Performance of multivariate process capability indices under normal and non-normal distributions.

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Abstract.

In the context of process capability analysis, the results of most processes are dominated by two or even more quality characteristics, so that the assessment of process capability requires that all of them are considered simultaneously. In recent years, many researchers have developed different alternatives of multivariate capability indices using different approaches of construction.

In this paper, four of them are compared through the study of their ability to correctly distinguish capable processes from incapable processes under a diversity of simulated scenarios, defining suitable minimum desirable values that allow decide whether the process meets or does not meet specifications. In this sense, properties analyzed can be seen as sensitivity and specificity, assuming that a measure is sensitive if it can detect the lack of capability when it actually exists, and specific if it correctly identifies capable processes. Two indices based on ratios of regions and two based on the principal component analysis have been selected for the study. The scenarios take into account several joint distributions for the quality variables, normal and non-normal, several numbers of variables and different levels of correlation between them, covering a wide range of possible situations.

The results showed that one of the indices has better properties across most scenarios, leading to right conclusions about the state of capability of processes and making it a recommendable option for its use in real-world practice.

1. Introduction.

Over the past thirty years, a wide variety of measures have been developed to compare the actual performance of a process with its specifications; these measures are called capability indices.

In most situations, the process under study is assumed to have only one characteristic of interest. For such situations there are plenty of available capability indices. However, the success of a process is most commonly tied to two or more characteristics. In these cases, they need to be evaluated simultaneously in order to assess the global capability of the process. This is currently possible thanks to the advances in multivariate analysis techniques that have occurred in recent decades. The advantage of this treatment compared with the usual approach that analyze individually each variable, is that the correlation structure among variables could be incorporated and taken into account in the analysis.

Depending on the approach adopted to build multivariate capability indices, the proposals can be classified in four groups: 1) indices based on ratios of a tolerance region to a process region (Hubele *et al.*[1], Taam *et al.*[2], Shahriari *et al.*[3,4]); 2) indices defined making use of the principal component analysis (Wang *et al.*[5,6], Xekalaki and Perakis[7], Shinde and Khadse[8], Tano and Vännman[9]); 3) indices based on the probability of producing items out of specifications (Pal[10], Chen[11]); 4) other approaches such as those based on extensions of univariate capability indices (Chen *et al.*[12]) and the proposals of multivariate capability vectors (Hubele *et al.*[1], Shahriari *et al.*[3,4]).

Most of these proposals are based on the assumption of multivariate normality for the underlying distribution, hence the regions of natural variation of processes are considered ellipsoidal in shape, not having found in literature studies considering how the indices are affected by moderate departures from this assumption. Besides, most studies comparing the behavior of various indices have done it by setting out the comparisons from a particular perspective, considering real-world situations or proposing simulation studies on very specific scenarios (Wang *et al.*[13], García *et al.*[14], Pan and Lee[15]).

Therefore, the purpose of this research was to carry out a comparative study involving simultaneously some of the available proposals to build capability indices, in its original and modified versions, in order to evaluate their ability to correctly indicate the actual status of the process regarding the imposed specifications, through a diversity of scenarios. The scenarios considered take into account variations in the underlying multivariate distribution of data, the number of variables and the levels of correlation among them.

Hypothetical processes with different states of capability were defined on each scenario. The indices under study were calculated for those processes, noting whether the values lead to correctly conclude about the state of process capability. Although all of the capability indices are measured on a quantitative scale, as a practical rule, it is always defined a cutoff point from which decide whether the process meets or does not meet the specifications. This cutoff point is used to classify processes on a dichotomous scale and in this sense, properties analyzed can be seen as properties of sensitivity and specificity, assuming that a measure is sensitive if it can detect the lack of capability when it actually exists, and it is considered specific if it has the ability to classify a process as capable when it is capable.

Two options among those based on ratios of regions and other two based on the principal component analysis have been selected for the study. The first two are the initial proposal due to W. Taam *et al.*[2], the MC_{pm} index, and the improved version proposed by H. Shahriari and M. Abdollahzadeh[4], the NMC_{pm} index. The principal component based indices considered are the original proposal by F. K. Wang and J. Chen[5], the $MC_{pm}^{(W)}$ index, and the proposal by R. L. Shinde and K. Khadse[8], which is supposed to improve the former, the Mp_2 index.

The following two sections make a brief methodological review of these indices.

2. Indices based on ratios of relative volumes.

Let $\boldsymbol{X}$ be a $px1$ vector representing p quality characteristics from a multivariate normal distribution, with $px1$ mean vector $\boldsymbol{\mu}$ and pxp variance-covariance matrix $\boldsymbol{\Sigma}$. Let define $\boldsymbol{LSL}$, $\boldsymbol{USL}$ and $\boldsymbol{T}$ as the $px1$ vectors containing lower specification limits, upper specification limits and target values for each quality characteristic, respectively. This means that specifications are given individually for each variable.

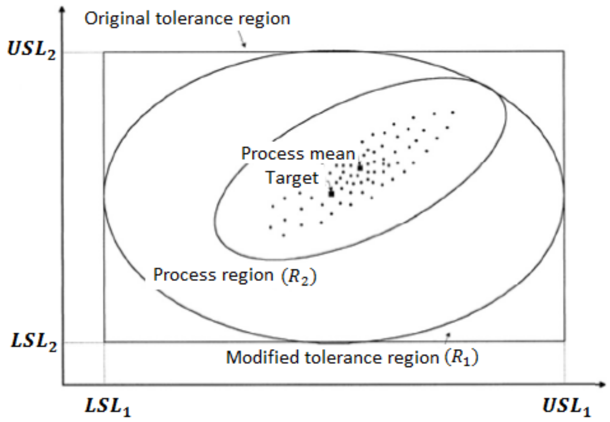
The index proposed by Taam *et al.*[2], MC_{pm}, is defined as a ratio of two volumes. The numerator is the volume of the modified tolerance region (R_1) and the denominator is the volume of the scaled $(1-\alpha)100$ percent process region (scaled by the process mean square error) (R_2).

The tolerance limits of each characteristic taken together form a hyper-rectangular tolerance region in the p-dimensional space of the variables, whereas the process region, under the assumption of multivariate normality, is ellipsoidal in shape. In order to compare both regions, the authors proposed to modify one of them so that both regions have the same geometric shape. These authors' proposal consists in modifying the hyper-rectangular tolerance region considering the largest ellipsoid that is centered at the target and completely within the original tolerance region. Hence, R_1 is the ellipsoid centered in $\boldsymbol{T}$ with semi-axes of length $\frac{(USL_i-LSL_i)}{2}$ $\forall\, i=1,\dots,p$. Meanwhile, region R_2 is represented by the quadratic form $(\boldsymbol{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}_{\boldsymbol{T}}^{-\boldsymbol{1}}(\boldsymbol{X}-\boldsymbol{\mu})\leq k(p)$ where $\boldsymbol{\Sigma}_{\boldsymbol{T}}=E(\boldsymbol{X}-$

$\boldsymbol{T})(\boldsymbol{X}-\boldsymbol{T})'$ is the mean squared error matrix from the process and $k(p)$ is the $(1-\alpha)100$th percentile of χ^2 distribution with p degrees of freedom (Figure 1). The index is then computed as:

$$MC_{pm} = \frac{Vol(R_1)}{Vol(R_2)} = \frac{Vol(modified\ tolerance\ region)}{Vol\big((\boldsymbol{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}_T^{-1}(\boldsymbol{X}-\boldsymbol{\mu}) \leq \chi^2_{1-\alpha,p}\big)} \tag{1}$$

Figure 1: Example of the modified tolerance region for Taam et al.'s method, with $p = 2$.



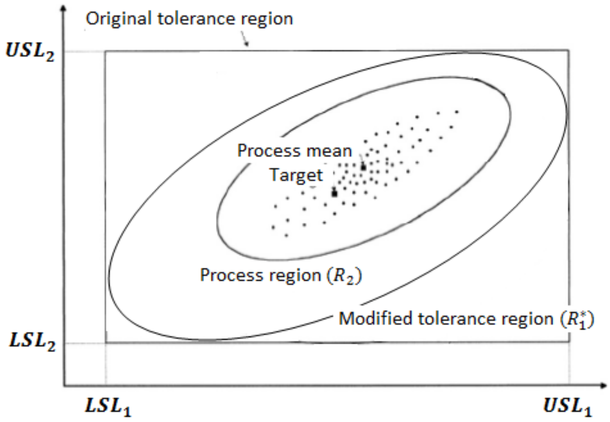
From its construction, this index is a measure of actual, not only potential, process capability. In fact, it is possible to re-write equation (1) as a product of two terms: one representing the process variability relative to the modified tolerance region, and the other reflects the process deviation from target (Taam *et al.*[2]).

From index definition it is deduced that for processes centered at the target and having $100(1-\alpha)\%$ of their values inside the tolerance region, the index takes the value 1.

After Taam *et al.*'s proposal, several authors have suggested modifications to the MC_{pm} index, mainly arguing that the variance-covariance structure of the process is ignored when modifying the tolerance region. In fact, two processes having the same specifications but different correlation structures would be evaluated by MC_{pm} over the same modified tolerance region.

Shahriari and Abdollahzadeh[4] proposed the NMC_{pm} as an improved version of MC_{pm}, which differs from the first precisely in the method used to modify the tolerance region. In this new index, the modified tolerance region (R_1^*) is defined as the largest ellipsoid centered at target with its axes parallel to the axes of the process ellipsoid (R_2) and completely within the actual tolerance region. This way, the direction of the tolerance ellipsoid axes goes with the direction of the process ellipsoid axes, which depend completely on the process variance-covariance structure (Figure 2).

Figure 2: Example of the modified tolerance region for Shahriari et al.'s method, with $p = 2$.



Hence, region R_1 is determined by the expression: $(\boldsymbol{X}-\boldsymbol{T})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{X}-\boldsymbol{T}) \leq c^2$ where c is a constant to be determined so that the region is fully contained within the original tolerance region. It can be shown (Härdle and Simar[16]) that:

$$c = \min\left\{\frac{USL_i - T_i}{\sqrt{\sigma_{ii}}}, i = 1, 2, \ldots, p\right\} \tag{2}$$

where σ_{ii} is the ith diagonal element of $\boldsymbol{\Sigma}$ matrix.

Then, NMC_{pm} is obtained as a ratio, similar to (1), but considering in the numerator the volume of this new modified tolerance region R_1^* and taking the pth root to capture the number of quality characteristics and to obtain a measure comparable with any univariate index:

$$NMC_{pm} = \left[\frac{Vol(R_1)}{Vol(R_3)}\right]^{\frac{1}{v}} \times [1 + (\boldsymbol{\mu}-\boldsymbol{T})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\boldsymbol{T})]^{-\frac{1}{2}} = \left[\frac{Vol(R_1)}{Vol(R_3)}\right]^{\frac{1}{v}} \times \mathcal{D}^{-1} \tag{3}$$

Values equal to or greater than 1 signal processes with at least $(1-\alpha)100\%$ of products within tolerance region.

3. Indices based on principal component analysis.

Using principal component analysis (PCA) to derive one-dimensional measures of process capability in multivariate domains was an original idea of the authors Wang and Chen[5].

These authors proposed applying PCA technique to the process data to transform the original related measurements variables into a set of uncorrelated linear functions, namely the principal components (PC), $Y_i = \boldsymbol{u_i}'\boldsymbol{X}$, $i = 1, \dots, p$, where $\boldsymbol{u_i}$ is the ith eigenvector of the variance-covariance matrix related to its ith eigenvalue λ_i. Besides, this technique allows reducing the dimensionality of the problem, since the index is built considering only the number of principal components accounted for much of the total variability (commonly 80-90%), which can be identified using one of various criteria (Kaiser criterion, Scree test, percentage of variance criterion, etc. (Eriksson *et al.*[17]).

The transformation used to obtain the principal components, is also applied to the vectors containing the specification limits and target values ($LSL_{Y_i} = \boldsymbol{u}_i'\boldsymbol{LSL}$, $USL_{Y_i} = \boldsymbol{u}_i'\boldsymbol{USL}$, $T_{Y_i} = \boldsymbol{u}_i'\boldsymbol{T}$), so that the whole problem is "moved" to a new system of coordinate axes defined by the principal components.

If the original variables come from a multivariate normal distribution, the principal components will also be normally distributed, since they are linear combinations of normal variables, and so they are also independent due to the no correlation.

In this new context, the multivariate capability index is simply defined as the product of univariate measures of process capability for each of the v principal components identified as important and taking the vth root to capture the dimensionality of the problem:

$$MC_{pm}^{(W)} = \left(\prod_{i=1}^{v} C_{pm,Y_i} \right)^{1/v} \tag{4}$$

where C_{pm,Y_i} is the traditional C_{pm} univariate capability index (Kane[18]) for the ith principal component Y_i:

$$C_{pm,Y_i} = \frac{USL_{Y_i} - LSL_{Y_i}}{6 \cdot \sqrt{\sigma_{Y_i}^2 + \left(\mu_{Y_i} - T_{Y_i}\right)^2}} \tag{5}$$

with μ_{Y_i} y $\sigma_{Y_i}^2$ being the mean and variance of Y_i, respectively.

In addition, C_{pm,Y_i} can be replaced by C_{p,Y_i}, C_{pk,Y_i} or C_{pmk,Y_i}, in order to generate multivariate versions of the most popular univariate indices.

