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Petrov type of linearly perturbed type-D spacetimes

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Abstract

We show that a spacetime satisfying the linearized vacuum Einstein equations around a type-D background is generically of type I, and that the splittings of the principal null directions (PNDs) and of the degenerate eigenvalue of the Weyl tensor are non-analytic functions of the perturbation parameter of the metric. This provides a gauge-invariant characterization of the effect of the perturbation on the underlying geometry, without appealing to differential curvature invariants. This is of particular interest for the Schwarzschild solution, for which there are no signatures of the even perturbations on the algebraic curvature invariants. We also show that, unlike the general case, the unstable even modes of the Schwarzschild naked singularity deform the Weyl tensor into a type-II one.

Keywords: linear perturbations, black holes, Petrov classification

1. Introduction

It is a rather intricate problem to understand how the background geometry is affected by gravitational waves, here meaning solutions h_{ab} to the linearized vacuum Einstein's equation (LEE) around a vacuum solution g_{ab}:

$$\lim_{\epsilon \to 0} \frac{R_{ab}\left[g_{cd} + \epsilon h_{cd}\right]}{\epsilon} = 0,$$

or, more explicitly,

$$\nabla^c \nabla_c h_{ab} + \nabla_a \nabla_b \left(g^{cd} h_{cd}\right) - 2\nabla^c \nabla_{(a} h_{b)c} = 0, \tag{1}$$

which is partly due to the gauge issue of linearized gravity, i.e. the fact that, given a solution h_{ab} of (1) and an arbitrary vector field V^c, h'_{ab} defined by

$$h'_{ab} = h_{ab} + \pounds_V \, g_{ab}, \tag{2}$$

is also a solution, although physically equivalent to h_{ab}. We are only interested in the equivalence classes of solutions of (1) under the equivalence relation $h'_{ab} \sim h_{ab} + \pounds_V \, g_{ab}$, and in functionals of h_{ab} that are gauge-invariant, i.e., depend only on the equivalence class of h_{ab}. The linear perturbation δT of a tensor field T that is a functional of the metric transforms as $\delta T \to \delta T + \pounds_V \, T$ under the gauge transformation (2), therefore only constant scalar fields and tensor products of the identity map δ^a_b are gauge invariants.

A possibility explored in [1] is to parametrize the equivalent classes of metric perturbations in terms of the perturbations of a set of curvature scalar fields that vanish in the background. In vacuum, there are only four functionally independent *algebraic* invariants of the Weyl tensor C_{abcd} and its dual C^*_{abcd} (recall that left and right Hodge duals of C_{abcd} agree); these are

$$\begin{aligned} Q_+ &:= \tfrac{1}{48} C_{abcd} C^{abcd} \\ C_+ &:= \tfrac{1}{96} C_{ab}{}^{cd} C_{cd}{}^{ef} C_{ef}{}^{ab}, \\ Q_- &:= \tfrac{1}{48} C^*_{abcd} C^{abcd}, \\ C_- &:= \tfrac{1}{96} C^*_{ab}{}^{cd} C_{cd}{}^{ef} C_{ef}{}^{ab}. \end{aligned} \tag{3}$$

For the Schwarzschild black hole only Q_- and C_- vanish in the background, and it is shown in [1] that δQ_- parametrizes the space of odd (also called *vector*, see, e.g., [2]) metric perturbations, and that $\delta C_- \propto \delta Q_-$ and therefore adds no information. On the other hand, under even (or *scalar* [2]) perturbations, every gauge-invariant combination of the perturbed scalars in (3) vanishes identically. For this reason, the gauge-invariant combination $(9M - 4r)\delta Q_+ + 3r^3 \delta X$, which involves the *differential invariant* $X = (1/720)(\nabla_a C_{bcde})(\nabla^a C^{bcde})$ was added to δQ_- in [1] to parametrize the entire set of perturbations using geometrically meaningful quantities. A natural question to ask is what—if any—are the effects of the even perturbations on the curvature itself; in other words, do we really need to look at differential invariants to find a geometric signature of the perturbation? That contractions of products of the curvature tensor hide vital information in Lorentzian geometry is not a surprise: pp-waves are an extreme example of non-flat vacuum metrics for which every scalar made out of C_{abcd} and arbitrary covariant derivatives of it vanish [3].

The Weyl tensor of a generic metric is of type I in the Petrov classification. This means that the eigenvalue problem

$$\tfrac{1}{2} C^{ab}{}_{cd} X^{cd} = \lambda X^{ab}, \quad X^{ab} = X^{[ab]} \tag{4}$$

admits three *different* solutions with $\lambda_1 + \lambda_2 + \lambda_3 = 0$ or, equivalently, that the equation

$$k_{[e} C_{a]bc[d} k_{f]} k^b k^c = 0, \quad k^a k_a = 0, \tag{5}$$

admits four solutions spanning four different null *lines* (called principal null *directions* or PNDs). type-D spacetimes, instead, are characterized by the fact that the eigenvalue equation (4) admits three linearly independent solutions with $\lambda_1 = \lambda_2 = -\frac{1}{2}\lambda_3$, a condition that turns out to be equivalent to the existence of two so-called double PNDs, that is, two non-proportional null vectors satisfying an equation stronger than (5):

$$C_{abc[d} k_{e]} k^b k^c = 0, \quad k^a k_a = 0. \tag{6}$$

It should be stressed that equations (5) and (6), being homogeneous, do not define (null) tangent vectors at a point p of the spacetime (i.e., elements of $T_p M$), but instead one-dimensional subspaces through the origin of T_pM (this is why we talk of null *directions*), that is, points in the projective space $P(T_pM)$. We may think that type-I (D) spacetimes have four (two) smooth-valued functions that assign to every $p \in M$ a point in $P(T_pM)$. Now suppose that $g_{ab}(\epsilon)$ (ϵ in an open interval around zero) is a mono-parametric family of vacuum solutions of type I for $\epsilon \neq 0$ and type D for $\epsilon = 0$. The tensor

