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Weighted estimates for integral operators on local BMO type spaces

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We prove the weighted boundedness for a family of integral operators T_α on Lebesgue spaces and local BMO type spaces. To this end we show that T_α can be controlled by the Calderón operator and a local maximal operator. This approach allows us to characterize the power weighted boundedness on Lebesgue spaces.

1 Introduction

In this paper we will study the weighted boundedness for a family of integral operators on different spaces of functions. To be more specific, for $n \geq 1$ and $0 < \alpha < n$, we define T_α by

$$T_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^\alpha |x+y|^{n-\alpha}}\, dy.$$

These operators were introduced in [6]. In that article the authors proved the boundedness on $L^2(\mathbb{R})$ in order to get the main results concerning a family of maximal operators on the three dimensional Heisenberg group. In [3] it was proved that this family of operators is of type (p, p) for $1 < p < \infty$ and of weak type $(1, 1)$ on $\mathbb{R}^n$.

Weighted boundedness for a more general family of operators is studied in [7]. The authors used the unweighted results already known to prove the weighted boundedness of type (p, p) for $1 < p < \infty$ and of weak type $(1, 1)$ for a wide family of weights in A_p. They also proved an appropriate weighted estimate from a subset of $L^\infty(\omega^{-1})$ into $BMO(\omega)$. Additionally, the boundedness of this type of operators on the Hardy spaces H^p is studied in [8].

In this article we prove the weighted boundedness on Lebesgue spaces for the operators T_α defined above, and we do this in a different way that in [7]. We bound T_α with the Calderón operator and a local maximal operator, and then we use the known results for these. This method allows us to obtain at the same time the weighted and unweighted boundedness. This approach leads us to characterize the boundedness of T_α for power weights.

Finally, based on the articles [1] and [4], we prove the boundedness of T_α from a subset of a local $BMO(\omega)$ type space into $BMO(\omega)$. This result improves the one in [7].

In Section 2 we recall some definitions and preliminary results that will be needed in this paper. In Section 3 we state our results and we give the necessary definitions. Finally, in Section 4 we prove our results.

2 Preliminaries

It is well known that a weight is a locally integrable and non-negative function, and the Muckenhoupt class A_p, $1 \leq p < \infty$, is defined as the class of weights ω such that for all balls B

$$\left(\frac{1}{|B|}\int_B \omega\right)\left(\frac{1}{|B|}\int_B \omega^{-\frac{1}{p-1}}\right)^{p-1} < C, \quad \text{if } 1 < p < \infty, \tag{2.1}$$

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$$\frac{\omega(B)}{|B|} \leq C \operatorname*{ess\,inf}_{x \in B} \omega(x), \quad \text{if} \quad p = 1, \tag{2.2}$$

where $\omega(B) = \int_B \omega$. Also $A_\infty = \bigcup_{1 \leq p < \infty} A_p$.

Other weight classes were introduced to analyze the boundedness of different operators. In this article we define some of these classes.

In [2], the authors consider the classical Hardy operator P and its adjoint Q given by

$$Pf(x) = \frac{1}{|x|^n} \int_{|y| \leq |x|} f(y)\, dy, \quad Qf(x) = \int_{|y| \geq |x|} \frac{f(y)}{|y|^n}\, dy,$$

and the Calderón operator S defined as $S = P + Q$. They prove a characterization of the weights for which S is of weighted type (p, p), $1 < p < \infty$, and of weighted weak type $(1, 1)$, by a single condition. The weight class denoted $A_{p,0}$, consists of the weights satisfying the conditions (2.1) and (2.2) respectively, but only for balls centered at the origin.

In [5] the authors analyze weighted boundedness for a local maximal operator. For this, they introduce local balls as follows: for $0 < k < 1$, a ball $B(z, r)$ centered at z with radius r is a k-local ball if $0 < r \leq k|z|$. Denote by $\mathcal{O}_k$ the family of all these balls and define a local Hardy-Littlewood maximal operator by

$$M_{k,loc} f(x) = \sup_{B \in \mathcal{O}_k : x \in B} \frac{1}{|B|} \int_B |f|, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Then, they characterize the class of weights for which $M_{k,loc}$ is of weighted type (p, p), $1 < p < \infty$, and of weighted weak type $(1, 1)$. This class of weights denoted $A_{p,k,loc}$, consists of the weights satisfying the conditions (2.1) and (2.2) respectively, but only for k-local balls. After that, they also prove that $A_{p,k,loc} = A_{p,l,loc}$ for any $0 < k, l < 1$. Therefore, this class of weights is denoted by $A_{p,loc}$ and $A_{\infty,loc} = \bigcup_{1 \leq p < \infty} A_{p,loc}$.

Remark 2.1 In both articles [2] and [5], some known properties of A_p were also analyzed for $A_{p,loc}$ and $A_{p,0}$. Also, some counterexamples are provided.

Continuing with the analysis of the local maximal, the boundedness of $M_{k,loc}$ on a weighted local BMO space were investigated in [1] and [4]. For this, the authors define a ball $B(z, r)$ to be a sub-k-critical ball whenever $r < k|z|$, a k-critical ball if $r = k|z|$ and a supra-k-critical ball if $r > k|z|$ and does not contain zero. In this context, the weight is a locally integrable and non-negative function ω on $\mathbb{R}^n \setminus \{0\}$.

Definition 2.2 $BMO_{k,loc}(\omega)$ is the space of functions f locally integrable on $\mathbb{R}^n \setminus \{0\}$ that satisfy the conditions $\frac{1}{\omega(B)} \int_B |f - f_B| < C_1$ for all sub-k-critical balls B, and $\frac{1}{\omega(B)} \int_B |f| < C_2$ for all k-critical or supra-k-critical balls. Here $f_B = \frac{1}{|B|} \int_B f$.

In [1], the authors prove that $M_{k,loc}$ is bounded from $BMO_{k,loc}(\omega)$ into itself if and only if $\omega \in A_{1,loc}$.

3 Main results

For simplicity, we denote $M_{loc} = M_{\frac{1}{2},loc}$. Throughout this paper c and C will denote positive constants, not necessarily the same at each occurrence.

The following lemma shows that T_α is pointwise controlled by the Calderón operator and the local maximal operator M_{loc}.

Lemma 3.1 *Let $f \in L^1_{loc}(\mathbb{R}^n)$ be non-negative, then*

$$T_\alpha f(x) \leq C\big(Sf(x) + M_{loc} f(x) + M_{loc} f(-x)\big),$$

where C depends on α and n.

Using the characterization of weighted boundedness for S and M_{loc} proved in [2] and [5], we obtain.