As in the univariate case, the minimum desirable value for this index is 1, in which case the process is assumed to be working according to the required specifications.

Shinde and Khadse[8] made an observation to the method proposed by Wang *et al*[5], arguing that the formulas they used to obtain specification limits for the principal components are incorrect. With Wang et al.'s procedure, the tolerance region in the principal component space is obtained assuming that specification limits of different PCs are independent from each other, when actually, only the distribution of PCs are independent. Their specification limits are interrelated.

In fact, if the original tolerance region is the hyper-rectangle with edges are parallel to the coordinate axes $S = \{\boldsymbol{X} \in \mathbb{R}^{\boldsymbol{p}} : LSL_i \leq X_i \leq USL_i \ \forall \ i = 1, \dots, p\}$, then the tolerance region for the new variables Y_i corresponds to a new region in a rotated space which edges are no longer parallel to the coordinate axes, defined by $S_2 = \{\boldsymbol{Y} \in \mathbb{R}^{\boldsymbol{p}} : \boldsymbol{Y} = \boldsymbol{U'X} \text{ with } LSL_i \leq X_i \leq USL_i, \forall \ i = 1, \dots, p\}$, where $\boldsymbol{U} = (\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_p})$ is the orthonormal matrix of eigenvectors of $\boldsymbol{\Sigma}$. However, Wang and Chen's proposal considers that the specification region for the PCs is $S_1 = \{\boldsymbol{Y} \in \mathbb{R}^{\boldsymbol{p}} : \boldsymbol{u}_i' \boldsymbol{LSL} \leq Y_i \leq \boldsymbol{u}_i' \boldsymbol{USL}\}$ (Figures 3 and 4), which is the result of combining the projections of two points of the original region.

Figure 3: Examples of the original and modified tolerance region for PCs according to Wang and Chen's method ($p = 2$).

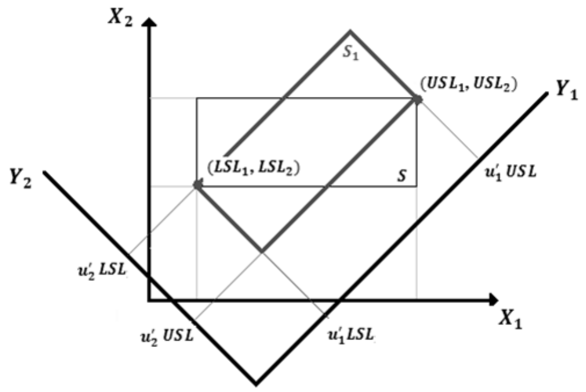
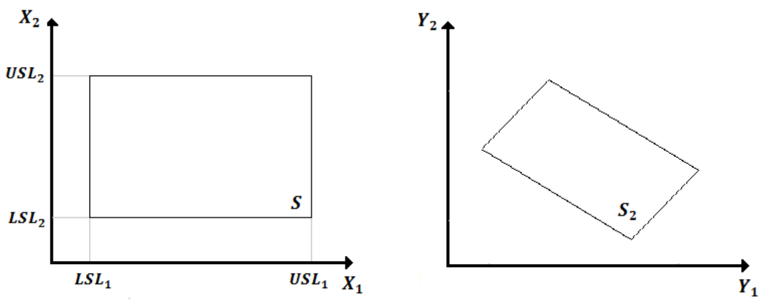


Figure 4: Examples of the original and modified tolerance region for PCs according to Shinde and Khadse's method ($p = 2$).



From this observation, the authors have proposed an alternative method for assessing multivariate process capability based on the empirical probability distribution of PCs. In the general case of p quality characteristics, the specification region S_2 is a really complex region, since it is defined by a set of $2p$ linear inequalities in p variables. Hence, authors propose probability-based indices:

$$Mp_1 = P\left(\boldsymbol{Y} \in S_2 : \boldsymbol{Y} \sim N_p\left(\boldsymbol{T_Y},\ \boldsymbol{\Sigma_Y} = diag(\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \dots, \boldsymbol{\lambda_p})\right)\right) \tag{6}$$

$$Mp_2 = P\left(\boldsymbol{Y} \in S_2 : \boldsymbol{Y} \sim N_p\left(\boldsymbol{\mu_Y}, \boldsymbol{\Sigma_Y} = diag(\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \dots, \boldsymbol{\lambda_p})\right)\right) \tag{7}$$

If $Mp_1 \geq 0.9973$ the process is potentially capable, and if $Mp_2 \geq 0.9973$ the process is actually capable.

If only the first v PCs are selected for further analysis, the specification region is then defined by taking $Y_r = E(Y_r)\ \forall\ r = v+1, \dots, p$:

$$S_2' = \{\boldsymbol{Y} \in \mathbb{R}^{\boldsymbol{v}} : \boldsymbol{Y} = \boldsymbol{U}'\boldsymbol{X} \text{ with } LSL_i \leq X_i \leq USL_i, \forall\ i = 1, \dots, p \text{ and } Y_r = E(Y_r)\ \forall\ r = v+1, \dots, p\}$$

Then, the indices are defined by means of the proportion of conforming products over this new reduced region:

$$Mp_1 = P\left(\boldsymbol{Y} \in S_2' : \boldsymbol{Y} \sim N_v\left(\boldsymbol{T_Y},\ \boldsymbol{\Sigma_Y} = diag(\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \dots, \boldsymbol{\lambda_v})\right)\right) \tag{8}$$

$$Mp_2 = P\left(\boldsymbol{Y} \in S_2' : \boldsymbol{Y} \sim N_v\left(\boldsymbol{\mu_Y},\ \boldsymbol{\Sigma_Y} = diag(\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \dots, \boldsymbol{\lambda_v})\right)\right) \tag{9}$$

The exact calculation of these indices involves the evaluation of multiple integrals on complicated regions hence authors propose to estimate them based on the empirical approach using simulation procedures.

4. Design of the comparative study.

The design of the hypothetical scenarios under which the indices were evaluated took into account in first place the probability distribution of the variables of interest. Three variants were chosen: multivariate normal distribution, mixture of multivariate normal distribution with three different contamination levels ($k = 0.05, 0.10$ and 0.15), and multivariate gamma distributions with different (slight) levels of marginal asymmetry. The choice of these three particular models is justified by the fact that they are able to represent situations commonly found in the context of quality improvement.

In addition, different number of variables ($p = 2, 3$ and 5) and different correlation levels, assuming a structure of equicorrelation among variables ($\rho_{ij} = \rho = 0.05t, t \in \mathbb{N},\ 1 \leq t \leq 19, \forall\ i \neq j$) were considered.

Variables with zero mean vector, $\boldsymbol{\mu_1} = \boldsymbol{0}$, and variance-covariance matrix $\boldsymbol{\Sigma_1}$ with elements $\sigma_{ij} = 1$ $\forall\ i = j$ and $\sigma_{ij} = \rho\ \forall\ i \neq j$ were assumed for the multivariate normal distribution case. Mixture of multivariate normal distributions were obtained combining the normal variables mentioned above with k percent of normal variables with mean vector $\boldsymbol{\mu_2} = \boldsymbol{0}$ and covariance matrix $\boldsymbol{\Sigma_2} = 10\ \boldsymbol{\Sigma_1}$, so that the resulting variables in the mixture have also zero mean vector and variance-covariance matrix $\boldsymbol{\Sigma} = (1 + 9\text{k})\, \boldsymbol{\Sigma_1}$. Finally, the multivariate gamma variables were obtained through the combination of marginal gamma variables, each with different level of asymmetry considering shape parameters $\boldsymbol{\alpha_i} = 15$ or 30 and scale parameters $\beta_i = 1\ \ \forall\, i = 1, \dots, p$.

Assuming as capable processes those which proportion of conforming products is at least 0.9973, for each scenario six sets of specifications were defined in order to generate capable processes (Situation I) as well as incapable processes. Five situations of non-capability were considered: no capability due to departures of the process mean from the target in one or more than one variables, no capability in process variability in one or more than one variables, and no capability in both mean and dispersion at the same time (situations II to VI, respectively). This comparative study has a total of 1710 scenarios under which each index is evaluated, so covering a wide spectrum of possible situations.

The parameters assumed for each simulated processes and the specifications (target values $\boldsymbol{T}$ and specification limits $\boldsymbol{SL}$) chosen in order to generate capable as well as incapable processes under each scenario, are detailed in Table 1(a to c).

It is important to mention that, under the multivariate gamma distribution, since it assumes only positive values, it was not possible, for $p = 3$ and $p = 5$, to define sets of symmetrical specifications generating capable processes in the sense adopted in this paper, i.e., tolerance regions covering at least 99.73 percent of data. Therefore, the analysis under this distribution is carried out assuming a more flexible definition of capable processes: a process is classified as capable if its proportion of conforming products is at least 0.9900.

Under each scenario, theoretical proportions of conforming items were obtained, which reflect the actual state of capability of the process according to the definition adopted. In order to use them as a reference value to compare with the multivariate capability indices MC_{pm}, NMC_{pm} and $MC_{pm}^{(W)}$, these proportions were turned into the traditional univariate C_p index. This way, in this new scale, capable processes are represented by values equal or greater than unity for normal and mixture of normal distributions, and equal or greater than 0.8586 for gamma distribution.

For Mp_2 index, since it is measured directly in terms of proportions of conforming products, capable processes are represented by reference values equal or greater than 0.9973 for normal and mixture of normal distributions, and by reference values equal or greater than 0.9900 for gamma distribution.

Finally, the population values of the four indices under study were obtained under each of the defined scenarios.

R 3.0.1 software was used to calculate the indices, with algorithms based on some functions already defined in the MPCI R-package (Santos Fernández and Scagliarini[19]). Multivariate gamma variables were generated using the algorithm proposed by Bustos *et al.*[20], which offers great flexibility allowing to generate variables with different marginal parameters and different correlations for each pair of variables as well, both positive and negative.

The following section shows the results obtained for the four studied indices in each scenario.

5. Results.

The nature of capability indices is quantitative; they measure process capability on a continuous scale. However, for the purposes of its application to a given problem it is necessary to define a cutoff value, which allows "dichotomizing" the decision about the state of the process, indicating whether it meets or does not meet specifications. This value represents the minimum desirable value of capability, and its choice depends on what is defined as a capable process in each case. Without a cutoff point that allows for this classification capability indices only serve as a means of comparison between processes, and would be useless for individual situations. Hence, suitable cutoff points for each index are discussed prior to the presentation of results.

5.1. Multivariate normal distribution.

5.1.1. Index MC_{pm}.

The authors of MC_{pm} index use the value 1 as cutoff point[2]. Indeed, it is possible to deduce that the value 1 does not necessarily signals capable processes. For a process centered on target, the value 1 only indicates that the modified tolerance region (R_1) and the process region (R_2) have the same volume, regardless of whether the process region is within the tolerance region or not. Figure 5 shows an example of this situation for $p = 2$. Nevertheless, because of the way this index is built it is not possible to define a cutoff point that avoids this type of situations, although they are not frequent unless the performance of the process is completely unknown. Hence, for the purposes of this work, value 1 is assumed as the cutoff point.

Figure 5: Illustration of an incapable process for which $MC_{pm} = 1$.

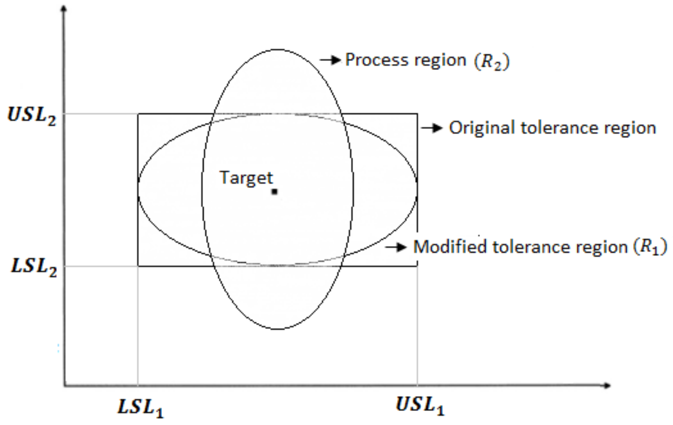


Figure 6 shows the population values of the indices under each scenario, the reference C_p values and the cutoff point. The main characteristic of this index is that it is highly affected by changes in the level of correlation among variables, showing a tendency to overvalue the real capability of process when correlation is high, and to undervalue it for low correlation levels. This feature becomes more evident as the number of variables under study increases and it can be seen as a consequence of a failure in the index definition, since it does not take into account the correlation among variables in modifying the tolerance region. This way, when correlation is high, the modified tolerance ellipsoid contains areas of rare observations, leading to overvaluations of the real capability (areas A and B in Figure 7, for $\rho = 0.80$). The volume of those areas will increase as the number of variables increases and therefore the overvaluation will be increased as well.

Regarding the ability of MC_{pm} index to distinguish between capable and incapable processes, this index is able to correctly classified capable processes when only two quality characteristics are evaluated. When $p \geq 3$ and correlation is low capable processes are wrongly classified as incapable.

On the other hand, the index is not always able to identify when the process is not meeting the specifications. In several situations of incapable processes, the index takes values greater than 1 leading to erroneous conclusions about the actual process capability.

From these results, it can be concluded that MC_{pm} is neither sensitive nor specific to distinguish processes by their state of capability. Table 2 shows an extract of the values represented in Figure 6.

