing (13), which holds in a nontrivial case (i.e. $n \geq 2$, $N \geq 2$, $1 \leq \tilde{m}_i \leq n-1$ for all $i \in \underline{N}$), with (16). Generically, we have $\tilde{p} = p$ and hence when $\tilde{m}_i \leq n-1$, we have that $N_D = N$ and it can be easily shown that (13) implies (16) but some systems may satisfy (16) and not (13). For example, a generic SLS with $n = 5$, $N = 2$, $\tilde{m}_1 = 4$, and $\tilde{m}_2 = 3$, for which $p = \tilde{p} = 2$, has $2p \not\geq n$ whereas $\tilde{p} \geq N_D = N = 2$.

If $N = 1$ (i.e. no switching), assumption ii) in Theorem 4 is equivalent to $\operatorname{rank}[\lambda_1 \mathrm{I} - A_1, B_1] = n$ for all $\lambda_1 \in \mathbb{C}$ because in this case $Q(\Lambda) = Q(\lambda_1) = [\lambda_1 \mathrm{I} - A_1, -b_1]$, where $\operatorname{img} B_1 = \operatorname{img} b_1$. This is nothing but the requirement that the pair (A_1, B_1) be controllable, according to a PBH test for controllability. In addition, the set $\{\mathcal{B}_1\}$ is trivially transverse, and hence the assumptions of Theorem 4 coincide with controllability of (A_1, B_1) when $N = 1$ (necessary and sufficient condition for arbitrary eigenvalue assigment by feedback). Also, if $N = 1$ then $\tilde{p} = \min\{\tilde{m}_1, n\}$ and hence the sufficient condition (16) is equivalent to $\tilde{m}_1 \geq 1$. The latter is a necessary and sufficient condition for the genericity of the controllability property, because controllability is known to be generic but the system should have at least 1 input for controllability to be possible. Consequently, the assumptions of Theorems 4 and 5 are necessary and sufficient when $N = 1$.

The condition $\operatorname{rank} Q(\Lambda) = Nn$ for all $\Lambda \in \mathbb{C}^N$ can thus be interpreted as a generalization of eigenvalue controllability to switched linear systems. Recall that an eigenvalue λ of the matrix A is (A, B)-controllable (or just controllable) if $\operatorname{rank}[\lambda \mathrm{I} - A, B] = n$ [17], [18], and that every eigenvalue of A is controllable if and only if (A, B) is controllable. To check whether $\operatorname{rank} Q(\Lambda) = Nn$ for all $\Lambda \in \mathbb{C}^N$ holds for a given system, one may regard Q as a polynomial matrix and perform elementary row and column operations in order to reduce it to the form $[\mathrm{I}\ \mathbf{0}]$. This is possible if and only if $\operatorname{rank} Q(\Lambda) = Nn$ for all $\Lambda \in \mathbb{C}^N$.

V. PROOF OF INTERMEDIATE RESULTS

A. Proof of Lemma 4

From Lemma 1, we have $B_i^r = U^* B_i$ and hence $\mathcal{B}_i^r = \operatorname{img} B_i^r = U^* \operatorname{img} B_i = U^* \mathcal{B}_i$. We have $\mathcal{B}_i^r + \mathcal{B}_j^r = U^*\mathcal{B}_i + U^*\mathcal{B}_j = U^*(\mathcal{B}_i + \mathcal{B}_j)$. By Lemma 3c), we have $\mathrm{d}(\mathcal{B}_i + \mathcal{B}_j) = n$ for all $i, j \in \underline{N}$ with $i \neq j$. Therefore, $\mathrm{d}(\mathcal{B}_i^r + \mathcal{B}_j^r) = n - 1$ because U satisfies $U^*U = \mathrm{I}_{n-1}$. Consequently, the sum of the subspaces in every subset of $\{\mathcal{B}_1^r, \mathcal{B}_2^r, \ldots, \mathcal{B}_N^r\}$, where $\mathcal{B}_i^r \subset \mathbb{C}^{n-1}$, has maximal dimension.

