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Characterizing N_+-perfect line graphs

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Abstract

The aim of this paper is to study the Lovász-Schrijver PSD operator N_+ applied to the edge relaxation of the stable set polytope of a graph. We are particularly interested in the problem of characterizing graphs for which N_+ generates the stable set polytope in one step, called N_+-perfect graphs. It is conjectured that the only N_+-perfect graphs are those whose stable set polytope is described by inequalities with near-bipartite support. So far, this conjecture has been proved for near-perfect graphs, fs-perfect graphs, and webs. Here, we verify it for line graphs, by proving that in an N_+-perfect line graph the only facet-defining subgraphs are cliques and odd holes.

Keywords: stable set polytope; N_+-perfect graphs; line graphs; PSD relaxation

1. Introduction

The context of this paper is the study of the stable set polytope. Our focus lies on N_+-perfect graphs: those graphs where a single application of the Lovász-Schrijver positive semidefinite (PSD) operator N_+ to the edge relaxation yields the stable set polytope.

The "stable set polytope" STAB(G) of a graph $G = (V, E)$ is defined as the convex hull of the incidence vectors of all stable sets of G (in a stable set all nodes are mutually nonadjacent).

Two canonical relaxations of STAB(G) are the "fractional" or "edge constraint stable set polytope"

$$\mathrm{ESTAB}(G) = \left\{ \mathbf{x} \in \mathbf{R}_+^{|V|} : x_i + x_j \le 1, ij \in E \right\},$$

and the "clique constraint stable set polytope"

$$\mathrm{QSTAB}(G) = \left\{ \mathbf{x} \in \mathbf{R}_+^{|V|} : \sum_{i \in Q} x_i \le 1, \ Q \subseteq V \text{ clique} \right\}$$

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(in a clique, all nodes are mutually adjacent, hence a clique and stable set share at most one node). We have

$$\mathrm{STAB}(G) \subseteq \mathrm{QSTAB}(G) \subseteq \mathrm{ESTAB}(G)$$

for any graph, where STAB(G) equals ESTAB(G) for bipartite graphs, and QSTAB(G) for perfect graphs only (Chvátal, 1975).

According to a famous characterization achieved by Chudnovsky et al. (2006), perfect graphs are precisely the graphs without chordless cycles C_{2k+1} with $k \geq 2$, termed as "odd holes," or their complements, the "odd antiholes" $\overline{C}_{2k+1}$ (the complement $\overline{G}$ has the same nodes as G, but two nodes are adjacent in $\overline{G}$ if and only if they are nonadjacent in G).

Perfect graphs turned out to be an interesting and important class with a rich structure and nice algorithmic behavior (Grötschel et al., 1988). However, solving the stable set problem for a perfect graph G by maximizing a linear objective function over QSTAB(G) does not work directly (Grötschel et al., 1981), but only via a detour involving a geometric representation of graphs (Lovász, 1979) and the resulting "theta-body" TH(G) introduced in Grötschel et al. (1988).

An orthonormal representation of a graph $G = (V, E)$ is a sequence $(\mathbf{u_i} : i \in V)$ of unit-length vectors $\mathbf{u_i} \in \mathbf{R}^N$, where N is some positive integer, such that $\mathbf{u_i}^T \mathbf{u_j} = 0$ for all $ij \notin E$. For any orthonormal representation of G and any additional unit-length vector $\mathbf{c} \in \mathbf{R}^N$, the corresponding orthonormal representation constraint is $\sum_{i \in V} (\mathbf{c}^T \mathbf{u_i})^2 x_i \leq 1$. TH$(G)$ denotes the convex set of all vectors $\mathbf{x} \in \mathbf{R}_+^{|V|}$ satisfying all orthonormal representation constraints for G. For any graph G, we have

$$\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{QSTAB}(G)$$

and approximating a linear objective function over TH(G) can be done with arbitrary precision in polynomial time (Grötschel et al., 1988). Most notably, the same authors proved a beautiful characterization of perfect graphs:

$$\begin{aligned} G \text{ is perfect} &\Leftrightarrow \mathrm{TH}(G) = \mathrm{STAB}(G) \\ &\Leftrightarrow \mathrm{TH}(G) = \mathrm{QSTAB}(G) \\ &\Leftrightarrow \mathrm{TH}(G) \text{ is polyhedral,} \end{aligned} \tag{1}$$

which even shows that optimizing a linear function on polyhedral TH(G) can be done in polynomial time.