Figure 6: Values of MC_{pm} index, reference values (C_p) and cutoff point for each scenario, under multivariate normal distribution.

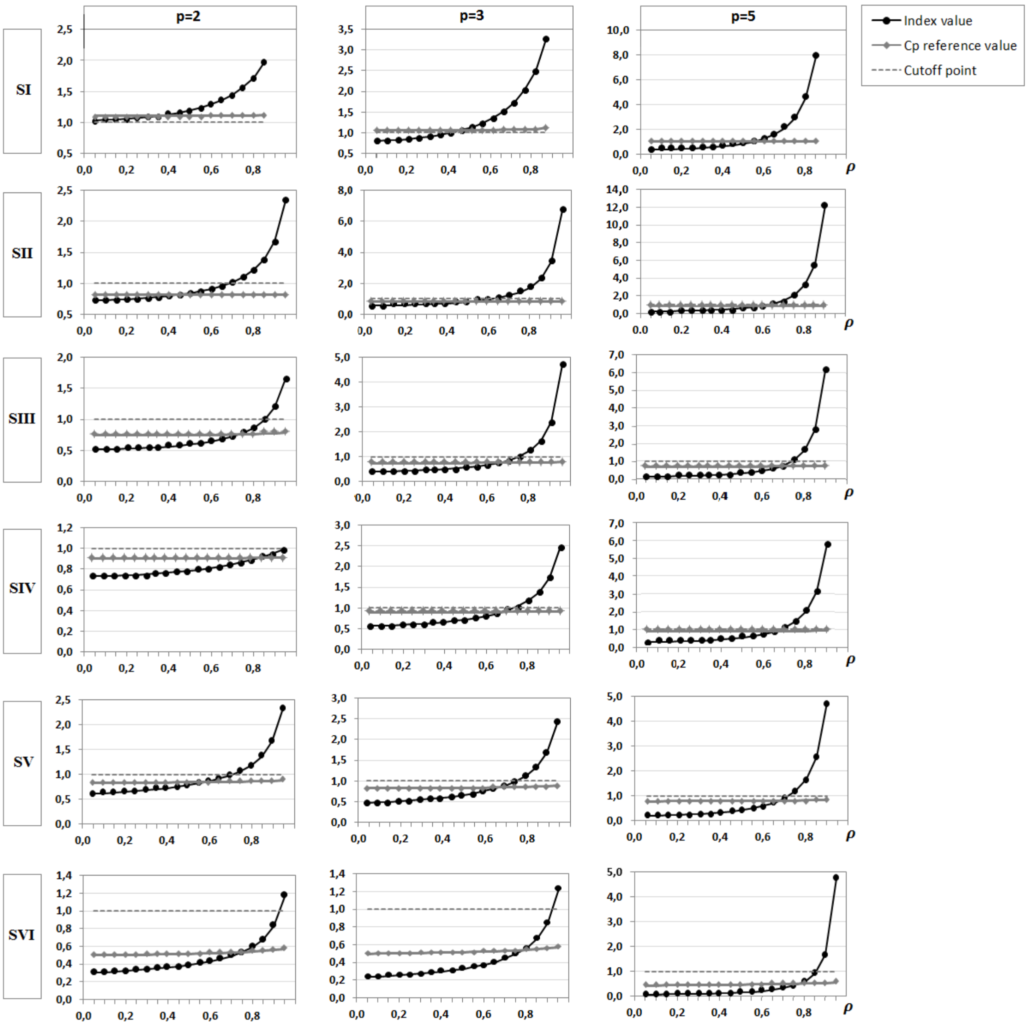
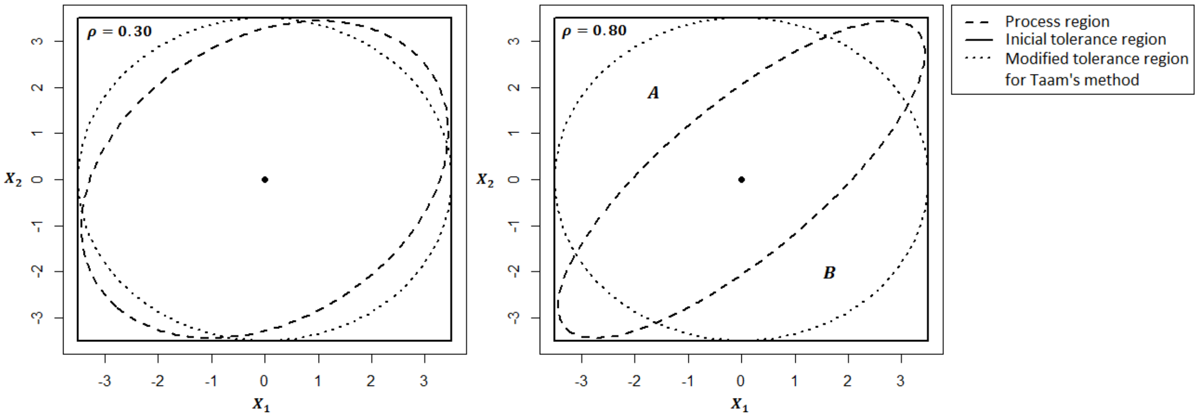


Figure 7: Example of process region and Taam et al.'s modified tolerance region for $p = 2$ and correlations $\rho = 0.30$ and $\rho = 0.80$.



5.1.2. Index NMC_{pm}.

This index definition does take into account the structure of correlation among variables in modifying the tolerance region, so it seems reasonable to assume the value 1 as the limit value to classify the processes. Indeed, when the index takes the value 1 then at least $100(1-\alpha)\%$ of process data falls inside the tolerance region. However, values lower than 1 do not necessarily indicate lack of capability, at least in the sense adopted in this paper. In fact, the analysis reveals that the index can take values less than 1 even when the proportion of conforming products is greater than 0.9973. This is indeed what happens in Situation I for $p = 3$ and $p = 5$ (Figure 4.5). In fact, a process may have more than $100(1-\alpha)\%$ of conforming products, but the $100(1-\alpha)\%$ process region can overflow the modified tolerance region. Figure 8 shows an example of this situation for a bivariate process, in which the 0.9973 process region overflows the modified tolerance region, leading to an index value lower than 1 ($NMC_{pm} = 0.9304$), eventhough the process has 99.76% of conforming products, according to the initial tolerance region.

Since the index assumes isodensity ellipsoids for representing the process region, the ellipsoid of probability 0.9973 may have to exceed the tolerance region, although the proportion of conforming products according to the initial tolerance region is greater than 0.9973. This means that by using the value 1 as cutoff point, this index classify as capable those processes whose ***central*** $100(1-\alpha)\%$ of data fall inside the tolerance region, which is a more demanding rule than the assumed for this work.

The index values obtained in Situation I (shown in Table 3) correspond to processes with the following proportion of conforming products: 0.9990 for $p = 2$, 0.9985 for $p = 3$ and 0.9977 for $p = 5$, and therefore these could be used as the cutoff points for $\alpha = 0.0010$, 0.0015 and 0.0023 for $p = 2, 3$ y 5 respectively, if it can be assumed that every product within the original tolerance region is conforming regardless of its probability of occurrence. In order to be consistent with the definition of capable process that has been adopted for this work, suitable cutoff points for $\alpha = 0.0027$ were deduced. That is, for each of the processes defined in Table 1(a), specifications limits proportional to the original ones were sought such that, under multivariate normal distribution, they lead to 99.73% of conforming product. Those limits were then used to compute the index and the resulting value was assumed as the cutoff point. Resulting values (**$\boldsymbol{\Phi}_{\boldsymbol{p};\,\boldsymbol{\alpha}}$**) are shown in Table 3.

Figure 9 shows that the index has the ability of correctly classify both capable and incapable processes, so it can be said that NMC_{pm} index is a measure both sensitive and specific to distinguish between capable and incapable processes.

Figure 8: Modified tolerance region and 99.73% process region for a process with $NMC_{pm} < 1$.

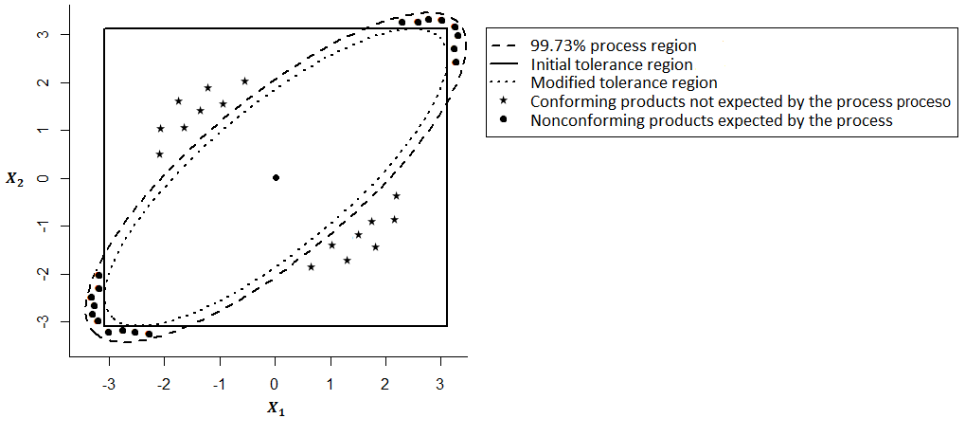
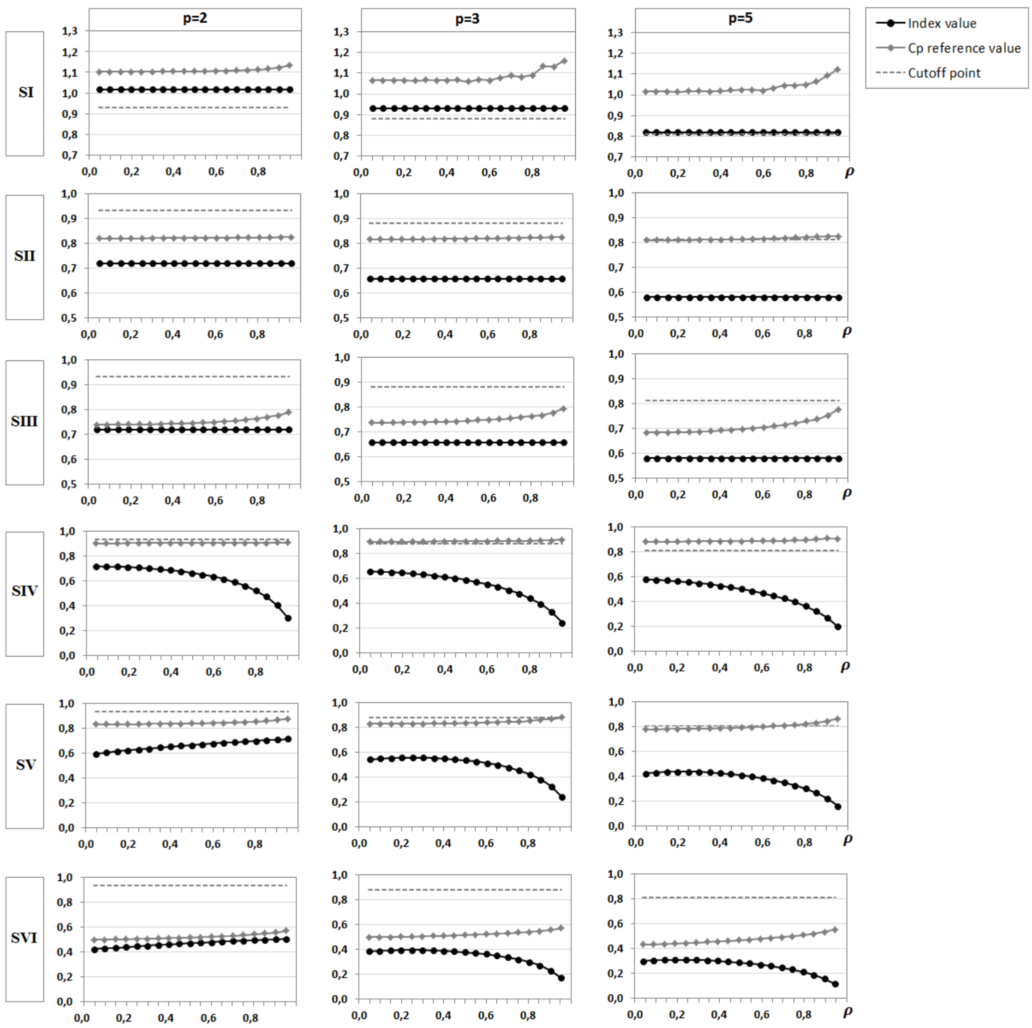


Figure 9: Values of NMC_{pm} index, reference values (C_p) and cutoff point for each scenario, under multivariate normal distribution.



