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Ultimate bound minimisation by state feedback in discrete-time switched linear systems under arbitrary switching

Rahmat Heidari [a,*], Julio H. Braslavsky [b,1], Maria M. Seron [a,2], Hernan Haimovich [c,3]

[a] Priority Research Centre for Complex Dynamic Systems and Control (CDSC), School of Electrical Engineering and Computer Science, The University of Newcastle, Callaghan NSW 2308, Australia

[b] Australian Commonwealth Scientific and Industrial Research Organisation (CSIRO), Energy Flagship, Newcastle, NSW 2300, Australia

[c] CIFASIS-CONICET, Departamento de Control, Facultad de Ciencias Exactas Ingeniería y Agrimensura, Universidad Nacional de Rosario, Argentina

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ABSTRACT

We present a novel state feedback design method for perturbed discrete-time switched linear systems. The method aims at achieving (a) closed-loop stability under arbitrary switching and (b) minimisation of ultimate bounds for specific state components. Objective (a) is achieved by computing state feedback matrices so that the closed-loop subsystem evolution matrices generate a solvable Lie algebra (namely, they are all upper triangular in a common coordinate basis). Previous results derived an iterative algorithm that computes the required feedback matrices, and established conditions under which this procedure is possible. Based on these conditions, objective (b) is achieved by exploiting available degrees of freedom in the iterative algorithm.

1. Introduction

In the last decade there has been increasing research activities in the areas of stability and stabilisability of switched systems; see, for example, [1–3]. An example of switched systems is a system with time-varying dynamics, which switches within a known set of modes, or subsystems, indexed by a switching signal. A problem of interest is that of stability under arbitrary switching, which consists in obtaining conditions that guarantee stability of the switched system for every switching signal. Finding these conditions in general is not a simple task except for special cases, such as when the subsystems are pairwise commutative, symmetric or normal [1, Chapter 2]. A well-known necessary and sufficient condition for exponential stability under arbitrary switching is the existence of a common Lyapunov function (CLF) for all

* Corresponding author. Tel.: +61 49217845.
E-mail addresses: heidari.rahmat@gmail.com (R. Heidari), julio.braslavsky@csiro.au (J.H. Braslavsky), maria.seron@newcastle.edu.au (M.M. Seron), haimo@fceia.unr.edu.ar (H. Haimovich).

1 Tel.: +61 249606071.

2 Tel.: +61 249216810.

3 Tel.: +54 3414808547.

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subsystems [4]. As a CLF might be rather complex and difficult to find, most developed work focuses on the existence of a common quadratic Lyapunov function (CQLF).

While most efforts on stability and stabilisation of switched systems deal with asymptotic stability of the origin (as equilibrium point of the system), it might not be possible to achieve asymptotic stability in some situations, such as when the switched system is subject to non-vanishing disturbances. In these situations, one may seek practical stability of the system, in the sense that the system trajectories are required to ultimately lie inside a bounded region around the origin. The *ultimate bounds* of the states of the system characterise such region, which should be sufficiently small for good asymptotic disturbance attenuation performance.

Estimates of the state ultimate bounds can be obtained through the use of level sets of suitable Lyapunov functions (see for example [5, Section 9.2]). This approach is applicable to a general class of nonlinear systems, but may produce overly conservative bounds in linear systems, since the structure of the system is generally lost in the Lyapunov function [6]. Tighter estimates of ultimate bounds in linear systems can be obtained through a *componentwise* analysis technique proposed in [6,7], which preserves system structure and dispenses with Lyapunov functions.

The problem of reducing the effect of disturbances by feedback has been long studied and a number of robust control methods exist for the minimisation of ultimate bounds in linear systems (e.g., [8–10]), although typically relying on Lyapunov analysis and induced norms (such as l_2 and l_∞). Compared to the componentwise approach proposed in this paper, induced norm minimisation approaches may be conservative in the sense that they might not yield the best results for some specific state components representing a meaningful or a physical quantity. An example projecting this idea is studied in [11]. The minimisation of componentwise ultimate bounds by feedback design has been studied in [12] for linear time-invariant (LTI) systems. The authors in [12] have shown that arbitrarily small ultimate bounds can be guaranteed in continuous-time systems by assigning closed-loop eigenvalues with arbitrarily large negative real part when disturbances are "matched" to the control input (that is, disturbances in the span of the system's control input matrix). For discrete-time systems, however, there is a fundamental limitation in rendering these ultimate bounds arbitrarily small, depending on the way the disturbance affects the state equations. In this regard, the problem of ultimate bound minimisation for discrete-time LTI systems has recently been studied in [13], where conditions were derived so that the ultimate bound on one (or more) state components can be minimised to its least possible value via eigenvalue–eigenvector assignment.

Ultimate boundedness of switched systems subject to uncertainty and disturbances has been the focus of attention recently. Necessary and sufficient conditions were derived in [14] for autonomous switched linear systems to have a finite disturbance attenuation level under arbitrary switching. The authors also provide sufficient conditions under which disturbance attenuation can be attained under a dwell-time switching constraint. Similar results with dwell-time switching constraints have been obtained for switched Euler–Lagrange systems in [15]. In [16], the authors present sufficient conditions on the existence of a CLF for a continuous-time switched linear system subject to parameter uncertainties to achieve uniformly ultimate boundedness under arbitrary switching. The stability and componentwise state ultimate bounds of autonomous switched systems under arbitrary switching have been analysed in [17–19], where an iterative algorithm that derives a CQLF is proposed.

The present paper examines the problem of feedback design for practical stabilisation with ultimate bound minimisation. An iterative algorithm is proposed to obtain the smallest possible bounds for specific state components in discrete-time switched linear systems under arbitrary switching. The methodology extends an algorithm from [20], which iteratively seeks a set of stabilising state feedback gains that render the closed-loop subsystem matrices simultaneously upper-triangular after a change of coordinates common to all subsystems. This closed-loop upper-triangular structure is a desirable property, since then the stable closed-loop subsystem matrices will generate a solvable Lie-algebra, which guarantees the existence of a CQLF [21].

The results in this paper improve on existing results from [20], which address stabilisation of switched linear systems under arbitrary switching. The work in [20] is one of the few available works on feedback control design in the switched system context [22,23]. The present paper deals with discrete-time switched linear systems in the presence of non-vanishing bounded disturbances, in contrast with [20], where no disturbance affects the system. The algorithm from [20] is modified in the current paper by imposing additional structure to the closed-loop subsystems to achieve disturbance attenuation by minimising componentwise state ultimate bounds.

The first contribution of this paper is to derive conditions in terms of eigenstructure of the perturbed switched system in order for the trajectories of one or more components of the state to lie within the smallest possible bound in at most one time step. Next, we extend the results of [20] by exploiting the available degrees of freedom in the iterative triangularisation algorithm by imposing a set of conditions on the common eigenvectors at each iteration of the algorithm. The main contribution of the paper is an eigenstructure assignment procedure embedded in the extended version of the aforementioned algorithm such that the resulting stabilising feedback laws achieve the minimum possible ultimate bound for one or more states of the switched system under arbitrary switching. The results in this paper build upon preliminary work communicated in the conference paper [24].

The layout of the remainder of the paper is as follows. In Section 2, structural conditions on the system matrices for one or more ultimate bound components to be the minimum possible are presented. In Section 3, ultimate bound minimisation is addressed through iterative eigenstructure assignment, which is performed via modifications in the iterative algorithm of [20]. In Section 4, a numerical example shows the effectiveness of the proposed algorithm and finally, Section 5 concludes the paper.

Notation. The index set $\{1, 2, \ldots, N\}$ is denoted by $\underline{N}$. The kernel (null space) of a matrix or linear map $A : \mathcal{X} \to \mathcal{Y}$ is denoted by $\operatorname{Ker} A$ and its image, $\operatorname{Im} A$. For $x \in \mathbb{C}^{n\times m}$, its jth row is denoted by $x_{(j,:)}$, its transpose by x', its conjugate transpose by x^* and its Moore–Penrose generalised inverse by $x^\dagger$. Given a set of indices $\mathcal{J} \subset \{1, \ldots, n\}$, $x_{(\mathcal{J},:)}$ denotes the matrix formed by the rows of x with indices in $\mathcal{J}$ and $x_{(\mathcal{J},k)}$ denotes the column vector constructed by the elements of the kth column of x with indices in $\mathcal{J}$. Also, given another set of indices $\mathcal{Z} \subset \{1, \ldots, n\}$, $x_{(\mathcal{J},\mathcal{Z})}$ is a matrix formed by the elements of x with indices (j, z) such that $j \in \mathcal{J}$ and $z \in \mathcal{Z}$. The cardinality of a set $\mathcal{J}$ is denoted by $\#\mathcal{J}$. If $\mathcal{S}$ is a vector space, then $\mathrm{d}(\mathcal{S})$ denotes the dimension of $\mathcal{S}$. For a column vector $x \in \mathbb{C}^n$, x_j denotes its jth component, and $x_{\mathcal{J}}$ denotes its elements with their indices in $\mathcal{J}$. An eigenvalue $\lambda \in \mathbb{C}$ is *stable* if $|\lambda| < 1$. By extension, a matrix is stable if all its eigenvalues are stable. Two pairs of numbers (a, b) and (c, d) are distinct if $a \neq c$ and $b \neq d$. Absolute values and inequalities are taken componentwise. A nonnegative vector means that all its elements are nonnegative. For vectors $x, \mathbf{x} \in \mathbb{R}^n$, $|x| \leq \mathbf{x}$ means that $|x_i| \leq \mathbf{x}_i$ for $i = 1, \ldots, n$ where $\mathbf{x}_i \geq 0$.

2. Problem statement: tightest ultimate bounds by feedback

The problem of ultimate bound minimisation for discrete-time switched systems is studied in this section. Consider a perturbed discrete-time switched linear system

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u_{\sigma(k)}(k) + H_{\sigma(k)}d(k) \tag{1}$$

where the switching function $\sigma(\cdot)$ takes values in a finite index set $\underline{N}$, $x \in \mathbb{R}^n$, and for all $i \in \underline{N}$, $u_i \in \mathbb{R}^{m_i}$, the matrices $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_i}$ and $H_i \in \mathbb{R}^{n\times z}$ are known and B_i have full column rank. The disturbance variable $d \in \mathbb{R}^z$ is componentwise bounded with a known bound $|d(\cdot)| \leq \mathbf{d}$, where $\mathbf{d} \in \mathbb{R}^z$ is a nonnegative vector.

The state of an asymptotically stable system driven by a bounded non-vanishing disturbance will not converge to the origin in general, but rather to a bounded neighbourhood of the origin. The state ultimate bounds, defined as the set $\{x \in \mathbb{R}^n : x \leq \sup_\sigma \limsup_{k\to\infty} |x(k)|\}$, provide a componentwise characterisation of the smallest neighbourhood of the origin that will ultimately contain the perturbed state trajectories. This paper aims to minimise state ultimate bounds by feedback design.

Before we state the main problem more precisely, we define the sense in which the minimisation problem is studied. For discrete-time switched systems under arbitrary switching, there exists an inherent limitation to reducing the ultimate bound for any state component, which can never be smaller than the effect of the perturbation on that component.

Lemma 1 (*Lowest Ultimate Bound [13]*)**.** *For the switched system (4), the ultimate bound on the state vector's jth component is bounded below as*

$$\sup_{\sigma} \limsup_{k\to\infty} |x_j(k)| \geq \max_{i\in\underline{N}}\Big[\max_{|d|\leq\mathbf{d}} |[H_i]_{(j,:)}d|\Big] \doteq \mathbf{b}_j^{\min}. \tag{2}$$

Proof. From (4), since $x_j(k+1) = [A^{cl}_{\sigma(k)}]_{(j,:)}x(k) + [H_{\sigma(k)}]_{(j,:)}d(k)$, then the ultimate bound on the jth state component can never be smaller than that corresponding to the case when the jth row of A_i^{cl} is zero for every $i \in \underline{N}$. Then, the result follows from direct analysis of (4). ■

Note that $\mathbf{b}_j^{\min}$ in (2) is independent of the closed-loop matrices A_i^{cl}, $i \in \underline{N}$, because it corresponds to the case when the jth row of every $A_i^{cl} = A_i + B_iK_i$ is zero.