Theorem 3.2 *If $\omega \in A_{p,loc} \cap A_{p,0} = A_p$ and satisfies $\omega(-x) \leq C\omega(x)$ for almost all $x \in \mathbb{R}^n$, then the operator T_α is of weighted strong type (p, p) for $1 < p < \infty$, and of weighted weak type $(1, 1)$ if $p = 1$.*

An important class of weights is that of power weights of the form $\omega(x) = |x|^\beta$.

It can be seen that a power weight $\omega \in A_p$ if and only if $\omega \in A_{p,0}$. On the other hand, it is known that: for $1 < p < \infty$, then $\omega(x) = |x|^\beta \in A_p$ if and only if $-n < \beta < n(p-1)$, and $\omega(x) = |x|^\beta \in A_1$ if and only if $-n < \beta \le 0$.

It is easy to prove that $T_\alpha f(x) \ge cSf(x)$, where c only depends on the dimension n. Then, using this fact and Theorem 3.2, we obtain:

Corollary 3.3 *Let ω be a power weight and let $1 < p < \infty$. The operator T_α is of weighted strong type (p, p) if and only if $\omega \in A_p$. The operator T_α is of weighted weak type $(1, 1)$ if and only if $\omega \in A_1$.*

Next, consider T_α from the space

$$L^\infty(\omega^{-1}) = \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \left\| f\omega^{-1} \right\|_\infty = \|f\|_{L^\infty(\omega^{-1})} < \infty \right\}$$

into $BMO(\omega)$, where $BMO(\omega)$ denotes the space of functions $f \in L^1_{loc}(\mathbb{R}^n)$ such that $\frac{1}{\omega(B)} \int_B |f - f_B| \le C$ for all balls B.

For this, we first take into account the boundedness of the Hardy operator P and its adjoint Q from a space of functions containing $L^\infty(\omega^{-1})$.

Definition 3.4 Let ω be a weight, we define the set $BM_0(\omega)$ as the functions $f \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\frac{1}{\omega(B)} \int_B |f(x)|\, dx \le C_1,$$

for all balls B centered at zero.

If $\omega \in A_{1,0}$, we can define $\|f\|_{BM_0(\omega)} = \inf\{C_1\}$, where the infimum is taken over all constants C_1 satisfying Definition 3.4. In this way $\|\cdot\|_{BM_0(\omega)}$ is a norm on $BM_0(\omega)$. It is immediate that $\|f\|_{BM_0(\omega)} \le \|f\|_{L^\infty(\omega^{-1})}$ for all $f \in L^\infty(\omega^{-1})$, thus $L^\infty(\omega^{-1}) \subset BM_0(\omega)$.

On the other hand, since $|Qf(x)|$ and $|T_\alpha f(x)|$ are infinity for a constant function f, we cannot expect the boundedness of these operators over all the space $L^\infty(\omega^{-1})$. Therefore we define

$$L^\infty_0(\omega^{-1}) = \left\{ f \in L^\infty(\omega^{-1}) : |Qf(x_0)| < \infty \text{ for some } x_0 \right\}$$

and

$$E\big(BM_0(\omega)\big) = \left\{ f \in BM_0(\omega) : |Qf(x_0)| < \infty \text{ for some } x_0 \right\}.$$

Proposition 3.5 *If $\omega \in A_{1,0}$ then there exists $C > 0$ such that*

(i) $\|Pf\|_{L^\infty(\omega^{-1})} \le C\, \|f\|_{BM_0(\omega)}$ *for all* $f \in BM_0(\omega)$.
(ii) $\|Qf\|_{BMO(\omega)} \le C\, \|f\|_{BM_0(\omega)}$ *for all* $f \in E\big(BM_0(\omega)\big)$.

These properties for P and Q will allow us to give another proof of the next result, already proved in [7].

Theorem 3.6 *If $\omega \in A_1$ and satisfies $\omega(-x) \le C\omega(x)$ for almost all $x \in \mathbb{R}^n$, then there exists $c > 0$ such that $\|T_\alpha f\|_{BMO(\omega)} \le c\, \|f\|_{L^\infty(\omega^{-1})}$ for all $f \in L^\infty_0(\omega^{-1})$.*

Now, we introduce a suitable local BMO type space.

Definition 3.7 Let ω be a weight, then $BMO_0(\omega)$ is the space of functions $f \in L^1_{loc}(\mathbb{R}^n)$ satisfying

(i) $f \in BM_0(\omega)$.
(ii) $\frac{1}{\omega(B)} \int_B |f - f_B| \le C_2$, for all $B = B(x_0, r)$ with $0 < r < \frac{1}{8}|x_0|$.

If $\omega \in A_1$, we can define on $BMO_0(\omega)$ the norm

$$\|f\|_{BMO_0(\omega)} = \inf\{C_1 + C_2\},$$

where the infimum is taken over all constants C_1 and C_2 satisfying the conditions (i) and (ii) of Definition 3.7, respectively.

It is clear that $L^\infty(\omega^{-1}) \subset BMO(\omega) \cap BM_0(\omega) \subset BMO_0(\omega)$. Moreover, it can be seen without difficulty that

$$BMO(\omega) \cap BM_0(\omega) = BMO_0(\omega). \tag{3.1}$$

For the same reason as before, we define

$$E\big(BMO_0(\omega)\big) = \big\{f \in BMO_0(\omega) : |Qf(x_0)| < \infty \text{ for some } x_0 \in \mathbb{R}^n\big\}.$$

We now state our main result.

Theorem 3.8 *Let $f \in E\big(BMO_0(\omega)\big)$. If $\omega \in A_1$ and satisfies $\omega(-x) \le C\omega(x)$ for almost all $x \in \mathbb{R}^n$, then $T_\alpha f \in BMO(\omega)$. Moreover, there exists $c > 0$ independent of f such that $\|T_\alpha f\|_{BMO(\omega)} \le c\,\|f\|_{BMO_0(\omega)}$.*

This theorem provides an improvement of the result given by Theorem 3.6. Indeed, if $\omega \equiv 1$ the function defined on $\mathbb{R}$ by $h_0(x) = \log \frac{1}{|x-2|}$ for $x \in (1,3)$, and $h_0(x) = 0$ otherwise, satisfies $h_0 \in E\big(BMO_0(\omega)\big)$ but $h_0 \notin L^\infty(\omega^{-1})$.

Remark 3.9 Considering (3.1), we make a brief comparison between the $BM_0(\omega)$, $BMO_0(\omega)$ and $BMO(\omega)$ spaces in the 1-dimensional case with $\omega \equiv 1$.