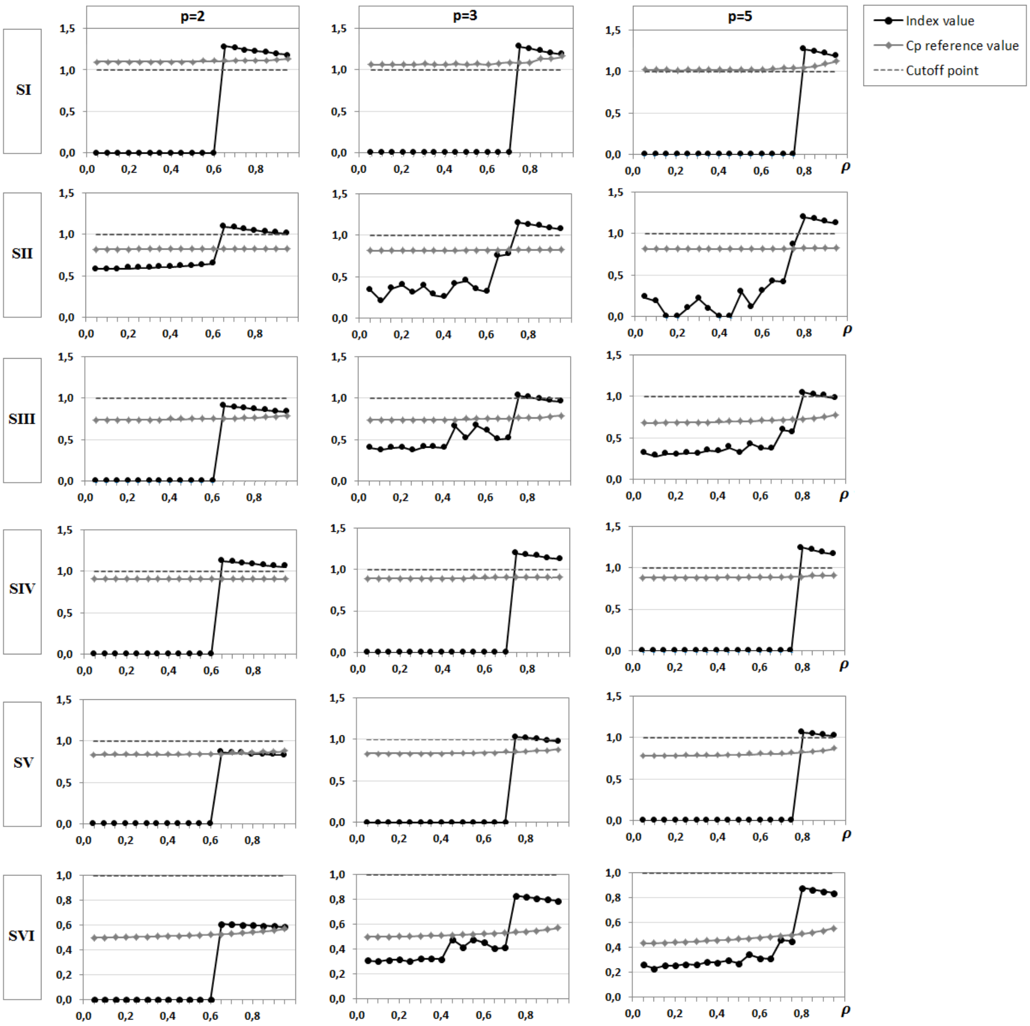
5.1.3. Index $MC_{pm}^{(W)}$.

Figure 10 shows the results of Wang et al's index. The first noticeable thing is that there are situations in which the index takes the value zero, and this is a consequence of the failure in transforming the tolerance region, as it has been highlighted by Shinde and Khadse[8]. In fact, Wang et al's method sometimes transforms the original tolerance region, causing null values for the univariate capability index for some of the principal components. This is the case, for example, of bivariate processes whose variables have the same dispersion and tolerances of the same width, in which both principal components has to be used. In these particular situations, the original tolerance region is a square region which is degenerated into a segment of straight line after applying the transformation suggested by Wang and Chen[5]. Then $LSE_{Y_i} - LIE_{Y_i} = 0$ for the degenerated dimension and so $C_{pm,Y_i} = 0$ and $MC_{pm}^{(W)} = 0$, no matter the real process capability.

Consider, for example, a bivariate normal process with mean vector $\boldsymbol{\mu} = \mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. The eigenvalues of $\boldsymbol{\Sigma}$ are $\lambda_1 = 1 + \rho$ and $\lambda_2 = 1 - \rho$, and the eigenvectors are $\boldsymbol{u}_1' = \frac{1}{\sqrt{2}}(1 \quad 1)$ and $\boldsymbol{u}_2' = \frac{1}{\sqrt{2}}(1 \quad -1)$. If components explaining more than 80% of total variability are retained, then both principal components will be retained whenever $\rho \leq 0.60$. In such case, $USL_{Y_2} - LSL_{Y_2} = \boldsymbol{u}_2'\boldsymbol{USL} - \boldsymbol{u}_2'\boldsymbol{LSL} = \frac{1}{\sqrt{2}}[(USL_1 - LSL_1) - (USL_2 - LSL_2)] = 0$.

Anyway, even when these kinds of situations are not the case, the index is not able to correctly classify capable and incapable processes. Since the index is based on the product of univariate indices for which a value equal or greater than one indicates that at least $100(1-\alpha)\%$ of process data is conforming, it is expected that this multivariate index leads to values lower than 1 whenever the multivariate process does not operate at the required quality level and this does not always happen in some of the situations of incapability proposed. An extract of the results are shown in Table 4.

Figure 10: Values of $MC_{pm}^{(W)}$ index, reference values (C_p) and cutoff point for each scenario, under multivariate normal distribution.



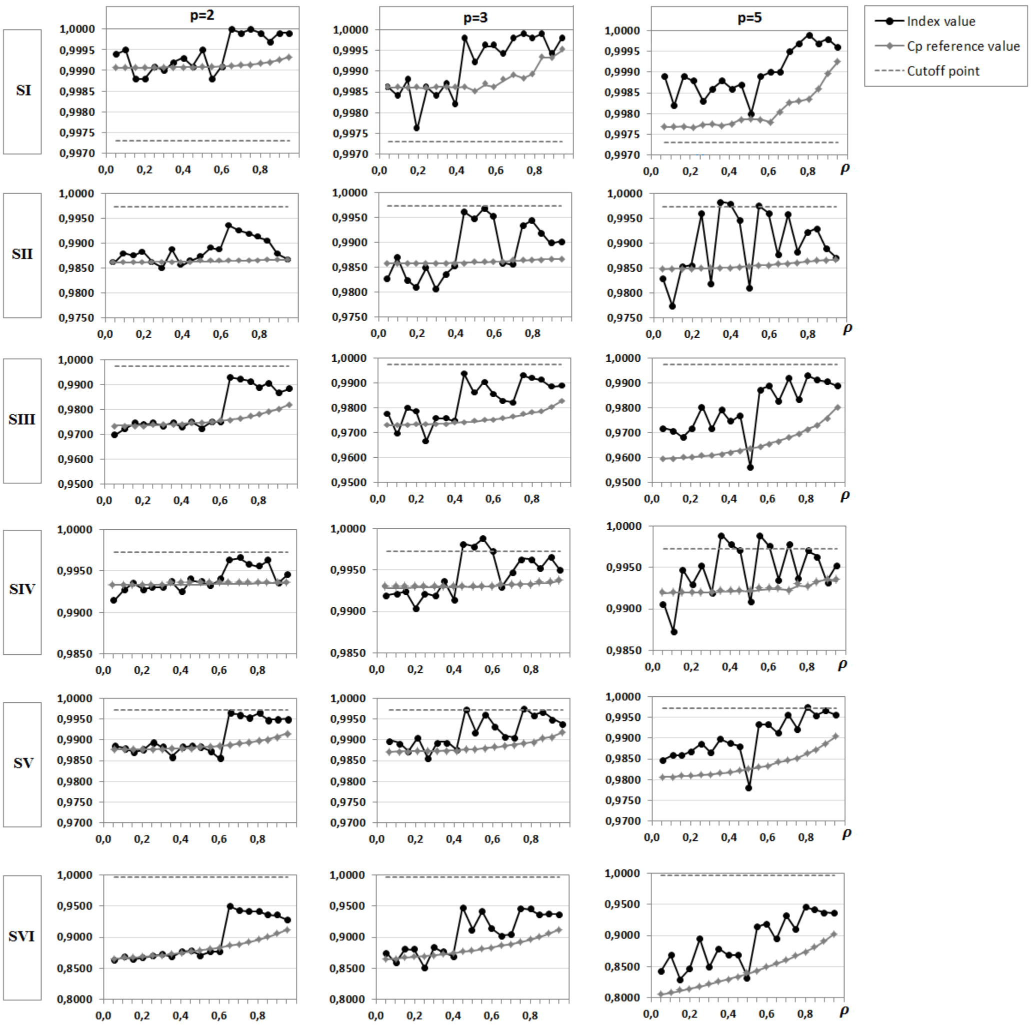
5.1.4. Index $\mathbf{Mp_2}$.

Unlike previous indices, this measure is expressed in terms of proportion of conforming products, so the choice of a cutoff point to differentiate capable and incapable processes depends exclusively on the desired quality level, being 0.9973 the value commonly used, and indeed is the value selected by the author of this index.

The reference values shown in Figure 11 represent the real proportion of conforming products under each scenario. For capable processes (Situations I) the index always takes values higher than the cutoff point (0.9973), correctly indicating that the process meets the specifications.

On the contrary, the index has problems to identify incapable processes, mainly when the lack of capability is mild or moderate. Situations II to V represent processes with mild anomalies in meeting the specifications, and in many of those scenarios the index take values higher than 0.9973 leading to a wrong conclusion about process capability. Moreover, this failure seems to worsen as the number of variables studied increases. Table 5 shows an extract of the numeric results.

Figure 11: Values of Mp_2 index, reference values (C_p) and cutoff point for each scenario, under multivariate normal distribution.



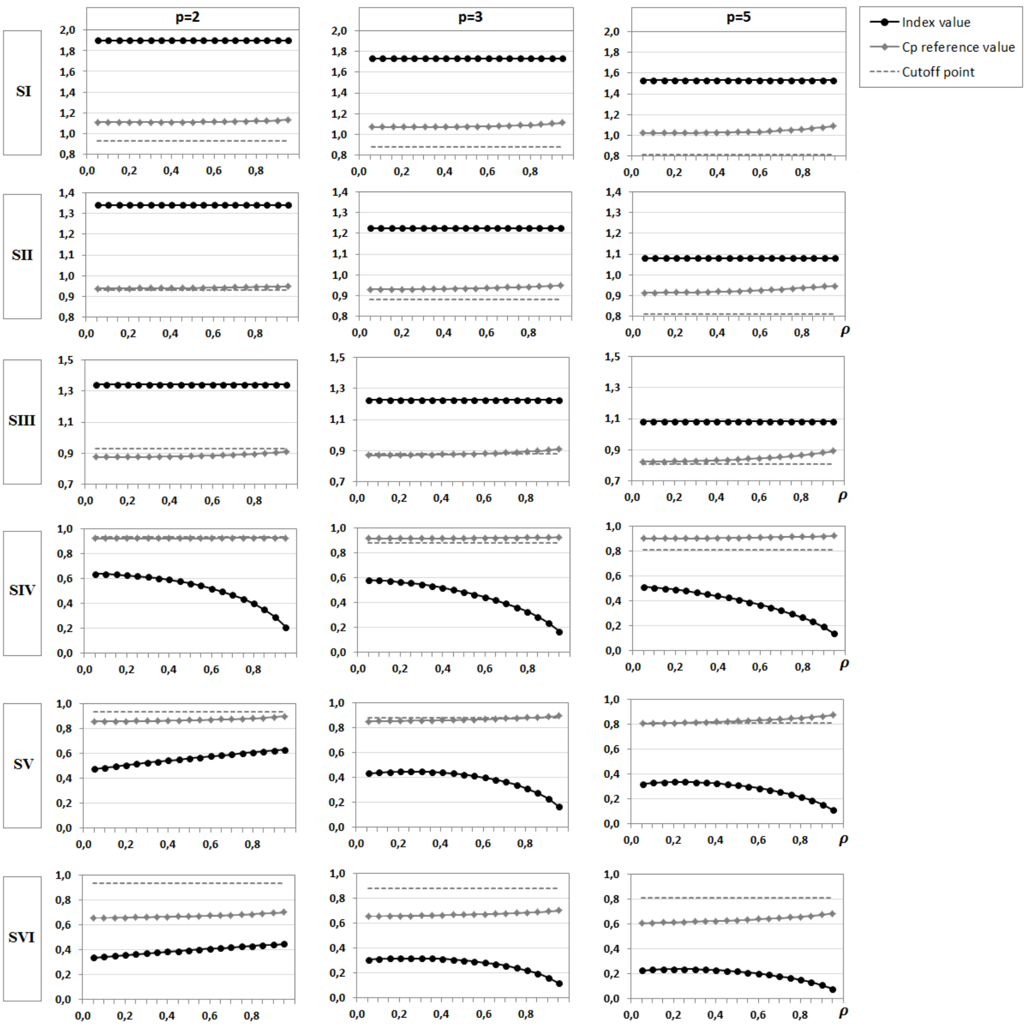
The results exposed until this point show that the initial proposals on both index construction approaches fail to identify the true process capability state. Meanwhile, subsequent proposals do show an improved performance, so they are more suitable alternatives to evaluate multivariate process capability. The effect of departures from normality is analyzed only for these recent proposals.

