Towards a polynomial equivalence between $\{k\}$-packing functions and k-limited packings in graphs

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Abstract. Given a positive integer k, the $\{k\}$-*packing function problem* ($\{k\}$PF) is to find in a given graph G, a function f of maximum weight that assigns a non-negative integer to the vertices of G in such a way that the sum of $f(v)$ over each closed neighborhood is at most k. This notion was recently introduced as a variation of the k-limited packing problem (kLP) introduced in 2010, where the function was supposed to assign a value in $\{0, 1\}$. For all the graph classes explored up to now, $\{k\}$PF and kLP have the same computational complexity. It is an open problem to determine a graph class where one of them is NP-complete and the other, polynomially solvable. In this work, we first prove that $\{k\}$PF is NP-complete for bipartite graphs, as kLP is known to be. We also obtain new graph classes where the complexity of these problems would coincide.
Keywords: computational complexity · $\mathcal{F}$-free graph· bipartite graph.

1 Basic definitions and preliminaries

All the graphs in this paper are simple, finite and undirected.

For a graph G, $V(G)$ and $E(G)$ denote respectively its vertex and edge sets. For any $v \in V(G)$, $N_G[v]$ is the *closed neighborhood* of v in G. For a given graph G and a function $f : V(G) \to \mathbb{R}$, we denote $f(A) = \sum_{v\in A} f(v)$, where $A \subseteq V(G)$. The *weight* of f is $f(V(G))$.

A graph H is *bipartite* if $V(G)$ is the union of two disjoint (possibly empty) independent sets called *partite sets* of G. Equivalently, bipartite graphs are defined as odd-cycle-free graphs, i.e graphs that have no induced odd-cycle.

A graph is *complete* if $E(G)$ contains all edges corresponding to any pair of distinct vertices from $V(G)$. The complete graph on n vertices is denoted by K_n.

Given G_1 and G_2 two graphs, the *strong product* $G_1 \otimes G_2$ is defined on the vertex set $V(G_1) \times V(G_2)$, where two vertices u_1v_1 and u_2v_2 are adjacent if and only if $u_1 = u_2$ and $(v_1, v_2) \in E(G_2)$, or $v_1 = v_2$ and $(u_1, u_2) \in E(G_1)$, or $(v_1, v_2) \in E(G_2)$ and $(u_1, u_2) \in E(G_1)$.

Given a graph G and a positive integer k, a set $B \subseteq V(G)$ is a *k-limited packing* in G if each closed neighborhood has at most k vertices of B [8]. Observe that a k-limited packing in G can be considered as a function $f : V(G) \to \{0, 1\}$ such that $f(N_G[v]) \leq k$ for all $v \in V(G)$. The maximum possible weight of a k-limited packing in G is denoted by $L_k(G)$. When $k = 1$, a k-limited packing in G is a *2-packing* in G and $L_k(G)$ is the known *packing number* of G, $\rho(G)$.

This concept is a good model for many utility location problems in operations research, for example the problem of locating garbage dumps in a city. In most of them, the utilities —garbage dumps— are necessary but probably obnoxious. That is why it is of interest to place the maximum number of utilities in such a way that no more than a given number of them (k) is near to each agent in a given scenario.

Class	k**LP**	$\{k\}$**PF**
General graphs	NP-c [4]	NP-c [6, 7]
Strongly chordal	P [3]	P [5]
Dually chordal	?	P [6, 7]
Doubly chordal	?	P [7]
P_4-lite	P [4]	P [5]
P_4-tidy	P [4]	P [5]
bounded tree-width	P [5]	P [5]
bounded clique-width	P [5]	P [5]
Split	NP-c [4]	NP-c [7]
Chordal	NP-c [4]	NP-c [7]
Bipartite	NP-c [4]	?

Table 1. “NP-c”, “P” and “?” mean NP-complete, polynomial and open problem, resp.

In order to expand the set of utility location problems to be modeled, the concept of $\{k\}$-packing function of a graph was introduced in [5] as a variation of a k-limited packing. Recalling the problem of locating garbage dumps in a given city, if a graph G and a positive integer k model the scenario, when dealing with $\{k\}$-packing functions we are allowed to locate more than one garbage dump in any vertex of G subject to there are at most k garbage dumps in each closed neighborhood. Formally, given a graph G and a positive integer k, a *$\{k\}$-packing function* of G is a function $f : V(G) \to \mathbb{Z}_+^0$ such that for all $v \in V(G)$, $f(N_G[v]) \leq k$. The maximum possible weight of a $\{k\}$-packing function of G is denoted by $L_{\{k\}}(G)$ [5].