We continue by induction on the number of subspaces in a subset. Let $I \subset \underline{N}$, let $C = \{\mathcal{B}_i^r : i \in I\}$, and let $c = \#C = \#I$. If $c = 2$, then C is transverse by Lemma 3a). Next, let $c < N$ and suppose that C is transverse. Let $j \in \underline{N}$ and $j \notin I$, and let $J = I \cup \{j\}$. Since $\{\mathcal{B}_i : i \in \underline{N}\}$ is transverse by assumption, then $\{\mathcal{B}_i : i \in J\}$ also is transverse. Since $p \geq 0$, then $p_J := n + \sum_{i\in J} m_i - n\#J \geq 0$. By Lemma 3d), then $\bigcap_{i\in I} \mathcal{B}_i + \mathcal{B}_j = \mathbb{C}^n$. We have

$$\bigcap_{i\in I} \mathcal{B}_i^r + \mathcal{B}_j^r = \bigcap_{i\in I} (U^*\mathcal{B}_i) + U^*\mathcal{B}_j \supset U^* \left(\bigcap_{i\in I} \mathcal{B}_i + \mathcal{B}_j \right),$$

and hence $\bigcap_{i\in I} \mathcal{B}_i^r + \mathcal{B}_j^r = \mathbb{C}^{n-1}$. By Lemma 3d), then $C \cup \{\mathcal{B}_j^r\}$ is transverse.

B. Proof of Lemma 5

From (11) and Lemma 3b), we have

$$\mathrm{d}(\mathcal{B}) = \max\{0, p\} = 0, \text{ and thus } \mathcal{B}^\perp = \sum_{i\in\underline{N}} \mathcal{B}_i^\perp = \mathbb{C}^n. \qquad (19)$$

Since $\mathrm{d}(\mathcal{B}_i) = m_i$, then $\mathrm{d}(\mathcal{B}_i^\perp) = n - m_i$. For each $i \in \underline{N}$, let $\mathcal{B}_i^\perp = \operatorname{Span}\{d_1^i, d_2^i, \ldots, d_{n-m_i}^i\}$. Since $p = 0$, then $\sum_{i\in\underline{N}}(n - m_i) = Nn - \sum_{i\in\underline{N}} m_i = n$. Define

$$\begin{aligned}
D_i &:= [d_1^i, d_2^i, \ldots, d_{n-m_i}^i] && \in \mathbb{C}^{n\times n-m_i},\\
D &:= [D_1, D_2, \ldots, D_N] && \in \mathbb{C}^{n\times n},\\
\mathcal{D} &:= \operatorname{blkdiag}(D_1, D_2, \ldots, D_N) && \in \mathbb{C}^{Nn\times n},\\
A &:= [(A_1)', (A_2)', \ldots, (A_N)'] && \in \mathbb{C}^{n\times Nn}.
\end{aligned}$$

By (19), it follows that the n columns of D are linearly independent, and hence D is invertible. Let $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{C}^n$ satisfy

$$(\lambda\mathrm{I} - D^{-1}A\mathcal{D})\alpha = 0, \quad \alpha \neq 0.$$

Partition the vector α so that

$$\begin{aligned}
&\alpha = \operatorname{col}[\alpha_1, \alpha_2, \ldots, \alpha_N], \quad \text{with } \alpha_i \in \mathbb{C}^{n-m_i}, \text{ and define}\\
&c_i := D_i\alpha_i, \qquad \beta := \operatorname{col}[c_1, c_2, \ldots, c_N] \in \mathbb{C}^{Nn}.
\end{aligned}$$

Since the columns of each D_i are linearly independent, and since $\alpha \neq 0$, it follows that the vector β is nonzero. Let $\Lambda \in \mathbb{C}^N$ be the vector all of whose components equal λ. We have

$$\beta' Q(\Lambda) = \left[\sum_{i\in\underline{N}} c_i'(\lambda\mathrm{I} - A_i),\ c_1'b_1\ ,\ \ldots\ ,\ c_N'b_N \right].$$

Since $c_i \in \mathcal{B}_i^\perp$, then $c_i'b_i = 0$ for every $i \in \underline{N}$. In addition,

$$\begin{aligned}
\left[\sum_{i\in\underline{N}} c_i'(\lambda\mathrm{I} - A_i) \right]' &= \sum_{i\in\underline{N}} (\lambda\mathrm{I} - (A_i)')D_i\alpha_i\\
&= D(\lambda\mathrm{I} - D^{-1}A\mathcal{D})\alpha = 0.
\end{aligned}$$

Consequently, we have $\beta' Q(\Lambda) = 0$ for a nonzero vector β.

