Research Article

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Standard cocycles: Variations on themes of C. Kassel's and R. Wilson's

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Abstract: Central extensions of Lie algebras can be understood and classified by means of 2-cocycles. The Lie algebras we are interested in are "twisted forms" (defined by Galois descent) of algebras of the form $\mathfrak{g} \otimes_k R$ with $\mathfrak{g}$ split finite-dimensional simple over a base field k of characteristic 0 and R a commutative unital and associative k-algebra (such algebras are ubiquitous in modern infinite-dimensional Lie theory). We introduce a special type of cocycle that we called *standard*. Our main result shows that any cocycle is cohomologous to a unique standard cocycle. As an application we give a precise description of the universal central extension of the twisted forms of $\mathfrak{g} \otimes_k R$ mentioned above. This yields a new proof of a classic theorem of C. Kassel [8]. For multiloop algebras, we obtain a "twisted" version of Kassel's result (which is due to R. Wilson [21] in the case of the affine Kac–Moody Lie algebras).

Keywords: Central extensions of Lie algebras, Galois descent, standard cocycle, multiloop algebras

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1 Introduction

If $\mathcal{L}$ is a perfect Lie algebra, the category of central extensions of $\mathcal{L}$ admits an initial perfect object $\widehat{\mathcal{L}}$ (which is then unique up to unique isomorphism), usually referred to as the *universal central extension*, or *universal covering algebra* of $\mathcal{L}$. In the case when $\mathcal{L} = \mathfrak{g} \otimes_k R$, where $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra over a field k of characteristic 0, and R is a commutative associative unital k-algebra, the explicit nature of $\widehat{\mathcal{L}}$ is described by a rather elegant result of Kassel [8], as we now recall.

Let $\Omega^1_{R/k}$ denote the R-module of Kähler differentials of the k-algebra R, and $d = d_{R/k} : R \to \Omega^1_{R/k}$ its corresponding universal derivation. By dR we will indicate the image of R under the map d. Thus defined dR is a k-subspace of $\Omega^1_{R/k}$ so that we can consider the corresponding canonical quotient map of k-spaces

$$\bar{\ } : \Omega^1_{R/k} \to \Omega^1_{R/k}/dR.$$

In what follows we will denote for convenience $\mathfrak{g} \otimes_k R$ by $\mathfrak{g}_R$, and identify $\mathfrak{g}$ with the subalgebra $\mathfrak{g} \otimes 1$ of $\mathfrak{g}_R$. Kassel's result asserts that

$$\widehat{\mathfrak{g}_R} = \mathfrak{g}_R \oplus \Omega^1_{R/k}/dR$$

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as a k-space with bracket given by

$$[x \otimes r, y \otimes s]_{\widehat{\mathfrak{g}_R}} = [x \otimes r, y \otimes s]_{\mathfrak{g}_R} + B_{\mathfrak{g}}(x, y)\overline{rds} = [x, y]_{\mathfrak{g}} \otimes rs + B_{\mathfrak{g}}(x, y)\overline{rds},$$

and

$$[\widehat{\mathfrak{g}_R}, \Omega^1_{R/k}/dR]_{\widehat{\mathfrak{g}_R}} = 0,$$

for all $x, y \in \mathfrak{g}$, $r, s \in R$, where $B_{\mathfrak{g}}$ denotes the Killing form of $\mathfrak{g}$.

The nature of the "universal cocycle" corresponding to Kassel's central extension is rather special, and it is an example of what will be called a *standard cocycle*. In the case of $\mathfrak{g}_R$ a cocycle $P \in Z^2(\mathfrak{g}_R, V)$ will be standard if it is of the form

$$P(x \otimes r, y \otimes s) = B_{\mathfrak{g}}(x, y)J(r, s)$$

for some k-bilinear form $J : R \times R \to V$.[1]

We will see in Section 6 that Kassel's theorem implies that every cocycle in $Z^2(\mathfrak{g}_R, V)$ is cohomologous to a standard cocycle. The aim of the present note is to provide an a priori proof of this fact, not only for $\mathfrak{g}_R$ but also for any of its twisted forms given by Galois descent. These are Lie algebras $\mathcal{L}$ over R such that $\mathcal{L} \otimes_R S \simeq \mathfrak{g}_R \otimes_R S$ for some finite Galois extension S/R. We first prove that the exact sequence

$$0 \longrightarrow B^2(\mathcal{L}, V) \longrightarrow Z^2(\mathcal{L}, V) \overset{\pi}{\longrightarrow} H^2(\mathcal{L}, V) \longrightarrow 0$$

is naturally isomorphic to the exact sequence

$$0 \longrightarrow \mathrm{Der}_R(\mathcal{L}, M) \longrightarrow \mathrm{Der}_k^{(-)}(\mathcal{L}, M) \overset{\dot{\eta}}{\longrightarrow} \mathrm{Der}_k^{(-)}(R, M) \longrightarrow 0,$$

where $M = \mathrm{Hom}_k(\mathcal{L}, V)$ and the superscript $^{(-)}$ denotes skew-symmetric derivations. In the last sequence $\dot{\eta}$ admits a natural section $\dot{\sigma}$. The corresponding section σ of π leads to the definition of standard cocycles for the twisted form $\mathcal{L}$: they are the elements of the k-linear subspace $Z^2_{\mathrm{st}}(\mathcal{L}, V) := \sigma(H^2(\mathcal{L}, V))$ of $Z^2(\mathcal{L}, V)$. In the untwisted case $\mathcal{L} = \mathfrak{g}_R$, these cocyles are exactly the ones mentioned above. The useful orthogonal decomposition $Z^2(\mathcal{L}, V) = Z^2_{\mathrm{st}}(\mathcal{L}, V) \bigoplus B^2(\mathcal{L}, V)$ and the general form of the standard cocycles (Theorem 4.2) are the main results of our paper. Once this is done, and a technical lifting of cocycles for twisted forms is established, Kassel's and Wilson's original results can be easily retrieved as particular cases of the explicit description of the universal central extension of multiloop algebras in terms of standard cocycles.

Our motivation for studying central extensions of twisted forms of $\mathfrak{g}_R$ comes from infinite-dimensional Lie theory (in which case R is a Laurent polynomial ring; see [1, 11] for further details). The twisted forms that appear are the so-called multiloop algebras based on $\mathfrak{g}$ that appear as the centerless cores of Extended Affine Lie Algebras (see [1, 4, 12]).

2 Recollections about derivations and central extensions

Throughout this work k will denote a field of characteristic 0. If $\mathcal{L}$ is a Lie algebra over k and V an $\mathcal{L}$-module, we use the standard notations $\Omega^m(\mathcal{L}, V)$, $C^m(\mathcal{L}, V)$, $Z^m(\mathcal{L}, V)$, $B^m(\mathcal{L}, V)$ and $H^m(\mathcal{L}, V)$ to denote the k-spaces of multilinear mappings (resp. alternating mappings, cocycles, coboundaries, cohomology) of degree m of $\mathcal{L}$ with values in V. Recall that the connecting morphism $\delta^m : \Omega^m(\mathcal{L}, V) \to \Omega^{m+1}(\mathcal{L}, V)$ is defined by

$$\begin{aligned}\delta^m\omega(a_1, \dots, a_{m+1}) = \sum_{1 \le i < j \le m+1} &(-1)^{i+j}\omega([a_i, a_j], a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{m+1}) \\ &+ \sum_{1 \le i \le m+1} (-1)^{i+1} a_i \cdot \omega(a_1, \dots, \hat{a}_i, \dots, a_{m+1}).\end{aligned} \tag{2.1}$$

By definition, $Z^m(\mathcal{L}, V)$ is the kernel of the restriction of δ^m to $C^m(\mathcal{L}, V)$, $B^m(\mathcal{L}, V)$ is the image of δ^{m-1}, and $H^m(\mathcal{L}, V) = Z^m(\mathcal{L}, V)/B^m(\mathcal{L}, V)$.

1 That P is a cocycle puts restrictions on the nature of J. More precisely, J needs to be a cyclic 1-cocycle, as we shall later explain. In the case of Kassel's cocycle $V = \Omega^1_{R/k}/dR$.

Remark 2.1. Assume V is a trivial $\mathcal{L}$-module. In this case the second summand of (2.1) vanishes. For $P \in C^2(\mathcal{L}, V)$ the 2-cocycle condition then takes the familiar form

$$P \in Z^2(\mathcal{L}, V) \iff P([a, b], c) + P([c, a], b) + P([b, c], a) = 0.$$

2.1 Central extensions

Assume $\mathcal{L}$ is a perfect Lie algebra over k and V is a trivial $\mathcal{L}$-module. Any cocycle $P \in Z^2(\mathcal{L}, V)$ leads to a central extension

$$0 \longrightarrow V \longrightarrow \mathcal{L}_P \xrightarrow{\pi} \mathcal{L} \longrightarrow 0$$

of $\mathcal{L}$ by V as follows: as a space $\mathcal{L}_P = \mathcal{L} \oplus V$, and the bracket $[\cdot, \cdot]_{\mathcal{L}_P}$ on $\mathcal{L}_P$ is given by

$$[x + u, y + v]_{\mathcal{L}_P} = [x, y]_{\mathcal{L}} + P(x, y) \quad \text{for } x, y \in \mathcal{L} \text{ and } u, v \in V.$$

In this situation, we will henceforth naturally identify $\mathcal{L}$ and V with subspaces of $\mathcal{L}_P$. The center of $\mathcal{L}_P$ is V, because the center of $\mathcal{L}$ is trivial and V is an abelian ideal of $\mathcal{L}_P$. Note that $\mathcal{L}$ is not in general a subalgebra of $\mathcal{L}_P$.

Definition 2.2. Given two central extensions

$$\operatorname{cext}(L, V, P) : 0 \longrightarrow V \longrightarrow \mathcal{L} \oplus V \xrightarrow{\pi} \mathcal{L} \longrightarrow 0$$

and

$$\operatorname{cext}(L, V', P') : 0 \longrightarrow V' \longrightarrow \mathcal{L} \oplus V' \xrightarrow{\pi} \mathcal{L} \longrightarrow 0,$$

a morphism of extensions is given by a Lie algebra morphism $\varphi : \mathcal{L} \oplus V \to \mathcal{L} \oplus V'$ over k such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i} & \mathcal{L} \oplus V & \xrightarrow{\pi} & \mathcal{L} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_{|V}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & V' & \xrightarrow{i'} & \mathcal{L} \oplus V' & \xrightarrow{\pi'} & \mathcal{L} & \longrightarrow & 0 \end{array}$$

commutes. To describe this situation we will adopt the terminology that $\varphi : \mathcal{L} \oplus V \to \mathcal{L} \oplus V'$ is a morphism over $\mathcal{L}$. The corresponding notion of isomorphism is clear.

Definition 2.3. Two central extensions $\operatorname{cext}(L, V, P)$ and $\operatorname{cext}(L, V, P')$ are equivalent if there is an isomorphism $\varphi : \mathcal{L} \oplus V \to \mathcal{L} \oplus V$ over $\mathcal{L}$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i} & \mathcal{L} \oplus V & \xrightarrow{\pi} & \mathcal{L} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & V & \xrightarrow{i} & \mathcal{L} \oplus V & \xrightarrow{\pi} & \mathcal{L} & \longrightarrow & 0 \end{array}$$

commutes.

Remark 2.4. From the definitions it follows that if $\varphi : \mathcal{L} \oplus V \to \mathcal{L} \oplus V'$ is a morphism over $\mathcal{L}$, then we have $-\delta\varphi_{|\mathcal{L}} + \varphi_{|V} \circ P = P'$. If φ is an equivalence, $-\delta\varphi_{|\mathcal{L}} + P = P'$. This yields the well-know relationship between classes of equivalence of extensions and the relevant second cohomology space $H^2(\mathcal{L}, V)$. In other words, the equivalence class of this extension depends only on the class of P in $H^2(\mathcal{L}, V)$, and this gives in fact a parametrization of all equivalence classes of central extensions of $\mathcal{L}$ by V (see for example [10] or [20] for details, as well as Neher's excellent survey [12]).