Let

$$f_0(x) = \begin{cases} \dfrac{1}{\sqrt{x-1}} & \text{if } x \in (1,2), \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g_0(x) = \begin{cases} \log \dfrac{1}{|x|} & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

It can be seen that $f_0 \in BM_0(\omega)$ but $f_0 \notin BMO(\omega)$, so $f_0 \notin BMO_0(\omega)$. After that, $g_0 \in BMO(\omega)$ but $g_0 \notin BM_0(\omega)$. Therefore, the inclusions $BMO_0(\omega) \subset BM_0(\omega)$ and $BMO_0(\omega) \subset BMO(\omega)$ are strict, and the $BM_0(\omega)$ and $BMO(\omega)$ spaces are different in the sense that there is no inclusion among them.

4 Proofs

In this section we use some properties of A_p weights, including the doubling condition and the reverse Hölder inequality (RHI), among others.

As usual, the Hardy-Littlewood maximal operator will be denoted by M. Given E, a subset of $\mathbb{R}^n$, we denote by E^c its complement in $\mathbb{R}^n$ and χ_E its characteristic function. $A(z, r, R)$, with $0 < r < R$, denotes the annulus centered at z with radii r and R.

Remark 4.1 If ω is a weight and E is a bounded measurable set in $\mathbb{R}^n$, then $\operatorname{ess\,inf}_E \omega \le \frac{1}{|E|}\int_E \omega$. Therefore, if $\omega \in A_1$ we have

$$\frac{\omega(B)}{|B|}\frac{|E|}{\omega(E)} \le C\left(\operatorname*{ess\,inf}_B \omega\right)\left(\operatorname*{ess\,inf}_E \omega\right)^{-1} \le C,$$

for all balls B and any set $E \subset B$ with positive measure.

Proof of Lemma 3.1 Let $x \ne 0$. We call $U_1 = B(0,|x|) \cap B^c\big(x, \frac{|x|}{2}\big) \cap B^c\big(-x, \frac{|x|}{2}\big)$ and $U_2 = B^c(0,|x|) \cap B^c\big(x, \frac{|x|}{2}\big) \cap B^c\big(-x, \frac{|x|}{2}\big)$. We decompose the operator T_α as follows

$$T_\alpha f(x) = T_\alpha\big(\chi_{U_1} f\big)(x) + T_\alpha\big(\chi_{U_2} f\big)(x) + T_\alpha\left(\chi_{B\left(x, \frac{|x|}{2}\right)} f\right)(x) + T_\alpha\left(\chi_{B\left(-x, \frac{|x|}{2}\right)} f\right)(x). \tag{4.1}$$

If $y \in U_1$ then $|x-y| \ge \frac{|x|}{2}$ and $|x+y| \ge \frac{|x|}{2}$, and so

$$T_\alpha\big(\chi_{U_1} f\big)(x) \le cPf(x). \tag{4.2}$$

If $y \in U_2$ then $|x-y| \ge \frac{|y|}{3}$ and $|x+y| \ge \frac{|y|}{3}$, and so

$$T_\alpha\big(\chi_{U_2} f\big)(x) \le cQf(x). \tag{4.3}$$

Now, if $y \in B\big(x, \frac{|x|}{2}\big)$ then $|x+y| > |x|$, and so

$$\begin{aligned}\int_{B\left(x,\frac{|x|}{2}\right)} \frac{f(y)}{|x-y|^{\alpha}|x+y|^{n-\alpha}}\,dy &\le \frac{1}{|x|^{n-\alpha}} \int_{B\left(x,\frac{|x|}{2}\right)} \frac{f(y)}{|x-y|^{\alpha}}\,dy \\ &= \frac{c}{|x|^{n-\alpha}} \sum_{j=1}^{\infty} \frac{1}{(2^{-j}|x|)^{\alpha}} \int_{2^{-j-1}|x|\le|x-y|<2^{-j}|x|} f(y)\,dy \\ &\le cM_{loc}f(x) \sum_{j=1}^{\infty} \frac{1}{2^{j(n-\alpha)}} \\ &= cM_{loc}f(x).\end{aligned} \tag{4.4}$$

Similarly, if $y \in B\big(-x, \frac{|x|}{2}\big)$ then $|x-y| > |x|$, and so

$$\int_{B\left(-x,\frac{|x|}{2}\right)} \frac{f(y)}{|x-y|^{\alpha}|x+y|^{n-\alpha}}\,dy \le cM_{loc}f(-x). \tag{4.5}$$

Therefore, using (4.2), (4.3), (4.4) and (4.5), and replacing in (4.1), we conclude the proof of the lemma. □

P r o o f o f P r o p o s i t i o n 3 . 5 . Part (i) is immediate. For part (ii), let $f \in E\big(BM_0(\omega)\big)$ and $\omega \in A_{1,0}$, then we want to prove that $Qf(x)$ is well defined for all $x \neq 0$. Indeed, let $x_0 \in \mathbb{R}^n$ such that $Qf(x_0)$ is finite. Then $|Qf(x)| \le Q|f|(x_0) < \infty$ for $|x| \ge |x_0|$. If $0 < |x| < |x_0|$, then

$$\begin{aligned}\big|Qf(x)\big| &= \left|\int_{|y|>|x|} \frac{f(y)}{|y|^n}\,dy\right| \\ &\le \left|\int_{|x_0|>|y|>|x|} \frac{f(y)}{|y|^n}\,dy\right| + \left|\int_{|y|>|x_0|} \frac{f(y)}{|y|^n}\,dy\right| \\ &\le \frac{1}{|x|^n} \int_{B(0,|x_0|)} |f(y)|\,dy + \big|Qf(x_0)\big| \\ &\le \|f\|_{BM_0(\omega)} \frac{1}{|x|^n}\,\omega\big(B(0,|x_0|)\big) + \big|Qf(x_0)\big| < \infty.\end{aligned}$$

Now, let $r > 0$ and $\nu \in \mathbb{R}^n$ such that $|\nu| = r$, then

$$\begin{aligned}\int_{B(0,r)} \big|Qf(x) - Qf(\nu)\big|\,dx &\le \int_{0<|x|<r} \int_{|x|<|y|<r} \frac{|f(y)|}{|y|^n}\,dy\,dx \\ &= \int_{0<|y|<r} \frac{|f(y)|}{|y|^n} \left(\int_{B(0,|y|)} dx\right) dy \\ &\le C\,\|f\|_{BM_0(\omega)}\,\omega\big(B(0,r)\big).\end{aligned}$$

Therefore,

$$\frac{1}{\omega(B(0,r))} \int_{B(0,r)} \big|Qf(x) - Qf(\nu)\big|\,dx \le C\,\|f\|_{BM_0(\omega)}\,. \tag{4.6}$$

