Generalized minor inequalities for the set covering polyhedron related to circulant matrices

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Abstract

We study the set covering polyhedron related to circulant matrices. In particular, our goal is to characterize the first Chvátal closure of the usual fractional relaxation. We present a family of valid inequalities that generalizes the family of minor inequalities previously reported in the literature and includes new facet-defining inequalities. Furthermore, we propose a polynomial time separation algorithm for a particular subfamily of these inequalities.

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1 Introduction

The *weighted set covering problem* can be stated as

$$(\text{SCP}) \quad \min\{c^T x : Ax \geq \mathbf{1}, x \in \{0,1\}^n\}$$

where A is an $m \times n$ matrix with $0, 1$ entries, $c \in \mathbb{Z}^n$, and $\mathbf{1} \in \mathbb{R}^m$ is the vector having all entries equal to one. The SCP is a classic problem in combinatorial optimization with important practical applications (crew scheduling, facility location, vehicle routing, to cite a few prominent examples), but hard to solve in general. One established approach to tackle this problem is to study the polyhedral properties of the set of its feasible solutions. [5, 11, 14, 15].

The *set covering polyhedron* $Q^*(A)$ is defined as the convex hull of all feasible solutions of SCP. Its *fractional relaxation* $Q(A)$ is the feasible region of the linear programming relaxation of SCP, i.e.,

$$Q(A) := \{x \in [0,1]^n : Ax \geq \mathbf{1}\}.$$

It is known that SCP can be solved in polynomial time if A belongs to the particular class of *circulant matrices* defined in the next section. Hence, it is natural to ask whether an explicit description in terms of linear inequalities can be provided for $Q^*(A)$ in this case, an issue that has been addressed in several recent studies by researchers in the field (see [2, 7, 8, 12] among others). For the related *set packing polytope* of circulant matrices,

$$P^*(A) := \text{conv}(\{Ax \leq \mathbf{1}, x \in \{0,1\}^n\})$$

such a description follows from the results published in [13].

Bianchi et al. introduced in [7] a family of facet-defining inequalities for $Q^*(A)$ which are associated with certain structures called circulant minors. Moreover, the authors presented in [8] two families of circulant matrices for which $Q^*(A)$ is completely described by this class of *minor inequalities*, together with the full-rank inequality and the inequalities defining $Q(A)$, usually denoted as boolean facets. The existence of a third family of circulant matrices having this property follows from previous results obtained by Bouchakour et al. [9] in the context of the dominating set polytope of some graph classes.

If an inequality $a^T x \leq b$ is valid for a polytope $P \subset \mathbb{R}^n$ and $a \in \mathbb{Z}^n$, then $a^T x \leq \lfloor b \rfloor$ is valid for the integer polytope $P_I := \text{conv}(P \cap \mathbb{Z}^n)$. This procedure is called Chvátal-Gomory rounding, and it is known that the system of all linear inequalities which can be obtained in this way defines a new polytope P', the *first Chvátal closure of P*. Moreover, iterating this

procedure yields P_I in a finite number of steps. An inequality is said to have *Chvátal rank* of t if it is valid for the t-th Chvátal closure of a polytope. All inequalities mentioned above have Chvátal rank less than or equal to one.

With the aim of investigating if the results in [8, 9] can be generalized to all circulant matrices, we have tried to characterize the first Chvátal closure of $Q(A)$ for any circulant matrix A. In particular, we addressed the question whether the system consisting of minor inequalities, boolean facets, and the full-rank inequality is sufficient for describing $Q'(A)$. We have obtained a negative answer to this question in the form of a new class of valid inequalities for $Q^*(A)$ which contains minor inequalities as a proper subclass. All inequalities from this class have Chvátal rank equal to one, and besides, some of them define new facets of $Q^*(A)$, as we show by an example.

This paper is organized as follows. In the next section, we introduce some notation and preliminary results required for our work. In Section 3 we describe our approach for computing the first Chvátal closure of $Q(A)$ and define the new class of *generalized minor inequalities*. A separation algorithm for a particular subclass of these is provided in Section 4. Finally, some conclusions and possible directions for future work are discussed in Section 5. A preliminary version of this article appeared without proofs in [16].

2 Notations, definitions and preliminary results

For $n \in \mathbb{N}$, let $[n]$ denote the additive group defined on the set $\{1, \ldots, n\}$, with integer addition modulo n. Throughout this article, if A is a $0, 1$ matrix of order $m \times n$, then we consider the columns (resp. rows) of A to be indexed by $[n]$ (resp. by $[m]$). In particular, addition of column (resp. row) indices is always considered to be taken modulo n (resp. modulo m). Two matrices A and A' are *isomorphic*, denoted by $A \approx A'$, if A' can be obtained from A by permutation of rows and columns. Moreover, we say that a row v of A is a *dominating row* if $v \geq u$ for some other row u of A, $u \neq v$.

