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# A new robust and most powerful test in the presence of local misspecification

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**ABSTRACT**
This article proposes a new test that is consistent, achieves correct asymptotic size, and is locally most powerful under local misspecification, and when any $\sqrt{n}$-estimator of the nuisance parameters is used. The new test can be seen as an extension of the Bera and Yoon (1993) procedure that deals with non maximum likelihood (ML) estimation, while preserving its optimality properties. Similarly, the proposed test extends Neyman's (1959) $C(\alpha)$ test to handle locally misspecified alternatives. A Monte Carlo study investigates the finite sample performance in terms of size, power, and robustness to misspecification.

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## 1. Introduction

A standard practice in applied econometrics is to start by estimating a small model and then checking whether departures away from it are supported or not by the data. Rao's (1948) score (henceforth, RS) or Lagrange multiplier tests are convenient since, unlike likelihood ratio and Wald tests, they require estimation of only the restricted model under the null hypothesis.

The performance of RS tests depends on how the model is estimated and on whether the alternative hypothesis is correctly specified. Consider a model consisting of a probability distribution characterized by three vectors of parameters: $\theta_1$, $\theta_2$, and $\theta_3$. Suppose that the primary interest is to test $H_0^2 : \theta_2 = \theta_{20}$ in a situation where $\theta_1$ can be easily estimated under the joint null $H_0^{23} : \theta_2 = \theta_{20},\ \theta_3 = \theta_{30}$. The properties of a test for $H_0^2$ derived in such context depend on (1) how $\theta_1$ is estimated and (2) whether $H_0^3 : \theta_3 = \theta_{30}$ holds.

When $\theta_1$ is estimated by maximum likelihood (ML) under the joint null $H_0^{23}$, the RS test for $H_0^2$ is consistent, has correct asymptotic size, and is locally most powerful when the alternative model is correctly specified, i.e., when $H_0^3$ holds and thus the only deviation away from the joint null is due to $H_0^2$ being false (see Rao and Poti, 1946; Rao, 1948; Cox and Hinkley, 1974; Bera and Bilias, 2001a). If any other $\sqrt{n}$-consistent estimator of $\theta_1$ under $H_0^{23}$ is used, Neyman's (1959) $C(\alpha)$ test is asymptotically equivalent to the RS and hence inherits all its optimality properties (see Smith, 1987; Bera and Bilias, 2001b).

When the alternative hypothesis is misspecified ($H_0^3 : \theta_3 \neq \theta_{30}$), both RS and $C(\alpha)$ tests reject $H_0^2$ spuriously, as shown by Davidson and MacKinnon (1987) and Saikkonen (1989).

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That is, they reject $H_0^2$ not because of being false but due to the fact that $H_0^3$ does not hold. For example, Bera et al. (2001) find that the standard Breusch and Pagan (1979) test for random effects in the error component model spuriously rejects its null under the presence of serial correlation. Bera and Yoon (1993) (henceforth, BY) propose a modification of the RS test for $H_0^2$ that is still based on the ML estimation of $\theta_1$ under $H_0^{23}$, but unlike *RS* and $C(\alpha)$ tests, is consistent, and has correct asymptotic size under local misspecification. The BY test can be shown to be asymptotically equivalent to a $C(\alpha)$ test and hence it is also locally most powerful. The BY principle has been successfully implemented in many econometric "model search" problems, for instance see Anselin et al. (1996), Godfrey and Veall (2000), Bera et al. (2001), Baltagi and Li (2001), and Montes-Rojas (2010, 2011).

The use of an ML estimator (MLE) is an obvious restriction on the applicability of BY tests. Bera et al. (2010) (henceforth, BMS) extended the BY principle to the GMM (generalized method of moments) framework, proposing a test that is consistent and has correct asymptotic size for any initial GMM estimator and under locally misspecified alternatives.

In Box's (1953) characterization, the $C(\alpha)$ and the BY tests possess the *robustness of efficiency* property (see Welsh, 1996, pp. 242–243), in the sense that both, size and power, are preserved with respect to the original RS test. On the contrary, the test suggested by BMS is only *validity robust*, since it preserves consistency and correct asymptotic size but not necessarily efficiency.

In this article we propose a new test that is still based on any $\sqrt{n}$-consistent estimator of $\theta_1$ and has the robustness of efficiency property under local misspecification. Consequently, the proposed test improves upon three existing strategies by (a) allowing for non ML estimation in the BY test, (b) allowing for locally misspecified alternatives in the $C(\alpha)$ procedure, and (c) restoring asymptotic efficiency of BMS test. Intuitively, the new test is derived by applying a double $C(\alpha)$-style correction that deals simultaneously with the non ML estimation and locally misspecified alternatives.

The practical relevance of the proposed tests relates to situations where simple estimators for relevant parameters are readily available, as compared to fully MLEs. Linear panel data error components models are one example of such scenario, where method-of-moments estimators of the variance components are much simpler to compute than MLEs. For example, Baltagi et al. (2001, 2002) consider a nested error components model $y_{ijt} = x'_{ijt}\beta + u_{ijt}$ with $u_{ijt} = \mu_i + \nu_{ij} + \epsilon_{ijt}$. Normality of the error components is assumed to develop a testing framework for the appropriate nested variance structure. An RS test for the presence of the random effect $\mu_i$ (or $\nu_{ij}$) being present requires the estimation of $\beta$ and the variance of $\nu_{ij}$ (or $\mu_i$) and $\epsilon_{ijt}$. Baltagi et al. (2001) suggest that, even though a fully MLE is available under normality, much simpler method-of-moments estimators of the nuisance parameters are very good competitors. Moreover, tests for the presence of either $\mu_i$ or $\nu_{ij}$ are also constructed as BY robust test for local misspecification of the random component in the level not being tested (Baltagi et al., 2002). This is a clear example of a situation where the tests proposed in this article can be very useful in practice, since they can be based on any consistent estimate, bypassing the need of initial ML estimation. We discuss a second example of least-squares and quantile regression models in the context of our Monte Carlo study (in Section 4) where the finite sample size and power of the tests are studied.

The rest of the article is organized as follows. In Section 2, we review the loss of efficiency associated with non ML estimation of $\theta_1$, the $C(\alpha)$ approach (that preserves size and power of RS tests) and a new intermediate "modified RS" test, that only restores size. We then show in Section 3 that, as in the case of the RS test, though being able to accommodate non MLEs of $\theta_1$, both strategies are negatively affected when the alternative hypothesis is misspecified.

We thus introduce the new tests that are resistant to non MLEs and locally misspecified alternatives. We complement our theoretical analysis with a Monte Carlo experiment in Section 4, which investigates the small sample performance of the tests. Section 5 concludes.

## 2. Testing with non MLEs

Consider the following parametric model for independent and identically distributed (iid) random samples.

**Assumption 1.** Parametric model:

(i) Let $\{z_i\}_{i=1}^n$ be a random sample of iid random vectors $z_i \in \mathcal{Z} \subset \Re^K$.