Now, we consider the ball $B(a,r)$ with $r \ge \frac{|a|}{2} > 0$ and $\nu \in \mathbb{R}^n$ satisfying $|\nu| = 3r$. Since $B(a,r) \subset B(0,3r)$, using (4.6) and Remark 4.1, we have

$$\begin{aligned}&\frac{1}{\omega(B(a,r))} \int_{B(a,r)} \big|Qf(x) - Qf(\nu)\big|\,dx \\ &\le \frac{\omega(B(0,3r))}{\omega(B(a,r))} \frac{1}{\omega(B(0,3r))} \int_{B(0,3r)} \big|Qf(x) - Qf(\nu)\big|\,dx \\ &\le C\,\|f\|_{BM_0(\omega)}\,.\end{aligned}$$

In contrast, if we consider $B(a,r)$ with $0<r<\frac{|a|}{2}$ and $\nu\in\mathbb{R}^n$ such that $|\nu|=|a|+r$, then $B(a,r)\subset B(0,|a|+r)$, $|x|>|a|-r$ for $x\in B(a,r)$, and $|a|-r>\frac{|a|}{2}$. Thus, using Remark 4.1

$$\begin{aligned}\frac{1}{\omega(B(a,r))}\int_{B(a,r)}\left|Qf(x)-Qf(\nu)\right|dx &= \frac{1}{\omega(B(a,r))}\int_{B(a,r)}\left|\int_{|x|<|y|<|a|+r}\frac{f(y)}{|y|^n}\,dy\right|dx\\ &\le \frac{1}{\omega(B(a,r))}\int_{B(a,r)}\int_{|a|-r<|y|<|a|+r}\frac{|f(y)|}{|y|^n}\,dy\,dx\\ &\le \frac{|B(a,r)|}{\omega(B(a,r))}\frac{1}{(|a|-r)^n}\int_{|y|<|a|+r}|f(y)|\,dy\\ &\le C\,\|f\|_{BM_0(\omega)}\left(\frac{|a|+r}{|a|-r}\right)^n\\ &\le C\,\|f\|_{BM_0(\omega)}\,.\end{aligned}$$

□

Proof of Theorem 3.6. Let $f\in L_0^\infty(\omega^{-1})$. By Proposition 3.5 and Lemma 3.1, we have $T_\alpha f\in L^1_{loc}(\mathbb{R}^n)$. We make a similar decomposition as we did in the proof of Lemma 3.1. Let $K_\alpha(x,y)=|x-y|^{-\alpha}\cdot|x+y|^{-n+\alpha}$ and $U=B\big(0,\frac{3}{2}|x|\big)\cap B^c\big(x,\frac{|x|}{2}\big)\cap B^c\big(-x,\frac{|x|}{2}\big)$, then

$$\begin{aligned}T_\alpha f(x)-Qf(x) &= T_\alpha(\chi_U f)(x)+T_\alpha\big(\chi_{B(x,\frac{|x|}{2})}f\big)(x)+T_\alpha\big(\chi_{B(-x,\frac{|x|}{2})}f\big)(x)\\ &\quad -\int_{|x|\le|y|<\frac{3}{2}|x|}\frac{f(y)}{|y|^n}\,dy+\int_{|y|\ge\frac{3}{2}|x|}\left(K_\alpha(x,y)-\frac{1}{|y|^n}\right)f(y)\,dy.\end{aligned}$$

In a similar manner as the proof of Lemma 3.1, we obtain

$$\begin{aligned}&\left|T_\alpha(\chi_U f)(x)+T_\alpha\big(\chi_{B(x,\frac{|x|}{2})}f\big)(x)+T_\alpha\big(\chi_{B(-x,\frac{|x|}{2})}f\big)(x)-\int_{|x|\le|y|<\frac{3}{2}|x|}\frac{f(y)}{|y|^n}\,dy\right|\\ &\le C\left(P|f|\left(\frac{3}{2}x\right)+M_{loc}|f|(x)+M_{loc}|f|(-x)\right)\\ &\le C\big(M|f|(x)+M|f|(-x)\big).\end{aligned}$$

By the mean value theorem and $\omega\in A_1$, we have

$$\begin{aligned}\left|\int_{B^c\left(0,\frac{3}{2}|x|\right)}f(y)\left(K_\alpha(x,y)-\frac{1}{|y|^n}\right)dy\right| &\le C|x|\int_{B^c\left(0,\frac{3}{2}|x|\right)}\frac{|f(y)|}{|y|^{n+1}}\,dy\\ &\le C\,\|f\|_{L^\infty(\omega^{-1})}\,|x|\int_{B^c\left(0,\frac{3}{2}|x|\right)}\frac{\omega(y)}{|y|^{n+1}}\,dy\\ &= C\,\|f\|_{L^\infty(\omega^{-1})}\sum_{i=0}^{\infty}|x|\int_{\frac{3}{2}|x|2^i\le|y|<\frac{3}{2}|x|2^{i+1}}\frac{\omega(y)}{|y|^{n+1}}\,dy\\ &\le C\,\|f\|_{L^\infty(\omega^{-1})}\sum_{i=0}^{\infty}\frac{|x|}{\big(2^i|x|\big)^{n+1}}\int_{|y|<\frac{3}{2}|x|2^{i+1}}\omega(y)\,dy\\ &\le C\,\|f\|_{L^\infty(\omega^{-1})}\,\omega(x).\end{aligned}$$

Now, using that ω satisfies $\omega(-x)\le C\omega(x)$ for almost all $x\in\mathbb{R}^n$, we obtain

$$\|T_\alpha f-Qf\|_{L^\infty(\omega^{-1})}\le C\left(\big\|M(|f|)\big\|_{L^\infty(\omega^{-1})}+\big\|(M|f|)^\vee\big\|_{L^\infty(\omega^{-1})}+\|f\|_{L^\infty(\omega^{-1})}\right),$$

where $(Mf)^\vee(x)=Mf(-x)$. Thus, by Proposition 3.5 we have proved the theorem. □

Before proceeding to the proof of Theorem 3.8 we state some results that we need.

In [1], the version on $\mathbb{R}^+ = (0, \infty)$ of the following two lemmas is established (see Lemmas 3.1 and 3.3). We claim that the same proofs, with some obvious modifications, can be adapted to this setting.