Given $N \subset [n]$, the *minor of A obtained by contraction of N*, denoted by A/N, is the submatrix of A that results after removing all columns with indices in N and all dominating rows. In this work, when we refer to *a minor* of A we always consider a minor obtained by contraction.

Let $n, k \in \mathbb{N}$ with $2 \leq k \leq n-2$, and $C^i := \{i, i+1, \ldots, i+(k-1)\} \subset [n]$ for every $i \in [n]$. With a little abuse of notation we will also use C^i to denote the incidence vector of this set. A *circulant matrix* C_n^k is the square matrix

of order n whose i-th row vector is C^i. Observe that $C^i = \sum_{j=i}^{i+k-1} e^j$, where e^j is the j-th canonical vector in $\mathbb{R}^n$.

A minor of C_n^k is called a *circulant minor* if it is isomorphic to a circulant matrix $C_{n'}^{k'}$. As far as we are aware, circulant minors were introduced for the first time in [12], where the authors used them as a tool for establishing a complete description of ideal and minimally nonideal circulant matrices. More recently, Aguilera [1] completely characterized the subsets N of $[n]$ for which C_n^k/N is a circulant minor. We review at next his main result, as some terms will be needed for the separation algorithm presented in Section 4.

Given C_n^k, the digraph $G(C_n^k)$ has vertex set $[n]$ and (i, j) is an arc of $G(C_n^k)$ if $j \in \{i+k, i+k+1\}$. We call arcs of the form $(i, i+k)$ *short* arcs and arcs of the form $(i, i+k+1)$ *long* arcs. Associated with any directed cycle D in $G(C_n^k)$, three parameters can be defined: its number n_2 of short arcs, its number n_3 of long arcs, and the number n_1 of turns around the set of nodes it makes. Hence, the relationship $n_1 n = n_2 k + n_3(k+1)$ must hold for the integers $n, k, n_1, n_2,$ and n_3. In our current notation, Theorem 3.10 of [1] states the following.

Theorem 2.1. *[1] Let n, k be integers verifying $2 \leq k \leq n-1$ and let $N \subset \mathbb{Z}_n$ such that $1 \leq |N| \leq n-2$. Then, the following are equivalent:*

1. *C_n^k/N is isomorphic to $C_{n'}^{k'}$.*

2. *N induces in $G(C_n^k)$ $d \geq 1$ disjoint simple dicycles $D_0, \ldots, D_{d-1}$, each of them having the same parameters n_1, n_2 and n_3 and such that $|N| = d(n_2 + n_3)$, $n' = n - d(n_2 + n_3) \geq 1$, and $k' = k - dn_1 \geq 1$.*

The structure of $Q^*(C_n^k)$ has been the subject of many previous studies. It is known that $Q^*(C_n^k)$ is a full dimensional polyhedron. Furthermore, for every $i \in [n]$, the constraints $x_i \geq 0$, $x_i \leq 1$ and $\sum_{j \in C^i} x_j \geq 1$ are facet defining inequalities of $Q^*(C_n^k)$ and we call them *boolean facets* [15]. We will denote by $\mathcal{S}_0$ the system of linear inequalities corresponding to boolean facets.

The *rank constraint* $\sum_{i=1}^n x_i \geq \lceil \frac{n}{k} \rceil$ is always valid for $Q^*(C_n^k)$ and defines a facet if and only if n is not a multiple of k [15]. In [7] the authors obtained another family of facet-defining inequalities for $Q^*(C_n^k)$ associated with circulant minors.

Lemma 2.2. *[7] Let $N \subset [n]$ such that $C_n^k/N \approx C_{n'}^{k'}$, and let $W = \{i \in N : i - k - 1 \in N\}$. Then, the inequality*

$$\sum_{i \in W} 2x_i + \sum_{i \notin W} x_i \geq \left\lceil \frac{n'}{k'} \right\rceil \tag{1}$$

is a valid inequality for $Q'(C_n^k)$. Moreover, if $2 \leq k' \leq n' - 2$, $\lceil \frac{n'}{k'} \rceil > \lceil \frac{n}{k} \rceil$ and $n' = 1 (\mathrm{mod}\, k')$, this inequality defines a facet of $Q^*(C_n^k)$.

Observe that the set W uniquely determines the set N, and hence the minor C_n^k/N associated with it. The authors termed (1) as the *minor inequality corresponding to W*. Moreover, the authors showed that every non boolean facet-defining inequality of $Q^*(C_n^3)$ (whose facetial structure had been previously characterized in [9]) is either the rank constraint or a minor inequality. Similarly, $Q^*(C_{2k}^k)$ and $Q^*(C_{3k}^k)$ are completely described by boolean facets and minor inequalities, for any $k \geq 2$ [8].