5.2. Mixture of multivariate normal distributions.

5.2.1. Index NMC_{pm}.

Figure 12 shows the results of NMC_{pm} index applied to variables with mixture of multivariate normal distributions, as well as the reference values and the cutoff points. It is shown that under capable processes (Situations S-I) the index overvalues the true process capability, as a consequence of the contamination in the underlying data distribution. The original multivariate normal variables were mixed with a certain percentage of variables also normal but with a greater dispersion, so creating a distribution with heavier tails than the normal. Under this mixed distribution, it is necessary to define a greater tolerance region in order to cover $100(1-\alpha)\%$ of data than it would be necessary under a normal distribution with the same covariance matrix. In addition, since the index assumes normality to compute the process region, it has lower volume than the true process region (see Figure 13), leading to an overvaluation of the true process capability.

Figure 12: Values of NMC_{pm} index, reference values (C_p) and cutoff point for each scenario, under mixture of multivariate normal distributions ($k = 0.10$).

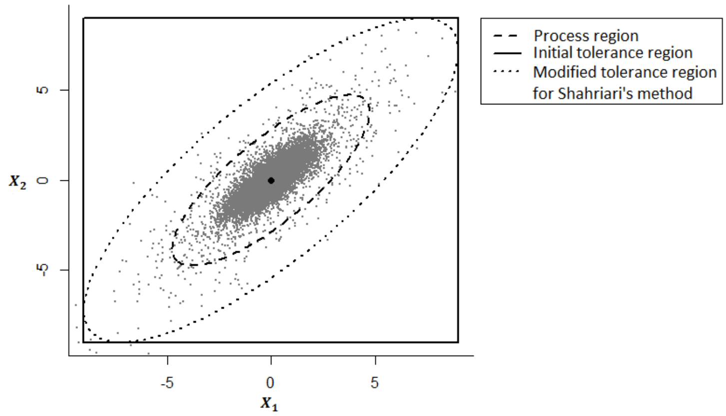


This overvaluation affects the ability of the index to identify incapable processes. The index does not detect processes having problems regarding its variability (Situations SII and SIII), since the overvalued values are great enough to wrongly conclude that the process is capable.

Similar results were found for $k = 0.05$ and $k = 0.15$, noting that the signaled effect becomes more noticeable as the contamination level increases.

It is then seen that departures from normality, even among the range of symmetric distributions, affects the ability of the index to correctly detect the true process capability state. An extract of numeric results is shown in Table 6.

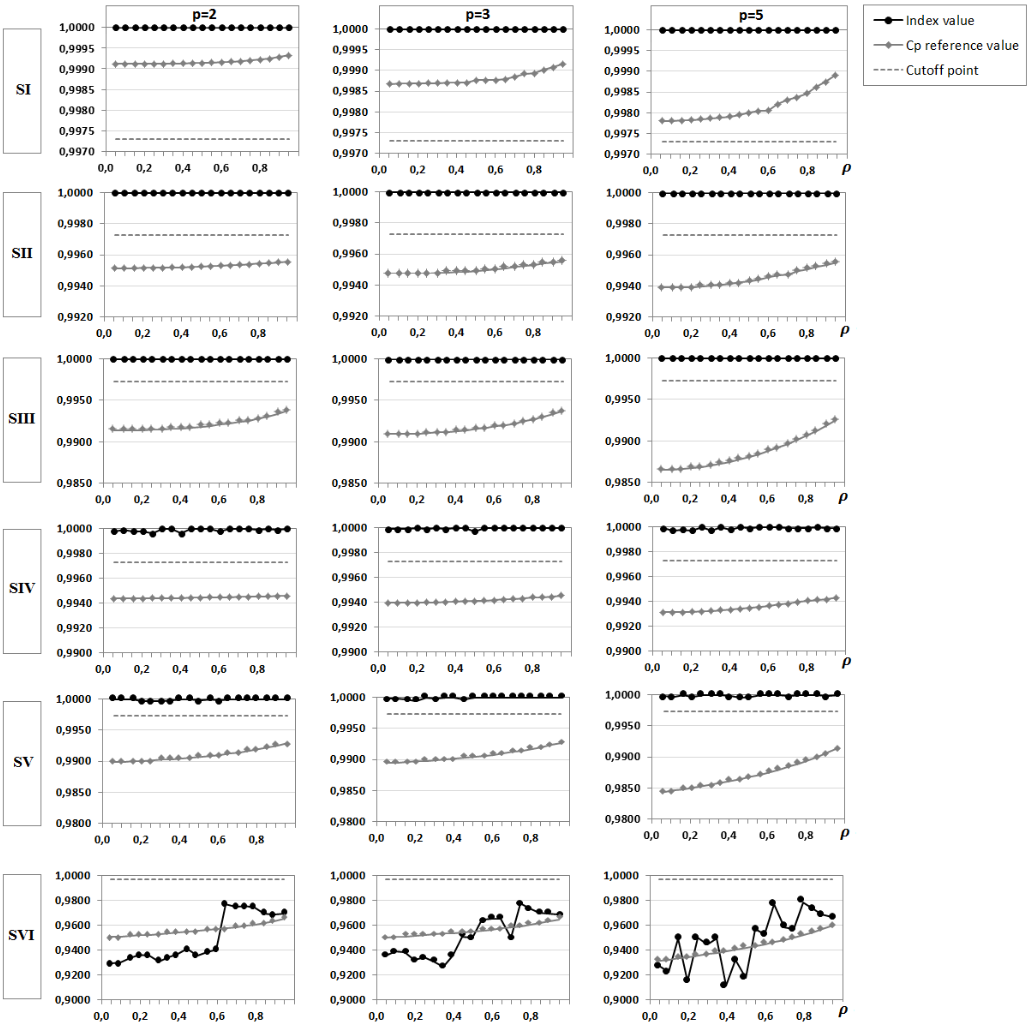
Figure 13: Example of process region and Shahriari et al's modified tolerance region for data coming from a mixture of multivariate normal distributions ($k = 0.10$ y $p = 2$).



5.2.2. Index Mp_2.

The index based on principal components is also affected if the underlying distribution has heavier tails than normal distribution, and the effect is similar to that observed for NMC_{pm} index. As a consequence of the departures from normality the true process capability is overvalued leading to the conclusion that the process is capable when actually it is not (Figure 14). The index does not detect the lack of capability unless the process is operating really far from its specifications (as in Situations S-VI). This is reasonable, since in the index computation it is assumed that principal components inherit the normality of original variables, which is not true in this case. So the probabilities are calculated under a more concentrated distribution than the real one, then the tolerance region is wide enough to allow movements of the process or increases in variability without being detected. This effect, again, is accentuated with increasing contamination levels. Table 7 shows an extract of the numeric results.

Figure 14: Values of Mp_2 index, reference values (C_p) and cutoff point for each scenario, under mixture of multivariate normal distributions ($k = 0.10$).



5.3. Multivariate gamma distribution.

5.3.1. Index NMC_{pm}.

Process regions are assumed to be ellipsoidal in shape to build this index, hence when the real distribution has a slight asymmetry the ellipsoidal $100(1-\alpha)\%$ process region will cover a zone where the process does not produce outcomes. Therefore the volume of the process region under the real distribution will be overvalued by the volume of that ellipsoid. However, the same happens with the modified tolerance region, since specifications are assumed to be symmetrical around the target. This way, the index is not affected by this kind of departures from normality (Figure 15).

Figure 16 shows the results of the index when applied on processes with multivariate gamma distributions, with mild levels of asymmetry. As explained before, the cutoff points were derived for $\alpha = 0.01$, i.e., a tolerance region proportional to the original one was sought such that it covers

99% of process data under a multivariate normal distribution with parameters defined in Table 1 (c), and then it was used to calculate the index value. The results are shown in Table 8.

Figure 15: Example of process region and Shahriari et al's modified tolerance region for data coming from a multivariate gamma distribution.

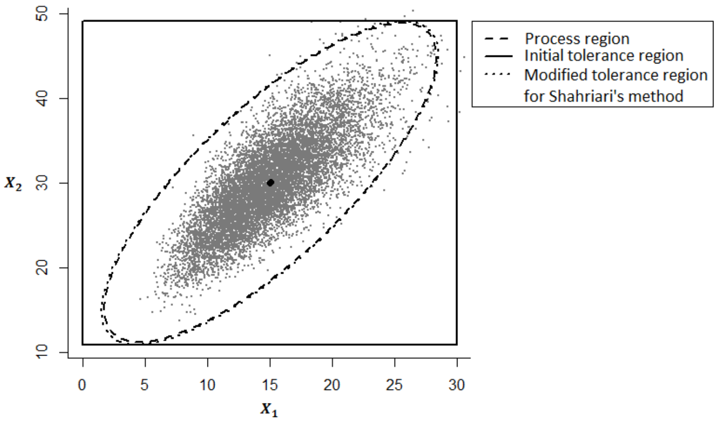
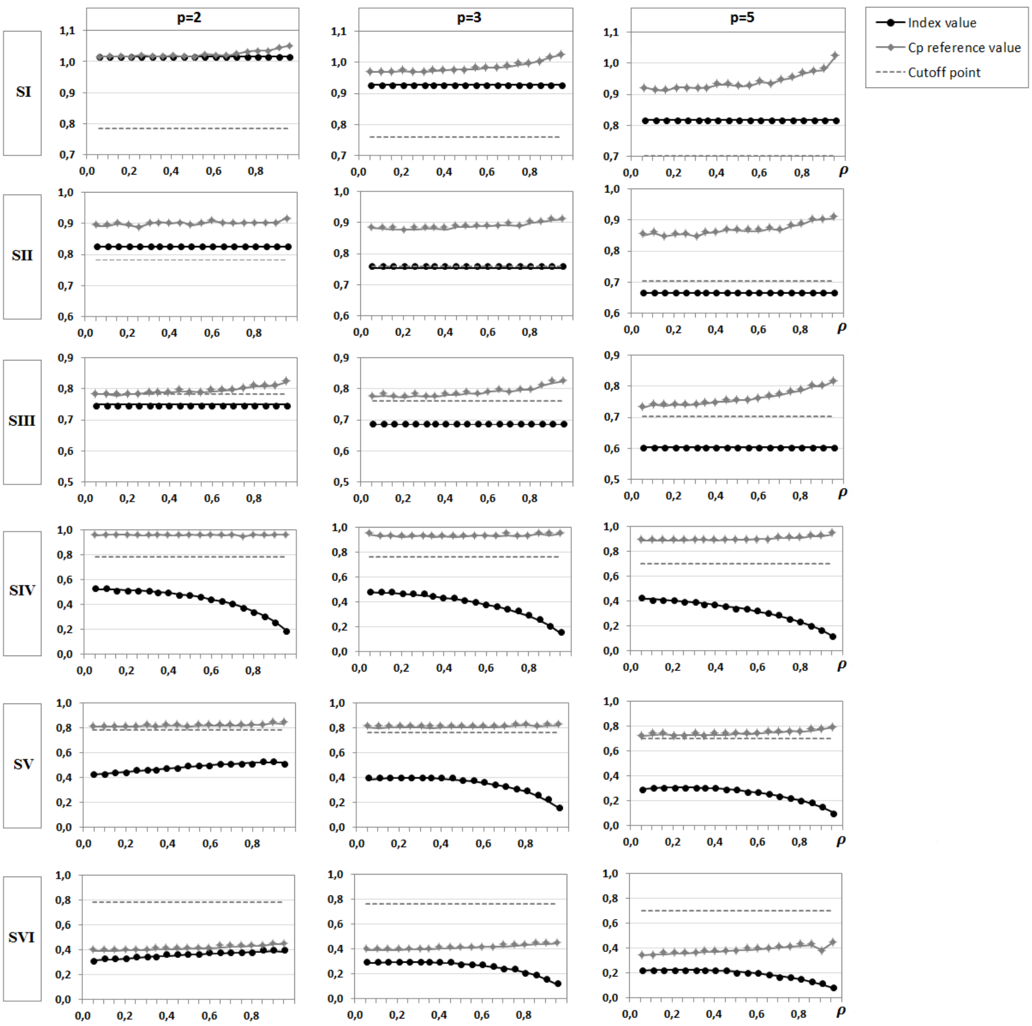


Figure 16: Values of NMC_{pm} index, reference values (C_p) and cutoff point for each scenario, under multivariate gamma distributions.