For all imperfect graphs G, it follows that STAB(G) does not coincide with any of the above relaxations. It is, thus, natural to study further relaxations and combinatorially characterize those graphs where STAB(G) equals one of them.

1.1. A linear relaxation and rank-perfect graphs

Rank-perfect graphs are introduced in Wagler (2000) in order to obtain a superclass of perfect graphs in terms of a further linear relaxation of STAB(G). As natural generalization of the clique constraints describing QSTAB(G), we consider rank constraints

$$\mathbf{x}(G') = \sum_{i \in G'} x_i \leq \alpha(G')$$

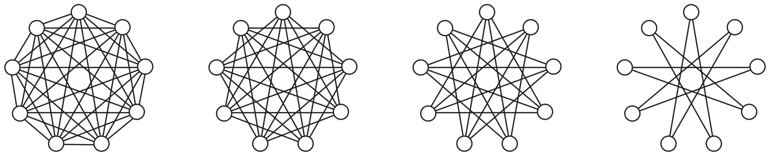

Fig. 1. The antiwebs A_9^k.

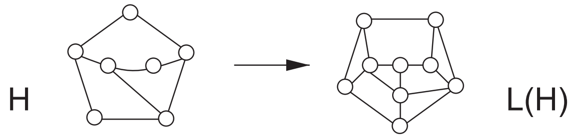

Fig. 2. A graph and its line graph.

associated with arbitrary induced subgraphs $G' \subseteq G$. By the choice of the right-hand side $\alpha(G')$, denoting the size of the largest stable set in G', rank constraints are obviously valid for STAB(G). The "rank constraint stable set polytope"

$$\mathrm{RSTAB}(G) = \left\{ \mathbf{x} \in \mathbf{R}^{|V|} : \sum_{i \in G'} x_i \ \leq \ \alpha(G'),\ G' \subseteq G \right\}$$

is a further linear relaxation of STAB(G). As clique constraints are special rank constraints (namely exactly those with $\alpha(G') = 1$), we immediately obtain

$$\mathrm{STAB}(G) \subseteq \mathrm{RSTAB}(G) \subseteq \mathrm{QSTAB}(G).$$

A graph G is "rank perfect" by Wagler (2000) if and only if STAB$(G) =$ RSTAB(G) holds. By definition, rank-perfect graphs include all perfect graphs (where rank constraints associated with cliques suffice). In general, by restricting the facet set to rank constraints associated with certain subgraphs only, several well-known graph classes are defined. For example, a graph is "near-perfect" (Shepherd, 1994) if the only rank constraints are associated with cliques and the whole graph. In the same line, a graph is "t-perfect" (Chvátal, 1975) ("h-perfect"; Grötschel et al., 1988) if rank constraints are associated with edges, triangles, and odd holes (cliques of arbitrary size and odd holes, respectively). Further classes of rank-perfect graphs are antiwebs (Wagler, 2004) and line graphs (Edmonds, 1965; Edmonds and Pulleyblank, 1974).

An "antiweb" A_n^k is a graph with n nodes $0, \ldots, n-1$ and edges ij if and only if $k \leq |i - j| \leq n - k$ and $i \neq j$. Antiwebs include all complete graphs $K_n = A_n^1$, all odd holes $C_{2k+1} = A_{2k+1}^k$, and their complements $\overline{C}_{2k+1} = A_{2k+1}^2$ (e.g., see Fig. 1).

As common generalization of perfect, t-perfect and h-perfect graphs as well as antiwebs, the class of "a-perfect graphs" was introduced in Wagler (2005) as those graphs whose stable set polytopes are provided by nonnegativity constraints and rank constraints associated with antiwebs only. Antiwebs are a-perfect by Wagler (2004), further examples of a-perfect graphs were presented in Wagler (2005).

A "line graph" is obtained by taking the edges of a graph as nodes and connecting two nodes if and only if the corresponding edges are incident (for illustration, see Fig. 2). Since matchings of the original graph correspond to stable sets of the line graph, the results on the matching polytope by Edmonds (1965), and Edmonds and Pulleyblank (1974) imply that line graphs are also rank perfect (for details, see Section 3).