$$h_{ab} = \lim_{\epsilon \to 0} \frac{g_{ab}(\epsilon) - g_{ab}(0)}{\epsilon}, \tag{7}$$

satisfies (1) and the four PNDs of $C_{abcd}(\epsilon)$ coalesce pairwise into two PNDs as $\epsilon \to 0$. We can choose solutions $k_1^a(\epsilon)$, $k_2^a(\epsilon)$, $k_3^a(\epsilon)$ and $k_4^a(\epsilon)$ of $k_{[e}C_{a]bc[d}(\epsilon)k_{f]}k^bk^c = 0$ such that $k_1(0) = k_2(0)$ and $k_3(0) = k_4(0)$ are the two non-proportional solutions of the type-D equation $C_{abc[d}(0)k_{e]}k^bk^c = 0$. Similarly we can label the eigenvalues $\lambda_1(\epsilon)$, $\lambda_2(\epsilon)$ and $\lambda_3(\epsilon)$ of (5) such that $\lambda_1(0) = \lambda_2(0) = -\frac{1}{2}\lambda_3(0)$. This suggests using either the splitting of type-D PNDs in $P(T_pM)$ into a pair of PNDs or the eigenvalue splitting

$$\pm\big(\lambda_2(\epsilon) - \lambda_1(\epsilon)\big) \tag{8}$$

to measure the distortion of the unperturbed $\epsilon = 0$ metric. Note that the sign ambiguity in (8) comes from the $1 \leftrightarrow 2$ freedom in labeling the two eigen-bivectors (4) that degenerate in the $\epsilon = 0$ limit. We have found that for $\epsilon \simeq 0$, these are appropriate, gauge-invariant estimates of the distortion of the geometry by a gravitational wave involving only the Weyl tensor.

The reason why we expect these quantities to be non-trivial is that the bulk of vacuum solutions of Einstein's equation are type I, while algebraically special solutions comprise a zero-measure subset $\mathcal{A}$. The curve $g_{ab}(\epsilon)$ will generically be transverse to $\mathcal{A}$ at $g_{ab}(0)$ (as assumed above); its 'tangent vector' h_{ab} will stick out of $\mathcal{A}$ (more on this in section 4.3).

A *speciality index* $\mathcal{S}$ was introduced in [6] based on the fact that a vacuum spacetime is algebraically special if and only if $C_+^2 = Q_+^3$; it was defined as $\mathcal{S} := C_+^2/Q_+^3$ and the departure of this index from unity is regarded as a measure of the 'degree of non-speciality' of a metric. Although useful in numerical evolutions of the full Einstein equation, $\mathcal{S} - 1$ vanishes identically in linear theory, suggesting the—wrong—conclusion that linearly perturbed algebraically special spacetimes remain special. This error was nicely clarified in [8], by taking advantage of the fact that a mono-parametric set of vacuum solutions within the Kasner family, with $g_{ab}(0)$ being type D, can be explicitly constructed. From the results in [8] it is to be expected that the PND splitting will be a power series in $\epsilon^{1/2}$. This non-analyticity on ϵ comes from the fact that the associated algebraic problem involves radicals that vanish for $\epsilon = 0$ [8].

Stationary electro-vacuum black holes have the algebraic symmetries of type D. Deviations from these metrics represent a number of very important astrophysical processes and, as argued above, will not be algebraically special. Thus, in the context of black hole perturbation theory it is important to study whether or not the perturbative techniques capture these algebraic aspects of the geometry. In numerical applications this question has been analyzed in [6, 7], and from the analytical side in [8] for the Kasner type-D solutions. In the present work we focus on the problem of finding expressions for the eigenvalue and PND splittings of the type-D background under *arbitrary* linear perturbations.

As an application, we consider the effects of linear perturbations of the Schwarzschild black hole solution and show that the even or scalar gravitational waves (those that do not leave a visible trace in the perturbed algebraic curvature invariants) do affect the Weyl tensor by turning it into type I, and that this effect is accounted for by the proposed gauge invariants above, and we also show that these invariants mix harmonic modes. As far as we are aware, these signatures of the black hole linear perturbation are analyzed in this form for the first time here.

Kerr black holes and Chandrasekhar algebraically special modes in Schwarzschild naked singularities are also briefly considered.

2. Petrov types

Although the simplest approach to the Petrov classification is accomplished using spinor methods, perturbation theory is much more tractable in tensorial language, which is the one we adopt in this paper. The Newman–Penrose equations will also be avoided, although complex methods and, in particular, a complex null tetrad will be used. For the sake of completeness, and to clarify some aspects of the linearized theory calculations, we briefly review the eigenvalue and PND approaches to the Petrov classification in this section.

2.1. Eigenvalue approach

The eigenvalue approach to the Petrov classification regards the Weyl tensor as a linear map $X^{cd} \to C^{ab}{}_{cd}X^{cd}$ in the space of rank-two antisymmetric tensors (also called 'bivectors'), and analyzes the eigenvalue equation (4). The six-dimensional bivector space is real isomorphic to the complex three-dimensional space of self-dual bivectors (SDB), which are those satisfying ${}^*S_{ab} \coloneqq \frac{1}{2}\epsilon^{ab}{}_{cd}S^{cd} = -\mathrm{i}S^{ab}$. The isomorphism is given by

$$X^{ab} \to X^{ab} + \frac{\mathrm{i}}{2}\epsilon^{ab}{}_{cd}X^{cd} =: \tilde{X}^{ab}, \tag{9}$$

its inverse being

$$X^{ab} = \frac{1}{2}\left(\tilde{X}^{ab} + \mathrm{c.c.}\right) \tag{10}$$

The space of SDBs, in turn, is isomorphic to $u_\perp^{\mathbb{C}}$, the complexification of the space of vectors orthogonal to a given unit time-like vector u^a. The isomorphism is

$$\tilde{X}^{ab} \to \tilde{X}^{ab}u_b =: X^a \tag{11}$$

and its inverse is

$$\tilde{X}^{ab} = 2u^{[a}X^{b]} + \mathrm{i}\epsilon^{ab}{}_{cd}u^cX^d. \tag{12}$$