(ii) Let the parametric family of models for the density of $z$ be given by $\{f(.|\theta) : \theta \in \Theta\}$ where $\Theta \subset \Re^p$ is a compact set that can be partitioned as $\Theta = \Theta_1 \times \Theta_2 \times \Theta_3$, subsets of $\Re^{p_1}$, $\Re^{p_2}$, and $\Re^{p_3}$, $p = p_1 + p_2 + p_3$, respectively, with typical element $\theta = (\theta_1', \theta_2', \theta_3')'$ and $f(.|\theta)$ is a density function to the measure $\upsilon(dz)$ for all $\theta \in \Theta$.

(iii) For some $\theta_0 \in int(\Theta)$, $\theta_0 = \text{argmax}_{\theta\in\Theta} E[l(z,\theta)]$ is unique, where $E[\cdot] = \int_{\mathcal{Z}} \cdot f(z,\theta_0)\upsilon(dz)$ and $\ell(z,\theta) = \ln \text{f}(z|\theta)$.

(iv) For each $\theta \in \Theta$, $\ell(.,\theta)$ is a Borel measureable function on $\mathcal{Z}$, and for each $z \in \mathcal{Z}$, $\ell(z,.)$ is a continuous function on $\Theta$.

Define $\ell(\theta) = \frac{1}{n}\sum_{i=1}^n \ell(z_i,\theta)$ as the log-likelihood. Let $d(z,\theta) = \partial l(z,\theta)/\partial\theta$ and $d(\theta) = \partial\ell(\theta)/\partial\theta$ be the score vectors (we will use $d_j(z,\theta)$ and $d_j(\theta)$ to denote the corresponding $p_j \times 1$ subvector $\partial l(\theta)/\partial\theta_j$, with $j = 1, 2, 3$). Moreover, let $J(z,\theta) = -\partial^2 l(z,\theta)/\partial\theta\partial\theta'$ be a $p \times p$ matrix of second partial derivatives, and

$$J(\theta) = -E\left[\frac{\partial^2 l(z,\theta)}{\partial\theta\partial\theta'}\right] = \begin{bmatrix} J_{11}(\theta) & J_{12}(\theta) & J_{13}(\theta) \\ J_{21}(\theta) & J_{22}(\theta) & J_{23}(\theta) \\ J_{31}(\theta) & J_{32}(\theta) & J_{33}(\theta) \end{bmatrix}$$

denote the information matrix. For notational convenience we write $J(\theta_0) = J$, i.e., we omit the dependence on $\theta$ when the functionals are evaluated at $\theta_0$.

**Assumption 2.** Scores and information matrix:

(i) $\ell(z,.)$ is twice continuously differentiable on int$(\Theta)$.

(ii) All elements in $\ell(z,\theta)$, $d(z,\theta)$, $d(z,\theta)d(z,\theta)'$, $J(z,\theta)$ are bounded in absolute value by a function $b(z)$ with $E[b(z)] < \infty$ for all $\theta \in \Theta$.

(iii) $J$ is positive definite.

Assumptions 1 and 2 provide sufficient conditions for identification, $\sqrt{n}$-consistency, and asymptotic normality of an MLE for iid random samples. These correspond to the assumptions of Theorems 13.1 and 13.2 in Wooldridge (2010) and Assumptions 1–9 for score functions in Newey (1985).

Consider first the problem of testing $H_0^2 : \theta_2 = \theta_{20}$ under the local alternative $H_A^2 : \theta_2 = \theta_{20} + \delta_2/\sqrt{n}$, $0 < \delta_2 < \infty$, and when $H_0^3 : \theta_3 = \theta_{30}$ holds. In this case the alternative hypothesis is said to be *correctly specified*, in the sense that $H_0^3$ holds, i.e., the only departure away from the joint null $H_0^{23} : \theta_2 = \theta_{20},\ \theta_3 = \theta_{30}$ is due to $\theta_2$ being different from $\theta_{20}$. Under this set up, the form of the optimal RS test statistic is given by

$$RS_{2\cdot1}(\theta) = n\, d_2(\theta)'\, J_{2\cdot1}^{-1}\, d_2(\theta) \tag{1}$$

where $J_{2\cdot1} = J_2 - J_{21}J_{11}^{-1}J_{12}$. Let $\hat{\theta} = (\hat{\theta}_1', \theta_{20}', \theta_{30}')'$, where $\hat{\theta}_1$ is the restricted MLE of $\theta_1$ under the joint null $H_0^{23}$. A standard result is that under $H_A^2$, and $H_0^3$, $RS_{2\cdot1}(\hat{\theta}) \xrightarrow{d} \chi^2_{p_2}(\lambda_{2\cdot1})$, as

$n \to \infty$, where the non centrality parameter is $\lambda_{2\cdot 1} = \delta_2' J_{2\cdot 1}\delta_2$. Thus under $H_0^2$, $RS_{2\cdot 1}(\hat{\theta})$ has, asymptotically, a *central* chi-squared distribution, ensuring its correct asymptotic size. Also, as mentioned in Section 1, $RS_{2\cdot 1}(\hat{\theta})$ is locally most powerful.

In certain contexts it might be difficult to obtain the MLE $\hat{\theta}_1$, while a $\sqrt{n}$-consistent estimator $\tilde{\theta}_1$ may be easily available. However, the use of a $\sqrt{n}$-consistent estimator other than the MLE affects the asymptotic properties of the RS test. Assume that an M-estimator $\tilde{\theta} = (\tilde{\theta}_1', \theta_{20}', \theta_{30}')'$ is available, defined as $\tilde{\theta}_1 = \operatorname{argmin}_{\theta_1 \in \Theta_1} \sum_{i=1}^n q(z_i, \theta_1, \theta_2, \theta_3)$ for $q(z, \theta)$ an objective function of the random vector $z$. Assume that a general estimating function $h_1(\theta) = \frac{1}{n}\sum_{i=1}^n h_1(z_i, \theta)$ exists, where $h_1(z, \theta) \equiv \partial q(z, \theta)/\partial\theta_1$, and that $\tilde{\theta}_1$ is the unique zero of $h_1(.)$ for all $n$ and for all $(\theta_2, \theta_3) \in (\Theta_2 \times \Theta_3)$. For example, the restricted MLE corresponds to $h_1(\theta_1, \theta_{20}, \theta_{30}) = d_1(\theta_1, \theta_{20}, \theta_{30})$, so in this case $\tilde{\theta}_1 = \hat{\theta}_1$. Define $H_1(\theta) = E[h_1(z, \theta)h_1(z, \theta)']$ and $B_1(\theta) = E[\partial h_1(z, \theta)/\partial\theta_1]$. For notational convenience we omit the dependence on $\theta$ when the functionals are evaluated at $\theta_0$. We consider the following assumptions:

**Assumption 3.** M-estimators:

(i) $\theta_{10} = \operatorname{argmin}_{\theta_1 \in \Theta_1} E[q(z, \theta_1, \theta_{20}, \theta_{30})]$ is unique and $E[h_1(z, \theta_1, \theta_{20}, \theta_{30})] = 0$ only if $\theta_1 = \theta_{10}$.

(ii) For each $\theta \in \Theta$, $(q(., \theta), h_1(., \theta))$ are Borel measureable functions on $\mathcal{Z}$, and for each $z \in \mathcal{Z}$, $(q(z, .), h_1(z, .))$ are continuous functions on $\Theta$.