1.2. A semidefinite relaxation and N_+-perfect graphs

In the early nineties, Lovász and Schrijver (1991) introduced the PSD operator N_+ that, applied over the edge relaxation ESTAB(G), generates the positive semidefinite relaxation $N_+(G)$ stronger than TH(G). We denote by $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n$ the vectors of the canonical basis of $\mathbf{R}^{n+1}$ (where the first coordinate is indexed zero), $\mathbf{1}$ is the vector with all components equal to 1, and S_+^n the space of $(n \times n)$ symmetric and positive semidefinite matrices with real entries. Given a convex set K in $[0, 1]^n$, let

$$\text{cone}(K) = \left\{ \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \in \mathbf{R}^{n+1} : \mathbf{x} = x_0 \mathbf{y}; \quad \mathbf{y} \in K \right\}.$$

Then, we define the set $M_+(K) =$

$$\begin{aligned} \{ Y \in S_+^{n+1} : \; & Y\mathbf{e}_0 = \text{diag}(Y), \\ & Y\mathbf{e}_i \in \text{cone}(K), \\ & Y(\mathbf{e}_0 - \mathbf{e}_i) \in \text{cone}(K), \; i = 1, \ldots, n \}, \end{aligned}$$

where diag(Y) denotes the vector whose ith entry is Y_{ii}, for every $i = 0, \ldots, n$. Projecting this lifting back to the space $\mathbf{R}^n$ results in $N_+(K) =$

$$\left\{ \mathbf{x} \in [0, 1]^n : \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} = Y\mathbf{e}_0, \text{ for some } Y \in M_+(K) \right\}.$$

Lovász and Schrijver (1991) proved that $N_+(K)$ is a relaxation of the convex hull of integer solutions in K and $N_+^n(K) = \text{conv}(K \cap \{0, 1\}^n)$, where $N_+^0(K) = K$ and $N_+^k(K) = N_+(N_+^{k-1}(K))$ for $k \geq 1$. In this work, we focus on the behavior of a single application of the N_+ operator to the edge relaxation of the stable set polytope of a graph.

In order to simplify the notation, we write $N_+(G) = N_+(\text{ESTAB}(G))$. In Lovász and Schrijver (1991), it is shown that

$$\text{STAB}(G) \subseteq N_+(G) \subseteq \text{TH}(G) \subseteq \text{QSTAB}(G).$$

Similar to the case of perfect graphs, the stable set problem can be solved in polynomial time for the class of graphs for which $N_+(G) = \text{STAB}(G)$ (Grötschel et al., 1981, 1988). We will call these graphs "N_+-perfect." A graph G that is not N_+-perfect is called "N_+-imperfect."

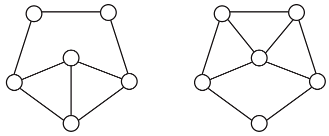

Fig. 3. The graphs G_{LT} (on the left) and G_{EMN} (on the right).

In Bianchi et al. (2011), the authors consider the characterization of N_+-perfect graphs similar to the characterizations of perfect graphs provided in (1). More precisely, they intend to find an appropriate polyhedral relaxation $P(G)$ of $\mathrm{STAB}(G)$ such that

$$\begin{aligned} G \text{ is } N_+ - \text{perfect} &\Leftrightarrow N_+(G) = \mathrm{STAB}(G) \\ &\Leftrightarrow N_+(G) = P(G). \end{aligned} \tag{2}$$

Following this line, the following conjecture has recently been proposed in Bianchi et al. (2013).

Conjecture 1 (Bianchi et al., 2013). *(N_+-perfect graph conjecture) The stable set polytope of every N_+-perfect graph can be described by facet-defining inequalities with near-bipartite support.*

"Near-bipartite" graphs, defined in Shepherd (1995), are those graphs in which removing all neighbors of an arbitrary node and the node itself leaves the resulting graph bipartite. Antiwebs and complements of line graphs are examples of near-bipartite graphs. Again from results in Lovász and Schrijver (1991), we know that graphs for which every facet-defining inequality of $\mathrm{STAB}(G)$ has a near-bipartite support is N_+-perfect. Thus, Conjecture 1 states that these graphs are the only N_+-perfect graphs. In particular, near-bipartite and a-perfect graphs are N_+-perfect.

In addition, it can be proved that every subgraph of an N_+-perfect graph is also N_+-perfect. This motivates the definition of "minimally N_+-imperfect graphs" as these N_+-imperfect graphs whose proper induced subgraphs are all N_+-perfect.