If $\Lambda^a{}_b$ is a Lorentz transformation of unit determinant

$$\Lambda^a{}_c\Lambda^b{}_dg^{cd} = g^{ab}, \quad \Lambda^a{}_p\Lambda^b{}_q\Lambda^c{}_r\Lambda^d{}_s\epsilon^{pqrs} = \epsilon^{abcd}, \tag{13}$$

one obtains from (13) that $(\Lambda^{-1})^a{}_b = \Lambda_b{}^a$ and

$$\Lambda^a{}_p\Lambda^b{}_q\epsilon^{pq}{}_{cd} = \epsilon^{ab}{}_{rs}\Lambda^r{}_c\Lambda^s{}_d. \tag{14}$$

Equation (14) implies that the unit-determinant Lorentz transformations commute with the map (9), as expected from

$$\frac{1}{2}X^{ab}X_{ab} = \tilde{X}^{ab}\tilde{X}_{ab} = -4X^aX_a. \tag{15}$$

The second equality above, obtained from (12) together with $u_aX^a = 0$, indicates that Λ acts on $u_\perp^{\mathbb{C}}$ as an $SO(3, \mathbb{C})$ transformation $X^a \to A^a{}_bX^b$. With the help of (12) and (11) we find that this gives an isomorphism between $SO(3,1)^\uparrow$ (the group of unit-determinant Lorentz transformations preserving time orientation, i.e., $\Lambda^a{}_bu_au^b < 0$ if u^c is time-like) and $SO(3, \mathbb{C})$:

$$A^a{}_k = \Lambda^0{}_0\, \Lambda^a{}_k - \Lambda^a{}_0\, \Lambda^0{}_k + \mathrm{i}\epsilon^{0a}{}_{ql}\, \Lambda^q{}_0\, \Lambda^l{}_k, \tag{16}$$

where a zero down (up) index means contraction with u^b $(-u_b)$. To gain some intuition on the isomorphisms (9) and (11), note that if X_{ab} is the electromagnetic tensor, then $X^a = E^a + \mathrm{i}B^a$, the electric and magnetic fields measured by an observer moving with velocity u^a. If $\Lambda^a{}_b u^b = u^a$, then Λ is a rotation, and (16) reduces to

$$A^a{}_k = u^a\, u_k + \Lambda^a{}_k \tag{17}$$

which belongs to $SO(3, \mathbb{R})$ and simply rotates $\vec{E}$ and $\vec{B}$ independently. Otherwise, Λ is a boost and (16) a complex rotation mixing $\vec{E}$ and $\vec{B}$, as expected.

We can replicate the above constructions for the self-dual piece of the Weyl tensor C_{abcd} by regarding it as an element of the symmetric tensor product of bivector space, and using the fact that left self-duality implies right self-duality. Define

$$\tilde{C}_{abcd} \coloneqq \frac{1}{2}\left(C_{abcd} + \frac{\mathrm{i}}{2}\epsilon_{ab}{}^{kl}C_{klcd}\right), \tag{18}$$

and introduce the map $Q : u_\perp^{\mathbb{C}} \to u_\perp^{\mathbb{C}}$

$$Q^a{}_c \coloneqq -\tilde{C}^a{}_{bcd}\, u^b\, u^d. \tag{19}$$

The above equation can again be inverted [4],

$$-\frac{1}{2}\tilde{C}_{abcd} = 4u_{[a}Q_{b][d}u_{c]} + g_{a[c}Q_{d]b} - g_{b[c}Q_{d]a} + \mathrm{i}\epsilon_{abef}u^e u_{[c}Q_{d]}{}^f + \mathrm{i}\epsilon_{cdef}u^e u_{[a}Q_{b]}{}^f, \tag{20}$$

and the eigenvalue problem (4) is easily seen to be equivalent to $\frac{1}{4}\tilde{C}^{ab}{}_{cd}\tilde{X}^{cd} = \lambda\tilde{X}^{ab}$, and also to

$$Q^a{}_b X^b = \lambda X^a, \quad X^a = \tilde{X}^{ab}u_b. \tag{21}$$

We should stress here that, although $Q^a{}_b$ and X^b were defined using a particular unit time-like vector u^c, the eigenvalue equation (21), being equivalent to (4), gives covariant information and therefore is fully meaningful. Let $\{e_o^a = u^a, e_1^a, e_2^a, e_3^a\}$ be an orthonormal tetrad $(g_{ab}e_\alpha^a e_\beta^b = \mathrm{diag}(-1, 1, 1, 1))$ adapted to u^a, and

$$k^a = \frac{e_0^a + e_3^a}{\sqrt{2}}, \quad l^a = \frac{e_0^a - e_3^a}{\sqrt{2}}, \quad m^a = \frac{e_1^a - \mathrm{i}e_2^a}{\sqrt{2}}, \quad \bar{m}^a = \frac{e_1^a + \mathrm{i}e_2^a}{\sqrt{2}}, \tag{22}$$

the related complex null tetrad. A basis of self-dual two-forms is [4]

$$U_{ab} = 2\bar{m}_{[a}l_{b]}, \quad V_{ab} = 2k_{[a}m_{b]}, \quad W_{ab} = 2\left(m_{[a}\bar{m}_{b]} + l_{[a}k_{b]}\right) \tag{23}$$