3   Computing the first Chvátal closure

In our attempt at finding a linear description of the first Chvátal closure of $Q(C_n^k)$, we use the following well-known result from integer programming:

Lemma 3.1. *Let $P = \{x \in \mathbb{R}^n \,:\, Ax \geq b\}$ be a nonempty polyhedron with A integral and $Ax \geq b$ totally dual integral. Then, $P' = \{x \in \mathbb{R}^n \,:\, Ax \geq \lceil b \rceil\}$.*

As we shall see below, if all vertices of P are known then a totally dual integral system describing the polyhedron can be computed from this information. This is the case for $Q(C_n^k)$, whose vertices have been completely characterized by Argiroffo and Bianchi [2].

Lemma 3.2 ([2]). *Let x^* be a vertex of $Q(C_n^k)$. Then one of the following statements holds:*

(i) x^* is integral.

(ii) $x^* = \frac{1}{k}\mathbf{1}$

(iii) There exists $N \subset [n]$ with $C_n^k/N \cong C_{n'}^{k'}$ and $\gcd(n', k') = 1$ such that

$$x_i^* := \begin{cases} \frac{1}{k'}, & \text{if } i \notin N, \\ 0, & \text{otherwise.} \end{cases}$$

In order to express $Q(C_n^k)$ via a totally dual integral system of linear inequalities, we use the method described below (see, e.g., [6, Ch. 8]). Given a polyhedral cone $K \subset \mathbb{R}^n$, consider the points in the lattice $L := K \cap \mathbb{Z}^n$. An *integral generating set* for L is a set $H \subseteq L$ having the property that every $x \in L$ can be written as a linear combination $\sum_{i=1}^k \alpha_i h_i$ of some elements $h_1, \ldots, h_k \in H$ with integral non negative coefficients $\alpha_1, \ldots, \alpha_k$.

The method consists in adding redundant inequalities to the original system $\mathcal{S}_0$ until the following property is verified: If $\{a_i^T x \geq b_i \; : \; i \in I(x^*)\}$ is the set of linear inequalities satisfied with equality by a vertex $x^* \in Q(C_n^{'k})$ and $K(x^*)$ is the cone generated by the set of vectors $H(x^*) := \{a_i \; : \; i \in I(x^*)\}$, then $H(x^*)$ is an integral generating set for $K(x^*) \cap \mathbb{Z}^n$.

This idea leads to the following procedure for computing $Q'(C_n^k)$:

1. Let $\mathcal{S} := \mathcal{S}_0$.

2. For every vertex x^* of $Q(C_n^k)$ do

   2.1 Compute an integral generating set $H(x^*)$ of $K(x^*) \cap \mathbb{Z}^n$.

   2.2 For all $a_i \in H(x^*)$, let $b_i := a_i^T x^*$ and add the inequality $a_i^T x \geq \lceil b_i \rceil$ to $\mathcal{S}$.

3. Return $\mathcal{S}$ as a linear description of $Q'(C_n^k)$.

Observe that the inequality $a_i^T x \geq b_i$ at step 2.2 is valid for $Q(C_n^k)$, since x^* minimizes $a_i^T x$ over this polyhedron for any $a_i \in K(x^*)$. Moreover, if x^* is integral, then the new inequality added to the system $\mathcal{S}$ is redundant, as $b_i \in \mathbb{Z}$. Therefore, new inequalities for $Q'(C_n^k)$ may only arise from integer generating sets $H(x^*)$ related to fractional vertices belonging to one of the two latter clases described in Lemma 3.2.

Firstly, we analyze all inequalities arising from the vertex $x^* = \frac{1}{k}\mathbf{1}$. The point $x^* = \frac{1}{k}\mathbf{1}$ is known to be a vertex of $Q(C_n^k)$ if and only if $\gcd(n, k) = 1$. In this case, $\{C^i x \geq 1 \; : \; i \in [n]\}$ are the inequalities of the original system $\mathcal{S}_0$ satisfied at equality by x^*. In order to find an integral generating set for $K(x^*) \cap \mathbb{Z}^n$ we need the following result.