(iii) $q(z, \theta)$ is twice continuously differentiable on $int(\Theta)$.

(iv) All elements in $q(z, \theta)$, $h_1(z, \theta)$, $h_1(z, \theta)h_1(z, \theta)'$, $h_1(z, \theta)d(z, \theta)'$, $\partial h_1(z, \theta)/\partial\theta$ are bounded in absolute value by a function $b(z)$ with $E[b(z)] < \infty$ for all $\theta \in \Theta$.

(v) $B_1$ is positive definite.

(vi) Let $w(z, \theta) = [h_1(z, \theta)' \;\; d(z, \theta)']'$, $E[w(z, \theta_0)w(z, \theta_0)']$ be positive definite.

Assumption 3, together with 1 and 2, guarantees identification, $\sqrt{n}$-consistency, and asymptotic normality of $\tilde{\theta}_1$ under $H_0^{23}$, given by $\sqrt{n}(\tilde{\theta}_1 - \theta_{10}) \xrightarrow{d} N(0_{p_1}, B_1^{-1}H_1B_1^{-1})$, as $n \to \infty$. See Wooldridge (2010, ch. 12) for a general discussion on M-estimators. Assumptions 1–3 correspond to Assumptions 1–9 in Newey (1985), in which case $\tilde{\theta}_1$ is defined as an Z-estimator based on the estimating function $h_1(z, \theta)$. Dependent random vectors (i.e., time-series) can be addressed with the use of the statistical model of Newey and West (1987). Moreover, the assumptions can be relaxed for non smooth log-likelihood or $q$-objective functions (e.g., quantile regression models) following Newey and McFadden (1994). For the sake of brevity, we do not discuss the standard regularity conditions for consistency of estimators for $J$, $B_1$, and $H_1$, and we assume that all matrices that need to be inverted in the construction of the statistics in this article are non singular.

$RS_{2\cdot 1}(\tilde{\theta})$ is no longer asymptotically chi-squared distributed, since it is based on an incorrect variance. The correct variance of $d_2(\tilde{\theta})$ is $V_{2\cdot 1} = J_2 - J_{21}B_1^{-1}H_1B_1^{-1}J_{12}$, which can be easily derived as in Newey and McFadden (1994) using the delta method. Consider the following modified RS test using the correct variance of the score function:

$$\widetilde{RS}_{2\cdot 1}(\theta) = n\, d_2(\theta)'\, V_{2\cdot 1}^{-1}\, d_2(\theta) \tag{2}$$

The following result establishes the consistency and asymptotic validity of this test, where $\theta$ is now replaced by a non MLE $\tilde{\theta}$.

**Theorem 1.** *Consider Assumptions 1–3. When $H_0^3$ is true and $H_A^2$ holds and $n \to \infty$,*

$$\widetilde{RS}_{2\cdot 1}(\tilde{\theta}) \xrightarrow{d} \chi^2_{p_2}(\tilde{\lambda}_{2\cdot 1})$$

*with $\tilde{\lambda}_{2\cdot 1} = \delta_2' V_{2\cdot 1}\delta_2$.*

**Proof.** In the Appendix. □

Though consistent and with correct asymptotic size, $\widetilde{RS}_{2\cdot 1}(\tilde{\theta})$ is less powerful than $RS_{2\cdot 1}(\hat{\theta})$. Note that $\lambda_{2\cdot 1} - \tilde{\lambda}_{2\cdot 1} = \delta_2'(J_{2\cdot 1} - V_{2\cdot 1})\delta_2$. The asymptotic efficiency of the MLE of $\theta_1$ implies that $J_1^{-1} - B_1^{-1}H_1B_1^{-1}$ is negative semi-definite, thus $J_{2\cdot 1} - V_{2\cdot 1}$ is positive semi-definite, and hence $\lambda_{2\cdot 1} - \tilde{\lambda}_{2\cdot 1} \geq 0$.

An optimal test for $H_0^2$ when any $\sqrt{n}$-consistent estimator of $\theta_1$ under $H_0^{23}$ is used can be based on Neyman's (1959) $C(\alpha)$ test statistic:

$$C_{2\cdot 1}(\theta) = n\ d_{2\cdot 1}(\theta)'\ J_{2\cdot 1}^{-1}\ d_{2\cdot 1}(\theta) \tag{3}$$

where $d_{2\cdot 1}(\theta) = d_2(\theta) - J_{21}J_{11}^{-1}d_1(\theta)$ is known as the *effective score*. A well-known result is that $C_{2\cdot 1}(\tilde{\theta})$ is asymptotically equivalent to $RS_{2\cdot 1}(\hat{\theta})$, and hence it has correct asymptotic size and is also locally most powerful. Intuitively, the $C(\alpha)$ test replaces the score of the test parameters $\theta_2$ by its projection on the orthogonal complement of the space spanned by the score of the nuisance parameters $\theta_1$, evaluated at $\tilde{\theta}$. And it does so in such a way that replacing the MLE $\hat{\theta}$ by $\tilde{\theta}$ does not lead to any loss in asymptotic power. It is relevant to remark that $C_{2.1}(\hat{\theta}) = RS_{2.1}(\hat{\theta})$, since $d_{2.1}(\hat{\theta}) = d_2(\hat{\theta})$ due to $d_1(\hat{\theta}) = 0$.

## 3. Testing under local misspecification

Suppose that $H_0^2$ is true but the alternative hypothesis is locally misspecified, that is, $H_A^3 : \theta_3 = \theta_{30} + \delta_3/\sqrt{n}$, $0 < \delta_3 < \infty$ holds. Davidson and MacKinnon (1987) and Saikkonen (1989) show that in such case $RS_{2\cdot 1}(\hat{\theta}) \xrightarrow{d} \chi^2_{p_2}(\lambda_{2/3\cdot 1})$, where $\lambda_{2/3\cdot 1} = \delta_3' J_{32\cdot 1}J_{2\cdot 1}^{-1}J_{23\cdot 1}\delta_3$ with $J_{23\cdot 1} = J_{23} - J_{21}J_{11}^{-1}J_{13} = J_{32\cdot 1}'$. That is, even when $H_0^2$ is true, $RS_{2\cdot 1}(\hat{\theta})$ has a *non central* chi-squared distribution due to $\theta_3 \neq \theta_{30}$, and hence leading to spurious rejections of $H_0^2$ due to misspecification and not to its falseness. Naturally this result affects Neyman's C($\alpha$) test alike, since it is asymptotically equivalent to $RS_{2\cdot 1}(\hat{\theta})$.