In Escalante et al. (2006) and Lipták and Tunçel (2003), it was proved that all the imperfect graphs with at most six nodes are N_+-perfect, except the two imperfect near-perfect graphs depicted in Fig. 3. The graph on the left is denoted by G_{LT} and the other graph is denoted by G_{EMN}. So, G_{LT} and G_{EMN} are the two smallest minimally N_+-imperfect graphs. Characterizing all minimally N_+-imperfect graphs within a certain graph class can be a way to attack Conjecture 1 for this class. Thus, Conjecture 1 has been already verified for near-perfect graphs by Bianchi et al. (2011), fs-perfect graphs (where the only facet-defining subgraphs are cliques and the graph itself) by Bianchi et al. (2013), and webs (the complements of antiwebs) by Escalante and Nasini (2014).

In this paper, we verify Conjecture 1 for line graphs. For this purpose, we present three infinite families of N_+-imperfect line graphs (Section 2) and show that all facet-defining subgraphs of a line graph different from cliques and odd holes contain one of these N_+-imperfect line graphs (Section 3). Finally, we note that the graphs in the three presented families are minimally N_+-imperfect and, in fact, the only minimally N_+-imperfect line graphs. Finally, we provide some concluding remarks and lines of further research.

2. Three families of N_+-imperfect line graphs

In this section, we provide three infinite families of N_+-imperfect line graphs. For this purpose, we apply an operation preserving N_+-imperfection to the two smallest N_+-imperfect graphs G_{LT} and G_{EMN} (note that both graphs are line graphs).

In Lipták and Tunçel (2003), the "stretching" of a node v is introduced as follows: Divide its neighborhood $N(v)$ into two nonempty, disjoint sets A_1 and A_2 (so that $A_1 \cup A_2 = N(v)$ and $A_1 \cap A_2 = \emptyset$). A stretching of v is obtained by replacing v by two adjacent nodes v_1 and v_2, joining v_i with every node in A_i for $i \in \{1, 2\}$, and either subdividing the edge $v_1 v_2$ by one node w or subdividing every edge between v_2 and A_2 with one node. Moreover, Lipták and Tunçel (2003) show that the stretching of a node preserves N_+-imperfection.

For this purpose, we will use the stretching of node v in the case of subdividing the edge $v_1 v_2$ by one node w. If $|A_1| = 1$ or $|A_2| = 1$, the stretching corresponds to the "3-subdivision" of an edge (i.e., when an edge is replaced by a path of length 3).

Next, we establish a connection between subdivisions of edges in a graph H and stretchings of nodes in its line graph $L(H)$. Let G be the line graph of H and consider an edge $e = u_1 u_2$ in H together with its corresponding node v in G. If e is a simple edge of H (i.e., if there is no edge parallel to e in H), then the neighborhood $N(v)$ of its corresponding node v in G is divided into two cliques, U_1 and U_2, representing the edges in H incident to e in u_1 and u_2, respectively. Accordingly, we call a stretching of a node v in a line graph "canonical," if these cliques U_1 and U_2 are selected as the partition of $N(v)$.

For illustration, see Fig. 4 that shows the graph $C_5 + c$ (a 5-hole with one chord c), graph $C_5 + E_3$ (obtained from $C_5 + c$ by subdividing c into a path E_3 of length 3), and their line graphs, where $L(C_5 + E_3)$ results from $L(C_5 + c)$ by a canonical stretching of the node corresponding to c.

In fact, we have in general:

Lemma 2. *Let e be a simple edge in H and v be the corresponding node in its line graph G. If H' is the graph obtained after a 3-subdivision of e in H then $L(H')$ is the canonical stretching of v in G.*

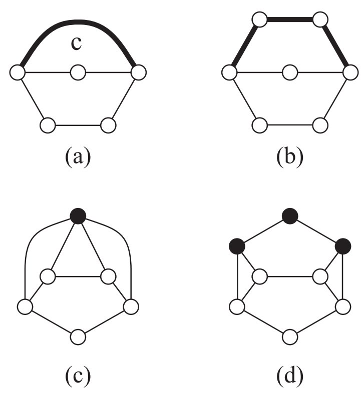

Fig. 4. This figure shows (a) the graph $C_5 + c$ (a 5-hole with one chord c), (b) the graph $C_5 + E_3$ (obtained from $C_5 + c$ by a 3-subdivision of c), (c) the line graph $L(C_5 + c)$, and (d) the line graph $L(C_5 + E_3)$, where $L(C_5 + E_3)$ results from $L(C_5 + c)$ by a canonical stretching of the (black-filled) node corresponding to c.