(note that the only non-zero contractions are $U_{ab}V^{ab} = 2$ and $W_{ab}W^{ab} = -4$). These can be used to expand

$$\frac{1}{2}\tilde{C} = \Psi_0 UU + \Psi_1(UW + WU) + \Psi_2(VU + UV + WW) + \Psi_3(VW + WV) + \Psi_4 VV \tag{24}$$

where UV stands for $U_{ab}V_{cd}$, etc, and

$$\begin{aligned} &\Psi_0 \coloneqq C_{abcd}k^a m^b k^c m^d, \quad \Psi_1 \coloneqq C_{abcd}k^a l^b k^c m^d \\ &\Psi_2 \coloneqq C_{abcd}k^a m^b \bar{m}^c l^d, \quad \Psi_3 \coloneqq C_{abcd}k^a l^b \bar{m}^c l^d \\ &\Psi_4 \coloneqq C_{abcd}\bar{m}^a l^b \bar{m}^c l^d. \end{aligned} \tag{25}$$

$u_{\perp}^{\mathbb{C}}$ is the complex span of $\{e_1^a, e_2^a, e_3^a\}$ and, in this basis,

$$Q^i{}_j = \begin{pmatrix} -\frac{1}{2}\Psi_0 + \Psi_2 - \frac{1}{2}\Psi_4 & \frac{\mathrm{i}}{2}(\Psi_4 - \Psi_0) & \Psi_1 - \Psi_3 \\ \frac{\mathrm{i}}{2}(\Psi_4 - \Psi_0) & \frac{1}{2}\Psi_0 + \Psi_2 + \frac{1}{2}\Psi_4 & \mathrm{i}(\Psi_1 + \Psi_3) \\ \Psi_1 - \Psi_3 & \mathrm{i}(\Psi_1 + \Psi_3) & -2\Psi_2 \end{pmatrix} \tag{26}$$

If Q has three distinct eigenvalues, then the algebraic type of the spacetime is I. If instead two of them are equal, say $\lambda_1 = \lambda_2 =: \lambda$, the space is of type II if $\dim(\ker(\mathbf{Q} - \lambda\mathbf{I})) = 1$ or type D if $\dim(\ker(\mathbf{Q} - \lambda\mathbf{I})) = 2$. Finally, in the case in which all three eigenvalues are identical (then necessarily equal to zero, since $Q^i{}_i = 0$), the Petrov type will be III, N or O, if $\dim(\ker(\mathbf{Q})) = 1$, 2 or 3, respectively. The matrix (26) representing $Q^i{}_j$ can be put in normal form (that is, diagonal or Jordan form) by acting on it with elements of $SO(3, \mathbb{C})$. As $SO(3, \mathbb{C})$ is isomorphic to $SO(3, 1)^{\uparrow}$ (cf equation (16)), any such transformation is uniquely associated with a (proper, orthochronous) Lorentz transformation on the spacetime, and this transformation, acting on the null tetrad above, produces what is called a *principal tetrad*. The transformation leading to the normal form of Q, and therefore the principal tetrad, is uniquely determined in the cases of Petrov types I, II and III (neither k^a nor l^a of the unique principal tetrad gives a PND in type-I spacetimes [5]). For a type-D space, however, there is a two-dimensional residual $U(1) \times \mathbb{R}_{>0}{}^{\times} = C^{\times}$ subgroup of $SO(3, 1)^{\uparrow} \simeq SO(3, \mathbb{C})$ of boost and rotations preserving the normal form (and thus the PNDs):

$$k^a \to \alpha k^a, \quad l^a \to \alpha^{-1} l^a, \quad m^a \to e^{\mathrm{i}\theta} m^a, \quad \bar{m}^a \to e^{-\mathrm{i}\theta} \bar{m}^a, \tag{27}$$

In this case, k^a and l^a are aligned along the repeated PNDs, i.e., they satisfy (6). Principal null tetrad components of tensors are said to carry spin weight s and boost weight q if under (27) they pick up a factor $e^{\mathrm{i}s\theta}\alpha^q$ (e.g., Ψ_3 has $s = q = -1$). Truly scalar fields, such as $Q_+ \propto \Re(\Psi_2{}^2)$, of course, carry zero weights.

2.2. Principal null directions

An alternative approach to the Petrov classification consists of studying the PNDs of the Weyl tensor, i.e. solving equation (5), which is equivalent to

$$k_{[e} \widetilde{C}_{a]bc[d} k_{f]} k^b k^c = 0. \tag{28}$$

Starting from a generic null tetrad with associated Weyl scalars (25) we find that (see (24))

$$\tfrac{1}{2} k_{[e} \widetilde{C}_{a]bc[d} k_{f]} k^b k^c = -\Psi_0 k_{[e} \bar{m}_{a]} \bar{m}_{[d} k_{f]}, \tag{29}$$

so the k^a vector of the tetrad is a PND if and only if the Ψ_0 component of the Weyl tensor in this tetrad vanishes. If we apply a null rotation (boost) to the given null tetrad around l^a,

$$\begin{aligned} &l^a \to l'^a = l^a, \\ &k^a \to k'^a = k^a + z\bar{z} l^a + \bar{z} m^a + z \bar{m}^a, \\ &m^a \to m'^a = m^a + z l^a, \end{aligned} \tag{30}$$

the resulting k'^a will sweep the S^2 set of null directions as z moves in the complex plane, avoiding only the l^a direction, which corresponds to $z = \infty$ (S^2 = complex plane plus point at infinity). So we can calculate $\Psi_0'(z)$ in the primed tetrad and solve the fourth-order equation $\Psi_0'(z) = 0$ to find the four PNDs. It can easily be checked using (25) that

Table 1. Petrov type according to the multiplicity of the roots of the polynomial in (31).