Problem statement. Assuming $\sigma(k)$ is known at time k, we are interested in designing a set of state feedback matrices $\{K_i\}_{i\in\underline{N}}$ such that the system (1) with

$$u_{\sigma(k)}(k) = K_{\sigma(k)}x(k) \tag{3}$$

is uniformly exponentially stable for every switching sequence $\{\sigma(k)\}$ (when $d = 0$) and minimises the ultimate bound of state components when $d \neq 0$. That is, the resulting perturbed closed-loop system

$$x(k+1) = A^{cl}_{\sigma(k)}x(k) + H_{\sigma(k)}d(k), \tag{4}$$

with $A_i^{cl} = A_i + B_iK_i$, admits a CLF and achieves the minimum possible ultimate bound $\mathbf{b}_j^{\min}$ as in (2), for one or more state components, $j \in \mathcal{J} \subset \{1, \ldots, n\}$, where $\mathcal{J}$ contains the indices of the desired state ultimate bounds to be minimised.

However, there are cases where the smallest ultimate bound is not attainable. One such case is singled out in the following lemma.

Lemma 2 (*Unfeasibility of Ultimate Bound Minimisation*)**.** *For the switched system (4) with (A_i, B_i) controllable for all $i \in \underline{N}$, suppose that for an arbitrary $j \in \{1, \ldots, n\}$, the jth row of at least one of the B_i matrices is identically zero. Then, the jth ultimate bound of the system cannot be minimised to its lowest value $\mathbf{b}_j^{\min}$ as in (2).*

Proof. To minimise the jth ultimate bound of the switched system (3), we need to achieve

$$[A_i^{cl}]_{(j,:)} = [A_i + B_iK_i]_{(j,:)} = [A_i]_{(j,:)} + [B_i]_{(j,:)}K_i = 0 \tag{5}$$

for all $i \in \underline{N}$. If for the kth subsystem $[B_k]_{(j,:)} = 0$, then for any choice of K_k we have $[A_k^{cl}]_{(j,:)} = [A_k]_{(j,:)} \neq 0$ by controllability and hence, the jth ultimate bound cannot be minimised to $\mathbf{b}_j^{\min}$ as in (2). ■

Lemma 2 establishes that to achieve the minimum ultimate bound for a state in the arbitrarily switched system (1) with controllable subsystems requires that the corresponding row in all the input matrices B_i be not identically zero.

Having the case of Lemma 2 ruled out, to minimise the ultimate bounds of the switched system, we give conditions in Lemma 3 on the structure of a common transformation matrix V and resulting transformed matrices M_i, such that

$$A_i^{cl} \doteq A_i + B_iK_i = VM_iV^{-1} \tag{6}$$

has its jth row equal to zero for all i and, hence, the jth ultimate bound of (4) is minimised.

Lemma 3 (*Closed-Loop Structure with Lowest Ultimate Bound in a Single State*). *Consider the discrete-time switched system (4) under arbitrary switching. For an arbitrary $j \in \{1, \ldots, n\}$, suppose that there exist feedback matrices K_i for all $i \in \underline{N}$ and an invertible transformation V such that $M_i = V^{-1}(A_i + B_iK_i)V$ are stable, upper triangular, and have the form*[4]

$$M_i = \begin{bmatrix} \Delta^{i,1}_{k\times k} & \delta^i_{k\times 1} & \Delta^{i,2}_{k\times(n-k-1)} \\ 0_{1\times k} & 0 & 0_{1\times(n-k-1)} \\ 0_{(n-k-1)\times k} & 0_{(n-k-1)\times 1} & \Delta^{i,3}_{(n-k-1)\times(n-k-1)} \end{bmatrix}, \tag{7}$$

where $\Delta^{i,1}$ and $\Delta^{i,3}$ are upper-triangular matrices, $\Delta^{i,2}$ is an arbitrary matrix, δ^i is an arbitrary vector, and the transformation matrix V is such that its jth row has a nonzero entry at the $(k+1)$th column and is zero everywhere else, that is,

$$V_{(j,:)} = \begin{bmatrix} 0_{1\times k} & V_{j,k+1} & 0_{1\times(n-k-1)} \end{bmatrix}, \quad V_{j,k+1} \neq 0. \tag{8}$$

Then the jth ultimate bound of the switched system is equal to its minimum possible value $\mathbf{b}_j^{\min}$ defined in (2).

Proof. Using (7) and (8), the jth row of the closed-loop matrix of each subsystems is

$$\begin{aligned}[A_i^{cl}]_{(j,:)} &= [A_i + B_iK_i]_{(j,:)} = [VM_iV^{-1}]_{(j,:)} = V_{(j,:)}M_iV^{-1} \\ &= \begin{bmatrix} 0 & V_{j,k+1} & 0 \end{bmatrix} \begin{bmatrix} \Delta^{i,1} & \delta^i & \Delta^{i,2} \\ 0 & 0 & 0 \\ 0 & 0 & \Delta^{i,3} \end{bmatrix} V^{-1} = 0_{1\times n}\end{aligned}$$

and hence, the ultimate bound on the jth state component is minimised to $\mathbf{b}_j^{\min}$ as in (2). ■

In the next lemma, the result in Lemma 3 is extended to the minimisation of more than one ultimate bound.

Lemma 4 (*Closed-Loop Structure with Lowest Ultimate Bounds Multiple States*). *Let $\mathcal{J} \subset \{1, \ldots, n\}$ with cardinality $\#\mathcal{J}$ contain indices of a set of desired states. Suppose that the matrices M_i and V in (6) satisfy*

- *M_i are upper triangular and stable and have $\#\mathcal{J}$ zero rows with indices in an arbitrary set $\mathcal{Z} \subset \{1, \ldots, n\}$ with $\#\mathcal{Z} = \#\mathcal{J}$.*
- *V is invertible and such that $V_{(\mathcal{J},\mathcal{Z})} \in \mathbb{C}^{\#\mathcal{J}\times\#\mathcal{J}}$ is invertible (for example, any permutation of the identity matrix) and $V_{(\mathcal{J},\{1,\ldots,n\}\setminus\mathcal{Z})} = 0$.*

Then the ultimate bounds of the switched system with their indices specified in the set $\mathcal{J}$ can be minimised to their minimum possible values $\mathbf{b}_{\mathcal{J}}^{\min}$ as in (2). ○

In [20] an algorithm that iteratively seeks feedback matrices K_i and the transformation V so that $M_i = V^{-1}(A_i+B_iK_i)V$ are stable and upper triangular was developed. In the next section, we modify this algorithm in order to achieve the additional conditions of Lemmas 3 and 4 and hence yield closed-loop matrices $A_i^{cl} = A_i + B_iK_i$ with the desired zero rows.

3. Stabilisation and ultimate bound minimisation by feedback

In this section, eigenstructure assignment guaranteeing stability and ultimate boundedness of a switched linear system under arbitrary switching is considered. We extend the iterative algorithm proposed in [20], which exploits state-feedback control and a common transformation to obtain subsystems with upper-triangular matrices.

[4] For clarity, the dimensions of some of the matrices are indicated as subscripts.

Algorithm ITBF: Iterative triangularisation and ultimate bound minimisation by feedback

Data: $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_i}$ for $i \in \underline{N}$, and $\mathcal{J}$
Output: $V \in \mathbb{C}^{n\times n}$, $K_i \in \mathbb{R}^{n\times n}$ for $i \in \underline{N}$
Initialisation: $A_i^1 \doteq A_i$, $B_i^1 \doteq B_i$, $K_i^0 \doteq 0$, $U_1 \doteq \mathrm{I}$, $\ell \leftarrow 0$, $\mathcal{J}^1 \doteq \mathcal{J}$, $m_i^1 = m_i$ for $i \in \underline{N}$
repeat

$$\ell \leftarrow \ell + 1, \quad n_\ell \leftarrow n - \ell + 1, \tag{9}$$

if $p_\ell = n_\ell + \sum_{i=1}^N m_i^\ell - Nn_\ell < n_\ell$ **then**

$$[\xi^\ell, \{F_i^\ell\}_{i=1}^N] \leftarrow \mathrm{SCEA}(\{A_i^\ell\}_{i=1}^N, \{B_i^\ell\}_{i=1}^N, \mathcal{J}^\ell), \tag{10}$$

if Procedure SCEA is unsuccessful **then**
The algorithm fails to achieve the minimum ultimate bounds;
else

$$A_i^{\ell,\mathrm{CL}} \doteq A_i^\ell + B_i^\ell F_i^\ell, \tag{11}$$

$$K_i^\ell \leftarrow K_i^{\ell-1} + F_i^\ell\Big(\prod_{r=1}^{\ell} U_r^*\Big), \tag{12}$$

$$V_{(:,\ell)} \leftarrow \Big(\prod_{r=1}^{\ell} U_r\Big)\,\xi^\ell. \tag{13}$$

end if
Construct a unitary matrix (15) with arbitrary set of distinct elements

$$\mathcal{J}^{\ell+1} \subset \{1, 2, \ldots, n_\ell - 1\}, \ \#\mathcal{J}^{\ell+1} = \#\mathcal{J} \tag{14}$$

where for each distinct pair $(j^\ell, j^{\ell+1})$, with $j^\ell \in \mathcal{J}^\ell$, $j^{\ell+1} \in \mathcal{J}^{\ell+1}$

$$[\xi^\ell | v_1^\ell | \cdots | v_{n_{\ell-1}}^\ell] \in \mathbb{C}^{n_\ell \times n_\ell}, \tag{15}$$

$$\begin{cases} v_{j^\ell,i}^\ell = 0 : i = 1, \ldots, n_\ell - 1, i \neq j^{\ell+1} \\ v_{j^\ell,j^{\ell+1}}^\ell \neq 0 \end{cases} \tag{16}$$

Assign (17)–(19):

$$U_{\ell+1} \leftarrow [v_1^\ell | \cdots | v_{n_{\ell-1}}^\ell], \tag{17}$$

$$A_i^{\ell+1} \leftarrow U_{\ell+1}^* A_i^{\ell,\mathrm{CL}} U_{\ell+1}, \tag{18}$$

$$B_i^{\ell+1} \leftarrow U_{\ell+1}^* B_i^\ell. \tag{19}$$

end if
until $p_\ell = n_\ell$;
Let $V^\ell = I_{n_\ell}$;
Select $\{\lambda_i^\ell, \ldots, \lambda_i^n\}$ stable with $\lambda_i^{\ell-1+j} = 0, \forall j \in \mathcal{J}^\ell$;
Construct $\Delta_i \doteq \mathrm{diag}\{\lambda_i^\ell, \ldots, \lambda_i^n\}$;
Compute

$$F_i^n \leftarrow (B_i^\ell)^{-1}(\Delta_i I_{n_\ell} - A_i^\ell); \quad K_i \leftarrow K_i^{\ell-1} + F_i^n\Big(\prod_{r=1}^{\ell} U_r^*\Big); \quad V_{(:,\ell:n)} \leftarrow \Big(\prod_{r=1}^{\ell} U_r\Big)V^\ell.$$

Fig. 1. Algorithm for iterative triangularisation and ultimate bound minimisation by feedback (ITBF).