Proof. Let $e = u_1u_2$ be a simple edge of H and H' be the graph obtained from H by replacing e by the path u_1, u_1', u_2', u_2.

The line graph $L(H')$ contains a node v_1 representing the edge u_1u_1' of H', a node w corresponding to the edge $u_1'u_2'$ of H', and a node v_2 for the edge $u_2'u_2$ of H'.

In $L(H')$, v_1 is adjacent to w and to a clique U_1 corresponding to all edges different from u_1u_1' incident to u_1, w has only v_1 and v_2 as neighbors, and v_2 is adjacent to w and to a clique U_2 corresponding to all edges different from $u_2'u_2$ incident to u_2.

All other nodes and adjacencies in $L(H')$ are same as in $L(H)$, hence $L(H')$ corresponds exactly to the graph obtained from $L(H)$ by the canonical stretching of the node v representing the edge e. □

This enables us to show the following theorem.

Theorem 3. *A line graph $L(H)$ is N_+-imperfect if H is*

- *an odd hole with one double edge,*
- *an odd hole with one chord,*
- *an odd hole with one odd path attached to nonadjacent nodes of the hole.*

Proof. Let $C_{2k+1} + d$ (resp. $C_{2k+1} + c$, resp. $C_{2k+1} + E_\ell$) denote the graph obtained from an odd hole C_{2k+1} with $k \geq 2$ by adding one edge d parallel to an edge of the hole (resp. adding one chord c to the hole, resp. attaching one path of length ℓ to two nonadjacent nodes of the hole).

To establish the N_+-imperfection of the three families, we first observe that the two minimally N_+-imperfect graphs G_{LT} and G_{EMN} are line graphs: indeed, we have $G_{LT} = L(C_5 + d)$ and $G_{EMN} = L(C_5 + c)$.

Clearly, the graph $C_{2k+3} + d$ can be obtained from $C_{2k+1} + d$ by 3-subdivision of a simple edge (not being parallel to d) of the hole. Thus, any odd hole with one double edge can be obtained from $C_5 + d$ by repeated 3-subdivisions of simple edges.

According to Lemma 2, their line graphs are obtained by repeated canonical stretchings of G_{LT}, which yields the first studied family of graphs.

Analogously, $C_{2k+3} + c$ can be obtained from $C_{2k+1} + c$ by 3-subdivision of an edge of the hole. Thus, any odd hole with one chord can be obtained from $C_5 + c$ by repeated 3-subdivisions of edges different from c. Moreover, applying repeated 3-subdivisions of the chord c yields graphs $C_{2k+1} + E_\ell$, where E_ℓ is a path of arbitrary odd length ℓ attached to two nonadjacent nodes of the hole at arbitrary distance.

According to Lemma 2, their line graphs are obtained by repeated canonical stretchings of G_{EMN}, which yields the two remaining families of graphs. G_{LT} and G_{EMN} are minimally N_+-imperfect and canonical stretchings preserve N_+-imperfection; this completes the proof. □

3. Characterizing N_+-perfect line graphs

A combination of results by Edmonds (1965) and Edmonds and Pulleyblank (1974) about the matching polytope implies the following description of the stable set polytope of line graphs.

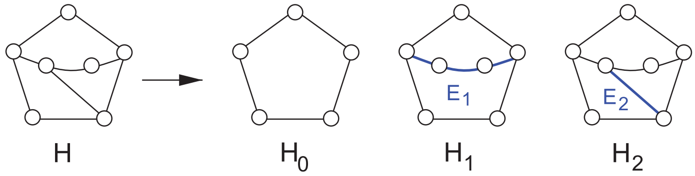

Fig. 5. A graph and an ear decomposition.

Theorem 4 (Edmonds, 1965; Edmonds and Pulleyblank, 1974). *If G is the line graph of a graph H, then* STAB(G) *is described by nonnegativity constraints, maximal clique constraints, and rank constraints*

$$x(L(H')) \leq \frac{|V(H')| - 1}{2} \tag{3}$$

associated with the line graphs of 2-connected hypomatchable induced subgraphs $H' \subseteq H$.

A graph H is "hypomatchable" if, for all nodes v of H, the subgraph $H - v$ admits a perfect matching (i.e., a matching meeting all nodes) and is "2-connected" if it remains connected after removing an arbitrary node.

