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Algebraic Treatment of $\mathcal{PT}$-Symmetric Coupled Oscillators

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Abstract The purpose of this paper is the discussion of a pair of coupled linear oscillators that has recently been proposed as a model of a system of two optical resonators. By means of an algebraic approach we show that the frequencies of the quantum-mechanical interpretation of the optical phenomenon are linear combinations of the classical frequencies. Consequently, if the classical frequencies are real, then the quantum-mechanical eigenvalues are also real.

Keywords PT symmetry · Coupled oscillators · Gain and loss · PT transitions · Algebraic treatment · Adjoint matrix representation

1 Introduction

In a recent paper Bender et al. [1] discussed the classical and quantum-mechanical versions of a system of two coupled linear oscillators one with gain and the other with loss. When the gain and loss parameters are equal the Hamiltonian derived from the classical equations of motion is $\mathcal{PT}$-symmetric and exhibits two $\mathcal{PT}$-transitions in terms of the coupling parameter. In the unbroken-$\mathcal{PT}$ region the classical frequencies are real otherwise they appear as pairs of complex conjugate numbers. This analysis is straightforward because one can derive analytical expressions for such frequencies from the equations of motion. The authors also obtained analytical expressions for the eigenvalues and eigenfunctions of the quantum-mechanical Hamiltonian and showed that both the eigenvalues of the quantum-mechanical Hamiltonian and the classical frequencies are real in the same region of model parameters. However, at first sight the analytical expressions of the quantum-mechanical eigenvalues on the one hand and the classical frequencies on the other do not resemble each other. This theoretical investigation was motivated by recent experiments on a $\mathcal{PT}$-symmetric system of two coupled optical resonators [2].

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The purpose of the present paper is to derive a clearer connection between the classical frequencies and the quantum-mechanical energies. In Section 2 we apply a well known algebraic method and show that the quantum-mechanical frequencies (spacing between eigenvalues) are linear combinations of the classical frequencies. We explicitly write the quantum-mechanical energies in terms of the classical frequencies and analyse the spectrum. Finally, in Section 3 we review the main results of the paper and draw conclusions.

2 Quantum-Mechanical Model

As indicated above, the analysis of the classical model is straightforward and here we focus on the quantal version in the case of equal gain and loss. The Hamiltonian operator derived from the classical equations of motion is given by [1]

$$H = p_x p_y + \gamma (y p_y - x p_x) + \left(\omega^2 - \gamma^2\right) xy + \frac{\epsilon}{2}\left(x^2 + y^2\right), \tag{1}$$

where p_x and p_y are the momenta conjugate to coordinates x and y: $[x, p_x] = [y, p_y] = i$. It is $\mathcal{PT}$ symmetric $\mathcal{PT}H\mathcal{PT} = H$ since it is invariant under the combined effect of parity $\mathcal{P} : \{x, y, p_x, p_y\} \rightarrow \{-y, -x, -p_y, -p_x\}$ and time reversal $\mathcal{T} : \{x, y, p_x, p_y\} \rightarrow \{x, y, -p_x, -p_y\}$ [1]. Note that we can also choose $\mathcal{P} : \{x, y, p_x, p_y\} \rightarrow \{y, x, p_y, p_x\}$ for exactly the same purpose. In the language of point-group symmetry such *parity* transformations are given by reflection planes σ_v and σ_v' perpendicular to the $x - y$ plane [3, 4]. Since the Hamiltonian H is also invariant under inversion $\hat{\imath} : \{x, y, p_x, p_y\} = \{-x, -y, -p_x, -p_y\}$ its eigenvectors $|\psi\rangle$ satisfy $\hat{\imath}\,|\psi\rangle = \pm\,|\psi\rangle$. This result is consistent with the solutions of the form $P_{mn}(x, y)e^{-(2axy+bx^2+cy^2)}$ obtained by Bender et al. [1] in the coordinate representation, where the polynomials $P_{mn}(x, y)$ satisfy $P_{mn}(-x, -y) = (-1)^m P_{mn}(x, y)$. Point-group symmetry proved to be useful for the discussion of broken- and unbroken-$\mathcal{PT}$ symmetry in some anharmonic oscillators [5, 6]. As far as we know Klaiman and Cederbaum [5] were the first to construct a non-Hermitian Hamiltonian with consecutive $\mathcal{PT}$ phase transitions in terms of the coupling parameter. Those authors coined the term space-time symmetry when $\mathcal{P}$ is different from $\hat{\imath}$ as in the present case.