3.1. The ITBF algorithm

In [20], conditions were given on the number of states n, the number of subsystems N, and the number of inputs of each subsystem m_i, $i \in \underline{N}$, so that the stabilising feedback matrices K_i and the simultaneous triangularisation transformation V will exist for almost every set of system parameters, i.e. for almost all possible entries of the matrices A_i and B_i, for all $i \in \underline{N}$. When these conditions are satisfied, [20] also shows that, in addition, the closed-loop eigenvalues for every subsystem can be arbitrarily selected. In this section, we modify the feedback design algorithm of [20] so that all available degrees of freedom are exploited to achieve minimum ultimate bounds through the selection of some closed-loop eigenvalues and the construction of a unitary matrix with specific properties.

Consider the discrete-time switched linear system (1) with state-feedback law (3), yielding the closed-loop system (4). The proposed modified algorithm is shown below as Algorithm ITBF in Fig. 1. This algorithm seeks feedback matrices K_i so that

1. the closed-loop matrices $A_i^{cl} = A_i + B_i K_i$ are stable and simultaneously triangularisable, and
2. for a selected set of state components with indices in $\mathcal{J} \subset \{1, 2, \ldots, n\}$, whose cardinality will be specified later, their corresponding ultimate bounds are minimised to their smallest values.

The Algorithm ITBF (shown in Fig. 1) is an extension of the algorithm in [20], where the main modifications are: (a) the set of state components to be minimised, namely $\mathcal{J} \subset \{1, \ldots, n\}$, has to be supplied as input data, (b) the common eigenvector assignment (CEA) procedure of [20] is replaced by the *structured* common eigenvector assignment (SCEA) procedure in (10), and (c) the unitary matrix construction (15) has to satisfy the additional constraints (16). As in [20], the proposed algorithm seeks feedback matrices K_i so that the closed-loop matrices A_i^{cl} in (4) are stable and simultaneously triangularisable, but with the additional requirement that the conditions in Lemma 4 are fulfilled.

A brief description of the algorithm is as follows. After initialisation, the algorithm iterates the following steps: common eigenvector computation for the internal subsystems identified by A_i^ℓ, B_i^ℓ (performed by procedure SCEA in (10)), state feedback and transformation update (performed at (12) and (13)), and internal matrices' update for the next iteration (at (15)–(19)). The internal subsystem matrices change dimensions during the execution of the algorithm because one state dimension is eliminated at each iteration. At start, the internal matrices for iteration $\ell = 1$ are set to coincide with the subsystem matrices: $A_i^1 = A_i, B_i^1 = B_i$. Then the SCEA procedure in (10) seeks a unit vector ξ^ℓ having specific structure (which will be explained later), and corresponding (internal) feedback matrices F_i^ℓ, so that ξ^ℓ is a feedback-assignable eigenvector common to all internal subsystems, with corresponding stable eigenvalues. That is, if Procedure SCEA is successful, then ξ^ℓ satisfies $\|\xi^\ell\| = 1$ and $(A_i^\ell + B_i^\ell F_i^\ell)\xi^\ell = \lambda_i^\ell \xi^\ell$ for some scalars λ_i^ℓ satisfying $|\lambda_i^\ell| < 1$ for all $i \in \underline{N}$.

3.2. Structural condition for simultaneous triangularisation by feedback

As noted above, successful satisfaction of the algorithm at each iteration requires the existence of the vector ξ^ℓ. Existence of such ξ^ℓ (without the additional requirement for ultimate bound minimisation) is ensured by the structural condition of [20], as we next explain. Define $m_i^\ell \doteq \operatorname{rank}(B_i^\ell) = \mathrm{d}(\operatorname{Im} B_i^\ell)$, and factor $B_i^\ell = b_i^\ell r_i^\ell$, where $r_i^\ell : \mathbb{R}^{m_i} \to \mathbb{R}^{m_i^\ell}$ has full row rank and $b_i^\ell : \mathbb{R}^{m_i^\ell} \to \mathbb{R}^{n_\ell}$ has full column rank. Note that $\operatorname{Im} B_i^\ell = \operatorname{Im} b_i^\ell$. Let Λ^ℓ be a vector with components $\lambda_i^\ell, i \in \underline{N}$, i.e.

$$\Lambda^\ell \doteq [\lambda_1^\ell\, \lambda_2^\ell\, \dots\, \lambda_N^\ell]', \tag{20}$$

and build the matrix

$$Q_\ell(\Lambda^\ell) \doteq [R_\ell(\Lambda^\ell)\ \ -B_\ell], \quad \text{where} \tag{21}$$

$$R_\ell(\Lambda^\ell) \doteq \begin{bmatrix} \lambda_1^\ell I - A_1^\ell \\ \vdots \\ \lambda_N^\ell I - A_N^\ell \end{bmatrix}, \qquad B_\ell \doteq \operatorname{blkdiag}\big[b_1^\ell, \dots, b_N^\ell\big],$$

and where blkdiag denotes block diagonal concatenation. The following result from [20] presents conditions for the existence of a common feedback assignable eigenvector based on properties of the matrix (21), and provides a mechanism to obtain the common eigenvector and the associated feedback matrices when the problem is solvable.

Lemma 5 (*Structural Condition [20]*). *Let*

$$p_\ell \doteq n_\ell + \sum_{i=1}^{N} m_i^\ell - N n_\ell. \tag{22}$$

Then,

(a) $\mathrm{d}(\operatorname{Ker} Q_\ell(\Lambda^\ell)) \geq p_\ell$ *for every choice of* Λ^ℓ *as in (20).*

(b) *A vector that can be assigned by feedback as a common eigenvector with corresponding eigenvalues* λ_i^ℓ *for* $i \in \underline{N}$ *exists if and only if* $\mathrm{d}(\operatorname{Ker} Q_\ell(\Lambda^\ell)) > 0$. *Consequently, if*

$$p_\ell > 0, \tag{23}$$

then a feedback-assignable common eigenvector exists for every choice of corresponding eigenvalues.

(c) *If* $Q_\ell(\Lambda^\ell) w^\ell = 0$ *with* $w^\ell \neq 0$ *partitioned as*

$$w^\ell \doteq [v'\ u_1'\ \cdots\ u_N']', \quad \text{then } v \neq 0, \text{ and} \tag{24}$$

$$(A_i^\ell + B_i^\ell F_i^\ell) v = \lambda_i^\ell v, \quad \text{for } i \in \underline{N}, \tag{25}$$

for every F_i^ℓ *satisfying* $r_i^\ell F_i^\ell v = u_i$. *For each* $i \in \underline{N}$ *one such* F_i^ℓ *is*

$$F_i^\ell = (r_i^\ell)^\dagger u_i v^\dagger, \tag{26}$$

where † *denotes the Moore–Penrose generalised inverse.* ○

If the structural condition (23) holds, the nullspace of $Q_\ell(\Lambda^\ell)$ is not empty and, thus, we can find $w^\ell \in \operatorname{Ker} Q_\ell(\Lambda^\ell)$. Suppose $\mathrm{d}(\operatorname{Ker} Q_\ell(\Lambda^\ell)) = \psi_\ell \geq p_\ell$, define $d_\ell \doteq n_\ell + \sum_{i=1}^N m_i^\ell$ and let $W^\ell \in \mathbb{C}^{d_\ell \times \psi_\ell}$ be a basis for $\operatorname{Ker} Q_\ell(\Lambda^\ell)$. Then, from Lemma 5(c), the vector $w^\ell \neq 0$ has the form

$$w^\ell = W^\ell \alpha^\ell \tag{27}$$

Procedure SCEA (Structured Common Eigenvector Assignment)

Input: $A_i^\ell \in \mathbb{R}^{n_\ell \times n_\ell}$, $B_i^\ell \in \mathbb{R}^{n_\ell \times m_i}$, for $i \in \underline{N}$ and $\mathcal{J}^\ell$
Output: ξ^ℓ with $\xi^\ell_{\mathcal{J}^\ell} = 0$, F_i^ℓ for $i \in \underline{N}$
Factor $B_i^\ell = b_i^\ell r_i^\ell$ with $b_i^\ell \in \mathbb{R}^{n_\ell \times m_i^\ell}$ and $m_i^\ell = \operatorname{rank}(B_i^\ell)$;
Select $\lambda_i^\ell \in \mathbb{R}$ stable and construct Λ^ℓ as in (20);
Construct $Q_\ell(\Lambda^\ell)$ as in (21)
Compute W^ℓ as in (27);
Select α^ℓ as in Lemma 9 satisfying Theorem 7;
if α^ℓ satisfying both Lemma 9 and Theorem 7 does not exist **then**
  Procedure SCEA is unsuccessful;
else
  Compute w^ℓ as in (27) and partition it as in (24);
  Compute F_i^ℓ as in (26)
  Determine $\xi^\ell = v/\|v\|$;
end if

Fig. 2. Procedure SCEA when the structural condition is satisfied.

where $\alpha^\ell \in \mathbb{C}^{\psi_\ell}$ is an arbitrary vector. Once (27) is obtained, a feedback-assignable common eigenvector ξ^ℓ provided by procedure SCEA (see Fig. 2) at iteration ℓ of algorithm ITBF can be computed by selecting the first n_ℓ components of w^ℓ to construct v (cf. (24)) and then letting

$$\xi^\ell = v/\|v\|. \tag{28}$$

The quantity p_1 given by (22) for the first iteration $\ell = 1$ is central to the solvability of the proposed algorithm. It also determines an upper bound on the number of ultimate bound components that can be minimised by appropriate use of the available degrees of freedom. These facts will be discussed in the following sections.

In the next subsection we provide a technical result that will be required to analyse the solvability of the ITBF algorithm at all iterations. Then, in Section 3.4 we analyse in detail the mechanism for ultimate bound minimisation embedded in the algorithm.

Remark 1. Note that the iterations in Algorithm ITBF (see Fig. 1) are divided into two categories. The first category iteratively runs with the structural condition $p_\ell < n_\ell$ while the second one deals with the case when $p_\ell = n_\ell$ for which the algorithm terminates in just one step. ○

3.3. Solvability of Algorithm ITBF

Solvability of the Algorithm ITBF builds on the solvability of the iterative algorithm proposed in [20]. In this section we first revisit some results from [20] that analyse the structural condition (23). Then we present a new result on how to exploit the available degrees of freedom in the iterative algorithm proposed in [20] to guarantee satisfaction of (23) at each iteration.

The structural condition (23) depends on m_i^ℓ, the rank of B_i^ℓ. At the first iteration $\ell = 1$, $m_i^\ell = m_i$ and thus, $p_1 = (1-N)n + \sum_{i=1}^N m_i$. At subsequent iterations, $m_i^{\ell+1}$ depends on the vector ξ^ℓ returned by Procedure SCEA as follows:

$$m_i^{\ell+1} = \begin{cases} m_i^\ell & \text{if } \xi^\ell \notin \operatorname{Im} B_i^\ell, \\ m_i^\ell - 1 & \text{if } \xi^\ell \in \operatorname{Im} B_i^\ell. \end{cases} \tag{29}$$

From (29), then $m_i^{\ell+1} = m_i^\ell - 1$ when $m_i^\ell = n_\ell$, because $\xi^\ell \in \mathbb{R}^{n_\ell} = \operatorname{Im} B_i^\ell$. The next key result from [20] follows from (22) and (29).

Lemma 6 (*[20]*). *Consider Algorithm ITBF at iteration ℓ and p_ℓ as in (22), with $m_i^\ell = \operatorname{rank}(B_i^\ell)$. Then, $p_{\ell+1} \geq p_\ell - 1$, with equality if and only if*

$$\xi^\ell \in \mathcal{B}^\ell, \quad \text{with } \mathcal{B}^\ell \doteq \bigcap_{i \in \underline{N}} \mathcal{B}_i^\ell \quad \text{and} \quad \mathcal{B}_i^\ell \doteq \operatorname{Im} B_i^\ell. \circ \tag{30}$$

According to Lemma 6, if at iteration ℓ, $p_\ell = 1$ and $\xi^\ell \in \bigcap_{i \in \underline{N}} \mathcal{B}_i^\ell$, then $p_{\ell+1} = p_\ell - 1 = 0$ and it is possible that no stable common eigenvector can be found; thus, the ITBF algorithm terminates unsuccessfully. We are thus interested in finding conditions to ensure that $p_\ell > 0$ will hold at every iteration. Following [20], we derive these conditions by considering certain subspaces associated with the internal subsystems (A_i^ℓ, B_i^ℓ).