C. Proof of Lemma 6

Since $p > 0$, then Lemma 2 ensures that for every $\Lambda \in \mathbb{C}^N$ as in (4), there exist $\tilde{v}$ and F_i satisfying $(A_i + B_iF_i)\tilde{v} = A_i^{\mathrm{CL}}\tilde{v} = \lambda_i\tilde{v}$ for all $i \in \underline{N}$. Let $v = \tilde{v}/\|\tilde{v}\|$, and $W = [v\ U] \in \mathbb{C}^{n\times n}$ be a unitary matrix with v as first column, i.e. so that $W^*W = \mathrm{I}_n$. Define $\bar{W} := \operatorname{blkdiag}(W, \ldots, W) \in \mathbb{C}^{Nn\times Nn}$, $X := \operatorname{blkdiag}(W, \mathrm{I}_{m_1}, \ldots, \mathrm{I}_{m_N})$, and

$$\bar{F} := \begin{bmatrix} \mathrm{I}_n & \mathbf{0} \\ \tilde{F} & \mathrm{I}_m \end{bmatrix}, \qquad \begin{aligned} \tilde{F} &:= \operatorname{col}[r_1F_1, \ldots, r_NF_N],\\ m &:= \textstyle\sum_{i\in\underline{N}} m_i, \end{aligned}$$

where r_i appear in the factorization $B_i = b_ir_i$ as explained in Section III-B. For each $\Lambda^r \in \mathbb{C}^N$, consider the matrix

$$\tilde{Q}(\Lambda^r) := \bar{W}^* Q(\Lambda^r)\bar{F}X.$$

The rank of $\tilde{Q}(\Lambda^r)$ coincides with that of $Q(\Lambda^r)$ because $\bar{W}$, X, and $\bar{F}$ are all invertible. Direct computation from the definitions shows that the matrix $\tilde{Q}(\Lambda^r)$ satisfies

$$\begin{aligned}
\tilde{Q}(\Lambda^r) &= \left[\bar{N}(\Lambda^r) \quad -\operatorname{blkdiag}(W^*b_1, \ldots, W^*b_N) \right],\\
\bar{N}(\Lambda^r) &:= \operatorname{col}[N_1(\lambda_1^r), \ldots, N_N(\lambda_N^r)],\\
N_i(\lambda^r) &:= \begin{bmatrix} \lambda^r - \lambda_i & \star \\ \mathbf{0} & U^*(\lambda^r\mathrm{I}_{n-1} - A_i^{\mathrm{CL}})U \end{bmatrix}.
\end{aligned}$$

The Nn rows of $\tilde{Q}$ are linearly independent, and hence the $N(n-1)$ rows that remain after removing the first row of each N_i must also be linearly independent. The remaining matrix has the form $\left[\mathbf{0} \quad R^r(\Lambda^r) \quad -\operatorname{blkdiag}(U^*b_1, \ldots, U^*b_N)\right]$, with R as in (5). Comparing this matrix with $Q^r(\Lambda^r)$ it follows that their ranks are equal, and hence $\operatorname{rank} Q^r(\Lambda^r) = N(n-1)$ for all $\Lambda^r \in \mathbb{C}^N$.

D. Proof of Proposition 2

The proof requires the following preliminary result.