Petrov type	PNDs
I	{1111}
II	{211}
D	{22}
III	{31}
N	{4}
O	{−}

$$\Psi_0'(z) = \Psi_0 + 4z\Psi_1 + 6z^2\Psi_2 + 4z^3\Psi_3 + z^4\Psi_4 \tag{31}$$

Generically (type-I spaces), there will be four different solutions $z_j, j = 1, 2, 3, 4$ corresponding to four PNDs. The special cases are those for which the polynomial (31) has repeated roots, and can be classified according to the partitions of 4 as shown in table 1. PNDs associated to roots with multiplicity higher than one are called *repeated PNDs*. According to the Goldberg–Sachs theorem these are tangent to shear-free geodesic congruences. Type-D spacetimes have two double roots, and the corresponding null directions satisfy the stronger equation (6). For a null tetrad with k^a and l^a aligned along the two repeated PNDs, equations (6) and (25) give

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0, \quad \Psi_2 \neq 0 \quad \text{(type D)}. \tag{32}$$

Similarly, different sets of Weyl scalars vanish for the other algebraically special spacetimes for adapted tetrads. Type O corresponds to conformally flat spaces, for which $C_{abcd} = 0$.

2.3. Physical interpretation

The characteristics of the different Petrov types, as well as the meaning of the Weyl scalars, can be realized by analyzing the geodesic deviation equation. The following interpretation is adapted from [15].

Let u^a be the unit future tangent vector to a congruence of time-like geodesics, X^a a geodesic deviation vector. Then

$$A^a := u^b\nabla_b\left(u^c\nabla_c X^a\right) = R^a{}_{bcd}u^b u^c X^d \tag{33}$$

measures the relative acceleration of neighboring test particles. Since the Riemann curvature tensor equals the Weyl tensor in vacuum, the contribution of the latter in (33), obtained by setting $R_{ab} = 0$, i.e. replacing $R^a{}_{bcd}$ with $C^a{}_{bcd}$, gives the effects of the gravitational field on the acceleration A^a. If we use $C^a{}_{bcd} = \tilde{C}^a{}_{bcd} + \text{c.c.}$ together with (24), and choose ${e_0}^a = u^a$ in (22), we find the following contributions to A^a:

$$\begin{aligned} A^a_{(4)} &= 2|\Psi_4|\left({p_2}^a p_{2d} - {p_1}^a p_{1d}\right)X^d, \\ \Psi_4 &= |\Psi_4|e^{2\mathrm{i}\alpha_4}, \;\; {p_1}^a - \mathrm{i}{p_2}^a = e^{\mathrm{i}\alpha_4}m^a \end{aligned} \tag{34}$$

$$\begin{aligned} A^a_{(3)} &= \sqrt{2}\,|\Psi_3|\left[\left(s_1^a + e_3^a\right)\left(s_{1d} + e_{3d}\right) - \left(s_1^a - e_3^a\right)\left(s_{1d} - e_{3d}\right)\right]X^d, \\ \Psi_3 &= -\left|\Psi_3\right|e^{\mathrm{i}\alpha_3}, \;\; {s_1}^a - \mathrm{i}{s_2}^a = e^{\mathrm{i}\alpha_3}m^a \end{aligned} \tag{35}$$

$$A^a_{(2)} = -\,2\Re\{\Psi_2\}\big(2e_3^a e_{3\,d} - I^a{}_d\big)X^d,$$
$$I^a{}_d \text{ the identity map in the 2-plane orthogonal to } e_0 \text{ and } e_3. \tag{36}$$

A wave with wave vector $k^a \propto u^a + e_3^a$ (equation (22)) is seen moving along the e_3^a direction by the observer with velocity u^a. The spatial vectors $p_1{}^a$, p_2^a in (34) constitute a pair of polarization axes, and the acceleration vector lies in the 2-plane spanned by them, which is orthogonal to $e_o^a = u^a$ and e_3^a. This property allows us to interpret of (34) as a *transverse mode*. By adapting the null tetrad so that k^a is along the fourfold PND of a type-N spacetime, we find that $\Psi_j = 0$ for $j \neq 4$ and that (33) reduces to (34).

The mode associated to Ψ_3 has a *longitudinal* polarization plane spanned by e_3^a and s_1^a, whereas the Ψ_2 mode produces a deformation of a sphere of free-falling particles to an ellipsoid by expanding it along the wave direction e_3, just like the effect of tidal forces on a sphere of particles under the influence of a Coulomb-like potential in Newtonian gravity. For this reason, Ψ_2 is usually interpreted as a *Coulomb term*. For type-D spacetimes, we can choose the tetrad as in (32), and find that this is the only contribution to A^a.

Note that the change $k^a \leftrightarrow l^a$ (which is equivalent to $e_3^a \leftrightarrow -e_3^a$) together with $m^a \leftrightarrow \bar{m}^a$ produces $\Psi_4 \leftrightarrow \Psi_0$, $\Psi_3 \leftrightarrow \Psi_1$ while keeping Ψ_2 unchanged. This implies that Ψ_4 and Ψ_0 are associated to transverse radiation propagating in opposite directions, and Ψ_3 and Ψ_1 to longitudinal waves in opposite directions. Therefore, we refer to Ψ_0 and Ψ_4 as in- and outgoing transverse radiation terms, respectively, whereas Ψ_1 and Ψ_3 are referred to as in- and outgoing longitudinal radiation terms, where e_3^a is defined as the outward direction.

The effects in type-I and -II and type-III spaces are a combination of those described above.