Lemma 4.2 *If $\omega \in A_{\infty,loc}$ then $BMO_{k,loc}(\omega) = BMO_{l,loc}(\omega)$ for any $0 < k, l < 1$, with norms and equivalence constants depending on ω, k and l.*

Lemma 4.3 *Let $\omega \in A_{p,loc}$ and $0 < k < 1$. For $1 \le r \le p'$, there exists a constant C_k depending on r, k and the $A_{p,k,loc}$ constant of ω, such that if $f \in BMO_{k,loc}(\omega)$ then*

$$\left(\frac{1}{\omega(B)}\int_B |f(x) - f_B|^r \omega^{1-r}(x)\,dx\right)^{\frac{1}{r}} \le C_k \|f\|_{BMO_{k,loc}(\omega)}$$

for all sub-k-critical balls B.

Remark 4.4 Let $\omega \in A_1$ and $f \in BMO_0(\omega)$. It is not difficult to see that $f \in BMO_{k,loc}(\omega)$ and $\|f\|_{BMO_{k,loc}(\omega)} \le C_k \|f\|_{BMO_0(\omega)}$, for $k = \frac{1}{8}$. By Lemma 4.2, this statement holds for all $0 < k < 1$.

Let $\psi \in \mathcal{C}^\infty(\mathbb{R})$ such that $0 \le \psi \le 1$, $supp(\psi) = \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $\psi(t) = 1$ for $t \in \left[-\frac{1}{4}, \frac{1}{4}\right]$. We define

$$V_\alpha f(x) = \int_{\mathbb{R}^n} \frac{\psi\left(|x|^{-1}|x-y|\right) f(y)}{|x-y|^\alpha |x+y|^{n-\alpha}}\,dy,\ x \neq 0.$$

If f is non-negative, then $V_\alpha f \le T_\alpha f$. Consequently, by Theorem 3.2, V_α is of type (p, p) with respect to the Lebesgue measure for $1 < p < \infty$.

The following lemma is the key to Theorem 3.8.

Lemma 4.5 *If $\omega \in A_1$, then V_α is bounded from $BMO_0(\omega)$ into itself.*

P r o o f. Let $f \in BMO_0(\omega)$ and let $t_0 = \frac{3}{4}$.

We will prove first that the bounded oscillation condition (ii) of Definition 3.7 holds for $V_\alpha f$. Let $B = B(x_0, r)$ with radius $0 < r < \frac{1}{8}|x_0|$, we want to prove

$$\frac{1}{\omega(B)}\int_B \left|V_\alpha f(x) - c\right| dx \le C \|f\|_{BMO_0(\omega)}, \tag{4.7}$$

for some constant c depending on f and B; and C depending on ω and n.

We decompose f as

$$f = (f - f_B)\chi_{2B} + (f - f_B)\chi_{(2B)^c} + f_B = f_1 + f_2 + f_3,$$

where $2B = B(x_0, 2r)$.

First, consider $V_\alpha f_1$. Since $\omega \in A_1$, then it satisfies RHI with exponent $s > 1$. Thus $\omega^s \in A_1$. We choose $q > 1$ and $\gamma > 1$ such that $s = \frac{\gamma q - 1}{\gamma - 1}$. Using that V_α is of type (q, q) for $1 < q < \infty$ and the Hölder inequality, we obtain

$$\begin{aligned}
\frac{1}{\omega(B)}\int_B \left|V_\alpha f_1\right| &\le \frac{|B|^{\frac{1}{q'}}}{\omega(B)}\left(\int_B \left|V_\alpha f_1\right|^q\right)^{\frac{1}{q}} \\
&\le \frac{|B|^{\frac{1}{q'}}}{\omega(B)}\left(\int_{2B} |f - f_B|^q \omega^{\frac{1-\gamma q}{\gamma}} \omega^{\frac{\gamma q - 1}{\gamma}}\right)^{\frac{1}{q}} \\
&\le \frac{|B|^{\frac{1}{q'}}}{\omega(B)}\left(\int_{2B} |f - f_B|^{\gamma q} \omega^{1-\gamma q}\right)^{\frac{1}{\gamma q}} \left(\int_{2B} \omega^{\frac{\gamma q - 1}{\gamma}\gamma'}\right)^{\frac{1}{\gamma' q}} \\
&\le \frac{|B|^{\frac{1}{q'}}}{\omega(B)}\left(\int_{2B} |f - f_{2B}|^{\gamma q} \omega^{1-\gamma q}\right)^{\frac{1}{\gamma q}} \left(\int_{2B} \omega^s\right)^{\frac{1}{\gamma' q}} \\
&\quad + \frac{|B|^{\frac{1}{q'}}}{\omega(B)}\left(\int_{2B} |f_{2B} - f_B|^{\gamma q} \omega^{1-\gamma q}\right)^{\frac{1}{\gamma q}} \left(\int_{2B} \omega^s\right)^{\frac{1}{\gamma' q}}.
\end{aligned} \tag{4.8}$$

Now, we analyze each term of the last expression in (4.8). Applying Lemma 4.3, the doubling property for the wights in A_p and Remark 4.4, we have

$$\begin{aligned}
&\frac{|B|^{\frac{1}{q'}}}{\omega(B)}\left(\int_{2B}|f-f_{2B}|^{\gamma q}\omega^{1-\gamma q}\right)^{\frac{1}{\gamma q}}\left(\int_{2B}\omega^{s}\right)^{\frac{1}{\gamma' q}}\\
&\quad\le C\frac{|B|^{\frac{1}{q'}}}{\omega(B)}\omega(2B)^{\frac{1}{\gamma q}}\,\|f\|_{BMO_{t_0,loc}(\omega)}\,|2B|^{\frac{1}{\gamma' q}}\left(\frac{\omega(2B)}{|2B|}\right)^{\frac{s}{\gamma' q}}\\
&\quad\le C|B|^{\frac{1}{q'}+\frac{1}{\gamma' q}-\frac{s}{\gamma' q}}\omega(B)^{-1+\frac{1}{\gamma q}+\frac{s}{\gamma' q}}\,\|f\|_{BMO_0(\omega)}\\
&\quad= C\,\|f\|_{BMO_0(\omega)}\,.
\end{aligned}\tag{4.9}$$