Lemma 7: Consider a matrix of the form

$$M(\Lambda) = \operatorname{col}[a^1(\lambda_1), \ldots, a^\ell(\lambda_\ell)], \qquad (20)$$

where $a^i(\lambda_i) \in \mathbb{C}^{q_i\times n}$ and the entries a_{jk}^i of a^i satisfy $a_{jk}^i(\lambda_i) = c_{jk}^i\lambda_i + d_{jk}^i$ for all $i \in \{1, \ldots, \ell\}$, $j \in \{1, \ldots, q_i\}$ and $k \in \{1, \ldots, n\}$. Let $\bar{q} = \sum_{i=1}^{\ell} q_i$ and consider the expressions:

$$\operatorname{rank} M(\Lambda) = \bar{q} \text{ for all } \Lambda \in \mathbb{C}^\ell, \qquad (21)$$

$$\operatorname{rank} M(\Lambda) < \bar{q} \text{ for some } \Lambda \in \mathbb{C}^\ell. \qquad (22)$$

Then, in the space of coefficients $c_{jk}^i, d_{jk}^i \in \mathbb{C}$ for all i, j, k,

i) If $n \geq \ell + \bar{q}$, then (21) holds generically.
ii) If $n \leq \ell + \bar{q} - 1$, then (22) holds generically.

Proof: Consider $\beta \in \mathbb{C}^{\bar{q}}$ and the product $\beta' M(\Lambda) \in \mathbb{C}^{1\times n}$. Analyzing whether $M(\Lambda)$ does not have full row rank is equivalent to analyzing whether the system of equations $\beta' M(\Lambda) = 0$ has a solution for some nonzero β. Consider a nonzero $\beta \in \mathbb{C}^{\bar{q}}$. Without loss of generality, suppose that the last component of β equals 1. Consider the equation $\beta' M(\Lambda) = 0$. This is a set of n equations whose unknowns consist of the remaining $\bar{q}-1$ entries of β and the ℓ components of Λ. The entries of $\beta' M(\Lambda)$ are multivariable polynomials in the components of β and in λ_i for $i=1,\ldots,\ell$. The monomials that form each of the components of $\beta' M(\Lambda)$ are of degree at most 1 in each of the variables separately, and the only monomials of (total) degree higher than 1 are of the form $\beta_j \lambda_i$.

i) We have more equations than unknowns and hence, generically, $\beta' M(\Lambda) = 0$ has no solution.

ii) We have at most the same number of equations as of unknowns. Therefore, the existence of at least one solution for $\beta' M(\Lambda) = 0$ is generic. ∎

Proof of Proposition 2: Multiply the last $\sum_{i\in\underline{N}} m_i$ columns of $Q(\Lambda)$ [recall (5)] by -1, and then reorder columns, yielding the matrix $\begin{bmatrix} B & R(\Lambda) \end{bmatrix}$. Partition each b_i and $L_i(\lambda_i) := \lambda_i \mathrm{I} - A_i$ as

$$b_i = \begin{bmatrix} b_i^{\mathrm{up}} \\ b_i^{\mathrm{dn}} \end{bmatrix}, \quad L_i(\lambda_i) = \begin{bmatrix} \lambda_i \mathrm{I} - A_i \end{bmatrix} = \begin{bmatrix} L_i^{\mathrm{up}}(\lambda_i) \\ L_i^{\mathrm{dn}}(\lambda_i) \end{bmatrix} \tag{23}$$

where $b_i^{\mathrm{up}} \in \mathbb{C}^{m_i \times m_i}$ and $L_i^{\mathrm{up}}(\lambda_i) \in \mathbb{C}^{m_i \times n}$. Without loss of generality, suppose that b_i^{up} is invertible. Reorder rows to reach

$$\begin{bmatrix} B^{\mathrm{up}} & L^{\mathrm{up}} \\ B^{\mathrm{dn}} & L^{\mathrm{dn}} \end{bmatrix}, \quad \text{with} \quad \begin{array}{l} B^k = \mathrm{blkdiag}(b_1^k, \ldots, b_N^k) \\ L^k = \mathrm{col}[L_1^k, \ldots, L_N^k] \end{array}$$

and $k \in \{\mathrm{up}, \mathrm{dn}\}$. The dependence of L_i^{up} and L_i^{dn} on λ_i has been omitted for ease of representation. Elementary column operations equivalent to multiplication on the right by the matrix $\mathrm{blkdiag}((b_1^{\mathrm{up}})^{-1}, \ldots, (b_N^{\mathrm{up}})^{-1}, \mathrm{I})$, yield