Bera and Yoon (1993) proposed the following modified test:

$$RS_{2\cdot 1}^*(\theta) = n\ d_2^*(\theta)'\ J_{2(3)\cdot 1}^{-1}\ d_2^*(\theta) \tag{4}$$

where $d_2^*(\theta) =\ d_2(\theta) - J_{23\cdot 1}J_{3\cdot 1}^{-1}d_3(\theta)$ and $J_{2(3)\cdot 1} = J_{2\cdot 1} - J_{23\cdot 1}J_{3\cdot 1}^{-1}J_{32\cdot 1}$. Their key result is that under $H_0^2$ and when $H_A^3$ holds, $RS_{2\cdot 1}^*(\hat{\theta}) \xrightarrow{d} \chi^2_{p_2}(0)$. That is, the BY test has asymptotic central chi-squared distribution even when $H_0^3$ is false (in a local sense); hence it does not lead to spurious rejections induced by misspecification. It is relevant to remark that both $RS_{2\cdot 1}^*(\hat{\theta})$ and $RS_{2\cdot 1}(\hat{\theta})$ are based on $\hat{\theta}$, the MLE of $\theta$ under the joint null $H_0^{23}$, and hence the use of the robustified test statistic shares all the computational advantages of the standard RS test. See Bera et al. (2009) for a geometrical interpretation of these results.

A quick inspection of the expressions of the $C(\alpha)$ and the BY test statistics respectively, in (3) and (4), suggests strong similarities between them, specially in terms of orthogonalization, i.e., in calculating the effective score. The most interesting fact is that the structure of orthogonalization is the same for replacing an MLE by a $\sqrt{n}$-consistent estimator of $\theta_1$, and for taking account of local misspecification relating to the parameter $\theta_3$.

Regarding power, the asymptotic distribution of $RS_{2\cdot 1}^*(\hat{\theta})$ under $H_A^2$ is non central chi-squared with non centrality parameter $\lambda_{2\cdot 1}^* = \delta_2' J_{2(3)\cdot 1}\delta_2$. Note that when $H_A^2$ and $H_A^3$ are true, $\lambda_{2\cdot 1}^* = \lambda_{2\cdot 13} + o_p(1/\sqrt{n})$, where $\lambda_{2\cdot 13}$ is the non centrality parameter of an RS test for $H_0^2$ when both $(\theta_1, \theta_3)$ are estimated by MLE. Consequently, the BY test restores consistency and correct asymptotic size under misspecified alternatives, with no power loss compared to the standard

RS that estimates $\theta_1$ and $\theta_3$ by MLE. Similarly, note that $(\hat{\theta}_1, \theta_{30})$ is trivially a $\sqrt{n}$-consistent estimator of $(\theta_1, \theta_3)$ under $H_0^2$ and $H_A^3$; hence $RS^*_{2.1}(\hat{\theta}) \overset{a}{=} C_{2\cdot 13}(\hat{\theta})$, where $C_{2\cdot 13}(\theta)$ is defined analogously as in (3).

Nevertheless, the BY test requires the use of the MLE for $\theta_1$. A simple modification that can handle any $\sqrt{n}$-consistent estimator for $\theta_1$ based on $h_1(\theta)$ is as follows. Define $B_{j\cdot 1} = J_j - J_{j1}B_1^{-1}H_1B_1^{-1}J_{1j}, \;\; j = 2, 3, 23, 32$, where the subindex 23 (similarly 32) is used to label the redefined parameter $\theta_{23} = (\theta_2', \theta_3')'$. In order to account for the effect of $H_A^3$ consider the adjusted score $\tilde{d}^*_{2\cdot 1}(\theta) = d_2(\theta) - B_{23\cdot 1}B_{3\cdot 1}^{-1}d_3(\theta)$. Now, following BY, consider the adjusted RS statistic:

$$\widetilde{RS}^*_{2\cdot 1}(\theta) = n\, \tilde{d}^*_{2\cdot 1}(\theta)'V_{2(3)\cdot 1}^{-1}\tilde{d}^*_{2\cdot 1}(\theta) \tag{5}$$

where $V_{2(3)\cdot 1} = B_{2\cdot 1} - B_{23\cdot 1}B_{3\cdot 1}^{-1}B_{32\cdot 1}$ is the variance of $\tilde{d}^*_{2\cdot 1}(\theta)$.

The next theorem establishes the properties of a locally size-robust "modified" BY test under non MLE estimation of $\theta_1$.

**Theorem 2.** *Consider Assumptions 1–3.*

*(i) When $H_0^2$ is true, but $H_A^3$ holds, as $n \to \infty$*

$$\widetilde{RS}_{2\cdot 1}(\tilde{\theta}) \xrightarrow{d} \chi^2_{p_2}(\tilde{\lambda}_{2/3\cdot 1})$$

*with $\tilde{\lambda}_{2/3\cdot 1} = \delta_3'(J_{23} - J_{21}B_1^{-1}H_1B_1^{-1}J_{13})'V_{2\cdot 1}^{-1}(J_{23} - J_{21}B_1^{-1}H_1B_1^{-1}J_{13})\delta_3$.*

*(ii) When $H_A^2$ and $H_A^3$ hold, as $n \to \infty$*

$$\widetilde{RS}^*_{2\cdot 1}(\tilde{\theta}) \xrightarrow{d} \chi^2_{p_2}(\tilde{\lambda}^*_{2\cdot 1})$$

*where $\tilde{\lambda}^*_{2\cdot 1} = \delta_2'V_{2(3)\cdot 1}\delta_2$.*

**Proof.** In the Appendix. □

The main result of this article is that a fully modified size and power robust test can be derived to accommodate non MLEs and misspecified alternatives. Define $d^*_{2\cdot 1}(\theta) = d_{2\cdot 1}(\theta) - J_{23\cdot 1}J_{3\cdot 1}^{-1}d_{3\cdot 1}(\theta)$ and

$$C^*_{2\cdot 1}(\theta) = n\, d^*_{2\cdot 1}(\theta)'J_{2(3)\cdot 1}^{-1}d^*_{2\cdot 1}(\theta) \tag{6}$$

where $d_{3.1}(\theta) = d_3(\theta) - J_{31}J_{11}^{-1}d_1(\theta)$ analogously as $d_{2.1}(\theta)$ in $C_{2.1}(\theta)$ in (3). The asymptotic properties of this new test are established in the following theorem.

**Theorem 3.** *Consider Assumptions 1–3. When $H_A^2$ and $H_A^3$ hold and $n \to \infty$*

$$C^*_{2\cdot 1}(\tilde{\theta}) \xrightarrow{d} \chi^2_{p_2}(\lambda^*_{2\cdot 1})$$

**Proof.** In the Appendix. □

The optimality of the new procedure is due to the fact the theorem implies that $C^*_{2\cdot 1}(\tilde{\theta})$ is asymptotically equivalent to $RS^*_{2\cdot 1}(\hat{\theta})$. This equivalence is analog to that between $RS_{2\cdot 1}(\hat{\theta})$ and $C_{2\cdot 1}(\tilde{\theta})$ in Section 2 when the alternative hypothesis is correctly specified. Consequently, this new test has both the robustness of validity and efficiency properties when a non MLE of $\theta_1$ is used and when the alternative hypothesis is locally misspecified. Also note that $C^*_{2\cdot 1}(\hat{\theta}) = RS^*_{2\cdot 1}(\hat{\theta})$. The improvement from $RS^*_{2\cdot 1}(\theta)$ to $C^*_{2\cdot 1}(\theta)$ is achieved by starting with $d_{2\cdot 1}(\theta)$ and $d_{3\cdot 1}(\theta)$ instead of $d_2(\theta)$ and $d_3(\theta)$, respectively, to take account of the fact that for the non MLE $d_1(\tilde{\theta}) \neq 0$. We can also view $C^*_{2\cdot 1}(\theta)$ as a modification of our initial C($\alpha$) statistic $C_{2\cdot 1}(\theta)$

in (3), by further orthogonalizing $d_{2.1}(\theta)$, now with respect to $d_{3.1}(\theta)$ to incorporate the fact that $d_3(\tilde{\theta}) \neq 0$. This duality goes back to our earlier observation that two orthogonalizations for taking care of the $\sqrt{n}$-consistent estimation of $\theta_1$ (as in C($\alpha$)) and for allowing for the possible local presence of $\theta_3$ (as in BY) are structurally the *same*.