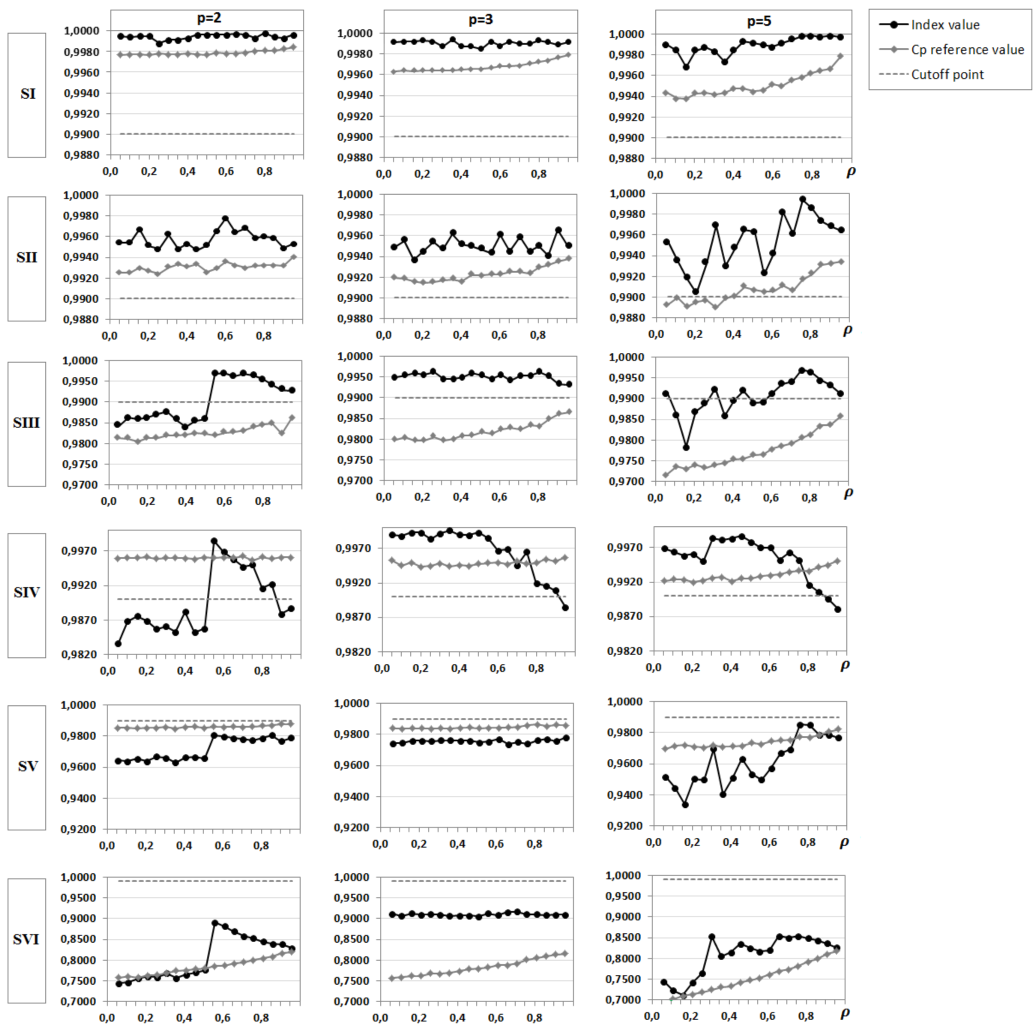


From Figure 16 it is seen that both capable and incapable processes are correctly identified by this index, so even when some of the variables are slightly asymmetric the index is still sensitive and specific to distinguish processes.

5.3.2. Index Mp_2.

In this case, the index has shown an erratic performance, sometimes overvaluing and undervaluing others. The advantage of this index is that it is expressed as proportion of conforming products so its values can be directly compared with the actual proportion of conforming products of the process to evaluate its ability to reflect the true process capability.

Figure 17: Values of Mp_2 index, reference values (C_p) and cutoff point for each scenario, under multivariate gamma distributions.



Under capable processes (Situations S-I), the index gives values greater than the real ones, but this does not represent a problem since the conclusion about the state of the process will be correct anyway. However, for the definition of capable process assumed under this distribution, processes in situations II and IV are also capable despite the anomaly considered and the index fails, giving false alarms in some of the scenarios of situation IV.

In addition, when processes are incapable because of an excessive variability (situation III) the index leads wrongly to the conclusion that the process is capable (Figure 17).

Therefore, this index is neither sensitive nor specific to distinguish capable processes from incapable ones. Table 9 shows an extract of the numeric results.

6. Discussion:

This research was intended to gain some insight into the performance of some of the several multivariate capability indices that have been developed in the past few years. Although there is a growing interest in this topic, there is no consensus about a methodology for assessing capability in the multivariate domain.

Among the available indices proposed in the literature we have selected four of them to be assessed for their ability to correctly signal the actual state of capability of a process.

Two of them are constructed following the ratio of relative volumes approach, MC_{pm} (Taam *et al.*[2]) and NMC_{pm} (Shahriari and Abdollahzadeh[4]). This type of indices can be thought as extensions of the traditional univariate indices, where ratios of interval lengths in univariate domain are turned into ratios of region volumes in the multivariate case. The attractiveness of this kind of indices is that they provide a procedure to partition the information in terms of process variation and target deviation, so helping users to define appropriate corrective actions when the indices indicate poor process capability.

The other two selected indices are based on the principal component analysis, $MC_{pm}^{(W)}$ (Wang and Chen[5]) and Mp_2 (Shinde and Khadse[8]). As it is already known, this multivariate statistical technique allows converting a set of observations of correlated variables into a set of values of linearly uncorrelated variables. Based on that, multivariate indices are defined using univariate process capability indices of the principal components. This approach is attractive because it offers the possibility of reduce the dimensionality of the data sets, so it is particularly useful in analyzing high-dimensional problems, even though each component has a more complex interpretation.

All these indices have been object of comparisons and discussion in several papers, between them or even with others available indices, however most of those comparisons were done in simpler situations, such as just bivariate problems, or considering data coming from a real-world problem (Wang *et al.*[13], García *et al.*[14], Shinde and Khadse[8], Pan and Lee[15]).

In our opinion, there was a need for practical rules that guide the choice of either index, taking into account the context and the characteristics of each particular problem. The large number of scenarios under which the four selected indices were evaluated in this work allows deriving certain patterns of their performances.

As a general result, we have found that the initial versions on both approaches, ratios of relative volumes and principal components, are not recommendable for their use in real-world practice. Even in the best scenarios (capable processes under multivariate normal distribution) they do not perform as it is desirable. Moreover, the construction of those indices has shown some failures that affect their performance, such us the case of Taam's index for which is not possible to deduce a cutoff value to decide whether a process meets specifications or not.

The newer indices, NMC_{pm} and Mp_2, have notably improved the performance of the former capability measures, although there still are some unsolved issues, like the cutoff points in Shahriari's index to distinguish between capable and incapable processes. We have found that NMC_{pm} is a sensitive and specific measure to identify the actual capability of processes. On the other hand, Shinde's index has shown to be specific but not always sensitive. It has the ability to correctly identify capable processes, but for incapable processes, the anomaly in process has to be very noticeable in order to be detected by this measure, if not, it would be unnoticed and the process will be wrongly classified as capable.

Both measures are defined under the assumption of multivariate normality. However it is very common in practice that processes do not follow an exact normal distribution. In order to investigate how these measures are affected by moderate departures from the multivariate normality, we have considered data coming from symmetrical distributions with heavier tails than normal, mixing two multivariate normal distributions, one more spread than the other; as well as data coming from distributions with certain level of asymmetry, assuming a multivariate gamma distribution. The idea behind the selection of these kinds of distributions was to introduce departures from normality moderate enough to be unnoticed in a practical situation. Otherwise, it would be meaningless to study normality dependent measures in those situations.

Regarding this, neither index has found robust under both kinds of departures from multivariate normality that we have considered.

When the underlying distribution is heavy-tailed, the index based on ratios of relative volumes is specific but not sensitive. NMC_{pm} has shown troubles to signal incapable processes, mainly when the source of lack of capability is related to excessive variability. However, when the departure from normality is due to a moderate asymmetry, the index does preserve its sensitivity and specificity to distinguish capable processes from incapable ones. On the other hand, Shinde's index is only specific under heavy-tailed distributions, and it loses sensitivity when applied under heavy-tailed or asymmetrical distributions. In those situations, as in the case of normal distribution, the anomaly in process has to be very noticeable in order to be detected by this measure.

Having briefly done this summary, it is reasonable to suggest the use of the index based on ratio of volumes instead of that based on principal components. Even so, careful should be taken with the distributional characteristics of variables under study, considering that this measure may not be reliable in cases of heavy-tailed distributions. Besides its performance, one advantage of NMC_{pm} is that its expression can be written as the product of two terms, separating the process performance regarding variability from the performance regarding centeredness. This way, the measure provides information about the source of lack of capability when processes are incapable.

Beyond NMC_{pm} has a better performance, Shinde's index might be preferred by operators who do not have a deep statistical knowledge, since its values have a direct meaning in terms of proportion of conforming products, while there is no way to translate NMC_{pm} index's values into proportions of conforming products. Furthermore, as its authors have remarked, the empirical approach used to estimate the index enables to consider other than hyper-rectangular specification region[8], widening its fields of application. In addition, as a general result, we have seen that increasing the number of variables involved in the analysis causes poorer index performances. The failures an index could have seem to be exaggerated as more variables are simultaneously studied, not to mention the greater complexity that carries a higher problem dimension. Situations involving more than 3 quality characteristics are not even possible to be plotted, so there is a really need for reliable capability measures applicable to that cases.

We have seen that there is a lot of researchers working on this topic, and every time more and more capability indices for the multivariate domain are available, but only few of them were subjected to comparative studies in order to prove their properties across the many factors arising in this kind of contexts. This was what encouraged and motivated the work done and presented in this paper, even though there is still much work to be done.

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<u>Table 1:</u> Parameters and specifications (T and SL) of simulated processes (S-I to S-VI).
Part a) Multivariate normal distribution case.

		$p = 2$	$p = 3$	$p = 5$
Parameters		$\mu = (0 \quad 0)^`$ $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$	$\mu = (0 \quad 0 \quad 0)^`$ $\Sigma = \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$	$\mu = (0 \quad 0 \quad 0 \quad 0 \quad 0)^`$ $\Sigma = \begin{pmatrix} 1 & \rho & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho & \rho \\ \rho & \rho & 1 & \rho & \rho \\ \rho & \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & \rho & 1 \end{pmatrix}$
S-I	T SL	μ $\pm 3.5\,(1 \quad 1)^t$	μ $\pm 3.5\,(1 \quad 1 \quad 1)^t$	μ $\pm 3.5\,(1 \quad 1 \quad 1 \quad 1 \quad 1)^t$
S-II	T SL	μ $\pm 3.5\left(1 \quad \frac{1}{\sqrt{2}}\right)^t$	μ $\pm 3.5\left(\frac{1}{\sqrt{2}} \quad 1 \quad 1\right)^t$	μ $\pm 3.5\left(\frac{1}{\sqrt{2}} \quad 1 \quad 1 \quad 1 \quad 1\right)^t$
S-III	T SL	μ $\pm 3.5\left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right)^t$	μ $\pm\ 3.5\left(\frac{1}{\sqrt{2}} \quad 1 \quad \frac{1}{\sqrt{2}}\right)^t$	μ $\pm 3.5\left(\frac{1}{\sqrt{2}} \quad 1 \quad \frac{1}{\sqrt{2}} \quad 1 \quad \frac{1}{\sqrt{2}}\right)^t$
S-IV	T SL	$(-1 \quad 0)^t$ $\pm 3.5\,(1 \quad 1)^t + T$	$(-1 \quad 0 \quad 0)^t$ $\pm 3.5\,(1 \quad 1 \quad 1)^t + T$	$(-1 \quad 0 \quad 0 \quad 0 \quad 0)^t$ $\pm 3.5\,(1 \quad 1 \quad 1 \quad 1 \quad 1)^t + T$
S-V	T SL	$(-1 \quad -1)^t$ $\pm 3.5\,(1 \quad 1)^t + T$	$(-1 \quad 0 \quad -1)^t$ $\pm 3.5\,(1 \quad 1 \quad 1)^t + T$	$(-1 \quad 0 \quad -1 \quad 0 \quad -1)^t$ $\pm 3.5\,(1 \quad 1 \quad 1 \quad 1 \quad 1)^t + T$
S-VI	T SL	$(-1 \quad -1)^t$ $\pm 3.5\left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right)^t + T$	$(-1 \quad 0 \quad -1)^t$ $\pm\ 3.5\left(\frac{1}{\sqrt{2}} \quad 1 \quad \frac{1}{\sqrt{2}}\right)^t + T$	$(-1 \quad 0 \quad -1 \quad 0 \quad -1)^t$ $\pm\ 3.5\left(\frac{1}{\sqrt{2}} \quad 1 \quad \frac{1}{\sqrt{2}} \quad 1 \quad \frac{1}{\sqrt{2}}\right)^t + T$

Table1-Part b) Mixture of multivariate normal distributions case.