$$\begin{bmatrix} \mathrm{blkdiag}(\mathrm{I}_{m_1}, \ \ldots \ , \mathrm{I}_{m_N}) & L^{\mathrm{up}} \\ \mathrm{blkdiag}\left(b_1^{\mathrm{dn}}(b_1^{\mathrm{up}})^{-1}, \ \ldots \ , b_N^{\mathrm{dn}}(b_N^{\mathrm{up}})^{-1}\right) & L^{\mathrm{dn}} \end{bmatrix}$$

and by elementary row and column operations, we reach

$$\left[\begin{array}{c|c} \mathrm{I} & \mathbf{0} \\ \hline \mathbf{0} & \tilde{L} \end{array}\right], \text{ with } \tilde{L} = \begin{bmatrix} L_1^{\mathrm{dn}} - b_1^{\mathrm{dn}}(b_1^{\mathrm{up}})^{-1} L_1^{\mathrm{up}} \\ \vdots \\ L_N^{\mathrm{dn}} - b_N^{\mathrm{dn}}(b_N^{\mathrm{up}})^{-1} L_N^{\mathrm{up}} \end{bmatrix}.$$

Note that for every $i \in \underline{N}$ for which $m_i = n$, we have that both b_i^{dn} and L_i^{dn} are empty. Removing these and renumbering the remaining subsystems, we have

$$\tilde{L} = \begin{bmatrix} L_1^{\mathrm{dn}} - b_1^{\mathrm{dn}}(b_1^{\mathrm{up}})^{-1} L_1^{\mathrm{up}} \\ \vdots \\ L_{N_D}^{\mathrm{dn}} - b_{N_D}^{\mathrm{dn}}(b_{N_D}^{\mathrm{up}})^{-1} L_{N_D}^{\mathrm{up}} \end{bmatrix}.$$

We next analyze the form of the elements in the bottom-right submatrix $\tilde{L}$, which has $\sum_{i\in\underline{N}}(n-m_i)$ rows and n columns. Define

$$a^i(\lambda_i) := L_i^{\mathrm{dn}}(\lambda_i) - b_i^{\mathrm{dn}}(b_i^{\mathrm{up}})^{-1} L_i^{\mathrm{up}}(\lambda_i). \tag{24}$$

The quantity a^i is an $n - m_i \times n$ matrix with entries $a_{jk}^i = c_{jk}^i \lambda_i + d_{jk}^i$. Generically in the space of entries of B_i, we have $m_i = \min\{\tilde{m}_i, n\}$, hence $\tilde{p} = p = n - \sum_{i=1}^N (n-m_i) = n - \sum_{i=1}^{N_D}(n-m_i)$, and (16) becomes equivalent to

$$n \geq N_D + \sum_{i=1}^{N_D} (n - m_i)$$

Applying Lemma 7 i) to $\tilde{L}(\Lambda)$, then $\operatorname{rank} \tilde{L}(\Lambda) = \sum_{i=1}^N (n-m_i) = Nn - \sum_{i\in\underline{N}} m_i$, and hence $\operatorname{rank} Q(\Lambda) = Nn$, for all $\Lambda \in \mathbb{C}^N$ holds generically in the space of coefficients c_{jk}^i, d_{jk}^i. Since these coefficients are rational functions of the entries of A_i, b_i, and since b_i is generically equal to the first $\min\{\tilde{m}_i, n\}$ columns of B_i, the result is established. ∎

VI. Conclusions

This note dealt with SLSs, i.e. switched linear systems under arbitrary switching and having control inputs. We have addressed the problem of existence of feedback matrices that render the closed-loop SLS simultaneously triangularizable and are able to achieve arbitrary placement of closed-loop eigenvalues. First, we derived novel sufficient conditions that ensure the existence of feedback matrices that render the closed-loop SLS simultaneously triangularizable, and that ensure the genericity of this property. Second, we derived sufficient conditions for the existence of feedback matrices that are able to, in addition to enabling simultaneous triangularization, place the closed-loop eigenvalues of every subsystem arbitrarily. We have also given conditions under which this latter property is valid for almost every set of system parameters. In this last case, the given sufficient conditions are less stringent, i.e. have wider applicability, than previously existing conditions and can be interpreted as an extension to SLSs of a PBH test for controllability.

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