## 4. Monte Carlo experiments

We present the results of a simple but illustrative empirical exploration of the costs and benefits of the alternative robustification strategies discussed earlier. Consider the following regression model:

$$y_i = \theta_1 x_{1i} + \theta_2 x_{2i} + \theta_3 x_{3i} + u_i, \quad i = 1, 2, \ldots, n \tag{7}$$

with

$$x_{1i} = a_i + e_{1i}, \quad x_{2i} = a_i + e_{2i}, \quad x_{3i} = a_i + e_{3i}$$

and

$$u_i, a_i, e_{1i}, e_{2i}, e_{3i} \sim \text{ iid } N(0, 1)$$

We use $\theta_1 = 1$, $n = 100$, and we consider 1000 replications. Results for other sample sizes are very similar qualitatively, and are available from the authors. All tests are based on a nominal size of 0.05.

Using the framework discussed in the previous sections, the joint null $H_0^{23} : \theta_2 = 0, \ \theta_3 = 0$ corresponds to a simple regression model, i.e., $y_i = \theta_1 x_{1i} + u_i$. The restricted MLE, $\hat{\theta} = (\hat{\theta}_1, 0, 0)$, is the ordinary least-squares (OLS) estimator of $\theta_1$ that regresses $y$ on $x_1$. In order to evaluate the performance of the tests under alternative consistent estimators, we have considered the 0.1 quantile regression estimator of $\theta_1$, $\tilde{\theta} = (\tilde{\theta}_1, 0, 0)$. The error term $u$ is generated independently of $x_1$, $x_2$, and $x_3$, and identically across all observations, which implies a simple location-shift model. Consequently, the quantile regression estimator for any quantile is consistent for $\theta_1$. We use the 0.1 quantile in order to produce a consistent though inefficient non MLE. Note that any quantile could have been selected, and that this particular estimator will be asymptotically efficient if $u$ follows an asymmetric Laplace distribution with location parameter at the 0.1 quantile of its distribution. When the data are generated using the asymmetric Laplace distribution, then a consistent but inefficient estimator is the OLS estimator. The score functions and the tests implemented below would then be based on the influence function of the quantile regression estimator at 0.1 quantile. The availability of a multitude of $\sqrt{n}$-consistent estimators can be viewed as a drawback of the $C(\alpha)$ test. While all will lead to asymptotic equivalent tests, their finite sample behavior could be quite different.

In this setup, the correlation between any pair of explanatory variables is 0.5; therefore, a test for $H_0^2 : \theta_2 = 0$ based on either $\hat{\theta}$ or $\tilde{\theta}$ will be affected by misspecification in $\theta_3$ (i.e., $\theta_3 \neq 0$). This is a simple omitted variable setup, where leaving $x_3$ out of the model affects both the estimate of $\theta_1$ and the test for $\theta_2$. A simple way to see this is to consider a Wald test statistic for $H_0^2$, which is based on the OLS estimate of $\theta_2$. This non robustness can also be seen from the non singularity of the matrix $J_{23\cdot 1}$.

The results that evaluate the performance of alternative tests, for different estimators and values of $\theta_2$ and $\theta_3$, are presented in Table 1. For part (a) we generated data using the joint null $\theta_2 = \theta_3 = 0$; for part (b) we considered $\theta_2 > 0$, $\theta_3 = 0$, and finally, part (c) is based on data with $\theta_2 = 0$, $\theta_3 > 0$. The first four columns present tests for the single hypothesis $H_0^2$ without any correction for whether $H_0^3$ is valid or not. $RS_{2\cdot 1}(\hat{\theta})$ is constructed using the restricted

**Table 1.** Monte Carlo simulations.

| $\theta_2$ | $\theta_3$ | $RS_{2\cdot1}(\hat{\theta})$ | $RS_{2\cdot1}(\tilde{\theta})$ | $\widetilde{RS}_{2\cdot1}(\tilde{\theta})$ | $C_{2\cdot1}(\tilde{\theta})$ | $RS^*_{2\cdot1}(\hat{\theta})$ | $RS^*_{2\cdot1}(\tilde{\theta})$ | $\widetilde{RS}^*_{2\cdot1}(\tilde{\theta})$ | $C^*_{2\cdot1}(\tilde{\theta})$ |
|---|---|---|---|---|---|---|---|---|---|
| (a) Size | | | | | | | | | |
| 0.00 | 0.0 | 0.051 | 0.120 | 0.059 | 0.043 | 0.043 | 0.083 | 0.046 | 0.038 |
| (b) Power in the $\theta_2$-direction | | | | | | | | | |
| 0.05 | 0.0 | 0.112 | 0.167 | 0.103 | 0.105 | 0.108 | 0.150 | 0.085 | 0.097 |
| 0.10 | 0.0 | 0.221 | 0.280 | 0.183 | 0.204 | 0.196 | 0.232 | 0.146 | 0.180 |
| 0.15 | 0.0 | 0.458 | 0.462 | 0.362 | 0.411 | 0.397 | 0.402 | 0.321 | 0.370 |
| 0.20 | 0.0 | 0.651 | 0.623 | 0.527 | 0.612 | 0.562 | 0.563 | 0.470 | 0.521 |
| 0.25 | 0.0 | 0.824 | 0.774 | 0.699 | 0.801 | 0.750 | 0.716 | 0.628 | 0.714 |
| 0.30 | 0.0 | 0.925 | 0.856 | 0.799 | 0.903 | 0.866 | 0.827 | 0.760 | 0.827 |
| 0.40 | 0.0 | 0.997 | 0.972 | 0.956 | 0.991 | 0.984 | 0.971 | 0.947 | 0.972 |
| 0.50 | 0.0 | 1.000 | 0.997 | 0.989 | 0.999 | 0.996 | 0.991 | 0.982 | 0.992 |
| 0.60 | 0.0 | 1.000 | 1.000 | 0.998 | 1.000 | 0.998 | 0.998 | 0.995 | 0.998 |
| 0.70 | 0.0 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.997 | 0.999 |
| (c) Robustness to $\theta_3$-misspecification | | | | | | | | | |
| 0.00 | 0.1 | 0.082 | 0.153 | 0.094 | 0.075 | 0.051 | 0.099 | 0.067 | 0.053 |
| 0.00 | 0.2 | 0.132 | 0.212 | 0.123 | 0.121 | 0.041 | 0.088 | 0.047 | 0.033 |
| 0.00 | 0.3 | 0.209 | 0.275 | 0.181 | 0.187 | 0.037 | 0.069 | 0.060 | 0.033 |
| 0.00 | 0.4 | 0.323 | 0.355 | 0.258 | 0.296 | 0.017 | 0.044 | 0.048 | 0.011 |
| 0.00 | 0.5 | 0.399 | 0.408 | 0.304 | 0.354 | 0.012 | 0.050 | 0.049 | 0.012 |
| 0.00 | 0.6 | 0.483 | 0.483 | 0.396 | 0.442 | 0.009 | 0.042 | 0.047 | 0.008 |
| 0.00 | 0.7 | 0.543 | 0.563 | 0.461 | 0.514 | 0.008 | 0.042 | 0.062 | 0.005 |
| 0.00 | 0.8 | 0.632 | 0.613 | 0.535 | 0.588 | 0.009 | 0.039 | 0.041 | 0.007 |
| 0.00 | 0.9 | 0.690 | 0.635 | 0.553 | 0.634 | 0.015 | 0.058 | 0.053 | 0.013 |
| 0.00 | 1.0 | 0.715 | 0.670 | 0.591 | 0.665 | 0.009 | 0.037 | 0.043 | 0.007 |