Due to the result in Lovász (1972), a graph H is hypomatchable if and only if there is a sequence $H_0, H_1, \ldots, H_k = H$ of graphs such that H_0 is a chordless odd cycle, and for $1 \leq i \leq k$, H_i is obtained from H_{i-1} by adding an odd path E_i that joins two (not necessarily distinct) nodes of H_{i-1} and has all internal nodes outside H_{i-1}. The odd paths $E_i = H_i - H_{i-1}$ are called "ears" for $1 \leq i \leq k$ and the sequence $H_0, H_1, \ldots, H_k = H$ an "ear decomposition" of H (for illustration, see Fig. 5). Moreover, we call an ear of length at least three "long," and "short" otherwise.

Hypomatchable graphs have an odd number of nodes, are nonbipartite and connected, but neither necessarily 2-connected (since an ear E_i may be attached to a single node of H_{i-1}) nor simple (since a short ear E_i may become an edge parallel to one edge of H_{i-1}).

However, if H is 2-connected, Cornuéjols and Pulleyblank (1983) proved that H admits an ear decomposition $H_0, H_1, \ldots, H_k = H$ with H_i 2-connected for every $0 \leq i \leq k$. If, in addition, H has at least five nodes, Wagler (2000) later proved that H admits an ear decomposition $H_0, H_1, \ldots, H_k = H$, where H_0 has at least five nodes and H_i is 2-connected for every $0 \leq i \leq k$. Since the latter result is a key property for our argumentation, we provide its proof for the sake of completeness.

Lemma 5 (Wagler, 2000). *Let H be a 2-connected hypomatchable graph and $|V(H)| \geq 5$. Then there is an ear decomposition $H_0, H_1, \ldots, H_k = H$ of H such that each H_i is 2-connected and H_0 is an odd cycle of length at least 5.*

Proof. Since H is 2-connected, it admits an ear decomposition $H_0, H_1, \ldots, H_k = H$ with H_i 2-connected for $0 \leq i \leq k$ by Cornuéjols and Pulleyblank (1983). We are ready if H_0 is an odd cycle of length ≥ 5, hence assume, for the sake of contradiction, that H_0 is a triangle.

From $|V(H)| > 3$ follows that there is an ear with at least three edges. Let $i \in \{1, \ldots, k\}$ be the smallest index such that E_i has length ≥ 3. Then $V(H_{i-1}) = V(H_0)$ holds and E_i has two distinct nodes $v, v' \in V(H_0)$ as endnodes (since H_i is 2-connected). Hence, $(H_0 - vv') \cup E_i, vv', E_1, \ldots, E_{i-1}$ is an ear decomposition of H_i starting with an odd cycle of length ≥ 5 and defining only 2-connected intermediate graphs. The ears $E_{i+1}, \ldots, E_k$ complete this ear decomposition to the studied decomposition of H. □

Using these results, we can provide the following characterization of 2-connected hypomatchable graphs.

Theorem 6. *If H is a 2-connected hypomatchable graph, then exactly one of the following conditions is true:*

- *H has only three nodes;*
- *H equals an odd hole;*
- *H contains one of the following subgraphs:*
  - *an odd hole with one double edge;*
  - *an odd hole with one chord;*
  - *an odd hole with one long ear, attached to nonadjacent nodes of the hole.*

Proof. Consider a 2-connected hypomatchable graph H and distinguish the following cases.

If H has more than three nodes, then H admits an ear decomposition $H_0, H_1, \ldots, H_k = H$, where H_0 has at least five nodes and H_i is 2-connected for every $0 \leq i \leq k$ by Wagler (2000).

If $H = H_0$, then H equals an odd hole.

If $H \neq H_0$, then H_1 equals one of the above-mentioned graphs:

- an odd hole with one double edge (if E_1 is a short ear attached to adjacent nodes of H_0);
- an odd hole with one chord (if E_1 is a short ear attached to nonadjacent nodes of H_0 or if E_1 is a long ear attached to adjacent nodes of H_0);
- an odd hole with one long ear E_1, attached to nonadjacent nodes of H_0 (otherwise).

(Recall that E_1 cannot be attached to a single node of H_0 since H_1 is 2-connected.) □

Combining Theorems 3 and 6, we can further prove:

Lemma 7. *If H is a 2-connected hypomatchable graph, then $L(H)$ is either a clique, an odd hole, or N_+-imperfect.*

Proof. Consider a 2-connected hypomatchable graph H and distinguish the following cases.