Since any k-limited packing in G can be seen as a $\{k\}$-packing function of G, it is clear to see that $L_k(G) \leq L_{\{k\}}(G)$. For K_3, $L_3(K_3) = L_{\{3\}}(K_3) = 3$. Nevertheless, for the following graph these numbers do not coincide:

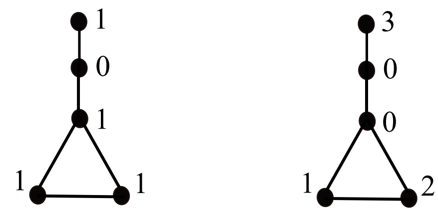

Fig. 1. A graph G with $L_3(G) = 4$ and $L_{\{3\}}(G) = 6$.

The above definitions induce the study —started in [3] and [5]— of the computational complexity of the following decision problems:

k**-LIMITED PACKING**, fixed $k \in \mathbb{Z}_+$ (kLP) [3]
Instance: (G, l), where G is a graph and $l \in \mathbb{Z}_+$.
Question: Does G have a k-limited packing of size at least l?

$\{k\}$**-PACKING FUNCTION**, fixed $k \in \mathbb{Z}_+$ ($\{k\}$PF) [5]
Instance: (G, l), where G is a graph and $l \in \mathbb{Z}_+$.
Question: Does G have a $\{k\}$-packing function of weight at least l?

Table 1 summarizes the already known results on the complexity of $\{k\}$PF in contrast with kLP, for fixed $k \in \mathbb{Z}_+$.

It is an open problem to determine a graph class where one of these problems is NP-complete and the other, polynomially solvable.

In Section 2 we prove that $\{k\}$PF is NP-complete on bipartite graphs, answering in this way one of the open questions in Table 1.

In Section 3 we obtain new graph classes where the complexity of kLP and $\{k\}$PF would coincide.

2 $\{k\}$-packing functions on bipartite graphs

As Table 1 shows, it is already known that kLP is NP-complete on bipartite graphs [4]. The proof is based on a reduction from a variation of the classical domination problem on a bipartite graph to kLP on a bipartite graph.

In this section we state that also $\{k\}$PF is NP-complete for bipartite graphs. In this case the proof consists in a reduction from $\{k\}$PF in a general graph to $\{k\}$PF in a bipartite graph.

We have:

Theorem 1. *For every fixed $k \in \mathbb{Z}_+$, $\{k\}$PF is NP-complete on bipartite graphs.*

Proof. Let $k \in \mathbb{Z}_+$ be fixed. It is already known that $\{k\}$PF is NP-complete for general graphs [6].

Let (G, l) be an instance of $\{k\}$PF. We build a bipartite graph B in the following way. Let

$$X = \{x_v : v \in V(G)\} \cup \{x\},\ Y = \{y_v : v \in V(G)\} \cup \{y\}$$

be the partite sets of B. Let also

$$E(B) = \bigcup_{v \in V(G)} \{(x_v, y_u) : u \in N_G[v]\} \cup \{(x, y') : y' \in Y\}.$$

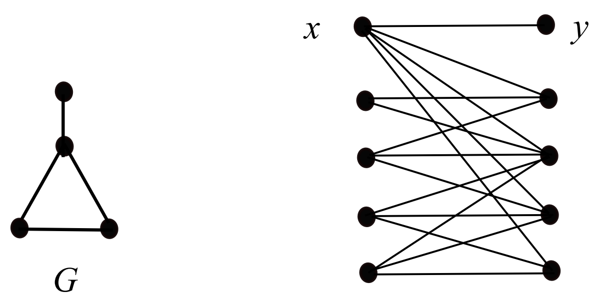

Fig. 2. Construction of B from a graph G in Theorem 1.

We will prove that

$$L_{\{k\}}(B) = L_{\{k\}}(G) + k.$$

On the one hand, let f be a $\{k\}$-packing function of G with weight $L_{\{k\}}(G)$. Consider the function $h : V(B) \to \mathbb{Z}_+^0$ defined as follows. For each $v \in V(G)$ let $h(x_v) = f(v)$ and $h(y_v) = 0$. Let also $h(x) = 0$ and $h(y) = k$. Notice that h is indeed a $\{k\}$-packing function of B with weight $L_{\{k\}}(G) + k$. Hence,

$$L_{\{k\}}(B) \geq L_{\{k\}}(G) + k.$$

On the other hand, let h be a $\{k\}$-packing function of B with weight $L_{\{k\}}(B)$. We can assume that h satisfies $h(x) = 0$ and $h(y_v) = 0$ for each $v \in V(G)$: if h does not satisfy these conditions, we can construct another $\{k\}$-packing function $\hat{h}$ of B of maximum weight, by defining $\hat{h}(x_v) = h(x_v)$ and $\hat{h}(y_v) = 0$ for each $v \in V(G)$, $\hat{h}(x) = 0$ and $\hat{h}(y) = \sum_{v \in V(G)} h(y_v) + h(x) + h(y)$. Now we construct a function $f : V(G) \to \mathbb{Z}_+^0$ by letting $f(v) = h(x_v)$ for each $v \in V(G)$. Clearly, f is a $\{k\}$-packing function of G with weight $L_{\{k\}}(B) - \hat{h}(y)$. Hence, $L_{\{k\}}(G) \geq L_{\{k\}}(B) - \hat{h}(y)$. Since $\hat{h}(y) = \sum_{v \in V(G)} h(y_v) + h(x) + h(y) = h(N_B[x]) \leq k$, it follows that