3. Linear perturbations

Let $g_{ab}(\epsilon)$ be a mono-parametric family of vacuum solutions with $g_{ab}(0) =: g_{ab}$ of type D. Assume $e_\alpha^a(\epsilon)$ is an orthonormal tetrad of the metric $g_{ab}(\epsilon)$, smooth in ϵ, and such that the associated null tetrad (22) has $k^a(0)$ and $l^a(0)$ aligned along the repeated PNDs of the type-D background g_{ab}, i.e. they satisfy (6). If $\Lambda(\epsilon)$ is a curve in $SO(3,1)^\uparrow$ with $\Lambda(0)$ the identity, then the tetrad $\tilde{e}_\beta^a(\epsilon) := \Lambda^\alpha{}_\beta(\epsilon)e_\alpha^a(\epsilon)$ satisfies this same condition (this is sometimes called the 'tetrad-gauge ambiguity'). In any case $\Psi_0(\epsilon) = (1/4)C(\epsilon)V(\epsilon)V(\epsilon)$ (using (24) and an obvious notation) and

$$\delta\Psi_0 := \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0}\Psi_0 = \tfrac{1}{4}(\delta C\ VV + C\delta VV + CV\delta V) = \tfrac{1}{4}\delta C\ VV, \tag{37}$$

since $C\delta VV = CV\delta V = \Psi_2 \delta V_{ab}V^{ab} = 0$. Equation (37) implies that $\delta\Psi_0$ is *tetrad-gauge invariant*. The reader can check that $\delta\Psi_2$ and $\delta\Psi_4$ are also tetrad-gauge invariant, with

$$\delta\Psi_4 = \tfrac{1}{4}\delta C\ UU. \tag{38}$$

Note from (23), (27), (37) and (38) that $\delta\Psi_0$ ($\delta\Psi_4$) has spin weight two and boost weight two (minus two and minus two respectively).

3.1. Perturbed eigenvalues

If we insert in (26) the first-order corrections to equation (32),

$$\Psi_2(\epsilon) = \Psi_2 + \epsilon\delta\Psi_2, \quad \Psi_i(\epsilon) = \epsilon\delta\Psi_i, \;\; i = 0, 1, 3, 4, \tag{39}$$

we find that the perturbed eigenvalues to order ϵ are

$$\lambda_1(\epsilon) = \Psi_2 + \left(\delta\Psi_2 - \sqrt{\delta\Psi_0\delta\Psi_4}\right)\epsilon, \tag{40}$$

$$\lambda_2(\epsilon) = \Psi_2 + \left(\delta\Psi_2 + \sqrt{\delta\Psi_0\delta\Psi_4}\right)\epsilon, \tag{41}$$

$$\lambda_3(\epsilon) = -2\Psi_2 - 2\delta\Psi_2\epsilon, \tag{42}$$

and the eigenvalue splitting (8) is

$$2\epsilon\ \sqrt{\delta\Psi_0\delta\Psi_4}, \tag{43}$$

the branch choice of the (complex) square root being responsible for the sign ambiguity anticipated in (8). It is important to emphasize that $\delta\Psi_0$ and $\delta\Psi_4$ are both free of the tetrad-gauge ambiguity and that they carry opposite spin and boost weights (see the discussion around equation (27)). Thus (43) is a well defined scalar field that carries information on the distortion of the Weyl tensor due to the perturbation, information that is missing, e.g., in the perturbed curvature scalars in the even sector of the Schwarzschild perturbations.

If either $\delta\Psi_0 = 0$ or $\delta\Psi_4 = 0$, the space degenerates into a type D or II, depending on the dimension of the eigenspace $\ker(\mathbf{Q} - \lambda_2\mathbf{I})$. We will comment on these *algebraically special perturbations* in section 4.3.

3.2. Splitting of the PNDs

Inserting (39) in (31) gives

$$P(z) := \delta\Psi_0\epsilon + 4\delta\Psi_1\epsilon z + 6\left(\Psi_2 + \epsilon\delta\Psi_2\right)z^2 + 4\delta\Psi_3\epsilon z^3 + \delta\Psi_4\epsilon z^4. \tag{44}$$

The equation to be solved is $P(z) = 0$ up to order ϵ. Note, however, that the solutions $z_{\pm}$ of the simpler equation $0 = \delta\Psi_0\epsilon + 6\Psi_2 z^2$,

$$z_{\pm} = \pm\sqrt{-\frac{\delta\Psi_0}{6\Psi_2}}\sqrt{\epsilon}, \tag{45}$$

satisfy

$$P(z_{\pm}) = 0 + \mathcal{O}\left(\epsilon^{3/2}\right), \tag{46}$$

i.e., they are (to order ϵ) two of the four solutions of $P(z) = 0$. Since $z_{\pm} \to 0$ with ϵ, these two solutions are easily guessed to be those related to the splitting of k^a into two different PNDs. Explicitly, in the leading order we have

$$k_{\pm}^{a}(\epsilon) := k^a \pm \epsilon^{1/2}\left[\sqrt{-\frac{\delta\Psi_0}{6\Psi_2}}\,\bar{m}^a + \left(\overline{\sqrt{-\frac{\delta\Psi_0}{6\Psi_2}}}\right)m^a\right]. \tag{47}$$

According to the discussion between equations (22) and (31) the other two solutions of $P(z) = 0$ should be near the unperturbed repeated PND l^a, which corresponds to $z = \infty$ in $S^2 = \mathbb{C} \cup \{\infty\}$, thus we expect the other two solutions to behave as an inverse power of ϵ (cf [8]). To obtain these, we can either switch to $x = 1/z$ or work with null rotations around l^a and solve the equation $\Psi_4 = 0$. In either case we arrive at

$$l^a_\pm(\epsilon) := l^a \pm \epsilon^{1/2}\left[\overline{\left(\sqrt{-\frac{\delta\Psi_4}{6\Psi_2}}\right)}\bar{m}^a + \sqrt{-\frac{\delta\Psi_4}{6\Psi_2}}\, m^a\right] \tag{48}$$

It is important to note that (47) has zero spin weight and boost weight one, and thus defines a PND, for which the overall scaling is irrelevant. Similarly (48) carries zero spin weight and boost weight minus one. The non-analytical character of the splitting, discussed in some detail in [8], can be avoided by a re-parametrization of $g_{ab}(\epsilon)$.