If H has only three nodes, then H has an ear decomposition $H_0, H_1, \ldots, H_k = H$, where H_0 is a triangle and all ears are short, becoming edges parallel to one edge of H_0. In this case, $L(H)$ is clearly a clique.

If H has more than three nodes, then H admits an ear decomposition $H_0, H_1, \ldots, H_k = H$, where H_0 is an odd hole with at least five nodes and H_i is 2-connected for every $0 \leq i \leq k$ by Wagler (2000).

If $H = H_0$, then H equals an odd hole and $L(H)$ is clearly an odd hole, too.

If $H \neq H_0$, then H_1 equals one of the graphs from proof of Theorem 6:

- an odd hole with one double edge;
- an odd hole with one chord;
- an odd hole with one long ear, attached to nonadjacent nodes of H_0.

According to Theorem 3, the line graph $L(H_1)$ is N_+-imperfect, hence $L(H)$ is N_+-imperfect. □

Combining Lemma 7 and the description of stable set polytopes of line graphs from Theorem 4 further yields the following characterization of N_+-perfect line graphs.

Theorem 8. *A line graph $L(H)$ is N_+-perfect if and only if all 2-connected hypomatchable induced subgraphs $H' \subseteq H$ either have only three nodes or are odd holes.*

Thus, the definition of *h*-perfect graphs finally implies the following characterization of N_+-perfect line graphs.

Corollary 9. *A line graph is N_+-perfect if and only if it is h-perfect.*

Since both class of line graphs and class of N_+-perfect graphs are hereditary (i.e., closed under taking induced subgraphs), we can derive also a characterization of minimally N_+-imperfect line graphs from a characterization of N_+-perfect line graphs.

In fact, combining the statements from Theorem 3, Theorem 4 by Edmonds (1965) and Edmonds and Pulleyblank (1974), Theorems 6 and 8 lead to the following characterization of minimally N_+-imperfect line graphs.

Corollary 10. *A line graph $L(H)$ is minimally N_+-imperfect if and only if H is an odd hole with one ear attached to distinct nodes of the hole.*

Proof. On the one hand, each of the line graphs $L(H)$ with $H \in \{C_{2k+1} + d, C_{2k+1} + c, C_{2k+1} + E_\ell\}$ from Theorem 3 has, by construction, the property that H is a 2-connected hypomatchable graph and admits an ear decomposition $H_0, H_1 = H$ with $H_0 = C_{2k+1}, k \geq 2$.

Removing any edge e from H yields a graph $H - e$, which is either an odd hole (if $e \in \{c, d\}$) or else not hypomatchable anymore. In all cases, $H - e$ is bipartite or contains at most one odd cycle so that $L(H - e)$ has cliques and odd holes as only facet-defining subgraphs.

Hence, $L(H)$ is N_+-imperfect by Theorem 3, but all proper induced subgraphs are *h*-perfect (and, thus, N_+-perfect). This implies that $L(H)$ is minimally N_+-imperfect for all graphs within the three studied families.

On the other hand, any minimally N_+-imperfect graph needs to have a full-support facet. Thus, any minimally N_+-imperfect line graph $L(H)$ is the line graph of a 2-connected hypomatchable graph H (by Theorem 4), which has more than three nodes and is different from an odd hole (by Theorem 6).

Then H admits an ear decomposition $H_0, H_1, \ldots, H_k = H$ with $k \geq 1$ where H_0 has at least five nodes and H_i is 2-connected for every $0 \leq i \leq k$ by Wagler (2000).

We conclude that $k = 1$ holds: since H_1 is 2-connected, $H_1 \in \{C_{2k+1} + d, C_{2k+1} + c, C_{2k+1} + E_\ell\}$ follows and, therefore, $L(H_1)$ is N_+-imperfect by Theorem 3. Also, since $L(H_1)$ is a node-induced subgraph of $L(H)$ and $L(H)$ is minimally N_+-imperfect, $L(H) = L(H_1)$.

Finally, H_1 has the stated property: it is an odd hole H_0 with one ear attached to distinct nodes of H_0. □

4. Conclusion and further results

In this paper, we addressed the problem of verifying Conjecture 1 for line graphs. For this purpose, we presented three infinite families of N_+-imperfect line graphs (Section 2) and showed that all facet-defining subgraphs of a line graph different from cliques and odd holes contain one of these N_+-imperfect line graphs (Section 3). Since cliques and odd holes are clearly near-bipartite, Corollary 9 shows that Conjecture 1 is true for line graphs.