Notes: Tests for $H_0^2 : \theta_2 = 0$. Robust tests consider potential local departures from $H_0^3 : \theta_3 = 0$. Empirical rejection rates based on a nominal size of 0.05. Sample size = 100, number of replications = 1000.

MLE; $RS_{2\cdot1}(\tilde{\theta})$ and $\widetilde{RS}_{2\cdot1}(\tilde{\theta})$ use a non MLE; and $C_{2\cdot1}(\tilde{\theta})$ is the $C(\alpha)$ test using a non MLE. Note that $C_{2\cdot1}(\hat{\theta}) = RS_{2\cdot1}(\hat{\theta})$ by definition of MLE. The last four columns present tests for the single hypothesis $H_0^2$ but correcting for local departures from $H_0^3$. $RS^*_{2\cdot1}(\hat{\theta})$ is the BY test using the restricted MLE; $RS^*_{2\cdot1}(\tilde{\theta})$ and $\widetilde{RS}^*_{2\cdot1}(\tilde{\theta})$ are the BY tests using a non MLE; and $C^*_{2\cdot1}(\tilde{\theta})$ is our proposed fully robust test using a non MLE. All test statistics are based on the score functions derived from the Gaussian log-likelihood. Therefore, each score is of the form $d_j(\theta) = \frac{1}{n}\sum_{i=1}^n x'_{ji}u_i(\theta)$, $j = 1, 2, 3$, where $u_i(\theta) = y_i - \theta_1 x_{1i} - \theta_2 x_{2i} - \theta_3 x_{3i}$. Each element in $J_{jh}$, $j, h = 1, 2, 3$, is estimated by the outer product of gradients $\frac{1}{n}\sum_{i=1}^n d_{ji}(z_i, \theta)d_{hi}(z_i, \theta)'$ where $d_{ji}(z_i, \theta) = x'_{ji}u_i(\theta)$, $j = 1, 2, 3$, $z_i = (y_i, x_{1i}, x_{2i}, x_{3i})$. Finally, $B_1^{-1}H_1B_1^{-1}$ is given by the variance of the 0.1-quantile regression estimator.

When $\theta_2 = \theta_3 = 0$ holds (part (a)), as expected, $RS_{2\cdot1}(\hat{\theta})$, $\widetilde{RS}_{2\cdot1}(\tilde{\theta})$, and $C_{2\cdot1}(\tilde{\theta})$ have correct empirical size, while $RS_{2\cdot1}(\tilde{\theta})$ has an empirical size that is more than twice of the nominal size and much larger than that of its counterparts implemented with the correct variance. Similar results are found for the BY statistics. That is, the size of $RS^*_{2\cdot1}(\tilde{\theta})$ is also quite high while that of $RS^*_{2\cdot1}(\hat{\theta})$, $\widetilde{RS}^*_{2\cdot1}(\tilde{\theta})$, and $C^*_{2\cdot1}(\tilde{\theta})$ is approximately correct.

Under correctly specified alternatives (part (b)), the highest power is achieved by the optimal RS test, $RS_{2\cdot1}(\hat{\theta})$, followed very closely by Neyman's $C_{2\cdot1}(\tilde{\theta})$. The tests robust to misspecification of $\theta_3$, $RS^*_{2\cdot1}(\hat{\theta})$ and $C^*_{2\cdot1}(\tilde{\theta})$, show less power than those of $RS_{2\cdot1}(\hat{\theta})$ and $C_{2\cdot1}(\tilde{\theta})$, consistent with the fact that $\lambda_{2\cdot1} \geq \lambda^*_{2\cdot1}$. This is the "robustification cost" for unnecessarily using a modified test. Nevertheless, it is interesting to highlight that, in this case, the power loss is minimal. A comparison of $RS^*_{2\cdot1}(\hat{\theta})$ and $C^*_{2\cdot1}(\tilde{\theta})$ shows that, as predicted by the theory, they have very similar power, suggesting that the power of the BY procedure can be successfully restored through a properly modified test based on a consistent, non MLE. Moreover, $\widetilde{RS}^*_{2\cdot 1}(\tilde{\theta})$ has less power than $C^*_{2\cdot 1}(\tilde{\theta})$.

Part (c) studies the effects of misspecification through $\theta_3$. As expected, all the non robust versions, $RS_{2\cdot 1}(\hat{\theta})$, $RS_{2\cdot 1}(\tilde{\theta})$, $\widetilde{RS}_{2\cdot 1}(\tilde{\theta})$, and $C_{2\cdot 1}(\tilde{\theta})$, have unwanted rejection for $H_0^2$, as $\theta_3$ increases, which is compatible with $\lambda_{2/3\cdot 1} > 0$. Nevertheless, the robustified versions ($RS^*_{2\cdot 1}(\hat{\theta})$, $\widetilde{RS}^*_{2\cdot 1}(\hat{\theta})$, and $C^*_{2\cdot 1}(\tilde{\theta})$) have rejection probabilities close to 0.05 or less. The empirical size of $RS^*_{2\cdot 1}(\hat{\theta})$ and $C^*_{2\cdot 1}(\tilde{\theta})$ reduces gradually as $\theta_3$ increases, possibly due to the fact that adjustments are designed only for local misspecifications, i.e., for $\theta_3$ values close to 0. We offer some intuitive explanation. In our setup $\theta_3 = \delta_3/\sqrt{n}$. For $n = 100$, choosing $\theta_3$ between 0.1 and 1.0, $\delta_3$ is allowed to vary from 1.0 to 10.0. Let us consider the case of our suggested test $C^*_{2\cdot 1}(\theta)$ which takes account of the presence of $\theta_3$ by indirectly estimating it through $d_3(\theta)$, evaluated at $\tilde{\theta}$. Since in our Monte Carlo design the explanatory variables have positive correlation (0.5), the components of the information matrix $J(\theta)$ will be positive. Thus the effective score $d^*_{2\cdot 1}(\theta)$ can be expected to be lower than $d_{2\cdot 1}(\theta)$ which again can be expected to be lower than the raw score $d_2(\theta)$. Thus for non local misspecification there could be some overcorrection for our Monte Carlo setup.