Definition 2.5. A central extension $\operatorname{cext}(\mathcal{L}, \widetilde{V}, \widetilde{P})$ is universal if for each extension $\operatorname{cext}(\mathcal{L}, V, P)$ there is a unique morphism $\widetilde{\varphi} : \mathcal{L} \oplus \widetilde{V} \to \mathcal{L} \oplus V$ over $\mathcal{L}$.

If such an extension exists, it is clearly unique up to unique isomorphism. On the other hand, it is well known that universal central extensions exist for perfect Lie algebras. (See for example [20] or [10].)

Remark 2.6. The group $\Lambda = \mathrm{GL}(V) \times \mathrm{Aut}_k(\mathcal{L})$ acts k-linearly on each $\Omega^m(\mathcal{L}, V)$ in a natural way: For $(\mu, \vartheta) \in \Lambda$ on $\omega \in \Omega^m(\mathcal{L}, V)$,

$$\omega^{(\mu,\vartheta)} = \mu^{-1} \circ \omega \circ (\underbrace{\vartheta \times \cdots \times \vartheta}_{m}).$$

This action stabilizes $C^m(\mathcal{L}, V)$ and commutes with the coboundary operator, so it induces an action of Λ on the cohomology spaces $H^m(\mathcal{L}, V)$ satisfying $[\omega]^{(\mu,\vartheta)} = [\omega^{(\mu,\vartheta)}]$ for all $[\omega] \in H^m(\mathcal{L}, V)$ and $(\mu, \vartheta) \in \Lambda$. Let $P, Q, \in Z^2(\mathcal{L}, V)$. It is easy to see that the two central extensions $\mathcal{L}_P$ and $\mathcal{L}_Q$ are isomorphic k-Lie algebras if and only if their cohomology classes $[P], [Q] \in H^2(\mathcal{L}, V)$ are in the same Λ-orbit. In other words, if and only if there exist $(\mu, \vartheta) \in \Lambda$ and $\alpha \in \Omega^1(\mathcal{L}, V) = C^1(\mathcal{L}, V)$ such that $Q^{(\mu,\vartheta)} - P = \delta\alpha$. Otherwise stated: the *isomorphic classes* of extensions of $\mathcal{L}$ by V are parametrized, as Lie algebras over k, by the orbit space $H^2(\mathcal{L}, V)/\Lambda$. As we mentioned above already (and this is well known – see for example [20] for a thorough coverage) the *equivalence classes* of extensions of $\mathcal{L}$ by V are parametrized by the cohomology space $H^2(\mathcal{L}, V)$. The isomorphism question and the action of Λ are not mentioned in the usual literature. It is important then to keep in mind that $\mathcal{L}_P$ and $\mathcal{L}_Q$ above could be central extensions of $\mathcal{L}$ by V which are not equivalent, yet $\mathcal{L}_P$ and $\mathcal{L}_Q$ are isomorphic as Lie algebras over k. For more details about this observation, see [17, Lemma 2.3].

2.2 Derivations

We will make extensive use of derivations for different types of algebras and rings. Let us begin by recalling the basic concepts and fixing some notation that will be used throughout the paper.

Let R be a commutative associative unital k-algebra and M an R-module. A k-derivation $D : R \to M$ is a k-linear map such that for all $r, t \in R$, $D(rt) = r \cdot D(t) + t \cdot D(r)$. We denote by $\mathrm{Der}_k(R, M)$ the k-module of derivations of R with values in M.

Next we turn to the Lie algebra counterpart. Let $\mathcal{L}$ be a Lie algebra over k and M an $\mathcal{L}$-module. A k-linear mapping $D : \mathcal{L} \to M$ is a derivation if for all $a, b \in \mathcal{L}$,

$$D([a, b]) = a \cdot D(b) - b \cdot D(a).$$

We denote by $\mathrm{Der}_k(\mathcal{L}, M)$ the k-module of derivations of $\mathcal{L}$ with values in M. Each $m \in M$ defines a derivation $D_m : \mathcal{L} \to M$ such that for $a \in \mathcal{L}$,

$$D_m(a) = a \cdot m.$$

These are the *inner derivations*

$$\mathrm{IDer}_k(\mathcal{L}, M) = \{D_m \in \mathrm{Der}_k(\mathcal{L}, M) : m \in M\}.$$

We remind the reader (even though this will not be used in our work) that the space $\mathrm{Der}_k(\mathcal{L}, M)/\,\mathrm{IDer}_k(\mathcal{L}, M)$ is nothing but $H^1(\mathcal{L}, M)$.

2.2.1 The case of $M = \mathrm{Hom}_k(\mathcal{L}, V)$

Let V be a k-space that we henceforth view as a *trivial* $\mathcal{L}$-module. Central to our work is the case of the space $\mathrm{Der}_k(\mathcal{L}, M)$ when M is the $\mathcal{L}$-module $\mathrm{Hom}_k(\mathcal{L}, V)$. We look at this case in some detail.

Recall that since V is trivial, the action of $\mathcal{L}$ on M is given by

$$(a \cdot \alpha)(b) = -\alpha([a, b])$$

for $\alpha \in M$ and all $a, b \in \mathcal{L}$. This is to say

$$D([a, b])(c) = (a \cdot D(b) - b \cdot D(a))(c) = -D(b)([a, c]) + D(a)([b, c]).$$

Inside $\mathrm{Der}_k(\mathcal{L}, M)$ we have the subspace $\mathrm{Der}_k^{(-)}(\mathcal{L}, M)$ of skew-symmetric derivations, namely, those $D \in \mathrm{Der}_k(\mathcal{L}, M)$ such that for all $a, b \in \mathcal{L}$,

$$D(a)(b) + D(b)(a) = 0.$$

It is clear that all inner derivations are skew-symmetric.

Given a k-bilinear form $P : \mathcal{L} \times \mathcal{L} \to V$, let us denote by $\partial_P : \mathcal{L} \to M$ the mapping sending $a \in \mathcal{L}$ to P_a, where $P_a(b) = P(a, b)$. Thus $\partial_P(a)(b) = P(a, b)$.

Lemma 2.7. *The map $\partial : P \mapsto \partial_P$ induces a k-space isomorphism $Z^2(\mathcal{L}, V) \to \mathrm{Der}_k^{(-)}(\mathcal{L}, M)$. Furthermore, we have $P \in B^2(\mathcal{L}, V)$ if and only if $\partial_P \in \mathrm{IDer}_k(\mathcal{L}, M)$.*

Proof. That ∂_P is a skew-symmetric derivation follows from the cocycle condition. The map given by $P \mapsto \partial_P$ is clearly k-linear and injective. Given $D \in \mathrm{Der}_k^{(-)}(\mathcal{L}, M)$, define $P : \mathcal{L} \times \mathcal{L} \to V$ by $P(a, b) = D(a)(b)$. Since D is skew-symmetric, $P \in Z^2(\mathcal{L}, V)$, while $\partial_P = D$ by definition. This establishes surjectivity.

Finally, we have $\partial_P \in \mathrm{IDer}_k(\mathcal{L}, M)$ if and only if $\partial_P = D_\alpha$ for some $\alpha \in M = \mathrm{Hom}_k(\mathcal{L}, V)$. Then

$$P(a, b) = \partial_P(a)(b) = D_\alpha(a)(b) = -\alpha([a, b]).$$

But this is precisely to say that $P \in B^2(\mathcal{L}, V)$. □

As a natural corollary we have the isomorphism of k-linear spaces

$$H^2(\mathcal{L}, V) = Z^2(\mathcal{L}, V)/B^2(\mathcal{L}, V) \xrightarrow{\simeq} \mathrm{Der}_k^{(-)}(\mathcal{L}, M)/\,\mathrm{IDer}_k(\mathcal{L}, M), \quad P + B^2(\mathcal{L}, V) \longmapsto \partial_P + \mathrm{IDer}_k(\mathcal{L}, M).$$

This gives a natural bijection between the set of equivalence classes of central extensions of $\mathcal{L}$ by V and the set of outer skew symmetric derivations:

$$\mathrm{Ext}_{\mathrm{cen}}(\mathcal{L}, V) \simeq H^2(\mathcal{L}, V) \simeq \mathrm{Der}_k^{(-)}(\mathcal{L}, M)/\,\mathrm{IDer}_k(\mathcal{L}, M) := \mathrm{ODer}_k^{(-)}(\mathcal{L}, M).$$

Let Δ be the inverse of the isomorphism ∂ of Lemma 2.7. We then have the following important commutative exact diagram of k-linear spaces (where the meaning of $\Delta_|$ and $\overline{\Delta}$ are clear):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{IDer}_k(\mathcal{L}, M) & \xrightarrow{i} & \mathrm{Der}_k^{(-)}(\mathcal{L}, M) & \xrightarrow{\dot\pi} & \mathrm{ODer}_k^{(-)}(\mathcal{L}, M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Delta_|} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \overline{\Delta}} & & \\
0 & \longrightarrow & B^2(\mathcal{L}, V) & \xrightarrow{i} & Z^2(\mathcal{L}, V) & \xrightarrow{\pi} & H^2(\mathcal{L}, V) & \longrightarrow & 0.
\end{array}$$

Thus, given a section

$$\dot\mu : \mathrm{ODer}_k^{(-)}(\mathcal{L}, M) \to \mathrm{Der}_k^{(-)}(\mathcal{L}, M)$$

of $\dot\pi$, the splitting (of k-linear spaces)

$$\mathrm{Der}_k^{(-)}(\mathcal{L}, M) = \dot\mu(\mathrm{ODer}_k^{(-)}(\mathcal{L}, M)) \oplus \mathrm{IDer}_k(\mathcal{L}, M)$$

induces a splitting

$$Z^2(\mathcal{L}, V) = \sigma(H^2(\mathcal{L}, V)) \oplus B^2(\mathcal{L}, V)$$

in the second sequence, where $\sigma = \Delta^{-1} \circ \dot\mu \circ \Delta$. Note that a nontrivial consequence of this is that the elements of $\sigma(H^2(\mathcal{L}, V))$ are, indeed, cocycles.

3 A natural section for Galois twisted forms of Lie algebras

The purpose of this section is to provide, for the type of twisted forms that we are considering, explicit formulas and identities that fall within the general framework of [13].

Let S/R be a finite Galois extension with Galois group $\Gamma = \{\gamma_1 = 1_\Gamma, \dots, \gamma_m\}$ (see [3], [9], or [7] for reference). Thus:

(1) The extension S/R is faithfully flat.
(2) Γ is a finite subgroup of $\mathrm{Aut}_R(S)$.
(3) The R-linear mapping

$$\phi : S \otimes_R S \to \overbrace{S \times \cdots \times S}^{m}$$

such that $\phi(s \otimes s') = (s\gamma_1(s'), \dots, s\gamma_m(s'))$ is an isomorphism.

Remark 3.1. Note that S is *not* assumed to be connected. In the terminology of [7] our Galois extension S/R, or rather the corresponding scheme morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, is a *revêtement principal de groupe de Galois* Γ. It is not a “revêtement galoisien” unless S is connected. In this case $\Gamma = \mathrm{Aut}_R(S)$.

Remark 3.2. The following consequences are well known (see [3] for details):
(1) $R = S^\Gamma = \{s \in S : \gamma(s) = s \text{ for all } \gamma \in \Gamma\}$.
(2) There exists a finite set $\{s_1, \dots, s_q, t_1, \dots, t_q\} \in S$ such that $\sum_{i=1}^q s_i t_i = 1$ and $\sum_{i=1}^q s_i \gamma(t_i) = 0$ for all $\gamma \neq 1_\Gamma$
(3) For each set as in (2) and for each $s \in S$, $s = m \sum_{i=1}^q \pi(st_i)s_i = m \sum_{i=1}^q \pi(ss_i)t_i$, where $\pi : S \to R$ is the usual average projection given by $\pi(s) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma(s)$.

Let N be an R-algebra. By means of the projection $\pi : S \to R$ above, we consider the R-linear map

$$1 \otimes \pi : N \otimes_R S \to N \otimes_R R \simeq N$$

such that $(1 \otimes \pi)(n \otimes s) = \pi(s)n$. Viewing $1 \otimes \pi$ as a k-linear map allows us to consider its V-transpose $(1 \otimes \pi)^* : \mathrm{Hom}_k(N, V) \to \mathrm{Hom}_k(N \otimes_R S, V)$, where $(1 \otimes \pi)^*(\varphi) = \varphi \circ (1 \otimes \pi)$.