Let $\mathcal{S}_i^\ell$ denote the set of vectors $v \in \mathcal{B}_i^\ell = \operatorname{Im} B_i^\ell$ for which there exist a matrix F_i^ℓ and a stable scalar λ so that

$$(A_i^\ell + B_i^\ell F_i^\ell)v = \lambda v. \tag{31}$$

By definition, $\mathcal{S}_i^\ell$ is the set of feedback-assignable eigenvectors for the internal subsystem (A_i^ℓ, B_i^ℓ), with associated stable eigenvalues, which are contained in $\mathcal{B}_i^\ell$.

For convenience, we reproduce the concept of transversality for a set of subspaces which is useful for our analysis (see [20] for further details).

Definition 1 (*[20]*)**.** Two subspaces $\mathcal{S}, \mathcal{T}$ of an ambient space $\mathcal{X}$ are *transverse* if the dimension of their intersection is minimal, given the dimensions of $\mathcal{S}$ and $\mathcal{T}$, i.e. when $\mathrm{d}(\mathcal{S} \bigcap \mathcal{T}) = \max\{0, \mathrm{d}(\mathcal{S}) + \mathrm{d}(\mathcal{T}) - \mathrm{d}(\mathcal{X})\}$. Equivalently, $\mathcal{S}$ and $\mathcal{T}$ are transverse when the dimension of their sum is maximal. To extend this definition to sets of subspaces let $\mathcal{S} = \{\mathcal{S}_1, \dots, \mathcal{S}_n\}$ be a set of subspaces of an ambient space $\mathcal{X}$. Then, we say that $\mathcal{S}$ is transverse when both the intersection of the subspaces in every subset of $\mathcal{S}$ has minimal dimension and the sum of the subspaces in every subset of $\mathcal{S}$ has maximal dimension.

Suppose $\{\mathcal{S}_i^\ell : i \in \underline{N}\}$ are transverse subspaces. Define the following quantities:

$$\rho_i^\ell \doteq \mathrm{d}(\mathcal{S}_i^\ell), \qquad q_\ell \doteq n_\ell + \sum_{i \in \underline{N}} \rho_i^\ell - N n_\ell, \tag{32}$$

$$\mathcal{S}^\ell \doteq \bigcap_{i \in \underline{N}} \mathcal{S}_i^\ell, \qquad \rho^\ell \doteq \mathrm{d}(\mathcal{S}^\ell). \tag{33}$$

From (33) and Lemma 6, if the common eigenvector ξ^ℓ lies within $\mathcal{S}^\ell$, then $\xi^\ell \in \mathcal{B}^\ell$ as in (30), and the structural condition $p_{\ell+1} = p_\ell - 1 > 0$ might not be satisfied. To avoid this situation and guarantee that the structural condition (23) continues to be satisfied at every iteration of the algorithm, we exploit the available degrees of freedom to choose the common eigenvector such that $\xi^\ell \notin \mathcal{S}^\ell$ and hence, $p_{\ell+1} \geq p_\ell$. This result, presented in Theorem 7, will come into use in the next section where we lay out the procedure for ultimate bound minimisation.

Theorem 7 (*Conditions for Solvability of Algorithm ITBF*)**.** *Let $\{\mathcal{S}_i^1 : i \in \underline{N}\}$ be transverse, $q_1 \geq 0$ and (A_i, B_i) be controllable for all $i \in \underline{N}$. Then,it is always possible to select α^ℓ in (27) such that $p_{\ell+1} \geq p_\ell - 1$, with equality if and only if $p_\ell = n_\ell$.*$\circ$

Before we proceed to the proof of the theorem, note that under the assumptions of Theorem 7, it was shown in [20, Theorem 2] that

- $p_\ell > 0$ for $\ell = 1, \dots, n$.
- There exist feedback gains K_i such that the set $\mathcal{Z} = \{(A_i + B_i K_i, B_i) : i \in \underline{N}\}$ consists of stable matrices and generates a solvable Lie-algebra. Hence, the closed-loop system admits a CQLF.

Namely, these are sufficient conditions to guarantee stabilisability of the switched system. Theorem 7 shows that without any further restrictions, it is possible to select feedback gains that, in addition to providing closed-loop stability, ensure that p_ℓ is non-decreasing for all iterations (until $p_\ell = n_\ell$ and remains equal to n_ℓ afterwards). This means that starting with 'enough freedom' (i.e., p_1 'large enough') to select the common eigenvector, this freedom is maintained for all iterations and hence lays the groundwork to tackle the ultimate bound minimisation problem, as we will show in Section 3.4.

As a preliminary result required to prove Theorem 7, the next lemma relates the quantities p_ℓ defined in (22) and ρ^ℓ defined in (33), and is central to the solvability of the proposed algorithm (SCEA Procedure) at all iterations.

Lemma 8. *Consider p_ℓ defined in (22), q_ℓ defined in (32), let $p_\ell > 0$, $\{\mathcal{S}_i^\ell : i \in \underline{N}\}$ be transverse and (A_i^ℓ, B_i^ℓ) be controllable. Then $p_\ell \geq \rho^\ell = \max\{0, q_\ell\}$, with $p_\ell = \rho^\ell$ if and only if $m_i^\ell = n_\ell$ for $i \in \underline{N}$.*

Proof. Recalling properties of transverse subspaces, the dimension of the intersection of transverse subspaces $\mathcal{S}^\ell$ satisfies

$$\rho^\ell = \max\{0, q_\ell\}. \tag{34}$$

Since $\mathcal{S}_i^\ell \subset \mathcal{B}_i^\ell$, then $\rho_i^\ell \leq m_i^\ell$ and $q_\ell \leq p_\ell$ which together with (34) and $p_\ell > 0$ yields $\rho^\ell \leq p_\ell$. Next, we prove that $\rho^\ell = p_\ell$ if and only if $m_i^\ell = n_\ell$ for $i \in \underline{N}$.

($\rightarrow$) If $\rho^\ell = p_\ell > 0$, then from (34) we have $\rho^\ell = q_\ell = p_\ell > 0$ which from (32) means $\rho_i^\ell = m_i^\ell$ for $i \in \underline{N}$. Controllability of (A_i^ℓ, B_i^ℓ) and the fact that ρ_i^ℓ is the number of controllability indices of (A_i, B_i) equal to 1 (see [20, Lemma 5]) then yield $m_i^\ell = n_\ell$, for $i \in \underline{N}$.

($\leftarrow$) If $m_i^\ell = n_\ell$ for all $i \in \underline{N}$, then $\rho_i^\ell = m_i^\ell$ for $i \in \underline{N}$ and thus, from (32) $p_\ell = q_\ell$. From (34) and the assumption $p_\ell > 0$, we have $\rho^\ell = q_\ell = p_\ell > 0$. ■

Now we are ready to proceed to the proof of Theorem 7.

Proof of Theorem 7. Here we show that by proper selection of α^ℓ in (27), p_ℓ is non-decreasing for all iterations until $p_\ell = n_\ell$ and remains equal to n_ℓ afterwards.

From Lemma 5(a), we have $\mathrm{d}(\mathrm{Ker}\, Q_\ell(\Lambda^\ell)) = \psi_\ell \geq p_\ell$. Thus, if the latter holds with equality, a basis for the nullspace of $Q_\ell(\Lambda^\ell)$ has the form (see (27))

$$W^\ell = \left[w_1^\ell \cdots w_{\psi_\ell}^\ell\right] = \begin{bmatrix} v_1 & \cdots & v_{\psi_\ell} \\ u_{11} & \cdots & u_{1\psi_\ell} \\ \vdots & & \vdots \\ u_{N1} & \cdots & u_{N\psi_\ell} \end{bmatrix}, \qquad \mathrm{rank}(W^\ell) = \psi_\ell \geq p_\ell, \tag{35}$$

where the partition of each vector follows from (24). From (24) and (27), the common eigenvector is determined as

$$v = \left[v_1 \ \cdots \ v_{\psi_\ell}\right]\alpha^\ell. \tag{36}$$

First, we show that the subspace generated by the v_r vectors is also of rank ψ_ℓ, i.e.

$$\mathrm{rank}\left(\left[v_1 \ \cdots \ v_k \ \cdots \ v_{\psi_\ell}\right]\right) = \psi_\ell. \tag{37}$$

This is shown by contradiction. Suppose that v_k, for some $k \in \{1, \ldots, \psi_\ell\}$, is a linear combination of other columns, that is

$$v_k = \sum_{r=1, r\neq k}^{\psi_\ell} v_r \gamma_r \tag{38}$$

where at least one coefficient γ_r is nonzero. Then, since W^ℓ in (35) is in the nullspace of $Q_\ell(\Lambda^\ell)$ we have

$$Q_\ell(\Lambda^\ell) w_r^\ell = 0, \quad r = 1, \ldots, \psi_\ell \tag{39}$$

and by replacing (21) and (35) in (39), for $i \in \underline{N}$ we obtain

$$(\lambda_i^\ell I - A_i^\ell) v_r - b_i^\ell u_{ir} = 0, \quad r = 1, \ldots, \psi_\ell. \tag{40}$$

For $r = k$, using (38) in (40) we obtain

$$\begin{aligned} (\lambda_i^\ell I - A_i^\ell) v_k - b_i^\ell u_{ik} &= \sum_{r=1, r\neq k}^{\psi_\ell} (\lambda_i^\ell I - A_i^\ell) v_r \gamma_r - b_i^\ell u_{ik} \\ &= b_i^\ell \sum_{r=1, r\neq k}^{\psi_\ell} u_{ir}\gamma_r - b_i^\ell u_{ik} = b_i^\ell \left(\sum_{r=1, r\neq k}^{\psi_\ell} u_{ir}\gamma_r - u_{ik} \right) = 0. \end{aligned} \tag{41}$$

Since the b_i^ℓ matrices have full column rank, (41) implies

$$\sum_{r=1, r\neq k}^{\psi_\ell} u_{ir}\gamma_r - u_{ik} = 0. \tag{42}$$

This means that $u_{ik} = \sum_{r=1, r\neq k}^{\psi_\ell} u_{ir}\gamma_r$, for $i \in \underline{N}$, which together with (38) yields

$$w_k^\ell = \sum_{r=1, r\neq k}^{\psi_\ell} w_r^\ell \gamma_r, \tag{43}$$

i.e. $\mathrm{rank}(W^\ell) < \psi_\ell$ which contradicts our assumption in (35). Hence, (37) holds. Therefore, the common eigenvector v in (24) can be chosen in a space of rank ψ_ℓ.

In [20, Theorem 2] it is proved that when $\{\mathcal{S}_i^1 \ : \ i \in \underline{N}\}$ is transverse, $q_1 \geq 0$ and (A_i, B_i) is controllable, then for $\ell = 1, \ldots, n$, $\{\mathcal{S}_i^\ell \ : \ i \in \underline{N}\}$ is transverse and (A_i^ℓ, B_i^ℓ) is controllable by induction. Hence, by Lemma 8, we know that $p_\ell \geq \rho^\ell$ with equality if and only if $m_i^\ell = n_\ell$ for $i \in \underline{N}$. We consider two cases, $p_\ell = \rho^\ell$ and $p_\ell > \rho^\ell$, separately as follows.