Lemma 3.3. *Let $x \in \mathbb{R}^n, b \in \mathbb{Z}^n$ be two vectors such that $\mathbf{0} \leq x < \mathbf{1}$ and $C_n^k x = b$, with $\gcd(n, k) = 1$. Then there exists $r \in \{0, 1, \ldots, k-1\}$ such that $x = \frac{r}{k}\mathbf{1}$ and $b = r\mathbf{1}$.*

Proof. Observe that x^* is the solution of the linear system

$$C^i x = \sum_{j=i}^{i+k-1} x_j = b_i, \quad \forall 1 \leq i \leq n.$$

Subtracting each equation from the previous one, and the first from the last, we obtain $x_i - x_{i+k} = b_i - b_{i+1}$, for $1 \leq i \leq n$. Moreover, from $\mathbf{0} \leq x < \mathbf{1}$ it follows $-1 < x_i - x_{i+k} < 1$ and, since b is integral, we must have $x_i - x_{i+k} = b_i - b_{i+1} = 0$. As a consequence, $b = r\mathbf{1}$ for some $r \in \mathbb{Z}$.

On the other hand, as $C_n^k \mathbf{1} = k\mathbf{1}$, we can write $b = C_n^k(\frac{r}{k}\mathbf{1})$ and substitute this expression in the original system. Since $\gcd(n,k) = 1$, the matrix C_n^k is invertible and we obtain $x = \frac{r}{k}\mathbf{1}$. Finally, from $\mathbf{0} \le x < \mathbf{1}$ we have $r \in \{0, 1, \ldots, k-1\}$. $\quad\square$

With this result we can compute an integral generating set for $K(x^*)$.

Theorem 3.4. *Let C_n^k be a circulant matrix such that $\gcd(n, k) = 1$ and consider the vertex $x^* = \frac{1}{k}\mathbf{1}$ of $Q(C_n^k)$. Then an integral generating set for $K(x^*) \cap \mathbb{Z}^n$ is given by $H(x^*) = \{C^1, C^2, \ldots, C^n, \mathbf{1}\}$.*

Proof. Let $b \in K(x^*) \cap \mathbb{Z}^n$. Then there exists a non negative vector $\hat{d} \in \mathbb{R}_+^n$ such that $b^T = \hat{d}^T C_n^k$. Moreover, as the (unordered) sets of row and column vectors of a circulant matrix are equal, this is equivalent to saying that there exists $d \in \mathbb{R}_+^n$ with $C_n^k d = b$. Let $x = d - \lfloor d \rfloor$. We have $\mathbf{0} \le x < \mathbf{1}$ and

$$C_n^k(x + \lfloor d \rfloor) = b \quad \Leftrightarrow \quad C_n^k x = b - C_n^k \lfloor d \rfloor.$$

Since $b - C_n^k \lfloor d \rfloor$ is integral, applying Lemma 3.3 yields $C_n^k x = r\mathbf{1}$, for some $r \in \{0, 1, \ldots, k-1\}$. Hence,

$$b = r\mathbf{1} + C_n^k \lfloor d \rfloor$$

and thus b can be written as a linear combination of the elements of $H(x^*)$ where all coefficients are non negative integers. $\quad\square$

When applying step 2.2. of the procedure described at the beginning of this section, the vectors C^i yield the inequalities $(C^i)^T x \ge 1$ from $\mathcal{S}_0$, while for the last vector we obtain the rank constraint of $Q^*(C_n^k)$:

Corollary 3.5. *If $\gcd(n, k) = 1$, then the inequality $\mathbf{1}^T x \ge \lceil \mathbf{1}^T x^* \rceil = \lceil \frac{n}{k} \rceil$ is valid for $Q'(C_n^k)$.*

On the other hand, given a vertex x^* corresponding to a circulant minor $C_{n'}^{k'}$, the task of finding an integral generating set for $K(x^*) \cap \mathbb{Z}^n$ turns out to be more complicated. We present here a partial result which is however sufficient for deriving a new class of facet-defining inequalities for $Q^*(C_n^k)$.

Consider a circulant minor $C_{n'}^{k'} \approx C_n^k / N$ of C_n^k, and let x^* be the corresponding vertex of $Q(C_n^k)$, defined as in Lemma 3.2(iii). From Lemma 2.1 and Lemma 2.4 in [1] it follows that $C_{n'}^{k'}$ can be obtained from C_n^k by deleting each column j with $j \in N$ and each row i with $i + 1 \in N$. Hence, $|C^i \setminus N| = k'$ holds for $i + 1 \notin N$ and x^* satisfies the following inequalities with equality:

$$(C^i)^T x \ge 1, \text{ for all } i \text{ such that } i + 1 \notin N; \qquad (2)$$
$$(e^j)^T x \ge 0, \text{ for all } j \in N. \qquad (3)$$

Observe that there might be other inequalities from $\mathcal{S}_0$ satisfied tightly by x^*. In the following we denote by K the subcone of $K(x^*)$ spanned by the normal vectors of the left-hand sides of (2) and (3).