		$p = 2$	$p = 3$	$p = 5$
Parameters		$\boldsymbol{\mu} = (0 \quad 0)`$ $\boldsymbol{\Sigma} = (1+9k)\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$	$\boldsymbol{\mu} = (0 \quad 0 \quad 0)`$ $\boldsymbol{\Sigma} = (1+9k)\begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$	$\boldsymbol{\mu} = (0 \quad 0 \quad 0 \quad 0 \quad 0)`$ $\boldsymbol{\Sigma} = (1+9k)\begin{pmatrix} 1 & \rho & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho & \rho \\ \rho & \rho & 1 & \rho & \rho \\ \rho & \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & \rho & 1 \end{pmatrix}$
S-I	$\boldsymbol{T}$ $\boldsymbol{SL}$	$\boldsymbol{\mu}$ $\pm 3.5k_0\sqrt{1+9k}\,(1 \quad 1)^t$	$\boldsymbol{\mu}$ $\pm 3.5k_0\sqrt{1+9k}\,(1 \quad 1 \quad 1)^t$	$\boldsymbol{\mu}$ $\pm 3.5k_0\sqrt{1+9k}\,(1 \quad 1 \quad 1 \quad 1 \quad 1)^t$
S-II	$\boldsymbol{T}$ $\boldsymbol{SL}$	$\boldsymbol{\mu}$ $\pm 3.5k_0\sqrt{1+9k}\,\left(1 \quad \frac{1}{\sqrt{2}}\right)^t$	$\boldsymbol{\mu}$ $\pm 3.5k_0\sqrt{1+9k}\,\left(\frac{1}{\sqrt{2}} \quad 1 \quad 1\right)^t$	$\boldsymbol{\mu}$ $\pm 3.5k_0\sqrt{1+9k}\,\left(\frac{1}{\sqrt{2}} \quad 1 \quad 1 \quad 1 \quad 1\right)^t$
S-III	$\boldsymbol{T}$ $\boldsymbol{SL}$	$\boldsymbol{\mu}$ $\pm 3.5k_0\sqrt{1+9k}\,\left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right)^t$	$\boldsymbol{\mu}$ $\pm\ 3.5k_0\sqrt{1+9k}\,\left(\frac{1}{\sqrt{2}} \quad 1 \quad \frac{1}{\sqrt{2}}\right)^t$	$\boldsymbol{\mu}$ $\pm 3.5k_0\sqrt{1+9k}\,\left(\frac{1}{\sqrt{2}} \quad 1 \quad \frac{1}{\sqrt{2}} \quad 1 \quad \frac{1}{\sqrt{2}}\right)^t$
S-IV	$\boldsymbol{T}$ $\boldsymbol{SL}$	$1.5\,k_0\sqrt{1+9k}\,(1 \quad 0)^t$ $\pm 3.5k_0\sqrt{1+9k}\,(1 \quad 1)^t + T$	$1.5\,k_0\sqrt{1+9k}\,(1 \quad 0 \quad 0)^t$ $\pm 3.5k_0\sqrt{1+9k}\,(1 \quad 1 \quad 1)^t + T$	$1.5\,k_0\sqrt{1+9k}\,(1 \quad 0 \quad 0 \quad 0 \quad 0)^t$ $\pm 3.5k_0\sqrt{1+9k}\,(1 \quad 1 \quad 1 \quad 1 \quad 1)^t + T$
S-V	$\boldsymbol{T}$ $\boldsymbol{SL}$	$1.5\,k_0\sqrt{1+9k}\,(1 \quad 1)^t$ $\pm 3.5k_0\sqrt{1+9k}\,(1 \quad 1)^t + T$	$1.5\,k_0\sqrt{1+9k}\,(1 \quad 0 \quad 1)^t$ $\pm 3.5k_0\sqrt{1+9k}\,(1 \quad 1 \quad 1)^t + T$	$1.5\,k_0\sqrt{1+9k}\,(1 \quad 0 \quad 1 \quad 0 \quad 1)^t$ $\pm 3.5k_0\sqrt{1+9k}\,(1 \quad 1 \quad 1 \quad 1 \quad 1)^t + T$
S-VI	$\boldsymbol{T}$ $\boldsymbol{SL}$	$1.5\,k_0\sqrt{1+9k}\,(1 \quad 1)^t$ $\pm 3.5k_0\sqrt{1+9k}\,\left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right)^t + T$	$1.5\,k_0\sqrt{1+9k}\,(1 \quad 0 \quad 1)^t$ $\pm\ 3.5k_0\sqrt{1+9k}\,\left(\frac{1}{\sqrt{2}} \quad 1 \quad \frac{1}{\sqrt{2}}\right)^t + T$	$1.5\,k_0\sqrt{1+9k}\,(1 \quad 0 \quad 1 \quad 0 \quad 1)^t$ $\pm 3.5k_0\sqrt{1+9k}\,\left(\frac{1}{\sqrt{2}} \quad 1 \quad \frac{1}{\sqrt{2}} \quad 1 \quad \frac{1}{\sqrt{2}}\right)^t + T$

$k_0 = 1.96936$ for $k = 0.05$, $k_0 = 1{,}86551$ for $k = 0.10$ y $k_0 = 1.77060$ for $k = 0.15$. These values were determined so that the individual specifications over each variable contain approximately 99.95% of process data, under capable processes.

Table1-Part c) Multivariate gamma distribution case.

		$p = 2$	$p = 3$	$p = 5$
Parameters		$\boldsymbol{\mu} = (15 \quad 30)`$ $\boldsymbol{\Sigma} = 15\begin{pmatrix} 1 & \sqrt{2}\rho \\ \sqrt{2}\rho & 2 \end{pmatrix}$	$\boldsymbol{\mu} = (15 \quad 30 \quad 30)`$ $\boldsymbol{\Sigma} = 15\begin{pmatrix} 1 & \sqrt{2}\rho & \sqrt{2}\rho \\ \sqrt{2}\rho & 2 & 2\rho \\ \sqrt{2}\rho & 2\rho & 2 \end{pmatrix}$	$\boldsymbol{\mu} = (15 \quad 15 \quad 30 \quad 30 \quad 30)`$ $\boldsymbol{\Sigma} = 15\begin{pmatrix} 1 & \rho & \sqrt{2}\rho & \sqrt{2}\rho & \sqrt{2}\rho \\ \rho & 1 & \sqrt{2}\rho & \sqrt{2}\rho & \sqrt{2}\rho \\ \sqrt{2}\rho & \sqrt{2}\rho & 2 & 2\rho & 2\rho \\ \sqrt{2}\rho & \sqrt{2}\rho & 2\rho & 2 & 2\rho \\ \sqrt{2}\rho & \sqrt{2}\rho & 2\rho & 2\rho & 2 \end{pmatrix}$
S-I	$\boldsymbol{T}$ $\boldsymbol{SL}$	$\boldsymbol{\mu}$ $\pm 3.5k_{0\alpha}\sqrt{15}(1 \quad \sqrt{2})` + T$	$\boldsymbol{\mu}$ $\pm 3.5k_{0\alpha}\sqrt{15}(1 \quad \sqrt{2} \quad \sqrt{2})` + T$	$\boldsymbol{\mu}$ $\pm 3.5k_{0\alpha}\sqrt{15}\ (1 \quad 1 \quad \sqrt{2} \quad \sqrt{2} \quad \sqrt{2})` + T$
S-II	$\boldsymbol{T}$ $\boldsymbol{SL}$	$\boldsymbol{\mu}$ $\pm 3.5\sqrt{15}(k_{0\alpha} \quad k_{\alpha}\sqrt{2})` + T$	$\boldsymbol{\mu}$ $\pm 3.5\sqrt{15}\ (k_{0\alpha} \quad k_{0\alpha}\sqrt{2} \quad k_{\alpha}\sqrt{2})^t + T$	$\boldsymbol{\mu}$ $\pm 3.5\sqrt{15}\ (k_{0\alpha} \quad k_{0\alpha} \quad k_{0\alpha}\sqrt{2} \quad k_{0\alpha}\sqrt{2} \quad k_{\alpha}\sqrt{2})^t + T$
S-III	$\boldsymbol{T}$ $\boldsymbol{SL}$	$\boldsymbol{\mu}$ $\pm 3.5k_{\alpha}\sqrt{15}(1 \quad \sqrt{2})` + T$	$\boldsymbol{\mu}$ $\pm 3.5\sqrt{15}\ (k_{\alpha} \quad k_{0\alpha}\sqrt{2} \quad k_{\alpha}\sqrt{2})^t + T$	$\boldsymbol{\mu}$ $\pm 3.5\sqrt{15}\ (k_{\alpha} \quad k_{0\alpha} \quad k_{\alpha}\sqrt{2} \quad k_{0\alpha}\sqrt{2} \quad k_{\alpha}\sqrt{2})^t + T$
S-IV	$\boldsymbol{T}$ $\boldsymbol{SL}$	$\boldsymbol{\mu} + 1.5k_{0\alpha}\sqrt{15}\ (1 \quad 0)`$ $\pm 3.5k_{0\alpha}\sqrt{15}(1 \quad \sqrt{2})` + T$	$\boldsymbol{\mu} + 1.5k_{0\alpha}\sqrt{15}\ (1 \quad 0 \quad 0)^t$ $\pm 3.5k_{0\alpha}\sqrt{15}(1 \quad \sqrt{2} \quad \sqrt{2})^t + T$	$\boldsymbol{\mu} + 1.5k_{0\alpha}\sqrt{15}(1 \quad 0 \quad 0 \quad 0 \quad 0)^t$ $\pm 3.5k_{0\alpha}\sqrt{15}\ (1 \quad 1 \quad \sqrt{2} \quad \sqrt{2} \quad \sqrt{2})^t + T$
S-V	$\boldsymbol{T}$ $\boldsymbol{SL}$	$\boldsymbol{\mu} + 1.5k_{0\alpha}\sqrt{15}\ (1 \quad \sqrt{2})`$ $\pm 3.5k_{0\alpha}\sqrt{15}(1 \quad \sqrt{2})` + T$	$\boldsymbol{\mu} + 1.5k_{0\alpha}\sqrt{15}\ (1 \quad 0 \quad \sqrt{2})^t$ $\pm 3.5k_{0\alpha}\sqrt{15}(1 \quad \sqrt{2} \quad \sqrt{2})^t + T$	$\boldsymbol{\mu} + 1.5k_{0\alpha}\sqrt{15}\ (1 \quad 0 \quad \sqrt{2} \quad 0 \quad \sqrt{2})^t$ $\pm 3.5k_{0\alpha}\sqrt{15}\ (1 \quad 1 \quad \sqrt{2} \quad \sqrt{2} \quad \sqrt{2})^t + T$
S-VI	$\boldsymbol{T}$ $\boldsymbol{SL}$	$\boldsymbol{\mu} + 1.5k_{0\alpha}\sqrt{15}\ (1 \quad \sqrt{2})`$ $\pm 3.5k_{\alpha}\sqrt{15}(1 \quad \sqrt{2})` + T$	$\boldsymbol{\mu} + 1.5k_{0\alpha}\sqrt{15}\ (1 \quad 0 \quad \sqrt{2})^t$ $\pm 3.5\sqrt{15}\ (k_{\alpha} \quad k_{0\alpha}\sqrt{2} \quad k_{\alpha}\sqrt{2})^t + T$	$\boldsymbol{\mu} + 1.5k_{0\alpha}\sqrt{15}\ (1 \quad 0 \quad \sqrt{2} \quad 0 \quad \sqrt{2})^t$ $\pm 3.5\sqrt{15}\ (k_{\alpha} \quad k_{0\alpha} \quad k_{\alpha}\sqrt{2} \quad k_{0\alpha}\sqrt{2} \quad k_{\alpha}\sqrt{2})^t + T$

$k_{0\alpha} = 1.10657$ and $k_{\alpha} = 0.73697$ for $\alpha = 15$ y $k_{0\alpha} = 0.99738$ and $k_{\alpha} = 0.81063$ for $\alpha = 30$. The $k_{0\alpha}$ values were determined in order to obtain, under capable processes and for each individual variable, symmetric specification intervals containing a proportion of process data as close to 0.9995 as possible. The k_{α} values were determined so that the individual specification width for each variable is twice the distance between the process mean and the 0.02333th percentile of the corresponding marginal gamma distribution ($2(\mu - p_{0.0233\%,\alpha})$).

Table 2: Values of MC_{pm} index, reference values (C_p) and cutoff point for each scenario, under multivariate normal distribution.

		$p=2$			$p=3$			$p=5$		
Correlation ρ		**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**
$\Phi_{p;\,0{,}0027}$		**1,0000**	**1,0000**	**1,0000**	**1,0000**	**1,0000**	**1,0000**	**1,0000**	**1,0000**	**1,0000**
S-I	MC_{pm}	1,0408	1,1958	2,3758	0,8165	1,1384	4,8106	0,3875	0,8577	17,3172
	C_p	1,1036	1,1051	1,1237	1,0653	1,0598	1,1328	1,0152	1,0236	1,0936
S-II	MC_{pm}	0,7360	0,8456	1,6799	0,5773	0,8050	3,4016	0,2740	0,6065	12,2451
	C_p	0,8209	0,8219	0,8248	0,8170	0,8193	0,8248	0,8096	0,8138	0,8247
S-III	MC_{pm}	0,5204	0,5979	1,1879	0,4082	0,5692	2,4053	0,1370	0,3033	6,1225
	C_p	0,7399	0,7462	0,7773	0,7377	0,7445	0,7776	0,6839	0,6972	0,7522
S-IV	MC_{pm}	0,7341	0,7828	0,9493	0,5747	0,7200	1,7241	0,2719	0,5253	5,7585
	C_p	0,9042	0,9051	0,9080	0,8968	0,8980	0,9061	0,8833	0,8848	0,9082
S-V	MC_{pm}	0,6200	0,7828	1,6583	0,4835	0,6573	1,6858	0,2037	0,4289	4,7322
	C_p	0,8340	0,8398	0,8681	0,8298	0,8356	0,8673	0,7799	0,7923	0,8442
S-VI	MC_{pm}	0,3100	0,3914	0,8291	0,2417	0,3286	0,8429	0,0720	0,1516	1,6731
	C_p	0,4997	0,5167	0,5577	0,4992	0,5164	0,5578	0,4350	0,4669	0,5342

Table 3: Values of NMC_{pm} index, reference values (C_p) and cutoff points for each scenario, under multivariate normal distribution.