It is worth noting that the Hamiltonian (1) is also a symmetric operator $\langle H\psi\,|\varphi\rangle = \langle\psi\,|H\varphi\rangle$, which for brevity we formally express by means of the usual notation for Hermitian operators as $H^\dagger = H$. In particular, note that $(yp_y - xp_x)^\dagger = (p_y y - p_x x) = (yp_y - xp_x)$. Therefore, one expects real eigenvalues at least for some range of the model parameters ω, γ and ϵ where the above turnover relation holds. In fact, if ψ is an eigenfunction of H with eigenvalue E then it follows from $\langle H\psi\,|\psi\rangle = \langle\psi\,|H\psi\rangle$ that $E^* = E$.

In order to solve the eigenvalue equation $H\,|\psi\rangle = E\,|\psi\rangle$ in a way that clearly reveals the connection with the classical interpretation we resort to a well known algebraic method [7] (see also http://en.wikipedia.org/wiki/Ladder_operator). It is suitable when there exists a set of symmetric operators $\{O_1, O_2, \ldots, O_N\}$ that satisfy the commutation relations

$$[H, O_i] = \sum_{j=1}^{N} H_{ji} O_j. \tag{2}$$

We look for an operator of the form

$$Z=\sum_{i=1}^{N}c_iO_i, \tag{3}$$

such that

$$[H,Z]=\lambda Z. \tag{4}$$

The operator Z is important for our purposes because

$$HZ\left|\psi\right\rangle=(E+\lambda)Z\left|\psi\right\rangle. \tag{5}$$

It follows from equations (2), (3) and (4) that

$$(\mathbf{H}-\lambda\mathbf{I})\mathbf{C}=0, \tag{6}$$

where $\mathbf{H}$ is an $N\times N$ matrix with elements H_{ij}, $\mathbf{I}$ is the $N\times N$ identity matrix, and $\mathbf{C}$ is an $N\times 1$ column matrix with elements c_i. $\mathbf{H}$ is called the adjoint or regular matrix representation of H in the operator basis $\{O_1,O_2,\ldots,O_N\}$ [7]. In the case of an Hermitian operator we expect all the roots λ_i, $i=1,2,\ldots,N$ to be real. These roots are obviously the natural frequencies of the quantum-mechanical system (the quantum-mechanical frequencies are linear combinations of them). Here we apply the same approach to symmetric Hamiltonians because all the relevant equations are formally identical. If λ is real then it follows from equation (4) that

$$[H,Z^{\dagger}]=-\lambda Z^{\dagger}, \tag{7}$$

where $Z^{\dagger}$ is a linear combination like (3) with coefficients c_i^*. This equation tells us that if λ is a real root of $\det(\mathbf{H}-\lambda\mathbf{I})=0$, then $-\lambda$ is also a root. Obviously, Z and $Z^{\dagger}$ are a pair of annihilation-creation or ladder operators because, in addition to (5), we also have

$$HZ^{\dagger}\left|\psi\right\rangle=(E-\lambda)Z^{\dagger}\left|\psi\right\rangle. \tag{8}$$

Other authors have already applied Lie-algebraic methods to the Hamiltonian (1) without coupling ($\epsilon=0$) [8–10] but they were not interested in the quantum-mechanical frequencies.

