On the homotopy perturbation method for Boussinesq–like equations

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We comment on some analytical solutions to a class of Boussinesq–like equations derived recently by means of the homotopy perturbation method (HPM). We show that one may obtain exactly the same result by means of the Taylor series in the time variable. We derive more general results by means of travelling waves and argue that a curious superposition principle may not be of any mathematical or physical significance.

In a recent paper Yıldırım[1] solved a kind of generalized Boussinesq–like equations by means of the homotopy perturbation method (HPM). The purpose of this comment is the discussion of the results derived by that author and the conclusions drawn from them.

Yıldırım[1] investigated $B(m,n)$ equations of the form

$$u_{tt} = (u^m)_{xx} - (u^n)_{xxxx},\ m,n > 1 \tag{1}$$

were the subscripts denote derivatives of the solution $u(x,t)$ with respect to its arguments. However, in the examples Yıldırım changed his mind and considered particular cases of

$$u_{tt} = -(u^m)_{xx} + (u^n)_{xxxx} \tag{2}$$

Both equations exhibit completely different solutions and we mainly restrict our discussion to Eq. (2) in what follows.

Yıldırım applied HPM which looks like an ordinary perturbation theory with a fancy name. After introducing a perturbation parameter p, Yıldırım obtained a perturbation series of the form

$$u(x,t) = \sum_{j=0}^{\infty} v_j(x,t)p^j \tag{3}$$

and stated that its convergence was proved by He[2]. However, that reference does not contain any convergence proof. As far as we know, the proof of convergence was suggested to He by an anonymous referee of an earlier paper[3] but to our knowledge no HPM user has ever verified if his particular implementation of the approach already met those convergence criteria.

After substituting $p = 1$ in the resulting perturbation series (3) Yıldırım realized that it was merely the Taylor series about $t = 0$:

$$u(x,t) = \sum_{j=0}^{\infty} f_j(x)t^j \tag{4}$$

where $f_0(x) = u(x,0)$ and $f_1(x) = u_t(x,0)$. By inspection of the Taylor series Yıldırım[1] identified some exact analytical solutions to the nonlinear partial differential equations. It is worth noticing that Yıldırım chose the particular initial conditions that he knew would lead to such analytical solutions. It is unlikely that one may be able to sum the time series exactly for two arbitrary functions $u(x,0)$ and $u_t(x,0)$; therefore, in an actual physical problem one has to rely on the accuracy of truncated time series that most probably will provide a quite poor description of the dynamics of the problem beyond a relatively small neighborhood of $t = 0$.

Obviously, we may substitute Eq. (4) into Eq. (2) and obtain the coefficients $f_j(x)$ in a straightforward way. However, Yıldırım preferred the application of the lengthier HPM. To make this point clearer let us write Eq. (2) as $u_{tt} = F(u, u')$, where u' collects all the derivatives of $u(x,t)$ with respect to x, and F is a nonlinear function. Taking into account that $F(u, u')$ can be expanded in a Taylor series $F = F_0 + F_1 t + \ldots$ we easily derive a recurrence relation for the coefficients f_j:

$$f_{j+2} = \frac{F_j}{(j+1)(j+2)},\ j = 0, 1, \ldots \tag{5}$$

where, obviously, F_j depends on f_i, $i = 0, 1, \ldots, j$. Since we know f_0 and f_1 as already indicated above then we obtain all the other coefficients $f_j(x)$ with $j > 1$ in a straightforward way. Considering that this procedure is a textbook example of calculus, is there any necessity for the application of HPM or any other more or less elaborate method to obtain the same series?. May the reader answer this question by him/herself.

Yıldırım obtained a kind of travelling–wave solutions to the $B(m, n)$ equations and stated that "exact solutions with solitary patterns are of important significance". Yıldırım also identified a kind of superposition of waves that is unexpected in nonlinear problems: "This paper reveals first time that in some special cases, addition of two solitary solutions satisfies exactly the nonlinear equation. Such phenomenon requires further mathematical study and physical explanation." In what follows we analyze Yıldırım results as well as such phenomenon.

Following the standard procedure for obtaining travelling waves we write

$$u(x,t) = U(\xi),\ \xi = x \pm vt \tag{6}$$

so that Eq. (2) becomes

$$\left[v^2 U + U^m - (U^n)''\right]'' = 0 \tag{7}$$

where the prime indicates differentiation with respect to ξ. For present purposes it is sufficient to consider polynomial solutions of the form

$$U(\xi) = \sum_{j=-M}^{N} a_j e^{\alpha j \xi} \tag{8}$$

keeping in mind that other expressions (like, for example, rational functions) may also be suitable.

We first discuss the $B(2, 2)$ equation

$$u_{tt} + (u^2)_{xx} - (u^2)_{xxx} = 0 \tag{9}$$

and purposely leave the boundary conditions unspecified. In this particular case we have

$$U(\xi) = a_2 e^{\xi/2} + a_{-2} e^{-\xi/2} - \frac{2}{3} v^2 \tag{10}$$

where $\alpha = 1/4$ and a_2 and a_{-2} are arbitrary coefficients.