Consider the map $\widetilde{\rho}_N : \mathrm{Hom}_k(N, V) \times S \to \mathrm{Hom}_k(N \otimes_R S, V)$ defined by $\widetilde{\rho}_N(\varphi, s) = s \cdot ((1 \otimes \pi)^*(\varphi))$ with the natural action of S on $\mathrm{Hom}_k(N \otimes_R S, V)$. Since $\widetilde{\rho}_N$ is clearly bilinear and R-balanced, it induces an R-linear map $\rho_N : \mathrm{Hom}_k(N, V) \otimes_R S \to \mathrm{Hom}_k(\mathcal{L} \otimes_R S, V)$ such that $\rho_N(\varphi \otimes s) = s \cdot ((1 \otimes \pi)^*(\varphi)) \in \mathrm{Hom}_k(N \otimes_R S, V)$. Note that by definition

$$(\rho_N(\varphi \otimes s))(n \otimes s') = \varphi(\pi(ss')n).$$

Lemma 3.3. *The map* $\rho_N : \mathrm{Hom}_k(N, V) \otimes_R S \to \mathrm{Hom}_k(N \otimes_R S, V)$ *is an isomorphism of* S*-modules.*

Proof. It is easy to see that ρ_N is S-linear. We will explicitly describe its inverse

$$\nu : \mathrm{Hom}_k(N \otimes_R S, V) \to \mathrm{Hom}_k(N, V) \otimes_R S.$$

Let us fix a finite set $\{s_1, \dots, s_q, t_1, \dots, t_q\}$ as in (3.2). For $\psi \in \mathrm{Hom}_k(N \otimes_R S, V)$ and for each $i \in \{1, \dots, q\}$ set $\psi_i \in \mathrm{Hom}_k(N, V)$ by $\psi_i(n) = m\psi(n \otimes s_i)$. Having done this, we now define

$$\nu(\psi) = \sum_{i=1}^q \psi_i \otimes t_i,$$

where $\psi_i \in \mathrm{Hom}_k(N, V)$ and $\psi_i(n) = m\psi(n \otimes s_i)$. If we now evaluate $\rho_N(\nu(\psi)) = \rho_N(\sum_{i=1}^q \psi_i \otimes t_i)$ at $n \otimes s$, we get

$$\begin{aligned}\Big(\rho_N\Big(\sum_{i=1}^q \psi_i \otimes t_i\Big)\Big)(n \otimes s) &= \sum_{i=1}^q (\rho_N(\psi_i \otimes t_i))(n \otimes s) = \sum_{i=1}^q \psi_i(\pi(st_i)n) \\ &= \sum_{i=1}^q m\psi(\pi(st_q)n \otimes s_q) = \psi\Big(m \sum_{i=1}^q (n \otimes \pi(st_i)s_i)\Big) \\ &= \psi\Big(n \otimes m \sum_{q=1}^q \pi(st_i)s_i\Big) \\ &= \psi(n \otimes s) \quad \text{(by Remark 3.2).}\end{aligned}$$

On the other hand $\nu(\rho_N(\varphi \otimes s)) = \sum_{i=1}^q (\rho_N(\varphi \otimes s))_i \otimes t_i \in \mathrm{Hom}_k(N, V) \otimes_R S$. Here $(\rho_N(\varphi \otimes s))_i \in \mathrm{Hom}_k(N, V)$ and $((\rho_N(\varphi \otimes s))_i)(n) = m\rho_N(\varphi \otimes s)(n \otimes s_q) = m\varphi(\pi(ss_i)l) = m(\pi(ss_i) \cdot \varphi)(n)$. So

$$\begin{aligned}\nu(\rho_N(\varphi \otimes s)) &= \sum_{i=1}^q (\rho_N(\varphi \otimes s))_i \otimes t_i = \sum_{i=1}^q m(\pi(ss_q) \cdot \varphi) \otimes t_i \\ &= m \sum_{i=1}^q \varphi \otimes \pi(ss_i)t_i = \varphi \otimes m \sum_{i=1}^q \pi(ss_i)t_i \\ &= \varphi \otimes s \quad \text{(by Remark 3.2).}\end{aligned}$$

This finishes the proof that ν is the inverse map of ρ_N. □

Remark 3.4. If N is a Lie algebra, then ρ_N is a morphism of $N \otimes_R S$-modules with the obvious actions: for $s, s' \in S$, $n \in N$, $\varphi \in \mathrm{Hom}_k(N, V)$ and $\psi \in \mathrm{Hom}_k(N \otimes_R S, V)$ we have

$$(n \otimes s) \cdot (\varphi \otimes s') = -\varphi \circ \mathrm{ad}_N(n) \otimes ss' \in \mathrm{Hom}_k(N, V) \otimes_R S$$

and

$$(n \otimes s) \cdot (\psi) = -\psi \circ \mathrm{ad}_{N \otimes_R S}(n \otimes s) \in \mathrm{Hom}_k(N \otimes_R S, V).$$

We leave the details to the reader.

The Lie algebras that we are interested in are S/R forms of $\mathfrak{g} \otimes_k R$, where $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra over k. By definition, this is an R-Lie algebra $\mathcal{L}$ with the property that $\mathcal{L} \otimes_R S \simeq \mathfrak{g} \otimes_k S$ as S-Lie algebras. For our purposes $\mathcal{L}$ will also be thought as a Lie algebra over k.

Remark 3.5. The set of isomorphism classes of S/R-forms of the algebra $\mathfrak{g} \otimes_k R$ is measured by the pointed set $H^1(\Gamma, \mathbf{Aut}(\mathfrak{g})(S))$, where $\mathbf{Aut}(\mathfrak{g})$ is the algebraic k-group of automorphism of the algebra $\mathfrak{g}$. We recall that by definition $\mathbf{Aut}(\mathfrak{g})(S)$ is the (abstract) group of automorphisms $\mathrm{Aut}_S(\mathfrak{g} \otimes_k S)$ of the S-algebra $\mathfrak{g} \otimes_k S$. The group Γ acts on $\mathbf{Aut}(\mathfrak{g})(S)$ by functoriality on S, but one can also check the explicit nature of this action as follows: if $\gamma \in \Gamma$ and $f \in \mathbf{Aut}(\mathfrak{g})(S)$, then

$${}^{\gamma}f = (1 \otimes \gamma) \circ f \circ (1 \otimes \gamma^{-1}).$$

Without going into details, let us simply recall for future use that the S/R-form corresponding to a cocycle $u = (u_\gamma)_{\gamma \in \Gamma} \in Z^1(\Gamma, \mathbf{Aut}(\mathfrak{g})(S))$ is given by

$$\mathcal{L}_u = \{z \in \mathfrak{g} \otimes_k S : u_\gamma {}^{\gamma}z = z \text{ for all } \gamma \in \Gamma\}. \tag{3.1}$$

We therefore can (and henceforth will) always view $\mathcal{L}$ as an R-subalgebra of $\mathfrak{g} \otimes_k S$. Recall also that the S-algebra isomorphism $\theta : \mathcal{L} \otimes_R S \to \mathfrak{g} \otimes_k S$ is also explicit and natural. The multiplication map $S'' = S \otimes_R S \to S$ gives an S-algebra homomorphism $\Theta : \mathfrak{g} \otimes_k S \otimes_R S \to \mathfrak{g} \otimes_k S$. Our isomorphism θ is nothing but the restriction of Θ to $\mathcal{L} \otimes_R S$. Thus, if $l \in \mathcal{L}$ and we write $l = \sum x_i \otimes s_i \in \mathfrak{g} \otimes_k S$, then

$$\theta(l \otimes s) = \Theta\Big(\Big(\sum x_i \otimes s_i\Big) \otimes s\Big) = \sum x_i \otimes s_i s.$$

If we identify $\mathcal{L} \otimes_R 1 \simeq \mathcal{L} \subseteq \mathfrak{g} \otimes_k S$, then $\theta(l \otimes s) = sl$, where the “scalar multiplication” by s is given by the natural action of $s \in S$ on $\mathfrak{g} \otimes_k S$. We also observe that for $l \in \mathcal{L} \subseteq \mathfrak{g} \otimes_k S$ we have $\theta^{-1}(l) = l \otimes 1$.

The above considerations apply to any finite-dimensional algebra $\mathfrak{g}$. Our assumption on $\mathfrak{g}$ is crucial in the understanding of the inner derivations and centroid of $\mathcal{L}$.

Henceforth we fix a k-space V which we view as a trivial $\mathcal{L}$-module and set (to substantially trim down the size of the formulas to follow)

$$M = \mathrm{Hom}_k(\mathcal{L}, V).$$

This puts us exactly in the situation discussed in Section 2.2.1. We begin by recalling (explicitly) the nature of some relevant known isomorphisms.

Since $\mathcal{L}$ is a projective R-module of constant rank $\dim_k(\mathfrak{g})$, the trace of elements of $\mathrm{End}_R(\mathcal{L})$ is defined. Thus $\mathcal{L}$ has a Killing form $B_{\mathcal{L}}$ defined as usual by $B_{\mathcal{L}}(a, b) = \mathrm{tr}(\mathrm{ad}_{\mathcal{L}}(a) \circ \mathrm{ad}_{\mathcal{L}}(b))$. For a general discussion (with references) of bilinear forms of twisted algebras the reader can refer to [14].

Theorem 3.6. *The map* $\widetilde{\ }: \mathrm{Hom}_k(R, V) \to \mathrm{Ctd}(\mathcal{L}, M)$ *given by*

$$\widetilde{\varphi}(a)(b) = \varphi(B_{\mathcal{L}}(a, b))$$

is an R-linear isomorphism, where

$$\mathrm{Ctd}(\mathcal{L}, M) = \{\chi \in \mathrm{Hom}_k(\mathcal{L}, M) : \chi([a, b]) = a \cdot \chi(b) = \chi(a) \cdot b \text{ for all } a, b \in \mathcal{L}\}.$$

Proof. See [14, Remark 5.3]. □

Remark 3.7. The set $\mathrm{Ctd}(\mathcal{L}, M)$ is the *centroid* of the $\mathcal{L}$-module M. This concept can of course be defined for $\mathcal{L}$ and M arbitrary. Centroids play an essential role in our constructions.

Lemma 3.8. *The sequence*

$$0 \longrightarrow \mathrm{Der}_R(\mathcal{L}, M) \longrightarrow \mathrm{Der}_k(\mathcal{L}, M) \xrightarrow{\eta} \mathrm{Der}_k(R, \mathrm{Ctd}(\mathcal{L}, M)) \longrightarrow 0$$

is exact for the natural map $\eta : \mathrm{Der}_k(\mathcal{L}, M) \to \mathrm{Der}_k(R, \mathrm{Ctd}(\mathcal{L}, M))$ *such that for* $\delta \in \mathrm{Der}_R(\mathcal{L}, M)$, $r \in R$ *and* $l \in \mathcal{L}$,

$$(\eta(\delta)(r))(l) = \delta(rl) - r\delta(l). \tag{3.2}$$

Proof. See [13, Proposition 3.1]. □

Under the action of R on M we have

$$(\eta(\delta)(r))(l)(l') = \delta(rl)(l') - \delta(l)(rl').$$

Let us denote by $\varphi : \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V)) \to \mathrm{Der}_k(R, \mathrm{Ctd}(\mathcal{L}, M))$ the isomorphism induced by the isomorphism in Theorem 3.6, that is to say $\varphi(\delta)(r)(l)(l') = \delta(r)(B_{\mathcal{L}}(l, l'))$. It is well defined: since the morphism $\sim$ is R-linear, $\varphi(\delta)$ is actually a derivation. Now we can define

$$\dot{\eta} : \mathrm{Der}_k(\mathcal{L}, M) \to \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))$$

by

$$\dot{\eta} = \varphi^{-1} \circ \eta. \tag{3.3}$$

For each $\delta \in \mathrm{Der}_R(\mathcal{L}, M)$, $r \in R$ and $l, l' \in \mathcal{L}$ we thus have

$$(\eta(\delta)(r))(l)(l') = \varphi(\dot{\eta}(\delta))(r)(l)(l') = \dot{\eta}(\delta)(r)(B_{\mathcal{L}}(l, l')). \tag{3.4}$$

Lemma 3.9. *The sequence*

$$0 \longrightarrow \mathrm{Der}_R(\mathcal{L}, M) \longrightarrow \mathrm{Der}_k(\mathcal{L}, M) \xrightarrow{\dot{\eta}} \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V)) \longrightarrow 0$$

is exact.