Theorem 3.6. *Let x^* be a vertex of $Q(C_n^k)$ associated with a minor $C_n^k/N \approx C_{n'}^{k'}$, and let $W = \{i \in N : i - k - 1 \in N\}$. An integral generating set for $K \cap \mathbb{Z}^n$ is given by*

$$\left\{ C^i \: : \: i + 1 \notin N \right\} \cup \left\{ e^j \: : \: j \in N \right\} \cup \left\{ r\mathbf{1} + \sum_{j \in W} e^j \: : \: 1 \leq r \leq k' - 1 \right\}.$$

Proof. Let A be the square coefficient matrix of the system (2)-(3). Reordering the columns, we may assume that A has the following block form:

$$A = \left(\begin{array}{c|c|c} B^1 & B^2 & C \\ \hline I_1 & \mathbb{O} & \mathbb{O} \\ \hline \mathbb{O} & I_2 & \mathbb{O} \end{array} \right),$$

where the first $n_3 := |W|$ columns correspond to indices in W, the next $n_2 := |N - W|$ columns correspond to indices in $N - W$ and the last n' columns correspond to indices not in N. Moreover, $C := C_{n'}^{k'}$, and $I_1, I_2, \mathbb{O}$ are identity and zero matrices of the appropriate sizes.

Let $b \in K \cap \mathbb{Z}^n$. Then there exist $d \in \mathbb{R}_+^{n'}$, $f \in \mathbb{R}_+^{n_3}$, and $g \in \mathbb{R}_+^{n_2}$ with $b^T = (d^T, f^T, g^T)A$. Now consider the vectors $x := d - \lfloor d \rfloor$, $y := f - \lfloor f \rfloor$, $z := g - \lfloor g \rfloor$, and define:

$$\tilde{b}^T := b^T - \left(\lfloor d^T \rfloor, \lfloor f^T \rfloor, \lfloor g^T \rfloor \right) A. \tag{4}$$

Observe that $\tilde{b} \in \mathbb{Z}^n$ and $\tilde{b}^T = (x^T B^1 + y^T, x^T B^2 + z^T, x^T C)$. From Lemma 3.3 it follows that $x^T C = r\mathbf{1}^T$ and $x = \frac{r}{k'}\mathbf{1}$, for some $r \in \{0, \dots, k' - 1\}$. Moreover, if $r = 0$ then we must have $\tilde{b} = 0$ and from (4) we conclude that b^T is an integral conic combination of the row vectors of A, which finishes the proof.

Now assume $r \in \{1, \dots, k' - 1\}$. For $\ell \in \{1, 2\}$, we have $x^T B^\ell = \frac{r}{k'}\mathbf{1}^T B^\ell$. From the proof of Theorem 3 in [7], it follows that each column of B^1 has support equal to $k' + 1$ and each column of B^2 has support equal to k'. Thus,

$$\tilde{b}^T = \left(\frac{r(k' + 1)}{k'}\mathbf{1}^T + y^T, r\mathbf{1}^T + z^T, r\mathbf{1}^T \right).$$

Finally, since $\tilde{b}^T \in \mathbb{Z}^n$, $\mathbf{0} \leq y < \mathbf{1}$, and $\mathbf{0} \leq z < \mathbf{1}$, we must have $y = (1 - \frac{r}{k'})\mathbf{1}$, $z = \mathbf{0}$, and hence $\tilde{b}^T = ((r+1)\mathbf{1}^T, r\mathbf{1}^T, r\mathbf{1}^T) = r\mathbf{1}^T + \sum_{j \in W}(e^j)^T$. Then the statement of the theorem follows from (4). $\square$

Vectors in the first two sets of last theorem give rise to boolean inequalities after applying step 2.2. of the procedure from the beginning of this section. For the third set, we have $\left\lceil r\mathbf{1}^T x^* + \sum_{j \in W}(e^j)^T x^* \right\rceil = \left\lceil \frac{rn'}{k'} \right\rceil$, and hence we obtain:

Corollary 3.7. *Let $N \subset [n]$ be such that $C_n^k/N \approx C_{n'}^{k'}$ and $W = \{i \in N : i - k - 1 \in N\}$. The inequalities*

$$\sum_{i \in W}(r+1)x_i + \sum_{i \notin W} rx_i \geq \left\lceil \frac{rn'}{k'} \right\rceil, \tag{5}$$

with $r \in \{1, \ldots, k' - 1\}$, are valid for $Q'(C_n^k)$.