By means of HPM Yıldırım[1] derived two solutions for Eq. (9) with different boundary conditions and then combined them into a more general one of the form

$$u^Y(x,t) \ = \ a \cosh^2\left(\frac{\xi}{4}\right) + b \sinh^2\left(\frac{\xi}{4}\right),$$

$$a = -\frac{4}{3}Lv^2,$$
$$b = \frac{4}{3}(1-L)v^2 \tag{11}$$

Yıldırım[1] obtained other solutions by addition of a constant to the arguments of the hyperbolic functions. This strategy for generating new solutions is successful because if $U(\xi)$ is a solution to Eq. (7), then $U(\xi+\xi_0)$ is also a solution (with, of course, different boundary conditions). Eq. (11) is a particular case of our Eq. (10) with $a_2 = a_{-2}$ as follows from the fact that

$$u^Y(x,t) = \frac{a+b}{4}(e^{\xi/2}+e^{-\xi/2}) - \frac{2}{3}v^2 \tag{12}$$

At this point it is worth noticing that the solutions that we are discussing are unbounded and, therefore, of little or no physical utility, whatsoever. However, there seems to be some interest in them anyway[1].

As promised above we now address the Yıldırım's superposition principle that is embodied in the solution (11) to Eq. (9). Notice that $u^Y(x,0)$ and $u_t^Y(x,0)$ depend on the coefficients of the linear combination (through L). Therefore, the solutions with, say, $L=0$ and $L=1$ exhibit different boundary conditions and thereby correspond to different physical problems. In other words, Yıldırım's superposition principle is a linear combination of solutions to different problems. We do not think that such a property is of any physical relevance but, of course, we may be wrong.

In the case of the $B(3,3)$ equation

$$u_{tt} + (u^3)_{xx} - (u^3)_{xxx} = 0 \tag{13}$$

we have

$$U(\xi) = a_3e^{\xi/3} + a_{-3}e^{-\xi/3}$$
$$\alpha = \frac{1}{9}$$
$$a_3a_{-3} = -\frac{3}{8}v^2 \tag{14}$$

that leads to the two solutions obtained by Yıldırım when $a_3 = -a_{-3} = \pm\sqrt{6}v/4$.

Finally, we mention that equations (1) and (2) are related by the transformation $t \rightarrow it$ (or $v \rightarrow iv$ for the travelling waves). Therefore, we easily derive the solutions to Eq. (1) from the ones just obtained.

We think that it is unnecessary to consider other cases because equations (1) and (2) do not appear to have any physical application whatsoever. They are merely tailor–made toy problems for the application of approximate methods of doubtful utility. Besides, the results and discussion above are more than enough for making our point.

Summarizing:

- The HPM proposed by Yıldırım[1] to treat Boussinesq–like equations simply leads to the Taylor series of the travelling–wave solutions about $t=0$; consequently, we can obtain the same results more easily by means of the latter well known approach. Obviously, we may apply the Taylor–series approach with initial conditions different to those that lead to travelling waves, but in such more general cases we are not certain to obtain analytical solutions.

- Instead of resorting to the HPM, one can obtain more general solutions by means of the textbook method of travelling–waves reduction.

- Yıldırım's superposition principle is just the linear combination of solutions to models with different boundary conditions and, therefore, it is of questionable physical significance.
- Finally, we stress the fact that many applications of approaches like HPM (see below) have been restricted to tailor–made toy problems with exact analytical solutions. As in the case analysed here the authors commonly obtain Taylor series in a rather indirect but more fashionable way.

In earlier reports we have argued that some new applications of variational and perturbation approaches (VAPA) like HPM, homotopy analysis method (HAM) (quite similar to HPM), variation–iteration method (VIM) and Adomian decomposition method (ADM) are responsible for some of the poorest scientific papers ever published in supposedly respectable journals[4, 5, 6, 7, 8, 9, 10]. There seems to be a great interest in a new physics, or mathematical physics, based on Taylor expansions of nonlinear problems, solutions to the Schrödinger equation that are not square integrable, tortuous ways for obtaining well–known analytical results, laughable treatments of chemical reactions, etc[4, 5, 6, 7, 8, 9, 10, 11]. The most remarkable example is provided by a predator–prey model that predicts a negative number of rabbits[7]. The VAPA journals that publish such kind of *scientific* articles do not accept criticisms or comments on them. JMP has engrossed the distinguished list of VAPA journals.

Of course we may have a completely wrong point of view. For that reason in what follows we show the critical opinion of a prestigious anonymous referee:

Referee #1 (Comments to the Author):

This paper is a comment on Yildirim's paper, but the title is too large.

Yildirim's might have some demerits in one way or another, the homotopy perturbation method does work for the Boussinesq-like equation.

If the author wants to write a comment on the homotopy perturbation method, the author should base on Ji-Huan He's publications on HMP.

The method is an asymptotic method , though the method can also lead to a convergent series solution.

An elementary introduction to the method is given in the following editorial article: He JH. Recent development of the homotopy perturbation method , TOPOLOGICAL METHODS IN NONLINEAR ANALYSIS 31(2008) 205-209

The author suggests a straightforward method , that we can submit Eq.(4) into (2), and same result can be obtained . This is true , and such method becomes invalid for most nonlinear equations.

The author ignors the connection between his method and famous known method of separation of variables used by Joseph Fourier in 1822.

The author further assumes that the solution can be expressed in (8) with exp-polynomials, again this special suggestion works, but it does work for most nonlinear problems, the author should follow the exp-function method in this case.

The author gives a solution without considering the boundary conditions, this has only mathematical interesting, and has no physical meanings at all.

Such discuss should be directly addressed to DR.Yildirim, not suitable for publication.

I note that the author has already posted his comments on arXiv, this is enough.

This is an interesting example of the quality of the JMP refereeing service. We should be grateful to be enlightened this way.

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