Proof. The surjectivity of $\dot{\eta}$, and that $\mathrm{Ker}(\dot{\eta}) = \mathrm{Der}_R(\mathcal{L}, M)$ follow immediately from Lemma 3.8 and the definition of $\dot{\eta}$ given by (3.3) and (3.2). □

Let $N = \mathrm{Hom}_k(R, V)$, so that $N \otimes_R S = \mathrm{Hom}_k(R, V) \otimes_R S \simeq \mathrm{Hom}_k(S, V)$ by Lemma 3.3. Since S/R is étale for any element $\delta \in \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))$, there exists an unique $\widetilde{\delta} \in \mathrm{Der}_k(S, \mathrm{Hom}_k(R, V) \otimes_R S)$ such that

$$\widetilde{\delta}(r) = \delta(r) \otimes 1 \in N \otimes 1 \hookrightarrow N \otimes S.$$

(See [6, Chapter 0, Section 20]. The last injection follows from S/R being faithfully flat.) By considering the isomorphism $\rho_R = \rho$ of Lemma 3.3 we obtain an element $\widehat{\delta} = \rho \circ \widetilde{\delta} \in \mathrm{Der}_k(S, \mathrm{Hom}_k(S, V))$ such that

$$\rho(\widetilde{\delta}(r))(s) = \rho(\delta(r) \otimes 1)(s) = \delta(r)(\pi(s)),$$

where the last equality follows from the definition of ρ. Note that $\widehat{\delta}$ is an extension of δ in the sense that the restriction of $\widehat{\delta}$ to R takes values in $\mathrm{Hom}_k(R, V) = \mathrm{Hom}_k(R, V) \otimes 1 \hookrightarrow \mathrm{Hom}_k(R, V) \otimes_R S$ and coincides with δ. Such an extension is unique. We observe that for $s = r' \in R$ we have

$$\rho(\widetilde{\delta}(r))(r') = \delta(r)(r').$$

Observe that $\rho^{-1} \circ \widehat{\delta}(r) = \delta(r) \otimes 1$ for all $r \in R$. Furthermore,

$$\begin{aligned}(\rho^{-1} \circ \widehat{\delta})(r) = \delta(r) \otimes 1 &\iff \widehat{\delta}(r) = \rho(\delta(r) \otimes 1)\\ &\iff \widehat{\delta}(r)(s) = \rho(\delta(r) \otimes 1)(s) = \delta(r)(\pi(s)) \text{ for all } s \in S.\end{aligned}$$

Summing up:

Proposition 3.10. *For each* $\delta \in \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))$ *there exists a unique extension* $\widehat{\delta} \in \mathrm{Der}_k(S, \mathrm{Hom}_k(S, V))$ *such that*

$$\widehat{\delta}(r)(s) = \delta(r)(\pi(s)).$$

Now we can define

$$\dot{\sigma} : \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V)) \to \mathrm{Der}_k(\mathcal{L}, M),$$

which will be one of the key ingredients of the construction of standard cocycles as follows. Given an element $\delta \in \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))$, let $\widehat{\delta} \in \mathrm{Der}_k(S, \mathrm{Hom}_k(S, V))$ be its unique extension as described in Proposition 3.10.

We fix once and for all a k-basis $\{e_1, \dots, e_n\}$ of $\mathfrak{g}$. For $l = \sum_{i=1}^n e_i \otimes s_i(l)$ and $l' = \sum_{j=1}^n e_j \otimes s_j(l') \in \mathcal{L}$ define

$$\dot{\sigma}(\delta)(l)(l') = \sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j)\widehat{\delta}(s_i(l))(s_j(l')). \tag{3.5}$$

The verification that $\dot{\sigma}(\delta)$ is, indeed, a derivation is a straightforward computation that we omit here.

Lemma 3.11. *Let the notation be as above. Then $\dot{\sigma}$ is a section of $\dot{\eta}$.*

Proof. Let $r, r' \in R$ and $\delta \in \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))$. Since the Killing form is surjective, there exist $l, l' \in \mathcal{L}$ such that $r' = B_{\mathcal{L}}(l, l')$. Then

$$\begin{aligned}
(\dot{\eta} \circ \dot{\sigma})(\delta)(r)(r') &= (\dot{\eta} \circ \dot{\sigma})(\delta)(r)(B_{\mathcal{L}}(l, l')) = \eta(\dot{\sigma}(\delta))(r)(l)(l') && \text{(by (3.4))}\\
&= (\dot{\sigma}(\delta))(rl)(l') - (\dot{\sigma}(\delta))(l)(rl') && \text{(by (3.3))}\\
&= \sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j)\big(\widehat{\delta}(rs_i(l))(s_j(l')) - \widehat{\delta}(s_i(l))(rs_j(l'))\big)\\
&= \sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j)\big(\widehat{\delta}(r)(s_i(l)s_j(l')) + \widehat{\delta}(s_i(l))(rs_j(l')) - \widehat{\delta}(s_i(l))(rs_j(l'))\big)\\
&= \sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j)\widehat{\delta}(r)(s_i(l)s_j(l'))\\
&= \widehat{\delta}(r)\Big(\sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j)s_i(l)s_j(l')\Big)\\
&= \widehat{\delta}(r)(B_{\mathcal{L}}(l, l')) = \widehat{\delta}(r)(r') = \delta(r)(r').
\end{aligned}$$
□

Theorem 3.12. *We have* $\mathrm{Der}_k(\mathcal{L}, M) \simeq \mathrm{Der}_R(\mathcal{L}, M) \oplus \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))$.

Proof. This follows from the fact that the sequence in Lemma 3.9 is split exact. □

Theorem 3.13. *We have* $\mathrm{Der}_R(\mathcal{L}, M) \simeq \mathrm{IDer}_k(\mathcal{L}, M)$.

Proof. See [13, Proposition 4.2]. □

Corollary 3.14. *We have* $\mathrm{Der}_k(\mathcal{L}, M) \simeq \mathrm{IDer}_k(\mathcal{L}, M) \oplus \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))$, *and this decomposition admits as a section the map* $\dot{\sigma} : \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V)) \to \mathrm{Der}_k(\mathcal{L}, M)$ *discussed in Lemma 3.11.*

Proof. All the assertions follow immediately from Theorem 3.12, Theorem 3.13 and Lemma 3.11. □

Remark 3.15. Since all derivations in $\mathrm{IDer}_k(\mathcal{L}, M)$ are skew-symmetric, we have the decomposition

$$\mathrm{Der}_k^{(-)}(\mathcal{L}, M) = \mathrm{IDer}_k(\mathcal{L}, M) \oplus (\dot{\sigma}[\mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))])^{(-)},$$

where

$$(\dot{\sigma}[\mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))])^{(-)} = \dot{\sigma}[\mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))] \cap \mathrm{Der}_k^{(-)}(\mathcal{L}, M).$$

We will see in Section 5 that

$$\dot{\sigma}(\mathrm{Der}_k^{(-)}(R, \mathrm{Hom}_k(R, V))) = (\dot{\sigma}[\mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))])^{(-)}.$$

4 Standard cocycles

Recall the isomorphism $\Delta : \mathrm{Der}_k^{(-)}(\mathcal{L}, M) \to Z^2(\mathcal{L}, V)$ as it was defined after Lemma 2.7.

Definition 4.1. Consider the k-linear subspace of $Z^2(\mathcal{L}, V)$ defined by

$$Z^2_{\mathrm{st}}(\mathcal{L}, V) = \Delta((\dot{\sigma}[\mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))])^{(-)}).$$

The elements of $Z^2_{\mathrm{st}}(\mathcal{L}, V)$are called standard cocycles.

Because of the importance of the concept of standard cocycle let us take the time to go over the details of their definition. To a given $P \in Z^2(\mathcal{L}, V)$ we can attach by Lemma 2.7 an element $\partial_P \in \mathrm{Der}_k^{(-)}(\mathcal{L}, M)$ such that

$$\partial_P = \partial_P^{\mathrm{inn}} + \dot{\sigma}(\dot{\partial_P})$$

(Remark 3.15), where $\partial_P^{\mathrm{inn}} \in \mathrm{IDer}_k(\mathcal{L}, M)$, $\dot{\partial_P} \in \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))$ and $\dot{\sigma}(\dot{\partial_P}) \in \mathrm{Der}_k^{(-)}(\mathcal{L}, M)$. We can again appeal to Lemma 2.7 and associate to $\dot{\sigma}(\dot{\partial_P})$ an element $P_{\mathrm{st}} \in Z^2_{\mathrm{st}}(\mathcal{L}, V)$. Note that, by construction, to each cocycle $P \in Z^2(\mathcal{L}, V)$ corresponds a *unique* standard cocycle. This is the element which we have denoted by P_{st}.

Theorem 4.2. *Let V be a trivial $\mathcal{L}$-module. Then:*
(i) $Z^2(\mathcal{L}, V) = B^2(\mathcal{L}, V) \oplus Z^2_{\mathrm{st}}(\mathcal{L}, V)$.
(ii) *A cocycle $P \in Z^2(\mathcal{L}, V)$ is standard if and only if there exist a $\widehat{\delta} \in \mathrm{Der}_k(S, \mathrm{Hom}_k(S, V))$ such that for $l = \sum_{i=1}^n e_i \otimes s_i(l)$ and $l' = \sum_{j=1}^n e_j \otimes s_j(l')$,*

$$P(l, l') = \sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j)\widehat{\delta}(s_i(l))(s_j(l')).$$

If this is the case, $\widehat{\delta}$ satisfies

$$\sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j)\big(\widehat{\delta}(s_i(l))(s_j(l')) + \widehat{\delta}(s_j(l'))(s_i(l))\big) = 0. \tag{4.1}$$

Proof. (i) This follows from Corollary 3.14 and Definition 4.1.

(ii) By definition, we know that a cocycle $P \in Z^2(\mathcal{L}, V)$ is standard if and only if there exists an element $\dot{\partial} \in (\dot{\sigma}[\mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))])^{(-)}$ such that $\Delta(\dot{\partial}) = P$. By (3.5), for this to be the case it is necessary and sufficient that there exists a $\delta \in \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))$ such that for all $l, l' \in \mathcal{L}$,

$$P(l, l') = \dot{\partial}(l)(l') = \dot{\sigma}(\delta)(l)(l') = \sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j)\widehat{\delta}(s_i(l))(s_j(l')),$$

where $\widehat{\delta} \in \mathrm{Der}_k(S, \mathrm{Hom}_k(S, V))$ is the unique extension of δ. The skew-symmetry of P implies that the derivation $\widehat{\delta}$ satisfies equation (4.1).[2] □

Remark 4.3. The derivation $\widehat{\delta}$ in Theorem 4.2 is determined by P. Indeed, a straightforward calculation shows that for any cocycle P, for all $l, l' \in \mathcal{L}$ and all $r, r' \in R$ we have

$$P(rl, r'l') - P(r'l, rl') = \widehat{\delta}(r)(r'B_{\mathcal{L}}(l, l')) - \widehat{\delta}(r')(rB_{\mathcal{L}}(l, l')).$$

So, if we choose $r' = 1$, we have

$$P(rl, l') - P(l, rl') = \widehat{\delta}(r)(B_{\mathcal{L}}(l, l'). \tag{4.2}$$

Since $B_{\mathcal{L}}$ is onto R, we have that the restriction and co-restriction of $\widehat{\delta}$ to elements of R is determined by P, that is to say δ is determined by P, and then $\widehat{\delta}$ is also determined by P.