4. Applications

4.1. Gravitational perturbations of the Schwarzschild black hole

For the Schwarzschild solution

$$\mathrm{d}s^2 = -f\ \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\varphi^2\right), \quad f = 1 - \tfrac{2M}{r}, \tag{49}$$

we use the orthonormal tetrad

$$\begin{aligned} e_0^a &= \frac{1}{\sqrt{f}}\left(\frac{\partial}{\partial t}\right)^a, \quad e_3^a = \sqrt{f}\left(\frac{\partial}{\partial r}\right)^a, \\ e_1^a &= \frac{1}{r}\left(\frac{\partial}{\partial\theta}\right)^a, \quad e_2^a = -\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\phi}\right)^a, \end{aligned} \tag{50}$$

then k^a and l^a in (22) are along the repeated PNDs, only Ψ_2 is different from zero, and the outgoing direction introduced in the end of section 2.3 agrees with $\partial/\partial r$. The perturbed metric admits a series expansion in terms of S^2 harmonic tensors [2]. These are labeled $(\ell, m, \pm)$ and can be obtained by applying a differential operator to the standard spherical harmonic scalars $Y^{(\ell,m)}$. Their further classification into even (+, or scalar) and odd (–, or vector) types describes the behavior of these tensors under the discrete parity isometry $(\theta, \varphi) \to (\pi - \theta, \varphi + \pi)$. The linearized Einstein's equations reduce to two-dimensional wave equations for functions $\phi^\pm_{(\ell,m)}(t, r)$ (see [1, 2]). These are the Regge–Wheeler equation for $\phi^-_{(\ell,m)}$ (equation (7) in [2], where Φ corresponds to $\phi^-_{(\ell,m)}$) and the Zerilli equation for $\phi^+_{(\ell,m)}$ (equation (28) in [2], where Ψ corresponds to $\phi^+_{(\ell,m)}$). In terms of these potentials we have found that

$$\delta\Psi_0 = \sum_{(\ell,m,\pm)} A^\pm_{(\ell,m)}(t, r)\ Y_2^{(\ell,m)}(\theta, \phi), \tag{51}$$

$$\delta\Psi_4 = \sum_{(\ell,m,\pm)} B^\pm_{(\ell,m)}(t, r)\ Y_{-2}^{(\ell,m)}(\theta, \phi), \tag{52}$$

where

$$\begin{aligned} A^-_{(\ell,m)} = &-\frac{3iM}{8r^3}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\left[(M-r)\frac{\partial\phi^-}{\partial r} - r\left(\frac{r-3M}{r-2M}\right)\frac{\partial\phi^-}{\partial t}\right. \\ &\left. + r(2M-r)\frac{\partial^2\phi^-}{\partial r^2} - r^2\frac{\partial^2\phi^-}{\partial t\partial r} + \left(\frac{\ell(\ell+1)}{2} - \frac{3M}{r}\right)\phi^-\right] \end{aligned}$$

$$
\begin{aligned}
A^{+}_{(\ell,m)} &= -\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\left[\frac{2M-r}{2r^2}\frac{\partial^2\phi^+}{\partial r^2}-\frac{1}{2r}\frac{\partial^2\phi^+}{\partial r\partial t}\right.\\
&\left.+\frac{K(r)}{2r^3D(r)}\frac{\partial\phi^+}{\partial r}+\frac{L(r)}{2r^2(r-2M)D(r)}\frac{\partial\phi^+}{\partial t}+\frac{N(r)}{4r^4D(r)}\phi^+\right],
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
B^{-}_{(\ell,m)} &= -\frac{3\mathrm{i}M}{8r^3}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\left[(M-r)\frac{\partial\phi^-}{\partial r}+r\left(\frac{r-3M}{r-2M}\right)\frac{\partial\phi^-}{\partial t}\right.\\
&\left.+r(2M-r)\frac{\partial^2\phi^-}{\partial r^2}+r^2\frac{\partial^2\phi^-}{\partial t\partial r}+\left(\frac{\ell(\ell+1)}{2}-\frac{3M}{r}\right)\phi^-\right]\\
B^{+}_{(\ell,m)} &= -\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\left[\frac{2M-r}{2r^2}\frac{\partial^2\phi^+}{\partial r^2}+\frac{1}{2r}\frac{\partial^2\phi^+}{\partial r\partial t}\right.\\
&\left.+\frac{K(r)}{2r^3D(r)}\frac{\partial\phi^+}{\partial r}-\frac{L(r)}{2r^2(r-2M)D(r)}\frac{\partial\phi^+}{\partial t}+\frac{N(r)}{4r^4D(r)}\phi^+\right],
\end{aligned}
\tag{54}
$$

$\phi^\pm$ stands for $\phi^\pm_{(\ell,m)}$, $Y_s^{(\ell,m)}$ are the normalized spin weight s spherical harmonics on S^2 [2], and

$$D(r) = (\ell+2)(\ell-1)r + 6M \tag{55}$$

$$K(r) = (\ell+2)(\ell-1)(M-r)r - 6M^2, \tag{56}$$

$$L(r) = (\ell+2)(\ell-1)(3M-r)r + 6M^2, \tag{57}$$

$$
\begin{aligned}
N(r) &= (\ell+2)^2(\ell+1)\ell(\ell-1)^2r^3\\
&+ 6M(\ell+2)^2(\ell-1)^2r^2 + 36M^2(\ell+2)(\ell-1)r + 72M^3.
\end{aligned}
\tag{58}
$$

Note that the eigenvalue splitting (43), as well as the repeated PND splittings (47) and (48), being proportional to $\sqrt{\Psi_0}$ and/or $\sqrt{\Psi_4}$, will contain multiple harmonics even if the metric perturbation contains a single non-zero $\phi^\pm_{(\ell,m)}$.