For the second term, we use that $\omega^s\in A_{\gamma'}$, the doubling property and Remark 4.4. Then

$$\begin{aligned}
&\frac{|B|^{\frac{1}{q'}}}{\omega(B)}\left(\int_{2B}|f_{2B}-f_{B}|^{\gamma q}\omega^{1-\gamma q}\right)^{\frac{1}{\gamma q}}\left(\int_{2B}\omega^{s}\right)^{\frac{1}{\gamma' q}}\\
&\quad= \frac{|B|^{\frac{1}{q'}}}{\omega(B)}\left(\int_{2B}\omega^{1-\gamma q}\right)^{\frac{1}{\gamma q}}\left(\int_{2B}\omega^{s}\right)^{\frac{1}{\gamma' q}}|f_{2B}-f_B|\\
&\quad\le C\frac{|B|^{\frac{1}{q'}}}{\omega(B)}|2B|^{\frac{1}{q}}\frac{1}{|B|}\int_B|f-f_{2B}|\\
&\quad\le C|B|^{\frac{1}{q'}+\frac{1}{q}-1}\,\|f\|_{BMO_{t_0,loc}(\omega)}\\
&\quad\le C\,\|f\|_{BMO_0(\omega)}\,.
\end{aligned}\tag{4.10}$$

Next, consider $V_\alpha f_2$. Let $x\in B=B(x_0,r)$,

$$\begin{aligned}
\left|V_\alpha f_2(x)-V_\alpha f_2(x_0)\right| &\le \int_{\mathbb{R}^n}|f_2(y)|K_\alpha(x,y)\left|\psi\left(|x|^{-1}|x-y|\right)-\psi\left(|x_0|^{-1}|x_0-y|\right)\right|dy\\
&\quad+\int_{\mathbb{R}^n}|f_2(y)|\psi\left(|x_0|^{-1}|x_0-y|\right)\left|K_\alpha(x,y)-K_\alpha(x_0,y)\right|dy.
\end{aligned}\tag{4.11}$$

We again analyze each term.

We note that if $x\in B$, then $\frac{7}{8}|x_0|<|x|<\frac{9}{8}|x_0|$. If $\psi\left(|x|^{-1}|x-y|\right)\neq 0$ then $y\in B\left(x,\frac{1}{2}|x|\right)$. Furthermore, if $y\in B\left(x,\frac{1}{4}|x|\right)$, then $\psi\left(|x|^{-1}|x-y|\right)=1$. Thus, if $\left|\psi\left(|x|^{-1}|x-y|\right)-\psi\left(|x_0|^{-1}|x_0-y|\right)\right|\neq 0$, then $y\in\Omega=\left[B\left(x,\frac{1}{2}|x|\right)\cup B\left(x_0,\frac{1}{2}|x_0|\right)\right]\cap\left[B^c\left(x,\frac{1}{4}|x|\right)\cup B^c\left(x_0,\frac{1}{4}|x_0|\right)\right]$. Now, if $y\in\Omega\setminus B\left(x_0,\frac{1}{2}|x_0|\right)$, then $|y-x_0|\le|y-x|+|x-x_0|\le\frac{1}{2}|x|+r<\frac{11}{16}|x_0|$. And if $y\in B(x_0,\frac{1}{2}|x_0|)$, then $|y-x_0|<\frac{1}{2}|x_0|$. Hence $\Omega\subset B\left(x_0,\frac{11}{16}|x_0|\right)=B_0$. Next, applying the mean value theorem, we obtain

$$\begin{aligned}
&\left|\psi\left(|x|^{-1}|x-y|\right)-\psi\left(|x_0|^{-1}|x_0-y|\right)\right|\\
&\quad\le\left|\psi'(\xi)\right|\left|\frac{|x-y|}{|x|}-\frac{|x_0-y|}{|x_0|}\right|\\
&\quad\le C\left(\left|\frac{|x-y|}{|x|}-\frac{|x_0-y|}{|x|}\right|+\left|\frac{|x_0-y|}{|x_0|}-\frac{|x_0-y|}{|x|}\right|\right)\\
&\quad\le C\frac{r}{|x_0|}.
\end{aligned}$$

We also observe that if $y \in \Omega$, then $|x - y| > c|x_0|$ and $|x + y| > c|x_0|$. Therefore, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |f_2(y)| K_\alpha(x,y) \big|\psi\big(|x|^{-1}|x-y|\big) - \psi\big(|x_0|^{-1}|x_0 - y|\big)\big|\, dy \\
&\le C \frac{r}{|x_0|} \frac{1}{|x_0|^n} \int_{B_0} |f_2(y)|\, dy \\
&= C \frac{r}{|x_0|} \frac{1}{|B_0|} \int_{B_0} |f_2(y)|\, dy. \qquad (4.12)
\end{aligned}$$

Next, we choose a natural number j_0 such that $2^{j_0} r < d_0 |x_0| \le 2^{j_0+1} r$ where $d_0 = \frac{11}{16}$. We denote $L_j = B(x_0, 2^j r)$ for $j = 0, 1, 2, \ldots, j_0$, so $L_j \subset B_0$. Then

$$\begin{aligned}
|f - f_B| &\le |f - f_{B_0}| + |f_{B_0} - f_{L_{j_0}}| + \sum_{j=1}^{j_0} |f_{L_j} - f_{L_{j-1}}| \\
&\le |f - f_{B_0}| + \frac{C}{|L_{j_0}|} \int_{L_{j_0}} |f - f_{B_0}| + \sum_{j=1}^{j_0} \frac{C}{|L_{j-1}|} \int_{L_{j-1}} |f - f_{L_j}| \\
&\le |f - f_{B_0}| + \frac{C}{|B_0|} \int_{B_0} |f - f_{B_0}| + \sum_{j=1}^{j_0} \frac{C}{|L_j|} \int_{L_j} |f - f_{L_j}|. \qquad (4.13)
\end{aligned}$$

Since B_0 and L_j, $j = 0, 1, 2, \ldots, j_0$, are sub-t_0-critical balls, using (4.13), Remark 4.4 and $\frac{r}{|x_0|}(2 + j_0) \le (1 + d_0)$, we obtain

$$\begin{aligned}
C \frac{r}{|x_0|} \frac{1}{|B_0|} \int_{B_0} |f_2| &\le C_{t_0} \frac{r}{|x_0|} \|f\|_{BMO_0(\omega)} \left(2 \frac{\omega(B_0)}{|B_0|} + \sum_{j=1}^{j_0} \frac{\omega(L_j)}{|L_j|} \right) \\
&\le C \frac{r}{|x_0|} \|f\|_{BMO_0(\omega)} \left(\operatorname*{ess\,inf}_B \omega \right) (2 + j_0) \\
&\le C \|f\|_{BMO_0(\omega)} \operatorname*{ess\,inf}_B \omega. \qquad (4.14)
\end{aligned}$$