		$p=2$			$p=3$			$p=5$		
Correlation ρ		**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**
$\Phi_{p;\,0{,}0027}$		**0,9319**	**0,9319**	**0,9319**	**0,8824**	**0,8824**	**0,8824**	**0,8109**	**0,8109**	**0,8109**
S-I	NMC_{pm}	1,0176	1,0176	1,0176	0,9302	0,9302	0,9302	0,8203	0,8203	0,8203
	C_p	1,1036	1,1051	1,1237	1,0653	1,0598	1,1328	1,0152	1,0236	1,0936
S-II	NMC_{pm}	0,7196	0,7196	0,7196	0,6578	0,6578	0,6578	0,5800	0,5800	0,5800
	C_p	0,8209	0,8219	0,8248	0,8170	0,8193	0,8248	0,8096	0,8138	0,8247
S-III	NMC_{pm}	0,7196	0,7196	0,7196	0,6578	0,6578	0,6578	0,5800	0,5800	0,5800
	C_p	0,7399	0,7462	0,7773	0,7377	0,7445	0,7776	0,6839	0,6972	0,7522
S-IV	NMC_{pm}	0,5856	0,5314	0,2997	0,6548	0,5883	0,3334	0,5755	0,5023	0,2728
	C_p	0,9042	0,9051	0,9080	0,8968	0,8980	0,9061	0,8833	0,8848	0,9082
S-V	NMC_{pm}	0,4726	0,5314	0,5775	0,5508	0,5371	0,3260	0,4312	0,4101	0,2242
	C_p	0,8340	0,8398	0,8681	0,8298	0,8356	0,8673	0,7799	0,7923	0,8442
S-VI	NMC_{pm}	0,3342	0,3758	0,4083	0,3895	0,3798	0,2305	0,3049	0,2900	0,1585
	C_p	0,4997	0,5167	0,5577	0,4992	0,5164	0,5578	0,4350	0,4669	0,5342

Table 4: Values of $MC_{pm}^{(W)}$ index, reference values (C_p) and cutoff points for each scenario, under multivariate normal distribution.

		$p=2$			$p=3$			$p=5$		
Correlation ρ		**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**
$\Phi_{p;\,0{,}0027}$		**1,000**	**1,000**	**1,000**	**1,000**	**1,000**	**1,000**	**1,000**	**1,000**	**1,000**
S-I	$MC_{pm}^{(W)}$	0,0000	0,0000	1,1970	0,0000	0,0000	1,2076	0,0000	0,0000	1,2163
	C_p	1,1036	1,1051	1,1237	1,0653	1,0598	1,1328	1,0152	1,0236	1,0936
S-II	$MC_{pm}^{(W)}$	0,5848	0,6268	1,0217	0,2012	0,4496	1,0897	0,1804	0,2968	1,1451
	C_p	0,8209	0,8219	0,8248	0,8170	0,8193	0,8248	0,8096	0,8138	0,8247
S-III	$MC_{pm}^{(W)}$	0,0000	0,0000	0,8464	0,3773	0,5200	0,9718	0,2782	0,3236	1,0026
	C_p	0,7399	0,7462	0,7773	0,7377	0,7445	0,7776	0,6839	0,6972	0,7522
S-IV	$MC_{pm}^{(W)}$	0,0000	0,0000	1,0650	0,0000	0,0000	1,1416	0,0000	0,0000	1,1907
	C_p	0,9042	0,9051	0,9080	0,8968	0,8980	0,9061	0,8833	0,8848	0,9082
S-V	$MC_{pm}^{(W)}$	0,0000	0,0000	0,8355	0,0000	0,0000	0,9939	0,0000	0,0000	1,0312
	C_p	0,8340	0,8398	0,8681	0,8298	0,8356	0,8673	0,7799	0,7923	0,8442
S-VI	$MC_{pm}^{(W)}$	0,0000	0,0000	0,5908	0,3020	0,4154	0,7999	0,2296	0,2709	0,8500
	C_p	0,4997	0,5167	0,5577	0,4992	0,5164	0,5578	0,4350	0,4669	0,5342

Table 5: Values of Mp_2 index, reference values (C_p) and cutoff points for each scenario, under multivariate normal distribution.

		$p = 2$			$p = 3$			$p = 5$		
Correlation ρ		**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**
$\Phi_{p;\,0{,}0027}$		**0,9973**	**0,9973**	**0,9973**	**0,9973**	**0,9973**	**0,9973**	**0,9973**	**0,9973**	**0,9973**
S-I	**MP_2**	0,9995	0,9995	0,9999	0,9984	0,9992	0,9994	0,9982	0,9980	0,9998
	Reference	0,9991	0,9991	0,9993	0,9986	0,9985	0,9993	0,9977	0,9979	0,9990
S-II	**MP_2**	0,9880	0,9873	0,9878	0,9870	0,9947	0,9898	0,9774	0,9811	0,9890
	Reference	0,9862	0,9863	0,9867	0,9858	0,9860	0,9866	0,9848	0,9854	0,9866
S-III	**MP_2**	0,9723	0,9725	0,9866	0,9692	0,9859	0,9884	0,9708	0,9561	0,9905
	Reference	0,9736	0,9748	0,9803	0,9731	0,9745	0,9803	0,9598	0,9635	0,9760
S-IV	**MP_2**	0,9927	0,9936	0,9934	0,9922	0,9979	0,9966	0,9873	0,9908	0,9933
	Reference	0,9933	0,9934	0,9936	0,9929	0,9929	0,9934	0,9919	0,9921	0,9936
S-V	**MP_2**	0,9881	0,9884	0,9950	0,9893	0,9919	0,9951	0,9859	0,9781	0,9966
	Reference	0,9877	0,9882	0,9908	0,9872	0,9878	0,9907	0,9807	0,9825	0,9887
S-VI	**MP_2**	0,8679	0,8714	0,9350	0,8594	0,9129	0,9379	0,8679	0,8320	0,9368
	Reference	0,8662	0,8789	0,9057	0,8658	0,8787	0,9058	0,8081	0,8387	0,8910

Table 6: Values of NMC_{pm} index, reference values (C_p) and cutoff points for each scenario, under mixture of multivariate normal distribution ($k = 0.10$).

		$p = 2$			$p = 3$			$p = 5$		
Correlation ρ		**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**
$\Phi_{p;\,0{,}0027}$		**0,9246**	**0,9246**	**0,9246**	**0,8747**	**0,8747**	**0,8747**	**0,8051**	**0,8051**	**0,8051**
SI	**NMC_{pm}**	1,8984	1,8984	1,8984	1,7354	1,7354	1,7354	1,5303	1,5303	1,5303
	C_p	1,1084	1,1109	1,1272	1,0704	1,0768	1,1039	1,0210	1,0300	1,0751
SII	**NMC_{pm}**	1,3415	1,2263	1,0814	1,3415	1,2263	1,0814	1,3415	1,2263	1,0814
	C_p	0,9389	0,9409	0,9473	0,9300	0,9340	0,9462	0,9140	0,9218	0,9439
SIII	**NMC_{pm}**	1,3415	1,2263	1,0814	1,3415	1,2263	1,0814	1,3415	1,2263	1,0814
	C_p	0,8751	0,8813	0,9047	0,8699	0,8778	0,9046	0,8238	0,8382	0,8812
SIV	**NMC_{pm}**	0,6356	0,5609	0,2920	0,5789	0,4858	0,2357	0,5075	0,4080	0,1912
	C_p	0,9230	0,9242	0,9267	0,9152	0,9179	0,9241	0,9010	0,9068	0,9190
SV	**NMC_{pm}**	0,4860	0,5609	0,6240	0,4405	0,4249	0,2298	0,3297	0,3090	0,1544
	C_p	0,8577	0,8673	0,8915	0,8532	0,8632	0,8893	0,8076	0,8262	0,8663
SVI	**NMC_{pm}**	0,3434	0,3964	0,4410	0,3113	0,3002	0,1624	0,2330	0,2184	0,1091
	C_p	0,6559	0,6692	0,6961	0,6551	0,6685	0,6952	0,6089	0,6314	0,6741

Table 7: Values of Mp_2 index, reference values (C_p) and cutoff points for each scenario, under mixture of multivariate normal distribution ($k = 0.10$).

		$p = 2$			$p = 3$			$p = 5$		
Correlation ρ		**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**
$\Phi_{p;\,0{,}0027}$		**0,9973**	**0,9973**	**0,9973**	**0,9973**	**0,9973**	**0,9973**	**0,9973**	**0,9973**	**0,9973**
SI	**MP_2**	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
	Reference	0,9991	0,9987	0,9978	0,9991	0,9988	0,9980	0,9993	0,9991	0,9987
SII	**MP_2**	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
	Reference	0,9951	0,9947	0,9939	0,9952	0,9949	0,9943	0,9955	0,9955	0,9954
SIII	**MP_2**	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000
	Reference	0,9913	0,9909	0,9865	0,9918	0,9915	0,9881	0,9934	0,9933	0,9918
SIV	**MP_2**	0,9999	1,0000	0,9999	0,9999	0,9997	1,0000	0,9997	0,9999	0,9999
	Reference	0,9944	0,9944	0,9946	0,9940	0,9941	0,9944	0,9931	0,9935	0,9942
SV	**MP_2**	0,9999	0,9998	0,9999	0,9998	1,0000	1,0000	0,9997	0,9997	0,9998
	Reference	0,9899	0,9907	0,9925	0,9895	0,9904	0,9924	0,9846	0,9868	0,9906
SVI	**MP_2**	0,9300	0,9353	0,9684	0,9392	0,9500	0,9694	0,9224	0,9176	0,9684
	Reference	0,9509	0,9553	0,9632	0,9506	0,9551	0,9630	0,9323	0,9418	0,9568

Table 8: Values of NMC_{pm} index, reference values (C_p) and cutoff points for each scenario, under multivariate gamma distribution.

		$p = 2$			$p = 3$			$p = 5$		
Correlation ρ		**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**
$\Phi_{p;\,0,01}$		**0,7831**	**0,7831**	**0,7831**	**0,7603**	**0,7603**	**0,7603**	**0,7028**	**0,7028**	**0,7028**
SI	NMC_{pm}	1,0150	1,0150	1,0150	0,9278	0,9278	0,9278	0.8181	0.8181	0.8181
	C_p	1,0174	1,0153	1,0444	0,9702	0,9762	1,0153	0,9129	0,9250	0,9785
SII (*)	NMC_{pm}	0,8249	0,8249	0,8249	0,7541	0,7541	0,7541	0,6650	0,6650	0,6650
	C_p	0,8916	0,8928	0,9022	0,8817	0,8862	0,9070	0,8579	0,8665	0,9030
SIII	NMC_{pm}	0,7500	0,7500	0,7500	0,6856	0,6856	0,6856	0,6045	0,6045	0,6045
	C_p	0,7830	0,7903	0,8119	0,7781	0,7870	0,8200	0,7407	0,7542	0,8003
SIV (*)	NMC_{pm}	0,5224	0,4700	0,2581	0,4760	0,4100	0,2093	0,4178	0,3464	0,1702
	C_p	0,9604	0,9610	0,9615	0,9256	0,9313	0,9383	0,8901	0,8919	0,9246
SV	NMC_{pm}	0,4315	0,4872	0,5243	0,3915	0,3789	0,2135	0,3027	0,2851	0,1471
	C_p	0,8129	0,8141	0,8354	0,8007	0,8017	0,8213	0,7297	0,7394	0,7800
SVI	NMC_{pm}	0,3188	0,3600	0,3874	0,2893	0,2800	0,1578	0,2236	0,2106	0,1087
	C_p	0,3925	0,4092	0,4437	0,3909	0,4078	0,4399	0,3471	0,3814	0,3814

***(*)** Under this distribution, these situations correspond to capable processes (reference C_p value > 0.8586).*

Table 9: Values of Mp_2 index, reference values (C_p) and cutoff points for each scenario, under multivariate gamma distribution.

		$p = 2$			$p = 3$			$p = 5$		
Correlation ρ		**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**	**0,10**	**0,50**	**0,90**
$\Phi_{p;\,0,01}$		**0,9900**	**0,9900**	**0,9900**	**0,9900**	**0,9900**	**0,9900**	**0,9900**	**0,9900**	**0,9900**
SI	MP_2	0,9994	0,9996	0,9993	0,9992	0,9985	0,9989	0,9984	0,9991	0,9999
	Reference	0,9977	0,9977	0,9983	0,9964	0,9966	0,9977	0,9938	0,9945	0,9967
SII (*)	MP_2	0,9954	0,9952	0,9949	0,9955	0,9947	0,9966	0,9935	0,9963	0,9968
	Reference	0,9925	0,9926	0,9932	0,9918	0,9922	0,9935	0,9899	0,9907	0,9933
SIII	MP_2	0,9863	0,9859	0,9929	0,9954	0,9954	0,9934	0,9860	0,9890	0,9933
	Reference	0,9812	0,9823	0,9823	0,9804	0,9818	0,9861	0,9737	0,9763	0,9837
SIV (*)	MP_2	0,9868	0,9858	0,9879	0,9988	0,9993	0,9909	0,9964	0,9978	0,9896
	Reference	0,9960	0,9961	0,9961	0,9945	0,9948	0,9951	0,9924	0,9925	0,9945
SV	MP_2	0,9639	0,9658	0,9767	0,9745	0,9748	0,9757	0,9445	0,9533	0,9786
	Reference	0,9853	0,9854	0,9878	0,9837	0,9838	0,9863	0,9714	0,9735	0,9807
SVI	MP_2	0,7466	0,7762	0,8380	0,9066	0,9059	0,9097	0,7226	0,8249	0,8363
	Reference	0,7610	0,7804	0,8169	0,7591	0,7789	0,8131	0,7022	0,7475	0,8102

***(*)** Under this distribution, these situations correspond to capable processes (reference C_p value > 0.8586).*