As a consequence of the theorem and (4.2) a cocycle $P \in Z^2(\mathcal{L}, V)$ is R-balanced if and only if it is a coboundary.

In the case that $\mathcal{L}$ is untwisted standard cocycles have a very natural expression.

2 In Corollary 5.11 we will prove that equation (4.1) is equivalent to the skew-symmetry of $\widehat{\delta}$ and δ.

Corollary 4.4. *Let $P \in Z^2(\mathfrak{g} \otimes_k R, V)$. For P to be standard it is necessary and sufficient that there exists a k-bilinear mapping $J_P : R \times R \to V$ such that*

$$P(x \otimes r, y \otimes s) = B_{\mathfrak{g}}(x, y) J_P(r, s),$$

where J_P is a cyclic 1-cocycle, i.e. it is a k-bilinear, skew-symmetric mapping such that for all $r, s, t \in R$,

$$J_P(rs, t) + J_P(st, r) + J_P(\mathrm{tr}, s) = 0.$$

Furthermore, the 1-cocycle J_P is unique.

Proof. Since the Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g}$ is non-degenerate, there exists $(x_0, y_0) \in \mathfrak{g} \times \mathfrak{g}$ such that $B_{\mathfrak{g}}(x_0, y_0) = 1$. Then, for $l = x_0 \otimes r$ and $l' = y_0 \otimes r'$, we have

$$0 = B_{\mathfrak{g}}(x_0, y_0)(\widehat{\delta}(r)(r') + \widehat{\delta}(r')(r)) = \widehat{\delta}(r)(r') + \widehat{\delta}(r')(r).$$

So $\delta = \widehat{\delta}|_{R \times R}$ is indeed skew-symmetric, and the corresponding 1-cocycle is given by

$$J_P(r, r') = \delta(r)(r') = P(x_0 \otimes r, y_0 \otimes r').$$

This shows that P determines J_P. □

Remark 4.5. We observe that J_P also satisfies the identity

$$J_P(t, 1) = 0 = J_P(1, t).$$

Remark 4.6. Every cocycle $P \in Z^2(\mathfrak{g} \otimes_k R, V)$ is cohomologous to a unique standard cocycle P_{st}. (This is the untwisted case of Theorem 4.2.) The equivalence classes of central extensions of $\mathfrak{g} \otimes_k R$ by V are parametrized by $Z^2_{\mathrm{st}}(\mathfrak{g} \otimes_k R, V)$, or equivalently, by the space $Z^1(R, V)$ of cyclic 1-cocycles.[3]

Corollary 4.7. *If $P \in Z^2_{\mathrm{st}}(\mathfrak{g} \otimes_k S, V)$, then the restriction of P to $\mathcal{L} \times \mathcal{L}$ is an element of $Z^2_{\mathrm{st}}(\mathcal{L}, V)$.*

Proof. If $P \in Z^2_{\mathrm{st}}(\mathfrak{g} \otimes_k S, V)$, it follows from Corollary 4.4 that P is such that

$$P(x \otimes s, y \otimes s') = B_{\mathfrak{g}}(x, y) J_P(s, s') = B_{\mathfrak{g}}(x, y) \widehat{\delta}(s)(s')$$

for some skew-symmetric derivation $\widehat{\delta} \in \mathrm{Der}_k(S, \mathrm{Hom}_k(S, V))$. Now for $l = \sum_{i=1}^n e_i \otimes s_i(l)$ and $l' = \sum_{j=1}^n e_j \otimes s_j(l')$ we get

$$P(l, l') = \sum_{i=1}^{n} \sum_{j=1}^{n} B_{\mathfrak{g}}(e_i, e_j) \widehat{\delta}(s_i(l))(s_j(l')).$$

So $P|_{\mathcal{L} \times \mathcal{L}}$ is standard. □

Remark 4.8. The restriction map $\mathrm{res} : Z^2(\mathfrak{g} \otimes_k S, V) \to Z^2(\mathcal{L}, V)$ preserves the orthogonal decomposition of Theorem 4.2 (i).

5 Standard cocycles and Kähler differentials

Given an S/R-form $\mathcal{L}$ of $\mathfrak{g} \otimes_k R$ and a trivial $\mathcal{L}$-module V, we shall use the modules of Kähler differentials $\Omega^1_{S/k}$ and $\Omega^1_{R/k}$ to give a simple form in which to write standard cocycles, namely the elements of $Z^2_{\mathrm{st}}(\mathcal{L}, V)$.

5.1 Kähler differentials and Galois extensions

Throughout S/R will be a finite Galois extension as above. The S-module of Kähler differentials of the k-algebra S will be denoted by $\Omega^1_{S/k}$ and the corresponding universal derivation by $d_S : S \to \Omega^1_{S/k}$.[4]

3 For a detailed account of the cohomology of algebras of the form $\mathfrak{g} \otimes_k R$, see [15].
4 Our standard reference for this material is [6, Chapter 0, Section 20]. The notation employed therein is $d_{S/k} : S \to \Omega^1_{S/k}$. We will throughout write d_S instead of $d_{S/k}$ since no confusion is possible. Similar considerations apply to R/k.

Recall that for all S-module N there is an S-module isomorphism

$$\mathrm{Hom}_S(\Omega^1_{S/k}, N) \to \mathrm{Der}_k(S, N), \quad \varphi \mapsto \varphi \circ d_S. \tag{5.1}$$

Given that S/R is faithfully flat, the natural map $i : R \to S$ is injective and allow us to identify R with a k-subalgebra of S. Since $d_S \circ i \in \mathrm{Der}_k(R, \Omega^1_{S/k})$, we can use (5.1) to obtain a (unique) R-module morphism χ making the following diagram commute:

$$\begin{array}{ccc} R & \xrightarrow{i} & S \\ {\scriptstyle d_R}\downarrow & & \downarrow{\scriptstyle d_S} \\ \Omega^1_{R/k} & \xrightarrow{\chi} & \Omega^1_{S/k}. \end{array}$$

Consider the S-module morphism

$$\Omega^1_{R/k} \otimes_R S \to \Omega^1_{S/k}, \quad \omega \otimes s \mapsto s\chi(\omega).$$

If S/R is Galois (hence étale), this last morphism is an isomorphism of S-modules [6, Section 20, Corollary 20.5.8]. Since S/R is faithfully flat, it follows that χ is injective, and this allows us to identify $\Omega^1_{R/k}$ with an R-submodule of $\Omega^1_{S/k}$ Following this with identification of R with a subalgebra of S allows us to identify $d_R(R) \subset \Omega^1_{R/k}$ with $d_S(R) \subset \Omega^1_{S/k}$.

The Galois group Γ of S/R acts on $\Omega^1_{S/k}$ via ${}^\gamma(sd_S(t)) = {}^\gamma s d_S({}^\gamma t)$ and the invariants are exactly the elements of $\chi(\Omega^1_{R/k})$. (We can check this fact by Galois descent: for any R-module M we have that $(M \otimes_R S)^\Gamma = M \otimes_R 1$, and since $\Omega^1_{R/k} \otimes_R S \simeq \Omega^1_{S/k}$ our claim follows). On the other hand, it is easy to see that:

Lemma 5.1. *We have $d_S(S)^\Gamma = d_S(R) = \chi(d_R(R))$.*

Proof. The last equality was explained in the penultimate paragraph above. Under the identifications explained therein, the inclusion $d_S(R) \subset d_S(S)^\Gamma$ is immediate by the way that the Galois group acts on $\Omega^1_{S/k}$ (simply observe that $S^\Gamma = R$). To see that this inclusion is in fact an equality we appeal to a standard reasoning from abelian Galois cohomology. Let V be the kernel of the k-linear map $d_S : S \to \Omega^1_{S/k}$. Consider the exact sequence of abelian groups (in fact of k-spaces)

$$0 \longrightarrow V \longrightarrow S \xrightarrow{d_S} d_S(S) \longrightarrow 0.$$

It is clear from the way that the Galois group acts that this sequence is Γ-equivariant. Passing to cohomology yields

$$R = S^\Gamma \xrightarrow{d_S} d_S(S)^\Gamma \longrightarrow H^1(\Gamma, V).$$

Since $H^1(\Gamma, V)$ is a torsion group and also a k-vector space, we get $H^1(\Gamma, V) = 0$ (see [18, Chapter 1, Section 5]). It follows that $d_S(R)$ surjects onto $d_S(S)^\Gamma$. □

Let $\pi_S : \Omega^1_{S/k} \to \Omega^1_{S/k}/d_S(S)$ and $\pi_R : \Omega^1_{R/k} \to \Omega^1_{R/k}/d_R(R)$ be the canonical k-spaces maps. Since

$$(\pi_S \circ \chi)(d_R(r)) = 0,$$

there exists an unique $\tilde{\chi} : \Omega^1_{R/k}/d_R(R) \to \Omega^1_{S/k}/d_S(S)$ making the diagram

$$\begin{array}{ccc} \Omega^1_{R/k} & \xrightarrow{\chi} & \Omega^1_{S/k} \\ {\scriptstyle \pi_R}\downarrow & & \downarrow{\scriptstyle \pi_S} \\ \Omega^1_{R/k}/d_R(R) & \xrightarrow{\tilde{\chi}} & \Omega^1_{S/k}/d_S(S) \end{array}$$

commute. The map $\tilde{\chi}$ is injective: $\overline{\omega} \in \mathrm{Ker}(\tilde{\chi}) \Leftrightarrow (\pi_S \circ \chi)(\omega) = \overline{0} \Leftrightarrow \chi(\omega) \in d_S(S)$. But $\chi(\omega) = \chi({}^\gamma\omega) = {}^\gamma\chi(\omega)$ for all $\gamma \in \Gamma$, so $\chi(\omega) \in (d_S(S))^\Gamma$, then $\omega \in d_R(R)$. This allows us to identify $\Omega^1_{R/k}/d_R(R)$ with a k-subspace of $\Omega^1_{S/k}/d_S(S)$

Lemma 5.2. *The group Γ also acts in the quotient space $\Omega^1_{S/k}/d_S(S)$ via ${}^\gamma(\overline{sd_S(t)}) = \overline{{}^\gamma s d_S({}^\gamma t)}$. Under this action*

$$(\Omega^1_{S/k}/d_S(S))^\Gamma \simeq \Omega^1_{R/k}/d_R(R).$$

Proof. We leave to the reader to check that the action is well defined. Consider the Γ equivariant exact sequence of k-spaces

$$0 \to d_S(S) \to \Omega^1_{S/k} \to \Omega^1_{S/k}/d_S(S) \to 0.$$

By passing to cohomology and appealing to the last lemma we get

$$0 \to d_R(R) \to \Omega^1_{R/k} \to (\Omega^1_{S/k}/d_S(S))^\Gamma \to H^1(\Gamma, d_S(S)).$$

As already explained before $H^1(\Gamma, d_S(S))$ vanishes. The lemma now follows. □

5.2 The skew-symmetry problem

Lemma 5.3. *An element $\delta \in \mathrm{Der}_k(S, \mathrm{Hom}_k(S, V))$ is skew-symmetric if and only if for all $s \in S$, $\delta(s)(1) = 0$.*

Proof. For all $s, s' \in S$ we have $\delta(ss')(1) = \delta(s)(s') + \delta(s')(s)$. □

Lemma 5.4. *Let $\delta \in \mathrm{Der}_k(S, \mathrm{Hom}_k(S, V))$.*
(i) *There exists $\phi \in \mathrm{Hom}_k(\Omega^1_{S/k}, V)$ (not necessarily unique) such that for all $s, t \in S$, $\delta(s)(t) = \phi(td_S(s))$.*
(ii) *δ is skew-symmetric if and only if one (and then all of the) ϕ of* (i) *verifies the equation $\phi \circ d_S = 0$.*
(iii) *The reciprocal of* (i) *and* (ii) *holds: For all $\phi \in \mathrm{Hom}_k(\Omega^1_{S/k}, V)$, the mapping $\delta : S \to \mathrm{Hom}_k(S, V)$ given by $\delta(s)(t) = \phi(td_S(s))$ is an element of $\mathrm{Der}_k(S, \mathrm{Hom}_k(S, V))$, and this derivation is skew-symmetric if and only if $\phi \circ d_S = 0$.*