Therefore, by (4.12)

$$\begin{aligned}
&\frac{1}{\omega(B)} \int_B \int_{\mathbb{R}^n} |f_2(y)| K_\alpha(x,y) \left| \psi\left(\frac{|x-y|}{|x|} \right) - \psi\left(\frac{|x_0 - y|}{|x_0|} \right) \right| dy\, dx \\
&\le C_{t_0} \frac{|B|}{\omega(B)} \|f\|_{BMO_0(\omega)} \operatorname*{ess\,inf}_B \omega \\
&\le C \|f\|_{BMO_0(\omega)}. \qquad (4.15)
\end{aligned}$$

We now estimate the second term of (4.11). Applying the mean value theorem, we have $\big|K_\alpha(x,y) - K_\alpha(x_0,y)\big| \le C \frac{|x - x_0|}{|y - x_0|^{n+1}} < C \frac{r}{|y - x_0|^{n+1}}$. Thus

$$\begin{aligned}
&\frac{1}{\omega(B)} \int_B \int_{\mathbb{R}^n} |f_2(y)| \psi\big(|x_0|^{-1}|x_0 - y|\big) \big| K_\alpha(x,y) - K_\alpha(x_0,y) \big| dy\, dx \\
&\le C \frac{r|B|}{\omega(B)} \int_{B\left(x_0, \frac{1}{2}|x_0|\right)} \frac{|f_2(y)|}{|y - x_0|^{n+1}}\, dy \\
&= C \frac{r|B|}{\omega(B)} \int_{A\left(x_0, 2r, \frac{|x_0|}{2}\right)} \frac{|f(y) - f_B|}{|y - x_0|^{n+1}}\, dy.
\end{aligned}$$

We choose j_1 a natural number such that $2^{j_1}r < \frac{1}{2}|x_0| \le 2^{j_1+1}r$. We call $A_j = A\left(x_0, 2^{j-1}r, 2^j r\right)$, $L_j = B(x_0, 2^j r)$ with $j = 1, 2, \dots, j_1$ and $A = A\left(x_0, 2^{j_1}r, \frac{|x_0|}{2}\right)$. Then

$$\begin{aligned}
&C\frac{r|B|}{\omega(B)}\int_{A\left(x_0,2r,\frac{|x_0|}{2}\right)}\frac{|f(y)-f_B|}{|y-x_0|^{n+1}}\,dy\\
&\le C\frac{r|B|}{\omega(B)}\left(\sum_{j=1}^{j_1}\int_{A_j}\frac{|f(y)-f_B|}{|y-x_0|^{n+1}}\,dy+\int_A\frac{|f(y)-f_B|}{|y-x_0|^{n+1}}\,dy\right)\\
&\le C\frac{r|B|}{\omega(B)}\left(\sum_{j=1}^{j_1}\frac{1}{(2^j r)^{n+1}}\int_{A_j}|f-f_B|+\frac{1}{(2^{j_1}r)^{n+1}}\int_A|f-f_B|\right)\\
&\le C\frac{|B|}{\omega(B)}\left(\sum_{j=1}^{j_1}\frac{1}{2^j|L_j|}\int_{L_j}|f-f_B|+\frac{1}{2^{j_1}|B(x_0,\frac{|x_0|}{2})|}\int_{B(x_0,\frac{|x_0|}{2})}|f-f_B|\right).
\end{aligned}$$

Arguments similar to those leading to (4.13), (4.14) and (4.15) give

$$\begin{aligned}
&\frac{|B|}{\omega(B)}\left[\sum_{j=1}^{j_1}\frac{1}{2^j|L_j|}\int_{L_j}|f-f_B|+\frac{2^{-j_1}}{|B(x_0,\frac{|x_0|}{2})|}\int_{B(x_0,\frac{|x_0|}{2})}|f-f_B|\right]\\
&\le C\frac{|B|}{\omega(B)}\left[\sum_{j=1}^{j_1}\frac{1}{2^j|L_j|}\int_{L_j}\left(|f-f_{L_j}|+\sum_{i=1}^{j}|f_{L_i}-f_{L_{i-1}}|\right)\right.\\
&\quad\left.+\frac{1}{2^{j_1}|B(x_0,\frac{|x_0|}{2})|}\int_{B(x_0,\frac{|x_0|}{2})}\left(|f-f_{B(x_0,\frac{|x_0|}{2})}|+|f_{B(x_0,\frac{|x_0|}{2})}-f_B|\right)\right]\\
&\le C\,\|f\|_{BMO_{t_0,loc}(\omega)}\frac{|B|}{\omega(B)}\left[\sum_{j=1}^{j_1}\left(\frac{\omega(L_j)}{2^j|L_j|}+\frac{1}{2^j}\sum_{i=1}^{j}\frac{\omega(L_i)}{|L_i|}\right)\right.\\
&\quad\left.+\frac{\omega\big(B(x_0,\frac{|x_0|}{2})\big)}{2^{j_1}\big|B(x_0,\frac{|x_0|}{2})\big|}+\frac{1}{2^{j_1}|B|}\int_B\left|f-f_{B(x_0,\frac{|x_0|}{2})}\right|\right]\\
&\le C\,\|f\|_{BMO_0(\omega)}\left[\sum_{j=1}^{j_1}\left(\frac{1}{2^j}+\frac{j}{2^j}\right)+2\right].
\end{aligned}\tag{4.16}$$

Then by (4.11), (4.15) and (4.16) we have proved

$$\frac{1}{\omega(B)}\int_B\left|V_\alpha f_2(x)-V_\alpha f_2(x_0)\right|dx\le C\,\|f\|_{BMO_0(\omega)}.$$

Next, consider $V_\alpha f_3$.

$$\begin{aligned}
V_\alpha f_3(x)&=\int_{\mathbb{R}^n}\frac{\psi\left(|x|^{-1}|x-y|\right)f_3}{|x-y|^\alpha|x+y|^{n-\alpha}}\,dy\\
&=f_3\int_{\mathbb{R}^n}\frac{\psi\left(\left|\frac{x}{|x|}-z\right|\right)}{\left|\frac{x}{|x|}-z\right|^\alpha\left|\frac{x}{|x|}+z\right|^{n-\alpha}}\,dz\\
&=f_3\int_{\mathbb{R}^n}\frac{\psi\left(\left|R(e_1)-R(u)\right|\right)}{\left|R(e_1)-R(u)\right|^\alpha\left|R(e_1)+R(u)\right|^{n-\alpha}}\,du
\end{aligned}$$

$$= f_3 \int_{\mathbb{R}^n} \frac{\psi\big(|e_1 - u|\big)}{|e_1 - u|^\alpha |e_1 + u|^{n-\alpha}}\, du$$

$$= f_3 \int_{B(0,\frac{1}{2})} \frac{\psi\big(|\nu|\big)}{|\nu|^\alpha |2e_1 + \nu|^{n-\alpha}}\, d\nu = C f_3,$$

where we made the change of variables $z = R(u)$ with R the rotation such that $R(e_1) = x|x|^{-1}$, $e_1 = (1, 0, \dots, 0)$.