## 5. Final remarks

This article proposes a new test that is consistent, achieves correct asymptotic size, and is locally most powerful under local misspecification, and when any $\sqrt{n}$-estimator of the nuisance parameters is used. The new test can be seen as an extension of the Bera and Yoon (1993) procedure in order to deal with non ML estimation, while preserving its optimality properties. Similarly, the procedure can be viewed as extending the standard $C(\alpha)$ test (that by construction admits non MLEs) to handle locally misspecified alternatives. In many practical situations non ML strategies are favored to handle initial, restricted models, such as the case of dynamic panels and spatial panel models, where GMM estimators are usually preferred.


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## Appendix

**Proof of Theorem 1.** The asymptotic distribution follows from an application of Newey (1985, Theorem 2.3). Note that Assumptions 1–3 correspond to assumptions 1–9 in Newey (1985). Define the vector of functions $w(z,\theta) = [h_1(z,\theta)'\ d_2(z,\theta)']'$ and $w(\theta) = [h_1(\theta)'\ d_2(\theta)']'$. Also define the matrices $\Gamma = [\iota_{p_1}\ 0_{p_2}]$ and $\Pi = [0_{p_1}\ \iota_{p_2}]$, where $\iota_.$ is a vector of 1s and $0_.$ a vector of 0s. The estimating equations for $\theta_1$ can then be rewritten as

$$\Gamma E\left[w(z,\theta_1,\theta_{20},\theta_{30})\right] = 0 \ \text{ only if } \ \theta_1 = \theta_{10}$$

The specification test can be based on the testing equations

$$\Pi E\left[w(z,\theta_{10},\theta_2,\theta_{30})\right] = 0 \ \text{ only if } \ \theta_2 = \theta_{20}$$

We follow the notation in Bera, Montes-Rojas, and Sosa-Escudero (2010). Let $K = E[\partial w(z,\theta)/\partial\theta_1]_{\theta=\theta_0} = [B_1' \quad -J_{21}']'$, where the equality $E[\partial d_2(z,\theta)/\partial\theta_1]_{\theta=\theta_0} = -J_{21}$ follows from the information matrix equality,

$$V = E\left[w(z,\theta_0)\; w(z,\theta_0)'\right] = \begin{bmatrix} H_1 & V_{12} \\ V_{21} & J_{22} \end{bmatrix}$$

where $V_{12} = E[h_1(z,\theta_0)d_2(z,\theta_0)'] = V_{21}'$, $D = E[w(z,\theta_0)d_2(z,\theta_0)'] = [V_{12}'\; J_{22}']'$, and $P = I - K(\Gamma K)^{-1}\Gamma$.

Then under $H_A^2$, $\sqrt{n}\Pi w(\tilde{\theta}) \xrightarrow{d} N(\Pi PD\delta_2, \Pi PVP'\Pi')$, as $n\to\infty$, hence

$$n\; w(\tilde{\theta})'\Pi'(\Pi PVP'\Pi')^{-1}\Pi w(\tilde{\theta}) \xrightarrow{d} \chi^2_{p_2}(\tilde{\lambda}_{2\cdot 1})$$

as $n\to\infty$ with $\widetilde{\lambda}_{2\cdot 1} = (\Pi PD\delta_2)'(\Pi PVP'\Pi')^{-1}(\Pi PD\delta_2)$.

After some matrix algebra, we obtain

$$\begin{aligned} \Pi P &= [J_{21}\; B_1^{-1}\; \iota_{p_2}] \\ \Pi PV &= [J_{21}B_1^{-1}H_1 + V_{21}\; J_{21}B_1^{-1}V_{12} + J_{22}] \\ \Pi PVP'\Pi &= J_2 + J_{21}B_1^{-1}V_{12} + V_{21}B_1^{-1}J_{12} + J_{21}B_1^{-1}H_1B_1^{-1}J_{12} \end{aligned}$$

Thus, $\Pi PVP'\Pi' = V_{2\cdot 1}$. Moreover, $\Pi PD = J_{22} + J_{21}B_1^{-1}V_{12}$. Then, $\tilde{\lambda}_{2\cdot 1} = \delta_2'(J_{22} + J_{21}B_1^{-1}V_{12})'V_{2\cdot 1}^{-1}(J_{22} + J_{21}B_1^{-1}V_{12})\delta_2$. Finally note that by an application of the generalized information matrix equality (Newey and McFadden, 1994, p. 2163) $V_{12} = E[h_1(z,\theta_0)d_2(z,\theta_0)'] = -E[\partial h_1(z,\theta)/\partial\theta_2']_{\theta=\theta_0} = -E[h_1(z,\theta_0)h_1(z,\theta_0)']E[\partial h_1(z,\theta)/\theta_1]^{-1}_{\theta=\theta_0}E[d_1(z,\theta_0)d_2(z,\theta_0)'] = -H_1B_1^{-1}J_{12} = V_{21}'$. Thus, $\tilde{\lambda}_{2\cdot 1} = \delta_2'V_{2\cdot 1}\delta_2$. Finally, note that $\widetilde{RS}_{2\cdot 1}(\tilde{\theta}) = n\; w(\tilde{\theta})'\Pi'(\Pi PVP'\Pi')^{-1}\Pi w(\tilde{\theta})$. □

**Proof of Theorem 2.**

(i) The proofs follows from a modification of the proof of Theorem 1 where $d_2$ is replaced by $d_{23} = [d_2'\; d_3']'$. Consider a new partition of a three parameter space $(\theta_1,\theta_2,\theta_3)$ into $(\theta_1,(\theta_2,\theta_3))$. This is only notation to emphasize that the "block" $(\theta_2,\theta_3)$ is taken together. Thus 23 denotes this new redefined parameter $\theta_{23} = (\theta_2',\theta_3')'$. Define the matrix

$$J = \begin{bmatrix} J_{11} & J_{1,23} \\ J_{23,1} & J_{23} \end{bmatrix}$$

and the vector of functions $w(z,\theta) = [h_1(z,\theta)'\; d_{23}(z,\theta)']'$ and $w(\theta) = [h_1(\theta)'\; d_{23}(\theta)']'$. Also define the matrices $\Gamma = [\iota_{p_1}\; 0_{p_2}\; 0_{p_3}]$ and $\Pi = [0_{p_1}\; \iota_{p_2}\; 0_{p_3}]$. Moreover, define $V_{1,23} = E[h_1(z,\theta_0)\; d_{23}(z,\theta_0)'] = V_{23,1}'$.

Following the notation in Bera et al. (2010), let $K = E[\partial w(z,\theta)/\partial\theta_1]_{\theta=\theta_0} = [B_1 \quad -J_{23,1}]$:

$$V = E\left[w(z,\theta_0)w(z,\theta_0)'\right] = \begin{bmatrix} H_1 & V_{1,23} \\ V_{23,1} & J_{23,23} \end{bmatrix}$$

$D = E[w(z,\theta_0)\; d_{23}(z,\theta_0)'] = [V_{23,1}\; J_{23,23}]$, and $P = I - K(\Gamma K)^{-1}\Gamma$.