For $r = 1$, these inequalities are the minor inequalities described in [7]. Accordingly, we have called (5) as *generalized r-minor inequalities*. In some cases, generalized r-minor inequalities with $r > 1$ can be obtained from the addition of (classical) minor inequalities and the rank constraint, and are thus redundant for $Q'(C_n^k)$. However, this is not true in general. For instance, consider $W := \{6 + 5k : 0 \leq k \leq 10\}$ and $N := W \cup \{1\} \subset [59]$. One can verify that $C_{59}^4/N \approx C_{47}^3$ and the corresponding inequality (5) for $r = 2$ has the form $\sum_{i \in W} 3x_i + \sum_{i \notin W} 2x_i \geq 32$. Moreover, it can be shown that this inequality defines a facet of $Q^*(C_{59}^4)$. As a consequence we have the following result.

Theorem 3.8. *There are circulant matrices C_n^k for which minor inequalities, boolean facets, and the rank constraint are not enough to describe $Q'(C_n^k)$.*

4 Separation algorithms for generalized minor inequalities

A polynomial time algorithm to separate 1-minor inequalities associated with particular classes of circulant minors has been proposed by S. Bianchi et al. in [7]. In this section we extend some of their results to the case of generalized minor inequalities. More precisely, we address the separation problem for r-minor inequalities corresponding to circulant minors having parameters $d = n_1 = 1$.

Following the ideas presented in [7], let us first prove a technical lemma that will be required for our separation procedure.

Lemma 4.1. *Let $d, n_1 = 1, n_2$, and n_3 be the parameters associated with a circulant minor of C_n^k such that $n_3 = p \,(\mathrm{mod}\,(k-d))$ with $1 \le p < k - d$. Then*

$$\left\lceil \frac{rn'}{k'} \right\rceil = \left\lceil r \, \frac{n - d(n_2 + n_3)}{k - d} \right\rceil = \left\lceil r\frac{n}{k} + \left(\left\lceil \frac{rp}{k-d} \right\rceil - \frac{rp}{k-d} \right) \right\rceil + \frac{r}{k(k-d)} dn_3.$$

Proof. Let s be the nonnegative integer such that $n_3 = s(k-d) + p$. Since $n = n_1 n = k(n_2 + n_3) + n_3$ we have that

$$\left\lceil r \, \frac{n - d(n_2 + n_3)}{k - d} \right\rceil = \left\lceil r \, \frac{(k-d)(n_2 + n_3) + n_3}{k - d} \right\rceil = r \, (n_2 + n_3) + \left\lceil \frac{rn_3}{k-d} \right\rceil.$$

From $s = \frac{n_3 - p}{k-d}$ and $n_2 + n_3 = \frac{n - n_3}{k}$ it follows that

$$r \, (n_2 + n_3) + \left\lceil \frac{rn_3}{k-d} \right\rceil = r \, \frac{n - n_3}{k} + r \, \frac{n_3 - p}{k-d} + \left\lceil \frac{rp}{k-d} \right\rceil$$

$$= \left\lceil r\frac{n}{k} + \left(\left\lceil \frac{rp}{k-d} \right\rceil - \frac{rp}{k-d} \right) \right\rceil + \frac{r}{k(k-d)} dn_3$$

and the proof is complete. $\qquad\square$

Observe that if n_3 is a multiple of $k - d$, then the corresponding r-minor inequality is redundant for $Q(C_n^k)$, as the value $\frac{rn'}{k'}$ on the right-hand side is integer. Otherwise, if $W \subset [n]$ defines a minor of C_n^k with parameters $d, n_1 = 1, n_2$, and n_3, where $n_3 = p \,(\mathrm{mod}\,(k-d))$ and $1 \le p < k - d$, the previous lemma implies that the corresponding r-minor inequality can be written as

$$\sum_{i \in W} x_i + r \sum_{i=1}^{n} x_i \ge \alpha(d, p) + \beta(d) \, |W|$$

where

$$\alpha(d, p) = r\frac{n}{k} + \left(\left\lceil \frac{rp}{k-d} \right\rceil - \frac{rp}{k-d} \right), \quad \beta(d) = \frac{r}{k(k-d)},$$

or, equivalently,

$$\sum_{i \in W} (x_i - \beta(d)) \ge \alpha(d, p) - r \sum_{i=1}^{n} x_i. \tag{6}$$

Given C_n^k and two integer numbers d, p with $1 \le d \le k - 2$ and $1 \le p < k - d$, we define the function c^d on $\mathbb{R}$ by $c^d(t) := t - \beta(d)$ and the function $L^{d,p}$ on $\mathbb{R}^n$ by $L^{d,p}(x) := \alpha(d, p) - r \sum_{i=1}^{n} x_i$.