4.2. Gravitational perturbations of the Kerr black hole

The Teukolsky equations [9] are a set of separable partial differential equations for linear fields on type-D backgrounds. Two of them describe the behavior of $\delta\Psi_0$ and $\delta\Psi_4$ for the type-D background (e.g., a perturbed Kerr black hole), assuming a background null tetrad with k^a and l^a aligned along the repeated PNDs. Although the connection of these quantities with the corresponding metric perturbation is rather intricate [10], the Teukolsky equations—unlike the Regge–Wheeler and Zerilli equations for a Schwarzschild background—are particularly well suited to our purposes since they give the quantities needed in (43), (48) and (47).

It was shown in [11] that for well behaved (meaning satisfying suitable boundary conditions at the horizon and infinity) *non-stationary* black hole perturbations $\delta\Psi_0$ and $\delta\Psi_4$ uniquely determine each other. In particular, $\delta\Psi_0 = 0$ if and only if $\delta\Psi_4 = 0$, and this corresponds to a trivial perturbation. In view of (47) and (48), both repeated PNDs split and the perturbed metric is type I. Non-stationary perturbations are those relevant to the black hole stability issue and are the ones we focus on in this work.

4.3. Chandrasekhar algebraically special modes and Schwarzschild's naked singularity instability

In his 1984 paper [12], Chandrasekhar dealt with the problem of finding perturbations of black holes in the Kerr–Newman family such that one of $\delta\Psi_0$, $\delta\Psi_4$ vanishes while the other does not. It follows from the comments in the previous subsection that these perturbations do not satisfy appropriate boundary conditions at the horizon or at infinity of a Kerr–Newman black hole. The requirement that $\delta\Psi_0 = 0$ or $\delta\Psi_4 = 0$ leads to an algebraic condition (the vanishing of the Starobinsky constant) that gives a relation among the black hole parameters, the harmonic number of the perturbation and its frequency ω (perturbations behave as $\sim e^{\mathrm{i}\omega t}$ for pure modes). Although Chandrasekhar's algebraically special (AS) modes in the Kerr–Newman family diverge either at infinity or at the horizon of a black hole background, for naked singularities in the Kerr–Newman family some AS modes are indeed very relevant, as they satisfy appropriate boundary conditions both at infinity and at the singularity, and they grow exponentially with time (i.e., have a purely imaginary ω). The existence of these modes was indeed crucial for proving the linear instability of the negative-mass Schwarzschild solution and of the super-extreme Reissner–Nordström solution (for the super-extreme Kerr solution, however, none of the AS modes satisfies appropriate boundary conditions and other methods were required to establish its linear instability).

The instability of the Schwarzschild solution (49) with $M < 0$ is due to the existence of a family of even/scalar solutions of the Zerilli equation of the form

$$\phi^{+}_{(\ell,m)} = \frac{r(r-2M)^k}{(\ell+2)(\ell-1)r+6M}\exp\left(\frac{k(r-t)}{2M}\right), \quad k = \frac{(\ell+2)!}{6(\ell-2)!}. \tag{59}$$

These were found in [13], then recognized in [14] to agree with the AS modes in [12]. The facts that they behave properly for $r \in (0, \infty)$ (keep in mind that $M < 0$ and that $\ell \geqslant 2$ for non-stationary perturbations, for further details see [13]) and grow exponentially with time indicate that the naked sigularity is unstable. For the perturbations (59) we find

$$\delta\Psi_0 = 0, \quad \delta\Psi_4 = \frac{6k}{M^2}(r-2M)^{k-1}\left[(\ell+2)(\ell-1)+\frac{6M}{r}\right]\exp\left(\frac{k(r-t)}{2M}\right)Y_{-2}^{(\ell,m)} \tag{60}$$

According to (47) and (48) the double PND k^a will remain double whereas l^a will split, the perturbed spacetime being of type II according to table 1. Note, however, that this can only be accomplished by fine tuning the perturbation to restrict to the modes (59). A general perturbation will also contain the stable, oscillating modes, and the PNDs will split into four, that is, to type I.

5. Discussion

We have found explicit formulas for the splitting of the repeated PNDs of type-D spacetimes under gravitational perturbations, and also for the splitting of the repeated eigenvalue in (4). These are given in equations (47) and (48), and in (43). These are observable (gauge-invariant) effects of the perturbation on the background geometry that do not require derivatives of the metric higher than two, in contrast to differential invariants.

In view of (47), (48) and the results in [11], perturbed black holes within the Kerr–Newman family suffer a PND splitting to type I, except for stationary perturbations, which by the black hole uniqueness theorems are restricted to changes in the mass and/or angular momentum parameters, and therefore trivially keep the type-D structure. This gives sense to the notion that the 'tangent vector' h_{ab} of a curve $g_{ab}(\epsilon)$ at a black hole metric $g_{ab}(0)$ will 'stick out' of the set of algebraically special metrics. Note, however, that boundary conditions play a crucial role in these assertions: the example in section 4.3 shows the flow of the type-D Schwarzschild naked singularity to a type-II spacetime, triggered by the instability. This flow, however, can only occur for finely tuned initial conditions, allowing only the (infinitely many) modes (59). A generic perturbation will contain modes other than these and will therefore split the two repeated PNDs into four single PNDs.

As a final comment, the non-analytical behavior of the PNDs in the perturbation parameter (the dominant order in the perturbed PNDs is $\epsilon^{1/2}$) is to be expected from the polynomial character of the PND equation and the confluence of the solutions as $\epsilon \to 0$. This fact was clarified in [8], whose results are in total agreement with ours.

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