Then under $H_0^2$ and $H_A^3$, $\sqrt{n}\Pi w(\tilde{\theta}) \xrightarrow{d} N(\Pi PD[0_{p_2}\delta_3], \Pi PVP'\Pi')$, as $n\to\infty$, hence

$$n\; w(\tilde{\theta})'\Pi'(\Pi PVP'\Pi')^{-1}\Pi w(\tilde{\theta}) \xrightarrow{d} \chi^2_{p_2}(\tilde{\lambda}_{2/3\cdot 1})$$

as $n\to\infty$ with $\tilde{\lambda}_{2/3\cdot 1} = (\Pi PD\delta_3)'(\Pi PVP'\Pi')^{-1}(\Pi PD\delta_3)$.

After some algebra we obtain $\Pi PD = J_{23} + J_{21}B_1^{-1}V_{13}$ and $\Pi PVP'\Pi' = V_{2\cdot1}$. Thus, $\tilde{\lambda}_{2/3\cdot1} = \delta_3'(J_{23} + J_{21}B_1^{-1}V_{13})'V_{2\cdot1}^{-1}(J_{23} + J_{21}B_1^{-1}V_{13})\delta_3$. Moreover, note that by an application of the generalized information matrix equality (Newey and McFadden, 1994, p. 2163) $V_{13} = E[h_1 d_3'] = -E[\partial h_1/\partial\theta_3']_{\theta=\theta_0} = -H_1B_1^{-1}J_{13} = V_{31}'$. Finally, note that $\widetilde{RS}_{2\cdot1}(\tilde{\theta}) = w(\tilde{\theta})'\Pi'(\Pi PVP'\Pi')^{-1}\Pi w(\tilde{\theta})$.

(ii) The result follows from part (i) and Bera et al. (2010, Theorem 3). We need to modify the score function for $\theta_2$, $d_2$, to account for the local misspecification in $\theta_3$. This is done by considering the adjusted score for the score $d_2$ in the function: $w(z,\theta) = [h_1(z,\theta)'\ \tilde{d}_{2\cdot1}^*(z,\theta)'\ d_3(z,\theta)']'$ where $\tilde{d}_{2\cdot1}^*(z,\theta) = d_2(z,\theta) - B_{23\cdot1}B_{3\cdot1}^{-1}d_3(z,\theta)$, and $w(\theta) = [h_1(\theta)'\ \tilde{d}_{2\cdot1}^*(\theta)'\ d_3(\theta)']'$ where $\tilde{d}_{2\cdot1}^*(\theta) = d_2(\theta) - B_{23\cdot1}B_{3\cdot1}^{-1}d_3(\theta)$. Define $\Gamma$ and $\Pi$ as in part (i) and obtain $K$, $V$, $D$, and $P$ with the same procedure for the newly defined $w(z,\theta)$. Then under $H_0^2$ and $H_A^3$, $\sqrt{n}\Pi w(\tilde{\theta}) \xrightarrow{d} N(0_{p_2}, \Pi PVP'\Pi')$, that is, it recovers the zero mean of the testing function. Finally, after some algebra $\Pi PVP'\Pi' = V_{2(3)\cdot1}$, where $V_{2(3)\cdot1}$ accounts for the variance of $\widetilde{d_{2\cdot1}^*}(\tilde{\theta})$, and is given by $V_{2(3)\cdot1} = B_{2\cdot1} - B_{23\cdot1}B_{3\cdot1}^{-1}B_{32\cdot1}$, and the chi-squared distribution follows. Under $H_A^2$ and $H_A^3$, $\sqrt{n}\Pi w(\tilde{\theta}) \xrightarrow{d} N(\Pi PD\delta_2, \Pi PVP'\Pi')$, as $n\to\infty$; hence, $n\, w(\tilde{\theta})'\Pi'(\Pi PVP'\Pi')^{-1}\Pi w(\tilde{\theta}) \xrightarrow{d} \chi_{p_2}^2(\tilde{\lambda}_{2\cdot1}^*)$, with $\tilde{\lambda}_{2\cdot1}^* = (\Pi PD\delta_2)'(\Pi PVP'\Pi')^{-1}(\Pi PD\delta_2) = \delta_2'V_{2(3)\cdot1}\delta_2$. Finally, note that $\widetilde{RS}_{2\cdot1}^*(\tilde{\theta}) = n\, w(\tilde{\theta})'\Pi'(\Pi PVP'\Pi')^{-1}\Pi w(\tilde{\theta})$.

□

**Proof of Theorem 3.**

Define $d_{2\cdot1}^*(z,\theta) = d_{2\cdot1}(z,\theta) - J_{23\cdot1}J_{3\cdot1}^{-1}d_{3\cdot1}(z,\theta), w(z,\theta) = [h_1(z,\theta)'d_{2\cdot1}^*(z,\theta)'d_3(z,\theta)']'$, $d_{2\cdot1}^*(\theta) = d_{2\cdot1}(\theta) - J_{23\cdot1}J_{3\cdot1}^{-1}d_{3\cdot1}(\theta)$, $w(\theta) = [h_1(\theta)'\ d_{2\cdot1}^*(\theta)'\ d_3(\theta)']'$. Define $\Gamma$ and $\Pi$ as in Theorem 2, part (i), and obtain $K$, $V$, $D$, and $P$ with the same procedure for the newly defined $w(\theta)$.

Then under $H_0^2$ and $H_A^3$, $\sqrt{n}\Pi w(\tilde{\theta}) \xrightarrow{d} N(0_{p_2}, \Pi PVP'\Pi')$, as $n\to\infty$, that is, it recovers the asymptotic zero mean of the testing function. Moreover under $H_A^2$ and $H_A^3$, $\sqrt{n}\Pi w(\tilde{\theta}) \xrightarrow{d} N(\Pi PD\delta_2, \Pi PVP'\Pi')$, as $n\to\infty$; hence, $n\, w(\tilde{\theta})'\Pi'(\Pi PVP'\Pi')^{-1}\Pi w(\tilde{\theta}) \xrightarrow{d} \chi_{p_2}^2(\lambda_{2\cdot1}^*)$, as $n\to\infty$ with $\lambda_{2\cdot1}^* = (\Pi PD\delta_2)'(\Pi PVP'\Pi')^{-1}(\Pi PD\delta_2) = \delta_2'J_{2(3)\cdot1}\delta_2$, where $J_{2(3)\cdot1}$ accounts for the variance of $d_{2\cdot1}^*(\tilde{\theta})$, and is given by $J_{2(3)\cdot1} = J_{2\cdot1} - J_{23\cdot1}J_{3\cdot1}^{-1}J_{32\cdot1}$. Finally, note that $C_{2\cdot1}^*(\tilde{\theta}) = n\, w(\tilde{\theta})'\Pi'(\Pi PVP'\Pi')^{-1}\Pi w(\tilde{\theta})$. □