Then, inequality (6) can be written as

$$\sum_{i \in W} c^d(x_i) \geq L^{d,p}(x). \tag{7}$$

Following the same notation introduced in [7], let $\mathcal{W}(d,p)$ be the family of sets $W \subset [n]$ defining minors with parameters $d, n_1 = 1, n_2, n_3 = p \, (\mathrm{mod} \, (k - d))$. We are interested in the separation of generalized r-minor inequalities corresponding to sets $W \in \mathcal{W}(1,p)$.

To this end, given n, k let $K_n^k(p) = (V, A)$ be the digraph with set of nodes

$$V = \bigcup_{j \in [k-1]} V^j \cup \{t\}$$

where $V^j = \{v_i^j : i \in [n]\}$ and set of arcs defined as follows: first consider in A the arcs

- (v_1^1, v_l^2) for all l such that $k + 2 \leq l \leq n$ and $l = 2 \, (\mathrm{mod} \, k)$,

then consider in a recursive way for $j \in [k-1]$:

- for each $(v, v_i^j) \in A$, add (v_i^j, v_l^{j+1}) whenever l is such that $i + k + 1 \leq l \leq n$ and $l - i = 1 \, (\mathrm{mod} \, k)$,

and finally,

- for each $(v, v_i^p) \in A$, add (v_i^p, t) whenever i is such that $i \leq n - k$ and $n - i = 0 \, (\mathrm{mod} \, k)$.

Note that, by construction, $K_n^k(p)$ is acyclic. In Figure 1 we sketch the digraph $K_{59}^4(2)$ where only the arcs corresponding to a particular $v_1^1 t$-path are drawn.

For the proof of the next lemma, we need the following observation.

Remark 4.2. *Let $W = \{w_i : i \in [n_3]\}$ with $1 \leq w_1 < \cdots < w_{n_3} \leq n$, then W defines a circulant minor with parameters $d = n_1 = 1$ if and only if $w_{i+1} - w_i = 1 \, (mod \, k)$ and $w_{i+1} - w_i \geq k + 1$, for all $i \in [n_3]$.*

Now we can state our result.

Lemma 4.3. *There is a one-to-one correspondence between $v_1^1 t$-paths in $K_n^k(p)$ and subsets $W \in \mathcal{W}(1,p)$ with $1 \in W$.*
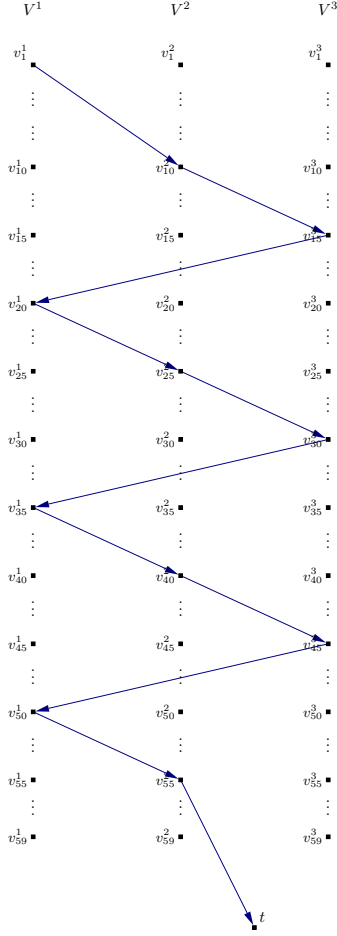
Figure 1: A $v_1^1 t$-path in the digraph $K_{59}^4(2)$.

Proof. Let $W \in \mathcal{W}(1, p)$ and assume that $W = \{i_j : j \in [n_3]\} \subset [n]$ with $1 = i_1 < i_2 < \ldots < i_{n_3} \le n$.

Then, by Remark 4.2, $i_{j+1} - i_j = 1 \pmod{k}$ and $i_{j+1} - i_j \ge k + 1$ for all $j \in [n_3]$. Thus, the set of nodes $P \cup \{t\}$ with

$$P := \left\{ v_{i_j}^\ell : i_j \in W, \ \ell = j \pmod{(k-1)} \right\}$$

induces a $v_1^1 t$-path in $K_n^k(p)$.

Conversely, let P be a $v_1^1 t$-path in $K_n^k(p)$. By construction, there exists a nonnegative integer s such that $|V(P) \cap V^j| = s + 1$ for all $j \in \{1, \ldots, p\}$ and $|V(P) \cap V^j| = s$ for $j \in \{p+1, \ldots, k-1\}$. Then, $|V(P) - \{t\}| = s(k-1) + p$.