When $p_\ell = \rho^\ell$, then $m_i^\ell = n_\ell$ for $i \in \underline{N}$ and all input matrices are invertible. We then have $p_\ell = \rho^\ell = n_\ell$ and hence, the dimension of $\mathcal{S}^\ell$ is n_ℓ (see (33)) and any common eigenvector $\xi^\ell \in \mathbb{C}^{n_\ell}$ is also in $\mathcal{S}^\ell$. Then, from Lemma 6, $p_{\ell+1} = p_\ell - 1 = n_\ell - 1$. On the other hand, the reduction of subsystems dimension in the next step (see (9)) yields $n_{\ell+1} = n_\ell - 1$ which results in $p_{\ell+1} = n_{\ell+1}$.

When $p_\ell > \rho^\ell$, from (37), it is always possible to select α^ℓ in (36) such that the resulting vector v is not in $\mathcal{S}^\ell$. From (28) we then have $\xi^\ell \notin \mathcal{S}^\ell$, hence $\xi^\ell \notin \mathcal{B}^\ell$ and (30) does not hold. Thus, from Lemma 6 we have $p_{\ell+1} > p_\ell - 1$. ■

3.4. Iterative ultimate bound minimisation

In [13], the problem of ultimate bound minimisation for non-switched discrete-time systems required a separate analysis depending on the number of control inputs. For single-input systems, eigenvalue assignment based on the roots of certain polynomials associated with the system matrices characterises the possibility of having the minimum ultimate bound. For multiple-input systems, on the other hand, the smallest ultimate bound on a state is achievable by eigenstructure assignment under certain conditions. The above analysis can be deployed in switched discrete-time systems as explained below.

For switched systems of the form (1), if the structural condition (23) holds at each iteration of Algorithm ITBF, the nullspace of the matrix $Q_\ell(\Lambda^\ell)$ defined in (21) is non-empty and thus, there exists a common eigenvector for all subsystems. At each iteration, $\psi_\ell(\geq p_\ell)$ represents the degrees of freedom to choose the common eigenvector such that a desirable property is satisfied. Indeed, as seen from (27), the vector α^ℓ needs to be selected to shape the common eigenvector ξ^ℓ given in (28) in a specific way. This is shown in the following result.

Lemma 9. *At iteration ℓ of Algorithm ITBF, consider $w^\ell = W^\ell\alpha^\ell$, $w^\ell \in \mathbb{C}^{d_\ell}$, where $W^\ell \in \mathbb{C}^{d_\ell\times\psi_\ell}$ is a basis for $\operatorname{Ker} Q_\ell(\Lambda^\ell)$, with the matrix $Q_\ell(\Lambda^\ell)$ defined in (21) and $\alpha^\ell \in \mathbb{C}^{\psi_\ell}$ an arbitrary vector. Let $\mathcal{J}^\ell \subset \{1, 2, \ldots, n_\ell\}$ be a subset of desired indices with cardinality $\#\mathcal{J}^\ell$. If $p_\ell > \#\mathcal{J}^\ell$, then the elements of the common eigenvector ξ^ℓ given in (28), with their indices specified in $\mathcal{J}^\ell$, can be made zero, that is $\xi^\ell_{\mathcal{J}^\ell} = W^\ell_{(\mathcal{J}^\ell,:)}\alpha^\ell/\|v\| = 0$.*

Proof. Since $\mathrm{d}(\operatorname{Ker} W^\ell_{(\mathcal{J}^\ell,:)}) \geq \psi_\ell - \#\mathcal{J}^\ell \geq p_\ell - \#\mathcal{J}^\ell \geq 1$, then there exists a nonzero nullspace in which α^ℓ can be determined. ■

The above lemma states that if $p_\ell > \#\mathcal{J}^\ell$ at each iteration of Algorithm ITBF, then by proper selection of $\alpha^\ell \neq 0$ such that

$$\alpha^\ell \in \operatorname{Ker} W^\ell_{(\mathcal{J}^\ell,:)}, \tag{44}$$

it is possible to assign common eigenvector so that the desired elements of the matrix V in (6) are zero. Otherwise, if $p_\ell \leq \psi_\ell \leq \#\mathcal{J}^\ell$, then, noting that $W^\ell \triangleq W^\ell(\Lambda^\ell)$ depends on the eigenvalue vector Λ^ℓ of the form (20), the desired zero elements could still be achieved by eigenvalue assignment provided the solution of the equation $W^\ell_{(\mathcal{J}^\ell,:)}(\Lambda^\ell) = 0$ has elements with magnitude smaller than one. This will generically not hold so we concentrate on the case $p_\ell > \#\mathcal{J}^\ell$. Since we consider the cases where p_ℓ is non-decreasing (see Theorem 7), hence, we make the following assumption.

Assumption 10. For the system (4), (6), the set $\mathcal{J} \subset \{1, \ldots, n\}$ that contains the indices of the desired state components for ultimate bound minimisation has cardinality $\#\mathcal{J} \leq p_1 - 1$, with p_1 defined in (22) for the first iteration.

Theorem 11 (*Minimisation of Ultimate Bounds by Feedback*)**.** *Consider the perturbed switched discrete-time system (1), and let $\{\mathcal{S}^1_i : i \in \underline{N}\}$ be transverse, $q_1 \geq 0$, $p_1 > 1$ and (A_i, B_i) be controllable for all $i \in \underline{N}$. Let $\mathcal{J} \subset \{1, 2, \ldots, n\}$, with cardinality $\#\mathcal{J}$ satisfying Assumption 10, contain the indices of the desired state components for ultimate bound minimisation. Suppose that at every iteration of Algorithm ITBF, there exists a vector $\alpha^\ell \in \mathbb{C}^{n_\ell}$ satisfying (44) which yields a $p_{\ell+1} \geq p_\ell - 1$ with equality if and only if $p_\ell = n_\ell$. Then, the ultimate bounds of system (1) with indices in $\mathcal{J}$ can be minimised to their minimum possible values $\mathbf{b}^{\min}_{\mathcal{J}}$ defined in (2) by executing Algorithm ITBF in Fig. 1.*

Proof. The assumption $q_1 \geq 0$ together with $\{\mathcal{S}^1_i : i \in \underline{N}\}$ being transverse and (A_i, B_i) being controllable for all $i \in \underline{N}$, satisfies the conditions in Theorem 7. From the assumption $p_1 > 1$ and Theorem 7, with the appropriate selection of α^ℓ, p_ℓ can be made non-decreasing at all iterations while $p_\ell < n_\ell$, thereby $p_\ell > 1$ for all those iterations. (When $p_\ell = n_\ell$, whether $n_\ell = 1$ or $n_\ell > 1$, no more iterations are necessary since the algorithm can be terminated in one step as explained later in the proof.) If the selection of α^ℓ as described in Theorem 7 is also compatible with condition (44), then ultimate bound minimisation can be achieved. In the remainder of the proof, the iterative ultimate bound minimisation is explained.

Let $\mathcal{J}^1 = \mathcal{J}$ with cardinality $\#\mathcal{J}$ contain the indices of the desired state components for ultimate bound minimisation. The aim is to iteratively apply Lemma 3 by constructing the blocks of the M_i matrices for $i \in \underline{N}$ through eigenvalue assignment and the columns of the matrix V through (13)–(17) to achieve the final matrices M_i and V, where M_i has the form (7) and the matrix V has rows of the form (8). With regard to the matrix M_i, the algorithm accomplishes the required upper triangular structure through (10)–(12), provided its diagonal entries are set as the desired eigenvalues. However, for the desired matrix V, since its columns are the result of a product of matrices (cf. (13)), the idea is to propagate the location of zero and nonzero elements in relevant rows of these matrices so that the end result is the jth row of V, $j \in \mathcal{J}^1$, having all zero elements except for one nonzero element different for each $j \in \mathcal{J}^1$.

The proof is divided into two parts. First, we analyse the ultimate bound minimisation problem at iterations with $p_\ell < n_\ell$. At these iterations, for arbitrary eigenvalues and by common eigenvector assignment, we determine successive columns of the common triangularising transformation matrix V with zero elements at places specified in $\mathcal{J}^1$. Next, we show that if at some iteration κ we reach $p_\kappa = n_\kappa$, then Algorithm ITBF can be terminated in just one more step.

At the first iteration, from (22) and considering that the B_i matrices are of full column rank ($m_i \leq n$), we have $p_1 \leq n$ with equality if and only if $m_i = n$ for all $i \in \underline{N}$. If $p_1 = n$, the algorithm can be terminated in one step as explained below. If $p_1 < n$, then, $p_1 > p_1 - 1 \geq \#\mathcal{J}$ admits the existence of enough degrees of freedom to execute the common eigenvector assignment as in (44). Since $U_1 = I_n$, to have $V_{(\mathcal{J}^1,1)} = 0$, Procedure SCEA needs to select the common eigenvector such that $\xi^1_{\mathcal{J}^1} = 0$ (see (13)). Then, we construct a unitary matrix as in (15) with ξ^1 as its first column, such that for all $j^1 \in \mathcal{J}^1$, the j^1th row of the unitary matrix will have $n - 1$ zero entries and one nonzero element. To this end, choose an arbitrary set of distinct elements $\mathcal{J}^2 \subset \{1, 2, \ldots, n-1\}$, $\#\mathcal{J}^2 = \#\mathcal{J}$. Pair each index $j^1 \in \mathcal{J}^1$ with an index $j^2 \in \mathcal{J}^2$ (where no two pairs have the same first element or the same second element), and let the j^1th row of the unitary matrix be zero except for a nonzero entry at the (j^2+1)th place. Thus, the matrix U_2 as in (17) will be such that for all $j^1 \in \mathcal{J}^1$, its j^1th row has $n_2-1=(n-1)-1$ zeros and a nonzero entry at component j^2. Then, if (j^1, j^2), (i^1, i^2) are two of those pairs, with $i^1, j^1 \in \mathcal{J}^1$ and $i^2, j^2 \in \mathcal{J}^2$ we have

$$U_1U_2 = I_n \begin{bmatrix} * & \cdots & * & \cdots & * & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & v^1_{j^1,j^2} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & v^1_{i^1,i^2} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \cdots & * & \cdots & * & \cdots & * \end{bmatrix}_{n\times n_2}$$

where $v^1_{j^1,j^2}$ and $v^1_{i^1,i^2}$ are nonzero and $*$ is a non-specified entry. Thus, the matrix U_1U_2 has $\#\mathcal{J}$ rows, with their indices in $\mathcal{J}^1$ (only two rows, with indices j^1 and i^1, are illustrated above), with just one nonzero entry at the place indexed by the second element of the pair, and otherwise zero.

For the second iteration of Algorithm ITBF (assuming that $p_2 < n_2$), since by assumption α^ℓ can be chosen to satisfy (44) and the condition in Theorem 7, we obtain $p_2 \geq p_1 > \#\mathcal{J}$ and hence, the common eigenvector assignment can be performed. For the second column of the matrix V to have zero elements at places specified in $\mathcal{J}^1$, i.e. $V_{(\mathcal{J}^1,2)} = 0$, from (13) and considering the nonzero entries of U_1U_2 in the intersections of rows with indices in $\mathcal{J}^1$ and columns with indices in $\mathcal{J}^2$, we need to have $\xi^2_{\mathcal{J}^2} = 0$ for the common eigenvector ξ^2. Accordingly, the unitary matrix U_3 as in (17) constructed using this ξ^2, will have $\#\mathcal{J}$ rows with indices $j^2 \in \mathcal{J}^2$ having $n_3 - 1$ zeros and one nonzero entry at their j^3th places, for $j^3 \in \mathcal{J}^3$, for an arbitrary set of distinct elements $\mathcal{J}^3 \subset \{1, 2, \dots, n_2 - 1\}$, $\#\mathcal{J}^3 = \#\mathcal{J}$ (that is, each index $j^2 \in \mathcal{J}^2$ is paired with an index $j^3 \in \mathcal{J}^3$, as explained above for the first iteration). For $i^1, j^1 \in \mathcal{J}^1$, $i^2, j^2 \in \mathcal{J}^2$ and $i^3, j^3 \in \mathcal{J}^3$ (correspondingly paired) we have

$$\prod_{r=1}^{3} U_r = (U_1)_{n\times n}(U_2)_{n\times n_2}(U_3)_{n_2\times n_3}$$

$$= I_n \begin{bmatrix} * & \cdots & * & \cdots & * & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & v^1_{j^1,j^2} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & v^1_{i^1,i^2} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \cdots & * & \cdots & * & \cdots & * \end{bmatrix} \begin{bmatrix} * & \cdots & * & \cdots & * & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & v^2_{j^2,j^3} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & v^2_{i^2,i^3} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \cdots & * & \cdots & * & \cdots & * \end{bmatrix} = \begin{bmatrix} * & \cdots & * & \cdots & * & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & \pi^2_{j^1,j^3} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & \pi^2_{i^1,i^3} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \cdots & * & \cdots & * & \cdots & * \end{bmatrix}_{n\times n_3},$$

where $\pi^2_{j^1,j^3} \doteq v^1_{j^1,j^2}\, v^2_{j^2,j^3}$.