By (4.8), (4.9), (4.10), (4.11), (4.15) and (4.16) we have

$$\frac{1}{\omega(B)} \int_B \big|V_\alpha f(x) - V_\alpha f_2(x_0) - V_\alpha f_3(x)\big| dx$$

$$\leq \frac{1}{\omega(B)} \int_B \big|V_\alpha f_1(x)\big| dx + \frac{1}{\omega(B)} \int_B \big|V_\alpha f_2(x) - V_\alpha f_2(x_0)\big|\, dx$$

$$\leq C \, \|f\|_{BMO_0(\omega)} .$$

Namely, (4.7) is proved.

Now we will prove that the bounded mean condition (i) of Definition 3.7 holds for $V_\alpha f$. Let $B = B(0, r)$, $r > 0$, we want to prove

$$\frac{1}{\omega(B)} \int_B \big|V_\alpha f(x)\big| dx \leq C \, \|f\|_{BMO_0(\omega)} ,$$

for some C depending on ω and n.

If $x \in B$ and $\frac{|x-y|}{|x|} < \frac{1}{2}$, then $|y| < 2r$, and

$$\frac{1}{\omega(B)} \int_B \big|V_\alpha f(x)\big| dx \leq \frac{1}{\omega(B)} \int_B \int_{\mathbb{R}^n} \frac{|f(y)|\psi\big(|x|^{-1}|x-y|\big)}{|x-y|^\alpha |x+y|^{n-\alpha}}\, dy\, dx$$

$$= \frac{1}{\omega(B)} \int_{\mathbb{R}^n} |f(y)| \int_B \frac{\psi\big(|x|^{-1}|x-y|\big)}{|x-y|^\alpha |x+y|^{n-\alpha}}\, dx\, dy$$

$$\leq \frac{1}{\omega(B)} \int_{2B} |f(y)| \int_B \frac{\psi\big(|x|^{-1}|x-y|\big)}{|x-y|^\alpha |x+y|^{n-\alpha}}\, dx\, dy.$$

The proof will be completed if $\int_B \frac{\psi\left(|x|^{-1}|x-y|\right)}{|x-y|^\alpha |x+y|^{n-\alpha}}\, dx \leq C$ for $0 < |y| < 2r$. To this end, let R be the rotation such that $R(e_1) = y|y|^{-1}$ and make the change of variables $R(u) = x|y|^{-1}$. Then, if $\psi\big(|u|^{-1}|u - e_1|\big) \neq 0$, it follows that $0 < c_0 < |u| < c_1$ and $|u + e_1| \geq c > 0$. Thus

$$\int_B \frac{\psi\big(|x|^{-1}|x-y|\big)}{|x-y|^\alpha |x+y|^{n-\alpha}}\, dx \leq \int_{A(0,c_0,c_1)} \frac{\psi\big(|u|^{-1}|u-e_1|\big)}{|u-e_1|^\alpha |u+e_1|^{n-\alpha}}\, du$$

$$\leq C \int_{A(0,c_0,c_1)} \frac{1}{|u-e_1|^\alpha}\, du \leq C.$$

□

Proof of Theorem 3.8. Let $f \in E\big(BMO_0(\omega)\big)$ and let $x \neq 0$. We define $U = B\big(0, \frac{3}{2}|x|\big) \cap B^c\big(x, \frac{|x|}{4}\big) \cap B^c\big(-x, \frac{|x|}{4}\big)$. Then

$$T_\alpha f(x) - V_\alpha f(x) - V_{n-\alpha} f(-x) - Q f(x)$$

$$= T_\alpha(\chi_U f)(x) - \int_{A\left(x, \frac{|x|}{4}, \frac{|x|}{2}\right)} \frac{\psi\big(|x|^{-1}|x-y|\big)}{|x-y|^\alpha |x+y|^{n-\alpha}} f(y)\, dy$$

$$- \int_{A\left(-x, \frac{|x|}{4}, \frac{|x|}{2}\right)} \frac{\psi\big(|x|^{-1}|x+y|\big)}{|x-y|^\alpha |x+y|^{n-\alpha}} f(y)\, dy$$

$$- \int_{|x|\le|y|<\frac{3}{2}|x|} \frac{f(y)}{|y|^n}\,dy + \int_{|y|\ge\frac{3}{2}|x|} \left(K_\alpha(x,y) - \frac{1}{|y|^n}\right) f(y)\,dy.$$

Since $\omega \in A_1$, then

$$\begin{aligned}
&\left| T_\alpha(\chi_U f)(x) - \int_{A\left(x,\frac{|x|}{4},\frac{|x|}{2}\right)} \frac{\psi\left(|x|^{-1}|x-y|\right)}{|x-y|^\alpha |x+y|^{n-\alpha}} f(y)\,dy \right.\\
&\left. - \int_{A\left(-x,\frac{|x|}{4},\frac{|x|}{2}\right)} \frac{\psi\left(|x|^{-1}|x+y|\right)}{|x-y|^\alpha |x+y|^{n-\alpha}} f(y)\,dy - \int_{|x|\le|y|<\frac{3}{2}|x|} \frac{f(y)}{|y|^n}\,dy \right|\\
&\le C \frac{1}{|x|^n} \int_{B\left(0,\frac{3}{2}|x|\right)} |f(y)|\,dy\\
&\le C\omega(x)\,\|f\|_{BMO_0(\omega)}\,.
\end{aligned}$$

Arguments similar to those in the proof of Theorem 3.6, imply

$$\begin{aligned}
\left| \int_{|y|\ge\frac{3}{2}|x|} \left(K_\alpha(x,y) - \frac{1}{|y|^n}\right) f(y)\,dy \right| &\le C \sum_{i=0}^{\infty} \frac{1}{2^i\left(2^i|x|\right)^n} \int_{|y|<\frac{3}{2}|x|2^{i+1}} |f(y)|\,dy\\
&\le C\omega(x)\,\|f\|_{BM_0(\omega)}\,.
\end{aligned}$$

Therefore,

$$\left\| T_\alpha f - V_\alpha f - (V_{n-\alpha} f)^\vee - Qf \right\|_{L^\infty(\omega^{-1})} \le C\,\|f\|_{BMO_0(\omega)}\,.$$

Thus, by Proposition 3.5, Lemma 4.5 and since $\omega(-x) \le C\omega(x)$ for almost all $x \in \mathbb{R}^n$, we have proved the theorem. □

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