Now, if we define

$$W = \left\{ i \in [n] : v_i^j \in V(P) \text{ for some } j \in [k-1] \right\}$$

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we have $|W| = s(k-1) + p$ and from Remark 4.2 it follows that $W \in \mathcal{W}(1,p)$. $\qquad\square$

Theorem 4.4. *Given C_n^k and $r \in \{1, \ldots, k-1\}$, the separation problem for r-minor inequalities corresponding to minors with parameters $d = n_1 = 1, n2$, and $n_3 = p \,(\mathrm{mod}\,(k-1))$ can be polynomially reduced to at most n shortest path problems in an acyclic digraph.*

Proof. Let $\hat{x} \in \mathbb{R}^n$. We will show that the problem of deciding if, given $j \in [n]$, there exists $W \in \mathcal{W}(1,p)$ with $j \in W$ and such that $\hat{x}$ violates the inequality (7) can be reduced to a shortest path problem. W.l.o.g we set $j = 1$.

Consider the digraph $K_n^k(p)$ and associate the cost $c^1(\hat{x}_i)$ with every arc $(v_l^j, v_i^{j+1}) \in A$, and the cost $c^1(\hat{x}_1)$ with every arc $(v_l^p, t) \in A$.

Clearly, if W is the subset of $[n]$ corresponding to a $v_1^1 t$-path P in $K_n^k(p)$, the length of P is equal to $\sum_{i \in W} c^1(\hat{x}_i)$.

Then, there exists $W \in \mathcal{W}(1,p)$ with $1 \in W$ and such that $\hat{x}$ violates the inequality (7), if and only if the length of the shortest $v_1^1 t$-path in $K_n^k(p)$ is less than $L^{1,p}(\hat{x})$. $\qquad\square$

Since $K_n^k(p)$ is acyclic, computing each shortest path from the last theorem can be accomplished in $O(|A|)$ time (see, e.g., [10, Theorem 2.18]). Moreover, from the definition of $K_n^k(p)$ it follows that each node has outdegree of order $O(\frac{n}{k})$ and that the graph contains $O(nk)$ nodes. Hence, each shortest path computation requires $O(n^2)$ time, and the separation problem of r-minor inequalities can be solved in $O(n^3)$ time for fixed $p, r \in \{1, \ldots, k-1\}$. Repeating this procedure for each possible value of p and r, we obtain the following result.

Theorem 4.5. *For a fixed k, the separation problem for r-minor inequalities corresponding to minors of C_n^k with parameters $d = n_1 = 1$ can be solved in polynomial time.*

5   Conclusions

We have presented a new class of valid inequalities for $Q^*(C_n^k)$ whose Chvátal rank is at most one. These inequalities strictly generalize the class of minor inequalities described in [7]. Moreover, some of these inequalities give rise to new facets of $Q^*(C_n^k)$, as shown by the example at the end of Section 3. Hence, despite of the results obtained for $Q^*(C_n^3)$ in [9] and $Q^*(C_{2k}^k), Q^*(C_{3k}^k)$ in [8], (classic) minor inequalities, together with boolean facets and the rank

constraint are not sufficient to provide a complete linear description of $Q^*(C_n^k)$ in general.

An apparently weaker problem consists in finding a complete linear description for the first Chvátal closure of $Q(C_n^k)$. However, as far as we are aware from literature, no circulant matrix C_n^k is known for which the Chvátal rank of $Q(C_n^k)$ is strictly larger than one. To complete the characterization of $Q'(C_n^k)$ following the path presented here, more work is still needed in order to fully characterize integral generating sets for the cones $K(x^*)$ associated with the fractional vertices defined in Lemma 3.2(iii). All computational experiments we have conducted so far support the conjecture that generalized minor inequalities, together with boolean facets and the rank constraint are sufficient for describing $Q'(C_n^k)$.

Besides of the search for complete linear descriptions of $Q'(C_n^k)$ and $Q^*(C_n^k)$, one line of future research could be the study of necessary and sufficient conditions for a generalized minor inequality to define a facet of the integer polytope, similar to the conditions presented in [7] for classic minor inequalities. Moreover, the separation problem for generalized minor inequalities corresponding to minors with parameters $d > 1$ or $n_1 > 1$ is an issue that requires further investigation.

Another possible line of research involves establishing analogies between set packing and set covering polyhedra. As mentioned in the introduction, a complete linear description for set packing polytope $P^*(C_n^k)$ related to circulant matrices (in fact, to the more general class of *circular* matrices) has been reported in [13]. The authors show that this polytope is described by nonnegativity constraints, clique inequalities and so-termed *clique family* inequalities. The first two families can be regarded as counterparts of boolean facets for the set covering polyhedron. Determining whether there is also an analogous class to clique family inequalities in the case of $Q^*(C_n^k)$, and how this class is related to the class of generalized minor inequalities could shed more light on the structure of this polyhedron. For instance, similarly to generalized minor inequalities, clique family inequalities have only two different coefficients, which are consecutive integers. First steps in this direction have been undertaken in [3, 4].

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