Following the same procedure for each subsequent iteration ℓ with $p_\ell < n_\ell$ and satisfying the conditions on α^ℓ, we have $p_\ell > \#\mathcal{J}$. At this iteration, the matrix $\prod_{r=1}^{\ell} U_r$ is of size $n \times n_\ell$ and has its rows with indices in $\mathcal{J}^1$ equal to zero except for their j^ℓth entries, $j^\ell \in \mathcal{J}^\ell$, $\mathcal{J}^\ell \subset \{1, 2, \dots, n_\ell\}$, $\#\mathcal{J}^\ell = \#\mathcal{J}$, that is, for $i^1, j^1 \in \mathcal{J}^1$ and $i^\ell, j^\ell \in \mathcal{J}^\ell$

$$\prod_{r=1}^{\ell} U_r = \begin{bmatrix} * & \cdots & * & \cdots & * & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & \pi^{\ell-1}_{j^1,j^\ell} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & \pi^{\ell-1}_{i^1,i^\ell} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \cdots & * & \cdots & * & \cdots & * \end{bmatrix}_{n\times n_\ell}, \qquad \pi^{\ell-1}_{j^1,j^\ell} = \prod_{i=1}^{\ell-1} v^i_{j^i,j^{i+1}}. \tag{45}$$

Thus, to have $V_{(\mathcal{J}^1,\ell)} = 0$, the common eigenvector assignment should satisfy $\xi^\ell_{\mathcal{J}^\ell} = 0$.

At iteration κ, with $p_\kappa = n_\kappa$, the same procedure yields a matrix $\prod_{r=1}^{\kappa} U_r$ of the form (45) with dimension $n \times n_\kappa$. Then since $p_\kappa = n_\kappa$ and from Lemma 8, all the B_i^κ matrices for $i \in \underline{N}$ have rank n_κ. That is, the control input matrices are invertible, and we can assign an arbitrary eigenvector matrix and select a diagonal matrix $\Delta_i^3 = \text{diag}\{\lambda_i^\kappa, \ldots, \lambda_i^n\}$ to complete the M_i matrices construction as follows

$$M_i = \begin{bmatrix} \Delta_i^1 & \Delta_i^2 \\ 0 & \Delta_i^3 \end{bmatrix}$$

where $\Delta_i^1 \in \mathbb{C}^{(\kappa-1)\times(\kappa-1)}$ is an upper triangular matrix with diagonal elements $\{\lambda_i^1, \ldots, \lambda_i^{\kappa-1}\}$ computed in the previous iterations and $\Delta_i^2 \in \mathbb{C}^{(\kappa-1)\times(n-\kappa+1)}$ is an arbitrary matrix.

We assign a common eigenvector matrix I_{n_κ}, that is, all remaining iterations from κ to n of Algorithm ITBF can be subsumed in one step by taking

$$V^\ell = I_{n_\kappa}.$$

Since the matrix $(\prod_{r=1}^{\kappa} U_r) \in \mathbb{C}^{n\times n_\kappa}$, as mentioned before, has $\#\mathcal{J}$ rows (with indices in $\mathcal{J}^1$) with one nonzero entry and otherwise zero, by multiplying this matrix with the common eigenvector matrix I_{n_κ}, the last $n-\kappa+1$ columns of the matrix V in (13) will be the matrix $\prod_{r=1}^{\kappa} U_r$ and thus,

$$V_{(:,\kappa:n)} = \left(\prod_{r=1}^{\kappa} U_r\right) V^\ell = \begin{bmatrix} * & \cdots & * & \cdots & * & \cdots & * \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & \pi_{j^1,j^\kappa}^{\kappa-1} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & \pi_{i^1,i^\kappa}^{\kappa-1} & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ * & \cdots & * & \cdots & * & \cdots & * \end{bmatrix},$$

$$V = \begin{bmatrix} V_{(:,1:\kappa-1)} & V_{(:,\kappa:n)} \end{bmatrix},$$

$$V_{(\mathcal{J}^1,:)} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & \pi_{j^1,j^\kappa}^{\kappa-1} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & \pi_{i^1,i^\kappa}^{\kappa-1} & \cdots & 0 \end{bmatrix}.$$

Then, the condition in Lemmas 3 and 4 for M_i, can be satisfied by assigning to zero the eigenvalues associated with the nonzero entries of the matrix $V_{(\mathcal{J}^1,:)}$, that is, the eigenvalues in Δ_i^3 satisfy $\lambda_i^{\kappa-1+j^\kappa} = 0, \forall j^\kappa \in \mathcal{J}^\kappa$. Note that in the matrix M_i, the row corresponding to the zero eigenvalue $\lambda_i^{\kappa-1+j^\kappa}, j^\kappa \in \mathcal{J}^\kappa$ is zero and hence, together with its corresponding nonzero entry in $V_{(j^1,:)}, j^1 \in \mathcal{J}^1$, the conditions of Lemma 3 are satisfied and the j^1th ultimate bound is minimised to its lowest possible value. ∎

3.5. Successful termination of Algorithm ITBF

As mentioned in Theorem 11, Algorithm ITBF is successful if at each iteration, α^ℓ that satisfies (44) also renders $p_{\ell+1} \geq p_\ell - 1$, with equality if and only if $p_\ell = n_\ell$ (see Theorem 7). The latter property is achieved if the vector α^ℓ satisfying (44) is such that the resulting common eigenvector ξ^ℓ as in (28) does not lie in the intersection subspace $\mathcal{S}^\ell$ unless $p_\ell = n_\ell = \rho^\ell$, in which case the minimisation problem is trivially solved in one step (see Remark 1) due to invertibility of all input matrices.

We next give sufficient conditions for the ℓth iteration of Algorithm ITBF ensuring the common eigenvector $\xi^\ell \in \mathbb{C}^{n_\ell}$ satisfying (44) is not inside the intersection subspace $\mathcal{S}^\ell$ with $\rho^\ell < n_\ell$.

Lemma 12. *At the ℓth iteration of Algorithm ITBF, consider p_ℓ defined in (22) and let $\mathcal{J}^\ell \subset \{1, 2, \ldots, n_\ell\}$, $\#\mathcal{J}^\ell \leq p_\ell - 1$ be the target zero elements of the common eigenvector ξ^ℓ. Let $S \in \mathbb{C}^{n_\ell \times \rho^\ell}$, $\rho^\ell < n_\ell$, be a basis of the intersection subspace $\mathcal{S}^\ell$, and from (35), consider the matrix $W^\ell_{(1:n_\ell,:)} \in \mathbb{C}^{n_\ell \times \psi_\ell}$ where W^ℓ is a basis for* Ker Q^ℓ*, with the matrix Q^ℓ defined in (21). Suppose that $\mathrm{d}(\text{Ker}\, S_{(\mathcal{J}^\ell,:)}) < \mathrm{d}(\text{Ker}\, W^\ell_{(\mathcal{J}^\ell,:)})$, then, a common eigenvector ξ^ℓ with $\xi^\ell_{\mathcal{J}^\ell} = 0$ can be found such that $\xi^\ell \notin \mathcal{S}^\ell$, namely, the intersection subspace can be avoided.*

Proof. From (24), (27), (36) and the matrix $W^\ell_{(1:n_\ell,:)}$, we have $v = W^\ell_{(1:n_\ell,:)}\alpha^\ell$ and thus, the common eigenvector (28) is

$$\xi^\ell = W^\ell_{(1:n_\ell,:)}\alpha^\ell / \|v\|. \tag{46}$$

For the condition $\xi^\ell \in \mathcal{S}^\ell$ to hold, there should exist a nonzero vector $\beta \in \mathbb{C}^{\rho^\ell}$ such that

$$S\beta = \xi^\ell = W^\ell_{(1:n_\ell,:)}\alpha^\ell/\|v\|. \tag{47}$$

For simplicity, reorder the rows of the matrices S and $W^\ell_{(1:n_\ell,:)}$, respectively, to form $\bar{S} = [S_1' \; S_2']'$ and $\bar{W} = [W_1' \; W_2']'$, such that $S_1 = S_{(\mathcal{J}^\ell,:)}$ and $W_1 = W^\ell_{(\mathcal{J}^\ell,:)}$. According to these reordered matrices, reorder β and α^ℓ (for simplicity we keep the same symbols). Then, combining (47) with (44) yields

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}\beta = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}\alpha^\ell/\|v\|, \quad S_1\beta = W_1\alpha^\ell = 0. \tag{48}$$

We divide the proof into two parts to obtain a sufficient condition to avoid (48).

(i) If $\mathrm{d}(\mathrm{Ker}\, S_1) = 0$, then there exists no $\beta \neq 0$ such that $S_1\beta = 0$ and hence, any α^ℓ in the nullspace of W_1 avoids the intersection space.

(ii) If $\mathrm{d}(\mathrm{Ker}\, S_1) > 0$, then $S_1\beta = W_1\alpha^\ell = 0$ is feasible and in order to avoid (48) we need to derive conditions preventing the occurrence of the equality $S_2\beta = W_2\alpha^\ell$.

Let $\Phi_{S_1} \in \mathbb{C}^{\rho^\ell \times \mathrm{d}(\mathrm{Ker}\, S_1)}$ be a basis of $\mathrm{Ker}\, S_1$. Then, we can choose $\beta \in \mathbb{C}^{\rho^\ell}$ in $\mathrm{Im}\, \Phi_{S_1}$ as $\beta = \Phi_{S_1}\eta_\beta$ with $\eta_\beta \in \mathbb{C}^{\mathrm{d}(\mathrm{Ker}\, S_1)}$ arbitrary. Also, let $\Phi_{W_1} \in \mathbb{C}^{p_\ell \times \mathrm{d}(\mathrm{Ker}\, W_1)}$ be a matrix of a set of basis vectors of $\mathrm{Ker}\, W_1$. The vector $\alpha^\ell \in \mathbb{C}^{p_\ell}$ can then be selected in $\mathrm{Im}\, \Phi_{W_1}$ as $\alpha^\ell = \Phi_{W_1}\eta_\alpha$ with $\eta_\alpha \in \mathbb{C}^{\mathrm{d}(\mathrm{Ker}\, W_1)}$ arbitrary.

The equality $S_2\beta = W_2\alpha^\ell$ then takes the form

$$S_2\Phi_{S_1}\eta_\beta = W_2\Phi_{W_1}\eta_\alpha. \tag{49}$$

Note that $W_2\Phi_{W_1}$ is not rank deficient. If otherwise, it means that $\mathrm{Im}\, \Phi_{W_1} \bigcap \mathrm{Ker}\, W_2 \neq \emptyset$, that is, $\mathrm{Ker}\, W_1 \bigcap \mathrm{Ker}\, W_2 \neq \emptyset$ which is in contradiction with $\bar{W}$ being full column rank (cf. (37)).