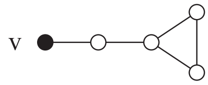

Fig. 6. A perfect (and, thus, joined a-perfect) graph with a node v such that removing v and its neighbor(s) leaves a nonbipartite graph.

In the following, we will discuss a reformulation of Conjecture 1.

As superclass of a-perfect graphs, "joined a-perfect graphs" were introduced in Coulonges et al. (2009) as those graphs whose only facet-defining subgraphs are complete joins of a clique and prime antiwebs (an antiweb A_n^k is "prime" if $k+1$ and n are relatively prime integers). The inequalities obtained from complete joins of antiwebs, called "joined antiweb constraints," are of the form

$$\sum_{i \le k} \frac{1}{\alpha(A_i)} x(A_i) + x(Q) \le 1,$$

where $A_1, \ldots, A_k$ are different antiwebs and Q is a clique (note that the inequalities are scaled to have the right-hand side equal to 1). We denote by $\mathrm{ASTAB}^*(G)$ the linear relaxation of $\mathrm{STAB}(G)$ obtained by all joined antiweb constraints. Then, a graph G is joined a-perfect if $\mathrm{STAB}(G)$ equals $\mathrm{ASTAB}^*(G)$.

In particular, Shepherd (1995) showed that the stable set polytope of a near-bipartite graph has only facet-defining inequalities associated with complete joins of a clique and prime antiwebs. Thus, every near-bipartite graph is joined a-perfect (but the converse is not true since there exist perfect graphs that are not near-bipartite, e.g., see Fig. 6).

Moreover, Conjecture 1 identifies N_+-perfect graphs and graphs for which its stable set polytope can be described by inequalities with near-bipartite support. It is known that, given a graph G, every facet-defining inequality of $\mathrm{STAB}(G)$ with support graph G' is a facet-defining inequality of $\mathrm{STAB}(G')$. Then, again by Shepherd's results (Shepherd, 1995), those graphs for which its stable set polytope can be described by inequalities with near-bipartite support are joined a-perfect graphs.

Taking this into account, Conjecture 1 can be reformulated as follows.

Conjecture 11. *Every N_+-perfect graph is joined a-perfect.*

The results of Lovász and Schrijver (1991) prove that joined a-perfect graphs are N_+-perfect, thus, the conjecture states that both graph classes coincide and $\mathrm{ASTAB}^*(G)$ shall be the studied polyhedral relaxation $P(G)$ of $\mathrm{STAB}(G)$ in (2).

In particular, in this paper we have proved that every N_+-perfect line graph is h-perfect. Then, combining these results, we obtain that a line graph is joined a-perfect if and only if it is h-perfect. However, it seems natural to consider a proof of the latter result independent of the N_+-operator.

Theorem 12. *A line graph is joined a-perfect if and only if it is h-perfect.*

Proof. A joined a-perfect graph has as only facet-defining subgraphs complete joins of a clique and prime antiwebs.

Following the same argumentation as in Shepherd (1995) that odd antiholes are the only prime antiwebs in complements of line graphs, we see that odd holes are the only prime antiwebs in line graphs.

Analogous arguments, as in Wagler (2004, 2005), yield that in a line graph, no complete join of a clique and odd holes or of two or more odd holes can occur: every such complete join would particularly contain an odd wheel W_{2k+1}, that is, the complete join of a single node and an odd hole C_{2k+1}.

The W_5 is one of the minimal forbidden subgraphs of line graphs by Beineke (1968), larger odd wheels contain a claw, another minimal forbidden subgraph of line graphs by Beineke (1968).

Thus, the only remaining facet-defining subgraphs in a joined a-perfect line graph are cliques and odd holes.

Conversely, an h-perfect line graph is clearly joined a-perfect. □

Our future lines of further research include:

- to look for new families of graphs where the conjecture holds (e.g., by characterizing the minimally N_+-imperfect graphs within the class);
- to find new subclasses of N_+-perfect or joined a-perfect graphs.

In all cases, the structural results would have algorithmic consequences since the stable set problem could be solved in polynomial time for the whole class or its intersection with N_+-perfect or joined a-perfect graphs by optimizing over $N_+(G)$.

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