Proof. (i) Given $\delta \in \mathrm{Der}_k(S, \mathrm{Hom}_k(S, V))$, by (5.1), we have a morphism $\phi_0 \in \mathrm{Hom}_S(\Omega^1_{S/k}, \mathrm{Hom}_k(S, V))$ such that $\delta = \phi_0 \circ d_S$, i.e. for all $s, t \in S$ $\delta(s)(t) = \phi_0(d_S(s))(t)$. Since ϕ_0 is S-linear, we have

$$\phi_0(sd_S(t))(u) = (s \cdot \phi_0)(d_S(t))(u) = \phi_0(d_S(t))(su).$$

To define ϕ we will consider first the linear mapping ϕ_1 defined in the free S-module with basis $\{ds : s \in S\}$ such that $\phi_1(tds) = \phi_0(d_S(s))(t) \in V$. This action factors through the quotient defining $\Omega^1_{S/k}$. Indeed,

$$\begin{aligned}
\phi_1(d(st) - sdt - tds) &= \phi_1(d(st)) - \phi_1(sdt) - \phi_1(tds) \\
&= \phi_0(d_S(st))(1) - \phi_0(d_S(t))(s) - \phi_0(d_S(s))(t) \\
&= \phi_0(sd_S(t))(1) + \phi_0(td_S(s))(1) - \phi_0(d_S(t))(s) - \phi_0(d_S(s))(t) \\
&= \phi_0(d_S(t))(s) + \phi_0(d_S(s))(t) - \phi_0(d_S(t))(s) - \phi_0(d_S(s))(t) = 0, \\
\phi_1(d(s+t) - ds - dt) &= \phi_1(d(s+t)) - \phi_1(ds) - \phi_1(dt) \\
&= \phi_0(d_S(s+t))(1) - \phi_0(d_S(s))(1) - \phi_0(d_S(t))(1) \\
&= \phi_0(d_S(s+t) - d_S(s) - d_S(t))(1) \\
&= \phi_0(0)(1) = 0.
\end{aligned}$$

It follows that ϕ_1 induces a k-linear morphism $\phi : \Omega^1_{S/k} \to V$ such that for all $s, t \in S$,

$$\phi(td_S(s)) = \phi_1(tds) = \phi_0(d_S(s))(t) = \delta(s)(t).$$

(ii) This follows immediately from Lemma 5.3: δ is skew-symmetric if and only if for all $s \in S$, $\delta(s)(1) = 0$ if and only if for all $s \in S$, $\phi(d_S(s)) = 0$

(iii) Given $\phi \in \mathrm{Hom}_k(\Omega^1_{S/k}, V)$, the mapping $\delta : S \to \mathrm{Hom}_k(S, V)$ given by $\delta(s)(t) = \phi(td_S(s))$ is an element of $\mathrm{Der}_k(S, \mathrm{Hom}_k(S, V))$:

$$\begin{aligned}
\delta(ss')(t) &= \phi(td_S(ss')) = \phi(tsd_S(s') + ts'd_S(s)) \\
&= \phi(tsd_S(s')) + \phi(ts'd_S(s)) = \delta(s')(st) + \delta(s)(s't) \\
&= s \cdot \delta(s')(t) + s' \cdot \delta(s)(t),
\end{aligned}$$

and again by Lemma 5.3 this derivation is skew-symmetric if and only if $\phi \circ d_S = 0$. □

Corollary 5.5. *The following statements hold.*

(i) *Given* $\delta \in \mathrm{Der}_k(S, \mathrm{Hom}_k(S, V))$ *skew-symmetric, there exists* $\Phi \in \mathrm{Hom}_k(\Omega^1_{S/k}/d_S(S), V)$ *(not necessarily unique) such that for all* $s, t \in S$, $\delta(s)(t) = \Phi(\overline{td_S(s)})$.

(ii) *Given* $\Phi \in \mathrm{Hom}_k(\Omega^1_{S/k}/d_S(S), V)$, *the mapping* $\delta : S \to \mathrm{Hom}_k(S, V)$ *given by* $\delta(s)(t) = \Phi(\overline{td_S(s)})$ *is a skew-symmetric derivation.*

Proof. (i) By Lemma 5.4 (i), there exists $\phi \in \mathrm{Hom}_k(\Omega^1_{S/k}, V)$ such that for all $s, t \in S$, $\delta(s)(t) = \phi(td_S(s))$. Since δ is skew-symmetric, it follows from Lemma 5.4 (ii) that $\phi \circ d_S = 0$. Then we have a well-defined morphism $\Phi \in \mathrm{Hom}_k(\Omega^1_{S/k}/d_S(S), V)$ such that for all $s, t \in S$,

$$\Phi(\overline{td_S(s)}) = \phi(td_S(s)) = \delta(s)(t).$$

Thus, δ is skew-symmetric if and only if one (and then all of the) ϕ of (i) is such that $\phi \circ d_S = 0$.

(ii) Given $\Phi \in \mathrm{Hom}_k(\Omega^1_{S/k}/d_S(S), V)$, we can define

$$\delta(s)(t) = \phi(td_S(s)) = \Phi(\overline{td_S(s)})$$

and by Lemma 5.4 (iii) it is a derivation. Since $\delta(s)(1) = \Phi(\overline{d_S(s)}) = \Phi(\overline{0}) = 0$, we have again by Lemma 5.4 (iii) that δ is skew-symmetric. □

Recall the average map

$$\overline{\pi} : \Omega^1_{S/k}/d_S(S) \to \Omega^1_{S/k}/d_S(S), \quad \overline{\pi}(\overline{\omega}) = \frac{1}{|\Gamma|}\sum_{\gamma \in \Gamma} {}^{\gamma}\overline{\omega}.$$

Since the elements in $\overline{\pi}(\Omega^1_{S/k}/d_S(S))$ are Γ-invariant, we actually have

$$\overline{\pi} : \Omega^1_{S/k}/d_S(S) \to (\Omega^1_{S/k}/d_S(S))^{\Gamma} \simeq \Omega^1_{R/k}/d_R(R).$$

Now we can prove the following theorem about extension of skew-symmetric derivations:

Proposition 5.6. *Let* $\delta \in \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))$ *be skew-symmetric. The unique extension*

$$\widehat{\delta} \in \mathrm{Der}_k(S, \mathrm{Hom}_k(S, V))$$

given in Proposition 3.10 is skew-symmetric.

Proof. Let $\delta \in \mathrm{Der}_k(R, \mathrm{Hom}_k(R, V))$ be skew-symmetric. By Corollary 5.5 (i) applied to R, there exists some $\Phi \in \mathrm{Hom}_k(\Omega^1_{R/k}/d_R(R), V)$ such that for all $r, r' \in S$,

$$\delta(r)(r') = \Phi(\overline{r' d_R(r)}).$$

So we can define $\widehat{\Phi} \in \mathrm{Hom}_k(\Omega^1_{S/k}/d_S(S), V)$ as the composition of the morphisms in the diagram

$$\begin{array}{ccccc} \Omega^1_{S/k}/d_S(S) & \xrightarrow{\overline{\pi}} & (\Omega^1_{S/k}/d_S(S))^{\Gamma} & \xrightarrow{\simeq} & \Omega^1_{R/k}/d_R(R) \\ & \searrow^{\widehat{\Phi}} & & & \downarrow \Phi \\ & & & & V. \end{array}$$

By Corollary 5.5 (ii), we obtain the skew-symmetric derivation $\check{\delta} \in \mathrm{Der}_k(S, \mathrm{Hom}_k(S, V))$ defined by

$$\check{\delta}(s)(t) = \widehat{\Phi}(\overline{td_S(s)}).$$

Since we can identify $\overline{r' d_S(r)} \in (\Omega^1_{S/k}/d_S(S))^{\Gamma}$ with $\overline{r' d_R(r)} \in \Omega^1_{R/k}/d_R(R)$, for all $r \in R$ and $t \in S$ we have

$$\check{\delta}(r)(t) = \widehat{\Phi}(\overline{td_S(r)}) = \Phi(\overline{\pi(t) d_R(r)}) = \delta(r)(\pi(t)) = \widehat{\delta}(r)(t).$$

By the uniqueness of the extension of δ, we have $\check{\delta} = \widehat{\delta}$. In particular, $\widehat{\delta}$ is skew-symmetric. □

5.3 Another expression for standard cocycles

From Theorem 4.2 and Lemma 5.4, for a standard cocycle P we have the expression

$$P(l,l')=\sum_{i=1}^{n}\sum_{j=1}^{n}B_{\mathfrak{g}}(e_i,e_j)\phi(s_j(l')d_S(s_i(l)))=\phi\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}B_{\mathfrak{g}}(e_i,e_j)s_j(l')d_S(s_i(l))\Big)$$

for $l=\sum_{i=1}^{n}e_i\otimes s_i(l)$ and $l'=\sum_{j=1}^{n}e_j\otimes s_j(l')\in\mathcal{L}$.

This last formula motivates the definition of the k-bilinear morphism

$$\dot{B}:(\mathfrak{g}\otimes_k S)\times(\mathfrak{g}\otimes_k\Omega^1_{S/k})\to\Omega^1_{S/k}\quad\text{such that}\quad\dot{B}(x\otimes s,y\otimes\omega)=B_{\mathfrak{g}}(x,y)s\omega$$

and the definition of the k-linear map

$$\partial:\mathfrak{g}\otimes_k S\to\mathfrak{g}\otimes_k\Omega^1_{S/k}\quad\text{such that}\quad\partial(x\otimes s)=x\otimes d_S(s).$$

Now we can rewrite the above formula: for $l=\sum_{i=1}^{n}e_i\otimes s_i(l)$ and $l'=\sum_{j=1}^{n}e_j\otimes s_j(l')\in\mathcal{L}$,

$$P(l,l')=\phi\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}B_{\mathfrak{g}}(e_i,e_j)s_j(l')d_S(s_i(l))\Big)=\phi\Big(\dot{B}\Big(\sum_{j=1}^{n}e_j\otimes s_j(l'),\sum_{i=1}^{n}e_i\otimes d_S(s_i(l)\Big)\Big)=\phi(\dot{B}(l',\partial(l))).\quad(5.2)$$

Some remarks about the existence and nature of these objects are in order:

- ∂ is well defined: The k-bilinear morphism $\mathfrak{g}\times S\to\mathfrak{g}\otimes_k\Omega^1_{S/k}$ given by $(x,s)\mapsto x\otimes d_S(s)$ induces a k-linear map $\partial:\mathfrak{g}\otimes_k S\to\mathfrak{g}\otimes_k\Omega^1_{S/k}$ satisfying $(x\otimes s)\mapsto x\otimes d_S(s)$.
- $\dot{B}$ is well defined: From the k-multilinear map $\mathfrak{g}\times S\times\mathfrak{g}\times\Omega^1_{S/k}\to\Omega^1_{S/k}$ given by $(x,s,y,\omega)\mapsto B_{\mathfrak{g}}(x,y)s\omega$ we obtain a k-linear map $\mathfrak{g}\otimes_k S\otimes_k\mathfrak{g}\otimes_k\Omega^1_{S/k}\to\mathfrak{g}\otimes_k\Omega^1_{S/k}$ such that $(x\otimes s\otimes y\otimes\omega)\mapsto B_{\mathfrak{g}}(x,y)s\omega$. This yields a k-bilinear map $\mathfrak{g}\otimes_k S\times\mathfrak{g}\otimes_k\Omega^1_{S/k}\to\mathfrak{g}\otimes_k\Omega^1_{S/k}$ such that $(x\otimes s,y\otimes\omega)\mapsto B_{\mathfrak{g}}(x,y)s\omega$.
- $\mathfrak{g}\otimes_k\Omega^1_{S/k}$ is an S-module, with S acting in $\Omega^1_{S/k}$.
- $\mathfrak{g}\otimes_k\Omega^1_{S/k}$ is a $\mathfrak{g}\otimes_k S$-module via $(x\otimes s)\cdot(y\otimes\omega)=[x,y]\otimes s\omega$. It is easy to see that this defines an action of the S-Lie algebra $\mathfrak{g}\otimes_k S$ on the S-module $\mathfrak{g}\otimes_k\Omega^1_{S/k}$.