Hence, $\mathrm{rank}(W_2\Phi_{W_1}) = \mathrm{d}(\mathrm{Ker}\, W_1)$. Then, if $\mathrm{d}(\mathrm{Ker}\, S_1) < \mathrm{d}(\mathrm{Ker}\, W_1)$, proper selection of η_α ensures that no η_β in $\mathrm{Ker}\, S_1$ can be found to satisfy (49). ∎

Remark 2. In Lemma 2, we singled out a case for which the ultimate bound minimisation is not feasible. When the jth row of at least one of the B_i matrices is zero, then the corresponding ultimate bound cannot be minimised to its minimum value $\mathbf{b}_j^{\min}$ as in (2). In the Appendix we show how this case causes Algorithm ITBF to break down at some iteration for which the conditions in Lemma 9 cease to hold. ○

4. Numerical example

Consider a switched system formed by two subsystems with matrices

$$A_1 = \begin{bmatrix} -1.1680 & 4.0080 & 0.4535 & -1.5970 & 2.0732 & -4.5139 \\ 2.1211 & 2.5822 & 1.6484 & -4.9292 & 0.0862 & 0.1569 \\ 3.5424 & 0.8391 & 3.1124 & -4.2657 & -0.4706 & -1.5780 \\ 1.9897 & -0.9946 & 3.0450 & -0.8680 & -3.1999 & -0.7216 \\ 4.5828 & -4.8787 & 0.8283 & -2.3990 & -3.7715 & 3.4032 \\ -4.7401 & -2.1672 & -2.7062 & 0.4180 & -0.2747 & -1.1410 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1.5627 & -1.5909 & -1.9712 & -2.9420 & 0.3049 & 1.4167 \\ 0.1561 & 4.6490 & 4.4860 & 1.6537 & -4.4574 & -1.6594 \\ 0.8228 & -4.1364 & -3.9620 & -3.1066 & -2.4372 & -1.6533 \\ 0.4060 & 4.6741 & 4.3958 & 1.9146 & -0.1129 & 2.3545 \\ 3.9448 & 0.5444 & -1.1935 & -2.2255 & -2.2387 & 4.5450 \\ 0.6211 & -0.1799 & -0.2757 & 2.5900 & -4.6122 & -2.9693 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -1.2420 & 2.3311 & -4.6076 & 0.9842 & -0.1453 \\ -4.2966 & 0.2976 & -1.0773 & -4.4021 & -0.1310 \\ -1.1432 & -2.2856 & -0.9735 & -4.7206 & 3.1869 \\ -4.5608 & 0.6943 & -4.0939 & 3.5796 & -0.0540 \\ -3.6490 & 4.1370 & -1.0558 & 4.3792 & 1.7705 \\ 3.8813 & 4.6013 & 4.3709 & 0.1758 & -0.1383 \end{bmatrix}, \qquad H_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} -4.6122 & -0.6965 & -1.4818 & -0.0505 \\ -1.9460 & 1.3177 & -4.5455 & -1.1453 \\ 0.9533 & 3.4591 & -2.9129 & -2.6257 \\ 4.8218 & -0.2982 & 4.8561 & -2.7867 \\ -3.4101 & 1.0458 & 4.1421 & 2.4541 \\ 3.7593 & 3.7462 & 4.1393 & -3.8830 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{d} = 1.$$

From (22) we can compute $p_1 = 3$. Hence, as in Assumption 10, we aim at minimising up to $p_1 - 1 = 2$ ultimate bounds of the switched system under arbitrary switching. For this 6th order system, we choose the 5th and 6th states for ultimate bound minimisation and thus, $\mathcal{J} = \{5, 6\}$, $\#\mathcal{J} = 2$.

At the first iteration, ($q_1 = 0$ and) $p_1 = 3 > \#\mathcal{J}$ and thus, for an arbitrary set of eigenvalues $\Lambda^1 = \begin{bmatrix}\lambda_1^1 & \lambda_2^1\end{bmatrix}' = \begin{bmatrix}0.0551 & 0.3846\end{bmatrix}'$ Procedure SCEA successfully results in $V_{(:,1)} = \xi^1 = \begin{bmatrix}-0.6042 & 0.6026 & -0.4930 & -0.1696 & 0 & 0\end{bmatrix}^T$ with $\alpha^1 = \begin{bmatrix}0.2242 & -0.9283 & 0.2968\end{bmatrix}^T$. The matrix U_2 as in (17) with $\mathcal{J}^2 = \{4, 5\}$ is

$$U_2 = \begin{bmatrix} 0.7968 & 0 & 0 & 0 & 0 \\ 0.4569 & 0.6543 & 0 & 0 & 0 \\ -0.3738 & 0.7151 & -0.3254 & 0 & 0 \\ -0.1286 & 0.2461 & 0.9456 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

At the next iteration, ($q_2 = 3$ and) $p_2 = 4 > \#\mathcal{J}$ and thus, for an arbitrary set of eigenvalues $\Lambda^2 = \begin{bmatrix}\lambda_1^2 & \lambda_2^2\end{bmatrix}' = \begin{bmatrix}0.4242 & 0.5993\end{bmatrix}'$, Procedure SCEA for $\mathcal{J}^2$ gives ξ^2 and α^2 as shown below $\xi^2 = \begin{bmatrix}0.4273 & 0.3156 & 0.8472 & 0 & 0\end{bmatrix}^T$ with $\alpha^2 = \begin{bmatrix}0.0723 & 0.5665 & 0.9188 & 0.9108\end{bmatrix}^T$. The second column of the matrix V in (13) is thus $V_{(:,2)} = I_6\, U_2\, \xi^2 = \begin{bmatrix}0.3405 & 0.4018 & -0.2097 & 0.8238 & 0 & 0\end{bmatrix}^T$. Also the matrix U_3 as in (17) with $\mathcal{J}^3 = \{3, 4\}$ is

$$U_3 = \begin{bmatrix} -0.9041 & 0 & 0 & 0 \\ 0.1492 & -0.9371 & 0 & 0 \\ 0.4005 & 0.3491 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

At the next iteration, $n_3 = 4$ and $m_1^3 = m_2^3 = 4$ and hence, the input matrices are invertible. In this case, the common eigenvector matrix associated with arbitrary eigenvalues can be taken to be the identity matrix. Since at this iteration with subsystems of order 4 we have $\mathcal{J}^3 = \{3, 4\}$, we need the last two eigenvalues of both subsystems to be zero. Therefore, assigning the remaining eigenvalues at

$$\Delta_1^3 = \operatorname{diag}\{\lambda_1^3, \lambda_1^4, \lambda_1^5, \lambda_1^6\} = \operatorname{diag}\{-0.2854, -0.5910, 0, 0\}$$
$$\Delta_2^3 = \operatorname{diag}\{\lambda_2^3, \lambda_2^4, \lambda_2^5, \lambda_2^6\} = \operatorname{diag}\{-0.9381, 0.3268, 0, 0\}$$

and computing the eigenvectors as in Part 2 of the proof of Theorem 11, $V^3 = I_4$, yields

$$V_{(:,3:6)} = I_6\, U_2\, U_3\, V^3 = \begin{bmatrix} -0.7204 & 0 & 0 & 0 \\ -0.3155 & -0.6131 & 0 & 0 \\ 0.3144 & -0.7837 & 0 & 0 \\ 0.5317 & 0.0995 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The resulting feedback gains, triangularising transformation matrix and upper triangular closed-loop matrices are

$$K_1 = \begin{bmatrix} -0.1709 & -1.2207 & 0.9183 & -0.8941 & -1.2457 & 0.8258 \\ 0.2483 & -0.2276 & 0.3587 & 0.5143 & -0.2647 & 0.2418 \\ 0.9193 & 1.7895 & -0.5789 & 0.1856 & 1.4308 & -0.7072 \\ -0.3254 & 0.7433 & 0.1064 & -0.6640 & 0.4182 & -0.4880 \\ -2.1678 & 0 & -0.0218 & 0.0633 & 0 & 0 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.2890 & -0.1019 & -0.2887 & 0.1020 & -0.3859 & -0.4916 \\ -0.8009 & -0.0091 & 0.2280 & 0.8341 & 1.1647 & -0.9145 \\ -0.1930 & -0.0747 & 0.0430 & -0.3186 & 0.1153 & -0.0082 \\ -0.5388 & -0.2334 & -0.0847 & 1.2309 & -0.3149 & -2.1316 \end{bmatrix},$$

$$V = \begin{bmatrix} -0.6042 & 0.3405 & -0.7204 & 0 & 0 & 0 \\ 0.6026 & 0.4018 & -0.3155 & -0.6131 & 0 & 0 \\ -0.4930 & -0.2097 & 0.3144 & -0.7837 & 0 & 0 \\ -0.1696 & 0.8238 & 0.5317 & 0.0995 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

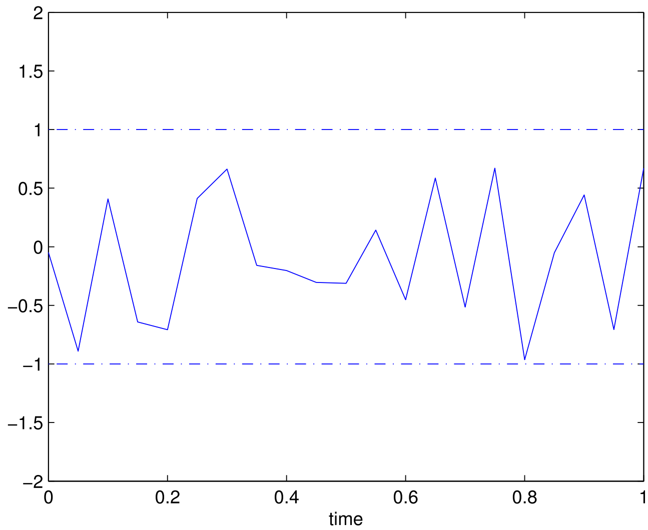

Fig. 3. Random bounded disturbance.