In the present case, the obvious choice $\{O_1,O_2,O_3,O_4\}=\{x,y,p_x,p_y\}$ leads to the matrix representation

$$\mathbf{H}=i\begin{pmatrix}\gamma & 0 & \epsilon & \omega^2-\gamma^2\\ 0 & -\gamma & \omega^2-\gamma^2 & \epsilon\\ 0 & -1 & -\gamma & 0\\ -1 & 0 & 0 & \gamma\end{pmatrix}, \tag{9}$$

with characteristic polynomial

$$\lambda^4+\lambda^2\left(4\gamma^2-2\omega^2\right)-\epsilon^2+\omega^4=0, \tag{10}$$

that is exactly the one that yields the classical frequencies [1]. Two of its roots are

$$\begin{aligned}\lambda_1&=\sqrt{\sqrt{\epsilon^2+4\gamma^4-4\gamma^2\omega^2}-2\gamma^2+\omega^2}\\ \lambda_2&=\sqrt{-\sqrt{\epsilon^2+4\gamma^4-4\gamma^2\omega^2}-2\gamma^2+\omega^2},\end{aligned} \tag{11}$$

and the other two ones are $\lambda_3 = -\lambda_1$ and $\lambda_4 = -\lambda_2$ in agreement with the more general equations (4) and (7). The operators Z_1 and Z_2 associated to λ_1 and λ_2 are creation or rising, while $Z_3 = Z_1^\dagger$ and $Z_4 = Z_2^\dagger$ are annihilation or lowering. The classical and quantal natural frequencies are exactly the same because the relevant Poisson brackets and commutators are similar: $i\{H, O_i\} \rightarrow [H, O_i]$. This result reveals why the condition for real classical frequencies

$$2\gamma\sqrt{\omega^2 - \gamma^2} < \epsilon < \omega^2 \tag{12}$$

is also the condition for real spectrum (unbroken-$\mathcal{PT}$ region) in the quantum-mechanical counterpart [1].

It is obvious that the algebraic method described above only applies to some exactly solvable problems. In particular, the form of the Poisson brackets and quantum-mechanical commutators is similar because the Hamiltonian operator (1) is a quadratic function of the coordinates and their conjugate momenta.

If we write the polynomial equation (10) as $\left(\lambda^2 - \lambda_1^2\right)\left(\lambda^2 - \lambda_2^2\right) = 0$ then we realize that

$$\begin{aligned} \lambda_1^2 + \lambda_2^2 &= 2\omega^2 - 4\gamma^2 \\ \lambda_1^2\lambda_2^2 &= \omega^4 - \epsilon^2. \end{aligned} \tag{13}$$

Throughout this paper we keep the parameter ω in order to facilitate the discussion of the results of Bender et al. However, it is worth noting that we can choose $\omega = 1$ without loss of generality as follows from the transformation $\{\lambda, a, \omega, \gamma, \epsilon\} \rightarrow \{\frac{\lambda}{\omega}, \frac{a}{\omega}, 1, \frac{\gamma}{\omega}, \frac{\epsilon}{\omega^2}\}$.

Bender et al. [1] derived the energies

$$E_{mn} = (m+1)a + (2n - m)\Delta, \tag{14}$$

where $m = 0, 1, \ldots$ and $n = 0, 1, \ldots, m$. In this equation a is a root of

$$4a^4 + 4a^2\left(2\gamma^2 - \omega^2\right) + \epsilon^2 + 4\gamma^2\left(\gamma^2 - \omega^2\right) = 0, \tag{15}$$

and

$$\begin{aligned} \Delta &= \sqrt{bc - \gamma^2}, \\ b &= c^* = \frac{\epsilon}{2(a + i\gamma)}. \end{aligned} \tag{16}$$

The expressions for a, b and c come from solving the eigenvalue equation $H\left|\psi_{00}\right\rangle = E_{00}\left|\psi_{00}\right\rangle$ in the coordinate representation with the ansatz $\psi_{00}(x, y) = e^{-(bx^2 + cy^2 + 2axy)}$, procedure that also yields $E_{00} = a$. If we write $\xi = a^2$ then we obtain the roots

$$\begin{aligned} \xi_1 &= \frac{\omega^2 - 2\gamma^2 - \sqrt{\omega^4 - \epsilon^2}}{2}, \\ \xi_2 &= \frac{\omega^2 - 2\gamma^2 + \sqrt{\omega^4 - \epsilon^2}}{2}. \end{aligned} \tag{17}$$