Proposition 5.7. *Let $\partial:\mathfrak{g}\otimes_k S\to\mathfrak{g}\otimes_k\Omega^1_{S/k}$ and $\dot{B}:(\mathfrak{g}\otimes_k S)\times(\mathfrak{g}\otimes_k\Omega^1_{S/k})\to\Omega^1_{S/k}$ be as above. Then:*

(i) *∂ is a derivation.*

(ii) *For all $a,b\in\mathfrak{g}\otimes_k S$ and $\varsigma\in\mathfrak{g}\otimes_k\Omega^1_{S/k}$ we have*

$$\dot{B}([a,b],\varsigma)=\dot{B}(a,b\cdot\varsigma).$$

(iii) *For all $a,b\in\mathfrak{g}\otimes_k S$,*

$$d_S(B_{\mathfrak{g}\otimes_k S}(a,b))=\dot{B}(a,\partial(b))+\dot{B}(b,\partial(a)).$$

Proof. The proof consists of straightforward computations that we omit. □

We can now rewrite Theorem 4.2.

Theorem 5.8. *Let V be a trivial $\mathcal{L}$-module and $P\in Z^2_{\mathrm{st}}(\mathcal{L},V)$ a standard cocycle. Then:*

(i) *There exists a (in general not necessarily unique) k-linear map $\phi:\Omega^1_{S/k}\to V$ such that for all $l,l'\in\mathcal{L}$ and $r\in R$,*

$$P(l,l')=\phi(\dot{B}(l',\partial(l)))\quad(5.3)$$

and

$$(\phi\circ d_S)(r)=0.\quad(5.4)$$

(ii) *Conversely, for any k-linear map $\phi:\Omega^1_{S/k}\to V$ satisfying* (5.4), *formula* (5.3) *defines a standard cocycle.*

Proof. The existence of ϕ has been seen in Lemma 5.4 and (5.2). Now, since P is skew-symmetric,

$$0=P(l,l')+P(l',l)=\phi(\dot{B}(l',\partial(l)))+\phi(\dot{B}(l,\partial(l')))=\phi(d_S(B_{\mathfrak{g}\otimes_k S}(l,l')))=\phi(d_S(B_{\mathcal{L}}(l,l'))),$$

where the last equality follows from Proposition 5.7 (iii). Since $B_{\mathcal{L}}$ is onto R, we have $(\phi\circ d_S)(r)=0$ for all $r\in R$.

Conversely, we have to prove first that formula (5.3) defines a cocycle. The skew-symmetry follows from the calculation above and formula (5.4). In order to prove the Jacobi identity, let $a, b, c \in \mathcal{L}$; then we have

$$\begin{aligned} &P([a,b],c) + P([b,c],a) + P([c,a],b) \\ &\quad = \phi(\dot{B}(c, \partial([a,b])) + \dot{B}(a, \partial([b,c])) + \dot{B}(b, \partial([c,a]))) \\ &\quad = \phi(\dot{B}(c, a\cdot\partial(b)) - \dot{B}(c, b\cdot\partial(a)) + \dot{B}(a, b\cdot\partial(c)) \\ &\qquad\quad - \dot{B}(a, c\cdot\partial(b)) + \dot{B}(b, c\cdot\partial(a)) - \dot{B}(b, a\cdot\partial(c))) \qquad \text{(by Proposition 5.7 (i))} \\ &\quad = \phi(\dot{B}([c,a], \partial(b)) - \dot{B}([c,b], \partial(a)) + \dot{B}([a,b], \partial(c)) \\ &\qquad\quad - \dot{B}([a,c], \partial(b)) + \dot{B}([b,c], \partial(a)) - \dot{B}([b,a], \partial(c))) \qquad \text{(by Proposition 5.7 (ii))} \\ &\quad = 2\phi(\dot{B}([c,a], \partial(b)) + \dot{B}([a,b], \partial(c)) + \dot{B}([b,c], \partial(a))). \end{aligned}$$

One the other hand,

$$\begin{aligned} \delta_S(B_{\mathcal{L}}([a,b],c)) &= \dot{B}([a,b], \partial(c)) + \dot{B}(c, \partial([a,b])) && \text{(by Proposition 5.7 (iii))} \\ &= \dot{B}([a,b], \partial(c)) + \dot{B}(c, a\cdot\partial(b)) - \dot{B}(c, b\cdot\partial(a)) && \text{(by Proposition 5.7 (i))} \\ &= \dot{B}([a,b], \partial(c)) + \dot{B}([c,a], \partial(b)) - \dot{B}([c,b], \partial(a)) && \text{(by Proposition 5.7 (ii))} \\ &= \dot{B}([a,b], \partial(c)) + \dot{B}([c,a], \partial(b)) + \dot{B}([b,c], \partial(a)). \end{aligned}$$

So

$$\begin{aligned} P([a,b],c) + P([b,c],a) + P([c,a],b) &= 2\phi(\dot{B}([c,a], \partial(b)) + \dot{B}([a,b], \partial(c)) + \dot{B}([b,c], \partial(a))) \\ &= 2\phi(d_S(\underbrace{B_{\mathcal{L}}([a,b],c)}_{\in R})) = 0. \end{aligned}$$

Finally, we have to prove that P is standard. Indeed, given $l = \sum_{i=1}^n e_i \otimes s_i(l)$ and $l' = \sum_{j=1}^n e_j \otimes s_j(l') \in \mathcal{L}$,

$$\begin{aligned} P(l,l') &= \phi(\dot{B}(l', \partial(l))) \\ &= \phi\Big(\dot{B}\Big(\sum_{j=1}^n e_j \otimes s_j(l'), \sum_{i=1}^n e_i \otimes d_S(s_i(l)\Big)\Big) \\ &= \phi\Big(\sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j) s_j(l') d_S(s_i(l))\Big) \\ &= \sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j) \phi(s_j(l') d_S(s_i(l))) \\ &= \sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j) \widehat{\delta}(s_i(l))(s_j(l')). \end{aligned}$$

By Theorem 4.2, P is standard. □

Corollary 5.9. *If $P \in Z^2_{\mathrm{st}}(\mathcal{L}, W)$ and $\psi \in \mathrm{Hom}_k(W, V)$, then $\psi \circ P \in Z^2_{\mathrm{st}}(\mathcal{L}, V)$.*

Proof. Follows immediately from the Theorem 5.8. □

We can establish some important facts about the skew-symmetry of the derivations involved in Theorem 4.2.

Proposition 5.10. *We have*

$$\dot{\sigma}(\mathrm{Der}_k^{(-)}(R, \mathrm{Hom}_k(R,V))) = (\dot{\sigma}[\mathrm{Der}_k(R, \mathrm{Hom}_k(R,V))])^{(-)}.$$

Proof. Let $\delta \in \mathrm{Der}_k(R, \mathrm{Hom}_k(R,V))$ and $\widehat{\delta} \in \mathrm{Der}_k(S, \mathrm{Hom}_k(S,V))$ its unique extension. If δ is skew-symmetric, then $\widehat{\delta}$ is also skew-symmetric (see Proposition 5.6). Then

$$\dot{\sigma}(\delta)(l)(l') + \dot{\sigma}(\delta)(l')(l) = \sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j)(\widehat{\delta}(s_i(l))(s_j(l')) + \widehat{\delta}(s_j(l'))(s_i(l))) = 0.$$

Reciprocally, if $\delta \in \mathrm{Der}_k(R, \mathrm{Hom}_k(R,V))$ is such that $\dot{\sigma}(\delta)$ is skew-symmetric, let us see that δ (then $\widehat{\delta}$) is skew-symmetric. By Lemma 5.4, there exists $\widehat{\phi} : \Omega^1_{S/k} \to V$ such that $\widehat{\phi}(t d_S(s)) = \widehat{\delta}(s)(t)$ and we can rewrite

$$\dot{\sigma}(\delta)(l)(l') = \widehat{\phi}(\dot{B}(l', \partial(l))).$$

Since $\dot{\sigma}(\delta)$ is skew-symmetric, $(\widehat{\phi} \circ d_S)(r) = 0$ for all $r \in R$ (see Theorem 5.8 and its proof). Now

$$\delta(r)(1) = \widehat{\delta}(r)(1) = \widehat{\phi}(d_S(r)) = 0;$$

then, by Lemma 5.3, δ is skew-symmetric. □

Corollary 5.11. *Equation* (4.1) *in Theorem 4.2 can be replaced by the skew-symmetry of δ (which in turn is equivalent to the skew-symmetry of $\widehat{\delta}$).*

6 Applications to universal central extension of Galois twisted forms

Let $\mathcal{L}$ be a twisted form of $\mathfrak{g} \otimes_k R$. By definition (see [4] for details that are relevant to the present paper), there exists a faithfully flat and finitely presented ring extension S/R such that $\mathcal{L} \otimes_R S \simeq \mathfrak{g} \otimes_k S$ as S-Lie algebras. Because $\mathbf{Aut}(\mathfrak{g})$ is smooth and affine, there is no loss of generality in assuming that S/R is étale. In the present paper we will only be interested in forms were S/R can be assumed to be Galois.[5]

Assume henceforth that S/R is Galois with Galois group Γ (see Section 3 for all relevant definitions and details). The action of Γ on $\Omega^1_{S/k}$ via ${}^{\gamma}(s d_S(t)) = {}^{\gamma}s d_S({}^{\gamma}t)$ passes to the quotient $\Omega^1_{S/k}/d_S(R)$, and Γ acts via

$${}^{\gamma}(\overline{\overline{s d_S(t)}}) = \overline{\overline{{}^{\gamma}s d_S({}^{\gamma}t)}},$$

where the double overline mean class in $\Omega^1_{S/k}/d_S(R)$.

Lemma 6.1. *We have*

$$(\Omega^1_{S/k}/d_S(R))^{\Gamma} \simeq \Omega^1_{R/k}/d_R(R).$$

Proof. The proof is similar to that of Lemma 5.2. □

Let us consider the k-linear map

$$\widehat{\phi} : \Omega^1_{S/k} \to \Omega^1_{S/k}/d_S(R), \quad \widehat{\phi}(t d_S(s)) = \overline{\overline{t d_S(s)}}. \tag{6.1}$$

Since clearly $(\widehat{\phi} \circ d_S)(r) = 0$ for all $r \in R$, by Theorem 5.8, we have the standard cocycle $\widehat{P} \in Z^2_{\mathrm{st}}(\mathcal{L}, \Omega^1_{S/k}/d_S(R))$ such that for all $l = \sum_{i=1}^n e_i \otimes s_i(l)$ and $l' = \sum_{j=1}^n e_j \otimes s_j(l') \in \mathcal{L}$,

$$\widehat{P}(l, l') = \widehat{\phi}(\dot{B}(l', \partial(l))) = \sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j)\overline{\overline{s_j(l') d_S(s_i(l))}}. \tag{6.2}$$

Remark 6.2. In terms of Theorem 4.2, we have the derivation

$$\widehat{\delta} \in \mathrm{Der}_k(S, \mathrm{Hom}_k(S, \Omega^1_{S/k}/d_S(R))) \quad \text{such that} \quad \widehat{\delta}(s)(t) = \overline{\overline{t d_S(s)}}.$$

Let us define an important k-subspace of $\Omega^1_{S/k}/d_S(R)$ that will appear in our new description of the universal central extension of $\mathcal{L}$. This is the space

$$W(\mathcal{L}) = \text{the } k\text{-linear span of } \widehat{P}(\mathcal{L} \times \mathcal{L}) \subseteq \Omega^1_{S/k}/d_S(R).$$

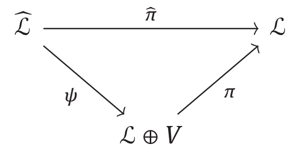
Theorem 6.3. *The Lie algebra $\widehat{\mathcal{L}} = \mathcal{L} \oplus W(\mathcal{L})$ with bracket*

$$[l + w, l' + w'] = [l, l']_{\mathcal{L}} + \widehat{P}(l, l')$$

is the universal central extension of $\mathcal{L}$.