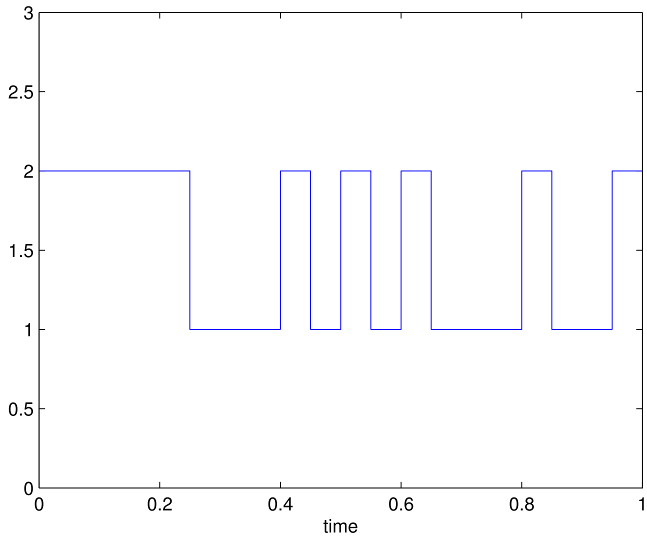

Fig. 4. First random switching signal.

$$M_1 = \begin{bmatrix} 0.0551 & 6.6301 & -6.4030 & 0.6186 & 4.3531 & 1.6601 \\ 0 & 0.4242 & 1.5813 & -1.3306 & -1.6077 & -3.5081 \\ 0 & 0 & -0.2854 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5910 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.3846 & 4.9649 & 3.7414 & -6.1443 & -2.7784 & -3.5492 \\ 0 & 0.5993 & -3.1425 & -8.1900 & -1.6449 & 6.7434 \\ 0 & 0 & -0.9381 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3268 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrices V and M_i satisfy the conditions of Lemmas 1 and 4. For a particular realisation of the disturbance signal shown in Fig. 3, and two random switching signals (Figs. 4 and 6), the trajectories of the perturbed switched system (Figs. 5 and 7) and zoomed trajectories of the perturbed switched system depicted in Fig. 8, corresponding to the initial condition $x(0) = [0.6146, 1.1240, 1.7603, 2.1086, 1.8297, 3.5015]'$, show that the 5th and 6th states are ultimately bounded within

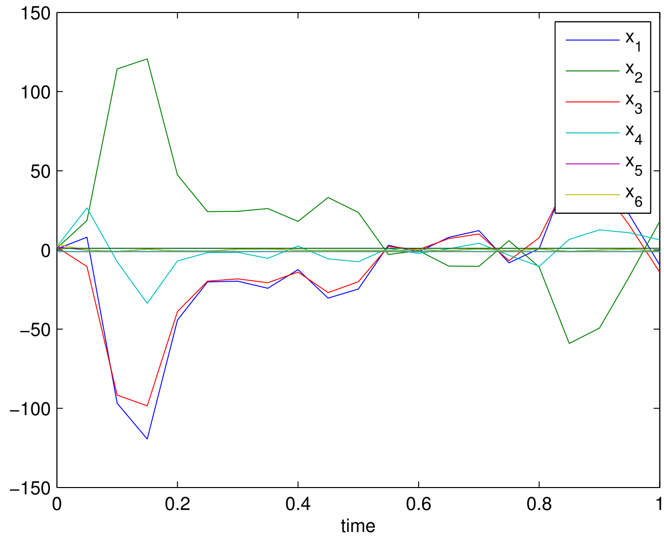

Fig. 5. Switched system trajectories under first random switching signal.

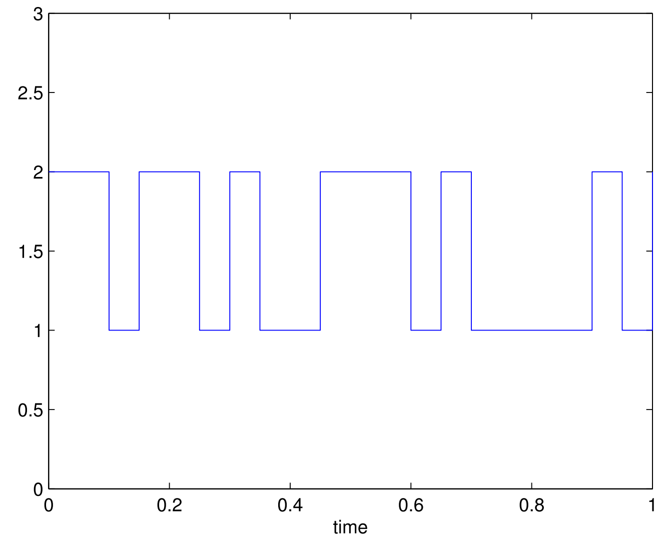

Fig. 6. Second random switching signal.

their smallest possible regions. As expected, the 5th and 6th state trajectories become equal after one time step since they correspond to $\max_i H_i(5)\mathbf{d} = \max_i H_i(6)\mathbf{d} = \mathbf{d}$, that is, only the disturbance.

5. Conclusion

This paper has considered discrete-time switched linear systems in the presence of non-vanishing perturbations and derived sufficient conditions to achieve the minimum ultimate bounds by state feedback for one or more components of the state of the closed-loop switched system under arbitrary switching. These conditions are expressed as design constraints on the eigenstructure of the subsystems. A constructive procedure to satisfy these constraints has been presented in the form of an iterative algorithm that simultaneously triangularises all closed-loop subsystem matrices and achieves the lowest ultimate bounds for the target states provided they are feasible.

The reliance of the proposed method on the construction of stable, simultaneously triangularisable closed-loop feedback matrices inherits the conservatism of this technique, which is only sufficient (and with a CQLF) for closed-loop stability under arbitrary switching. However, there is not much work available on control of switched systems under arbitrary switching, and the little available focuses on stabilisation. The present paper goes beyond stabilisation by integrating in the feedback design disturbance attenuation through the minimisation of state componentwise ultimate bounds. A distinctive feature of

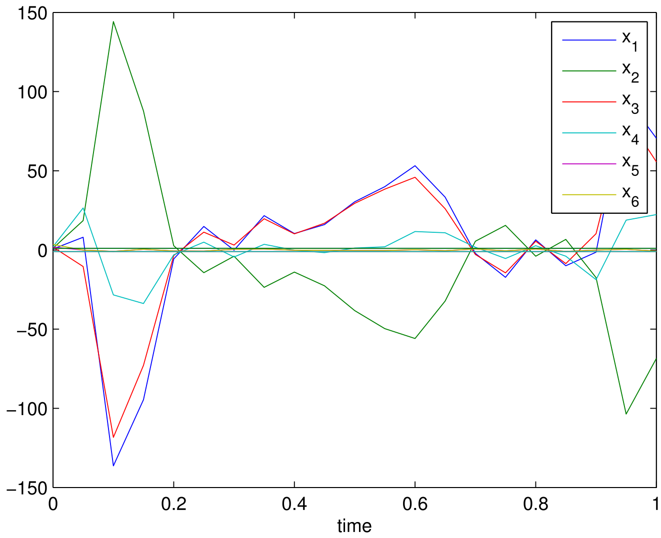

Fig. 7. Switched system trajectories under second random switching signal.

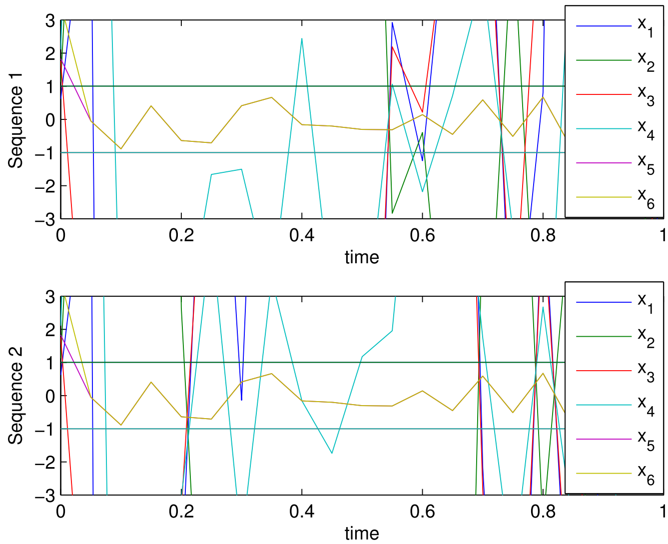

Fig. 8. Zoomed in trajectories under both random switching signals.

the proposed approach is that it is entirely analytic, which enables us to algebraically characterise when the problem can and cannot be solved in terms of the system structure.

Appendix. Unfeasibility of ultimate bound minimisation

Let the matrix B_1 of size $n \times m_1$ be of full column rank and its jth row be zero, that is

$$B_1 = \begin{bmatrix} * & \cdots & * \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ * & \cdots & * \end{bmatrix}. \tag{A.1}$$

From Lemma 2 we know that the jth ultimate bound cannot be minimised by any feedback control. Here we want to show where the ITBF algorithm breaks down.

At the first iteration, the common eigenvector assignment yields ξ^1 with $\xi^1_j = 0$. Then we should check whether $\xi^1 \in \operatorname{Im} B_1$ or not. In the worst case, if $m_1 = n - 1$, then the part of the matrix B_1 associated with the elements of ξ^1

excluding its jth element is square and thus, $\xi^1 \in \operatorname{Im} B_1$. Then, depending on whether or not $\xi^1 \in \operatorname{Im} B_i$ for all $i = 2, \dots, N$, we obtain $\xi^1 \in \mathcal{S}^1$ or $\xi^1 \notin \mathcal{S}^1$.

Assume that we have $m_1 < n - 1$ and $\xi^1 \notin \mathcal{S}^1$. Then, from (14)–(17), for $j^1 = j$ and an arbitrary j^2, the matrix U_2 is of the form

$$U_2 = \begin{bmatrix} * & \cdots & 0 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & v^1_{j^1,j^2} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ * & \cdots & 0 & \cdots & * \end{bmatrix}$$

which from (19) yields the input matrix B_1^2 for the second iteration with a zero j^2th row

$$B_i^2 = U_2^* B_i^1 = \begin{bmatrix} * & \cdots & 0 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & v^1_{j^1,j^2} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ * & \cdots & 0 & \cdots & * \end{bmatrix}^* \begin{bmatrix} * & \cdots & * \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ * & \cdots & * \end{bmatrix} = \begin{bmatrix} * & \cdots & * \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ * & \cdots & * \end{bmatrix}.$$

As it can be seen, the zero row of the matrix B_1 has been shifted to the input matrix of the second iteration such that the j^2th row of the matrix B_1^2 is zero.

At this iteration α^2 can be selected from (44) so that $\xi^2_{\mathcal{J}^2} = 0$. Again we need to check whether $\xi^2 \in \mathcal{S}^2$ which includes checking $\xi^2 \in \operatorname{Im} B_1^2$. Similar to the previous iteration, if $m_2 = n_2 - 1$, then it is certain that $\xi^2 \in \operatorname{Im} B_1^2$.

We suppose that up to the iteration ℓ, where $m_\ell = n_\ell - 1$, we have $\xi^\ell \notin \operatorname{Im} B_1^\ell$ and thus, $\xi^\ell \notin \mathcal{S}^\ell$. Computing the $U_{\ell+1}$ matrices as in (14)–(17) and determining the matrices B_i^ℓ from (19), similar to the above we see that the matrix B_1^ℓ has its j^ℓth row equal to zero. At this iteration we certainly have $\xi^\ell \in \operatorname{Im} B_1^\ell$ due to $m_\ell = n_\ell - 1$. However, having $\xi^\ell \in \mathcal{S}^\ell$ still needs checking whether $\xi^\ell \in \operatorname{Im} B_i^\ell$ for all $i = 2, \dots, N$. Suppose that $\xi^\ell \notin \operatorname{Im} B_i^\ell$ until the B_i matrices are invertible for all $i = 2, \dots, N$.

For the iteration $\ell + 1$, since $\xi^\ell \in \operatorname{Im} B_1^\ell$, we have $m_{\ell+1} = m_\ell - 1$ which together with $n_{\ell+1} = n_\ell - 1$ results in $m_{\ell+1} = n_{\ell+1} - 1$. Also, considering the shift of the zero row to the $B_1^{\ell+1}$ matrix, similar to the previous iteration, we will have $\xi^{\ell+1} \in \operatorname{Im} B_1^{\ell+1}$. The same situation happens at the subsequent iterations.

Continuing with Algorithm ITBF and considering the above arguments and assumptions, at some iteration l we obtain invertible B_i^l matrices for all $i = 2, \dots, N$ and B_1^l of size $n_l \times (n_l - 1)$. From the previous discussion we know that $\xi^l \in \operatorname{Im} B_1^l$. Together with invertibility of the other B_i matrices, the common eigenvector lies inside the intersection subspace $\mathcal{S}^l$, i.e. $\xi^l \in \mathcal{S}^l$ which results in $p_{l+1} = p_l - 1$. Thus the condition in Theorem 7 does not hold. However, we might still have $p_{l+1} > \#\mathcal{J}$ and the vector α^{l+1} could be selected as in (44) so that the algorithm continues without any problem. In this case, the next iterations of Algorithm ITBF will follow the same pattern as the lth iteration and reduce the quantity p_l until we reach $p_l \not> \#\mathcal{J}$, at which point the condition in Lemma 9 does not hold and the required α^l cannot be found. Therefore, Algorithm ITBF is unsuccessful.

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