Following Bender et al we write $a_1 = -\sqrt{\xi_1}$, $a_2 = \sqrt{\xi_1}$, $a_3 = -\sqrt{\xi_2}$, $a_4 = \sqrt{\xi_2}$. For concreteness, from now on we choose

$$a = a_2 = \frac{1}{2}\sqrt{2\omega^2 - 4\gamma^2 - 2\sqrt{\omega^4 - \epsilon^2}}. \tag{18}$$

So far, we have shown that the quantum-mechanical frequencies can be expressed as linear combinations of the classical frequencies; it only remains to rewrite the eigenvalues (14) in terms of the latter. One can easily verify that

$$\lambda_1 = \Delta + a,\ \lambda_2 = \Delta - a, \tag{19}$$

is consistent with (13). Thus, the expression for the energies becomes

$$\begin{aligned} E_{m,n} &= \frac{\lambda_1\,(2n+1)}{2} - \frac{\lambda_2\,(2m-2n+1)}{2} = \\ &= n\lambda_1 + (n-m)\lambda_2 + a. \end{aligned} \tag{20}$$

Bender et al. [1] showed numerically that a_2 and Δ are real and positive in the unbroken-$\mathcal{PT}$ region and concluded that the eigenvalues are also real and positive. However, this is not the case because $a - \Delta < 0$ and for every value of n $E_{mn} \rightarrow -\infty$ as $m \rightarrow \infty$. As an example, consider the model parameters $\omega = 1$, $\gamma = 0.05$ and $\epsilon = 0.5$ that lie the unbroken-$\mathcal{PT}$ region. In this case $\Delta \approx 0.964630863$ and $a \approx 0.2539434939$ that confirms what we have just said.

We can analyse those results by means of the algebraic method. If E_{00} were the lowest eigenvalue then both $Z_3\psi_{00}$ and $Z_4\psi_{00}$ would be expected to vanish. However, for the parameters chosen above we found that $Z_2\psi_{00}$ and $Z_3\psi_{00}$ vanish while $Z_1\psi_{00}$ and $Z_4\psi_{00}$ do not. Therefore, the eigenfunctions are given by

$$\psi_{nk} = Z_1^n Z_4^k \psi_{00} \tag{21}$$

with eigenvalues $E_{nk} = n\lambda_1 - k\lambda_2 + a$ which agree with (20) if $k = m - n$. The algebraic method clearly shows that the spectrum is unbounded from below.

3 Conclusions

The main purpose of this paper is to show that part of the mathematical analysis of some classical systems also applies to their quantum-mechanical counterparts. The underlying connection is that the frequencies of both interpretations are closely related because of the similarity between the Poisson brackets and commutators. Therefore, if the frequencies of the motion of the classical system are real then the quantum-mechanical eigenvalues are also real. The quantum-mechanical natural frequencies are the eigenvalues of the regular or adjoint matrix representation of the Hamiltonian operator in a suitable basis set of operators, whereas the corresponding eigenvectors provide the ladder operators. This well known algebraic approach applies to many exactly solvable problems [7] (see also http://en.wikipedia.org/wiki/Ladder_operator) and in particular for Hamiltonians that are quadratic functions of the coordinates and their conjugate momenta. Such Hamiltonians are suitable models for many physical problems like the one that motivated the paper by Bender et al. [1], among others [8–10].

In closing it is worth mentioning that the fact that the spectrum of the Hamiltonian (1) is not strictly positive, contrary to what Bender et al assumed, does not appear to be relevant to the interpretation of the physical data which is fitted by the classical (and also natural quantal) frequencies [2].

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