$$L_{\{k\}}(G) \geq L_{\{k\}}(B) - k.$$

□

3 A general result

Clearly, there are polynomial-time reductions from kLP ($\{k\}$PF) to $\{k\}$PF (kLP), since both problems are NP-complete in the general case. It is known a linear reduction from $\{k\}$PF to kLP that involves changes in the graph; more precisely, it is proved that $L_{\{k\}}(G) = L_k(G \otimes K_k)$, for every graph G and positive integer k [5]. This reduction is closed within some graph classes, for instance strongly chordal graphs and graphs with the parameter clique-width bounded by a constant. From these facts it is derived the polynomiality of $\{k\}$PF for strongly chordal graphs and graphs with the parameter clique-width bounded by a constant [5].

In this section we prove that the above reduction is closed within certain graph class defined by forbidden induced subgraphs, following the ideas of Theorem 9 in [2]. For this purpose, we consider the following definition:

Definition 1. *Let $\mathcal{F}$ be a family of graphs satisfying the following property: for every graph G in $\mathcal{F}$, $|V(G)| \geq 2$ and, for every $v \in V(G)$, no connected component of $G - v$ is complete. We call $\mathbf{G}$ the class of $\mathcal{F}$-free graphs.*

Some examples of graph classes in $\mathbf{G}$ are $\{house, hole, gem\}$-free graphs, $\{house, hole, domino\}$-free graphs and $\{house, hole, domino, sun\}$-free graphs. It is worth studying the complexities for the mentioned classes since they all have the parameter clique-width unbounded. For other examples, like distance-hereditary graphs which are $\{house, hole, domino, gem\}$-free graphs, the complexity of both problems is already known since they have the parameter clique-width bounded by a constant.

We can state and prove:

Theorem 2. *Consider the graph class $\mathbf{G}$ in Definition 1. For fixed positive integer k and graph G in $\mathbf{G}$, $G \otimes K_k \in \mathbf{G}$.*

Proof. Let k be a fixed positive integer and G be graph in $\mathbf{G}$. We will prove that $G \otimes K_k \in \mathbf{G}$, i.e. we will prove that $G \otimes K_k$ is $\mathcal{F}$-free. Let G' be a subgraph of $G \otimes K_k$ induced by V' with $|V'| \geq 2$. Then V' is the disjoint union of sets V'_{v_j} with $j \in J$, where $1 \leq |J| \leq |V(G)|$.

When $|J| = 1$, $G' = K_k$ and thus $G' \notin \mathcal{F}$. When $|J| \geq 2$, consider the subgraph G'' of G induced by the vertices $\{v_1, \ldots, v_{|J|}\}$. Since G is $\mathcal{F}$-free, there is a vertex v_r with $r \in J$ and such that $G'' - v_r$ has a complete connected component. From the definition of $G \otimes K_k$, it is not difficult to see that $G' - v'_r$ has a complete connected component, where v'_r is any vertex in V'_{v_r}. Therefore, $G' \notin \mathcal{F}$. Since G' is arbitrary, this proves that $G \otimes K_k$ is $\mathcal{F}$-free, concluding that $G \otimes K_k \in \mathbf{G}$. $\square$

As a corollary, knowing from [5] that $L_{\{k\}}(G) = L_k(G \otimes K_k)$ for every graph G and positive integer k, we have:

Corollary 1. *Consider the graph class* $\mathbf{G}$ *in Definition 1. Then, for fixed positive integer* k*,* $\{k\}PF$ *is solvable in polynomial time in the class* $\mathbf{G}$*, provided that* kLP *is solvable in polynomial time in* $\mathbf{G}$*. Besides, if* $\{k\}PF$ *is NP-complete in* $\mathbf{G}$*, then* kLP *is NP-complete in* $\mathbf{G}$*.*

4 Final remarks

It remains an open problem to know if there exists a graph class where one of the problems considered in this work is NP-complete and the other can be solved in polynomial time. Corollary 1 helps to keep working on this line of research. Besides, it is an open problem to determine the complexity of kLP for dually chordal graphs, as shown in Table 1, or at least for one of its maximal subclasses constituted by doubly chordal graphs (also shown in Table 1).

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