5 This assumption is superfluous in the case of Laurent polynomial rings $R = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ (which is the case that arises naturally in many areas of infinite-dimensional Lie theory). Indeed, by the main Isotriviality Theorem of [5], any twisted form $\mathcal{L}$ of $\mathfrak{g} \otimes_k R$, i.e. split by S/R étale, is necessarily split by a Galois extension of R.

Proof. Let $\mathcal{L}_P$ be a central extension $\mathcal{L}_P = \mathcal{L} \oplus V$, where $P \in Z^2(\mathcal{L}, V)$ is a cocycle, which may be supposed standard without loss generality (every cocycle is cohomological equivalent to a standard one, and isomorphic central extension correspond to cohomological equivalent cocycles). Let us prove that there is a unique morphism of Lie algebras $\psi : \widehat{\mathcal{L}} \to \mathcal{L} \oplus V$ such that the diagram



commutes, where π and $\hat{\pi}$ are the projections to $\mathcal{L}$. For a general linear mapping $\psi : \widehat{\mathcal{L}} \to \mathcal{L} \oplus V$ we write $\psi(l + w) = \psi_1(l + w) + \psi_2(l + w)$. Then a morphism of Lie algebra making the above diagram commute must verify $\pi \circ \psi = \hat{\pi}$. Thus $l = \hat{\pi}(l + w) = \pi(\psi(l + w)) = \psi_1(l + w)$ for all $l \in \mathcal{L}$. This is to say

$$\psi(l + w) = l + \psi_2(l + w) \tag{6.3}$$

and

$$\psi([l + w, l' + w]_{\widehat{\mathcal{L}}}) = [\psi(l + w), \psi(l' + w')]_{\mathcal{L} \oplus V}.$$

Taking into account (6.3) this yields

$$\begin{aligned} \psi([l, l']_{\mathcal{L}} + \widehat{P}(l, l')) &= [l + \psi_2(l + w), l' + \psi_2(l' + w')], \\ [l, l']_{\mathcal{L}} + \psi_2([l, l']_{\mathcal{L}} + \widehat{P}(l, l')) &= [l, l']_{\mathcal{L}} + P(l, l'), \\ \psi_2([l, l']_{\mathcal{L}} + \widehat{P}(l, l')) &= P(l, l'), \\ \psi_2([l, l']_{\mathcal{L}}) + \psi_2(\widehat{P}(l, l')) &= P(l, l'). \end{aligned}$$

But the mapping $(l, l') \mapsto \psi_2([l, l'])$ is obviously a coboundary in $\mathcal{L}$ with values in V, and $(l, l') \mapsto \psi_2(\widehat{P}(l, l'))$ is a standard cocycle, because $\widehat{P}$ is standard (see Corollary 5.9). Since Theorem 4.2 asserts that the decomposition of a cocycle as a sum of a coboundary and a standard part is unique, and P is standard, the last row is possible only if $\psi_2([l, l']) = 0$ for all $l, l' \in \mathcal{L}$. But L is perfect, so for all $l \in \mathcal{L}$,

$$\psi_2(l) = 0 \quad \text{and} \quad \psi_2(\widehat{P}(l, l')) = P(l, l'). \tag{6.4}$$

We may thus consider ψ_2 as a linear mapping $W(\mathcal{L}) \to V$ satisfying (6.4).

Summing up: we have proved that any morphism of Lie algebras $\psi : \widehat{\mathcal{L}} \to \mathcal{L} \oplus V$ making the diagram commute is necessarily of the form $\psi(l + w) = l + \Psi(w)$, where $\Psi : W(\mathcal{L}) \to V$ is a linear mapping satisfying $\Psi(\widehat{P}(l, l')) = P(l, l')$ for all $l, l' \in \mathcal{L}$. Thus the existence of a unique morphism $\psi : \widehat{\mathcal{L}} \to \mathcal{L} \oplus V$ of Lie algebras making the above diagram commute reduces to the existence of a unique linear mapping $\Psi : W(\mathcal{L}) \to V$ satisfying $\Psi(\widehat{P}(l, l')) = P(l, l')$ for all $l, l' \in \mathcal{L}$.

Existence. By Theorem 5.8, for $l, l' \in \mathcal{L}$ the standard cocycle P is of the form

$$P(l, l') = \phi(\dot{B}(l', \partial(l)))$$

for some k-linear map $\phi : \Omega^1_{S/k} \to V$ such that $(\phi \circ d_S)(r) = 0$. This induces a k-linear map

$$\Psi : \Omega^1_{S/k}/d_S(R) \to V \quad \text{such that} \quad \Psi(\overline{\overline{td_S(s)}}) = \Psi(\widehat{\phi}(td_S(s)) = \phi(td_S(s)),$$

where $\widehat{\phi}$ is as in (6.1). Now

$$\Psi(\widehat{P}(l, l')) = \Psi(\widehat{\phi}(\dot{B}(l', \partial(l))) = \phi(\dot{B}(l', \partial(l))) = P(l, l').$$

Then, the restriction of Ψ to $W(\mathcal{L})$ is the required morphism.

Uniqueness. Let us suppose that $\Psi : W(\mathcal{L}) \to V$ is a linear mapping satisfying $\Psi(\widehat{P}(l, l')) = 0$ for all $l, l' \in \mathcal{L}$. By the definition of $W(\mathcal{L})$, we have $\Psi = 0$. □

Remark 6.4. Just as in Section 5.1 we have the commutative diagram

$$\begin{array}{ccc} \Omega^1_{R/k} & \xrightarrow{\chi} & \Omega^1_{S/k} \\ {\scriptstyle \pi_R}\big\downarrow & & \big\downarrow{\scriptstyle \pi_S} \\ \Omega^1_{R/k}/d_R(R) & \xrightarrow{\tilde{\tilde{\chi}}} & \Omega^1_{S/k}/d_S(R) \end{array}$$

and the inclusion $\Omega^1_{R/k}/d_R(R) \hookrightarrow \Omega^1_{S/k}/d_S(R)$. It follows that

$$\Omega^1_{R/k}/d_R(R) \hookrightarrow W(\mathcal{L}).$$

Indeed given $\overline{\overline{r' d_R(r)}} \xrightarrow{\tilde{\tilde{\chi}}} \overline{\overline{r' d_S(r)}}$, we can choose $l, l' \in \mathcal{L}$ such that $B_{\mathcal{L}}(l, l') = r'$; then

$$\overline{\overline{r' d_S(r)}} = \overline{\overline{B_{\mathcal{L}}(l, l') d_S(r)}} = \hat{\delta}(r)(B_{\mathcal{L}}(l, l')) = \widehat{P}(rl, l') - \widehat{P}(l, rl') \in W(\mathcal{L}),$$

where the last equality is due to (4.2).

Corollary 6.5. *Kassel model is universal. That is to say the Lie algebra $\widehat{\mathfrak{g}_R} = \mathfrak{g} \otimes_k R \oplus \Omega^1_{R/k}/d_R(R)$ with bracket*

$$[x \otimes r + \xi, y \otimes s + \eta]_{\widehat{\mathfrak{g} \otimes_k R}} = [x, y] \otimes rs + B(x, y)\overline{rds}$$

is the universal central extension of $\mathfrak{g} \otimes_k R$.

6.1 Universal central extension of multiloop algebras

In this subsection we will consider multiloop algebras based on $\mathfrak{g}$. (See [16, Section 5]). Thus we have $R = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ with k algebraically closed (of characteristic 0). In this case S may be assumed to be of the form $S = k[t_1^{\pm 1/m}, \dots, t_n^{\pm 1/m}]$ for some positive integer m. The choice of a compatible set of primitive roots of unity in k determines an isomorphism of Galois group of S/R with $(\mathbb{Z}/m\mathbb{Z})^n$. The cocycle $u_\gamma \in \mathrm{Aut}_k(\mathfrak{g} \otimes_k S)$ defining $\mathcal{L}$ is actually in $\mathrm{Aut}_k(\mathfrak{g}) \hookrightarrow \mathrm{Aut}_k(\mathfrak{g} \otimes_k S)$, i.e. for $x \otimes s \in \mathfrak{g} \otimes_k S$ we have that $u_\gamma(x \otimes s) = v_\gamma \otimes s$ for some $v_\gamma \in \mathrm{Aut}_k(\mathfrak{g})$. We continue with the notation in the previous section: $\widehat{P} \in Z^2_{\mathrm{st}}(\mathcal{L}, \Omega^1_{S/k}/d_S(R))$ given by (6.2).

Lemma 6.6. *In the multiloop case, for all $l, l' \in \mathcal{L}$, $\widehat{P}({}^\gamma(l, l')) = {}^\gamma(\widehat{P}(l, l'))$, where Γ acts on $\mathcal{L} \subseteq \mathfrak{g} \otimes_k S \times \mathfrak{g} \otimes_k S$ via the twisted action.*

Proof. Indeed, for $l = \sum_{i=1}^n e_i \otimes s_i(l)$ and $l' = \sum_{j=1}^n e_j \otimes s_j(l') \in \mathcal{L}$,

$$\begin{aligned} \widehat{P}({}^\gamma(l, l')) &= \widehat{P}(u_\gamma({}^\gamma l), u_\gamma({}^\gamma l')) \\ &= \widehat{P}\Big(u_\gamma\Big({}^\gamma\Big(\sum_{i=1}^n e_i \otimes s_i(l)\Big)\Big), u_\gamma({}^\gamma\Big(\sum_{j=1}^n e_j \otimes s_j(l')))\Big) \\ &= \widehat{P}\Big(u_\gamma\Big(\sum_{i=1}^n e_i \otimes {}^\gamma s_i(l)), u_\gamma(\sum_{j=1}^n e_j \otimes {}^\gamma s_j(l')\Big)\Big) \\ &= \widehat{P}\Big(\sum_{i=1}^n v_\gamma(e_i) \otimes {}^\gamma s_i(l), \sum_{j=1}^n v_\gamma(e_j) \otimes {}^\gamma s_j(l')\Big) \\ &= \sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(v_\gamma(e_i), v_\gamma(e_j)) \overline{\overline{{}^\gamma s_j(l') d_S({}^\gamma s_i(l))}} \\ &= \sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j) \overline{\overline{{}^\gamma s_j(l') d_S({}^\gamma s_i(l))}} \\ &= {}^\gamma\Big(\sum_{i=1}^n \sum_{j=1}^n B_{\mathfrak{g}}(e_i, e_j) \overline{\overline{s_j(l') d_S(s_i(l))}}\Big) \\ &= {}^\gamma(\widehat{P}(l, l')). \end{aligned}$$

□

Remark 6.7. Cocycles of the form $u_\gamma(x \otimes s) = v_\gamma \otimes s$ for some $v_\gamma \in \mathrm{Aut}_k(\mathfrak{g})$ are usually called "constant" (because the action of the Galois group is trivial). As we have seen such cocycles allow the key identity

$$\widehat{P}({}^\gamma(l, l')) = {}^\gamma(\widehat{P}(l, l'))$$

for $l, l' \in \mathcal{L}$ to hold. Not all twisted forms of $\mathfrak{g} \otimes_k R$ are given by constant cocycles (the so-called Margaux algebra [4, Example 5.7] is one such example).

Theorem 6.8. *If $\mathcal{L}$ is a multiloop algebra based on $\mathfrak{g}$, then*

$$W(\mathcal{L}) \simeq \Omega^1_{R/k}/d_R(R).$$

Proof. We have already seen that $\Omega^1_{R/k}/d_R(R) \hookrightarrow W(\mathcal{L})$ (see Remark 6.4). On the other hand, all $(l, l') \in \mathcal{L} \times \mathcal{L}$ are Γ-invariant, so we have

$$\widehat{P}(l, l') = \widehat{P}({}^\gamma(l, l')) = {}^\gamma(\widehat{P}(l, l')).$$

So $\widehat{P}(l, l') \in (\Omega^1_{S/k}/d_S(R))^\Gamma \simeq \Omega^1_{R/k}/d_R(R)$ and therefore $W(\mathcal{L}) \hookrightarrow \Omega^1_{R/k}/d_R(R)$. □

Other descriptions of the universal central extension of multiloop algebras can be found in [2] and [19].

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