DEGREES IN AUSLANDER-REITEN COMPONENTS WITH ALMOST SPLIT SEQUENCES OF AT MOST TWO MIDDLE TERMS

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ABSTRACT. We consider A to be an artin algebra. We study the degrees of irreducible morphisms between modules in Auslander-Reiten components Γ having only almost split sequences with at most two indecomposable middle terms, that is, $\alpha(\Gamma) \leq 2$. We prove that if $f : X \to Y$ is an irreducible epimorphism of finite left degree with X or Y indecomposable, then there exists a module $Z \in \Gamma$ and a morphism $\varphi \in \Re^n(Z, X)\backslash\Re^{n+1}(Z, X)$ for some positive integer n such that $f\varphi = 0$. In particular, for such components if A is a finite dimensional algebra over an algebraically closed field and $f = (f_1, f_2)^t : X \to Y_1 \oplus Y_2$ is an irreducible morphism then we show that $d_l(f) = d_l(f_1)+d_l(f_2)$. We also characterize the artin algebras of finite representation type with $\alpha(\Gamma_A) \leq 2$ in terms of a finite number of irreducible morphisms with finite degree.

INTRODUCTION

Let A be an artin algebra. The representation theory of A deals with the study of the module category, $\operatorname{mod} A$, of finitely generated A-modules. An important tool in the study of $\operatorname{mod} A$ is the Auslander-Reiten theory, based on irreducible morphisms and almost split sequences. A morphism $f : X \to Y$ is said to be *irreducible* provided it does not split and whenever $f = gh$, then either h is a split monomorphism or g is a split epimorphism. It is known that if $f : X \to Y$ is an irreducible morphism with X or Y indecomposable then f belongs to the radical $\Re(X, Y)$ and not to its square $\Re^2(X, Y)$.

The theory of degrees of an irreducible morphism in a module category was developed by Liu in [12]. Using this concept he described the Auslander-Reiten components of an artin algebra of infinite representation type.

Recently, the concept of degree (1.1) has shown to be an important tool to solve many problems. In particular, by [7] we are able to determine if a finite dimensional algebra over an algebraically closed field is of finite representation type by computing the degree of a finite number of irreducible morphisms. Moreover, in [4] for an algebra of finite representation type the minimal lower bound $m \geq 1$, such that $\Re^m(\operatorname{mod} A)$ vanishes, was given. This bound was determined in terms of the right and the left degree of irreducible morphisms, not depending on the maximal length of the indecomposable modules. This result was extended by the authors in [8] where they found the nilpotency of the radical of a module category for an artin algebra.

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In [6, 7], by using degree theory the authors studied the relation between the powers of the radical of a module category and the composition of irreducible morphisms between indecomposable modules in a finite dimensional algebra over an algebraically closed field. More recently, the degrees and composition of irreducible morphisms in almost pre-sectional paths were studied in [5] in the context of artin algebras.

The aim of this work is to continue the study of the degree theory for irreducible morphisms between modules in mod A. The key fact for the finiteness of the left degree of an irreducible morphism is that the kernel of the irreducible morphism does not lie in the infinite power of the radical of the module category, whenever we consider A to be a finite dimensional algebra over an algebraically closed field. It is a natural question whether the same result holds true for artin algebras. In general we do not have an answer yet, but in this work we show that this is the case for irreducible morphism between modules in Auslander-Reiten components Γ with $\alpha(\Gamma) \leq 2$ and also for irreducible morphisms in generalized standard Auslander-Reiten components with length.

The main results proven in this work are the following theorems.

Theorem A. *Let A be an artin algebra and $\Gamma \subset \Gamma_A$ satisfying $\alpha(\Gamma) \leq 2$. Let $f : X \to Y$ be an irreducible epimorphism with $X \in \Gamma$ or $Y \in \Gamma$. If $d_l(f) < \infty$ then there is a positive integer n, a module $Z \in \Gamma$ and a morphism $\varphi : Z \to X$ with $\varphi \in \Re^n(Z, X) \backslash \Re^{n+1}(Z, X)$ such that $f\varphi = 0$.*

Theorem B. *Let A be an artin algebra where all the Auslander-Reiten components Γ of Γ_A are such that $\alpha(\Gamma) \leq 2$. Then, the following conditions are equivalent.*

(a) *A is finite representation type.*
(b) *For every non-simple indecomposable injective A-module I, the irreducible morphism $I \to I/socI$ has finite left degree.*
(c) *For every non-simple indecomposable projective A-module P, the irreducible morphism $radP \to P$ has finite right degree.*
(d) *For every irreducible epimorphism $f : X \to Y$ with X or Y indecomposable, the left degree of f is finite.*
(e) *For every irreducible monomorphism $f : X \to Y$ with X or Y indecomposable, the right degree of f is finite.*

The text is organized as follows. In Section 1 we recall some preliminary results and we extend the theory of degrees of irreducible morphisms to irreducible morphisms with non-indecomposable domain or codomain in generalized standard Auslander-Reiten components with length. In section 2 we study the degrees of irreducible morphisms in Auslander-Reiten components Γ, with $\alpha(\Gamma) \leq 2$ and we prove Theorem A. Section 3 is devoted to prove Theorem B.

We shall state only the results for the left degrees of irreducible morphisms. We observe that dual results hold true for the right degree in all the cases. We shall refrain from stating them since they can be easily obtained.

1. PRELIMINARIES

Throughout this paper A will be an artin algebra, $\operatorname{mod} A$ the category of finitely generated left A-modules and $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ consisting of one representative of each isomorphism class of indecomposable A-modules. By $\Re$ we denote the Jacobson radical of $\operatorname{mod} A$.

We denote by Γ_A the Auslander-Reiten quiver of $\operatorname{mod} A$ and by τ and τ^- the Auslander-Reiten translations DTr and TrD, respectively. We do not distinguish between an indecomposable module X in $\operatorname{mod} A$ and the corresponding vertex $[X]$ in Γ_A. By $\epsilon(X)$ we denote the almost split sequence ending in a non-projective indecomposable module X and by $\alpha(X)$ the number of indecomposable direct summands of the middle term of $\epsilon(X)$. We denote by $\epsilon'(X)$ and $\alpha'(X)$ the dual notions, respectively.

Given $X, Y \in \operatorname{mod} A$, the ideal $\Re_A(X,Y)$ is the set of all the morphisms $f: X \to Y$ such that, for each $M \in \operatorname{ind} A$, each $h: M \to X$ and each $h': Y \to M$ the composition $h'fh$ is not an isomorphism. In particular, if $X, Y \in \operatorname{ind} A$ then $\Re_A(X,Y)$ is the set of all the morphisms $f: X \to Y$ which are not isomorphisms. Inductively, the powers of $\Re_A(X,Y)$ are defined. By $\Re_A^{\infty}(X,Y)$ we denote the intersection of all powers $\Re_A^i(X,Y)$ of $\Re_A(X,Y)$, with $i \geq 1$.

Following [8], we say that the depth of a morphism $f: M \to N$ in $\operatorname{mod} A$ is infinite if $f \in \Re^{\infty}(M,N)$; otherwise, the depth of f is the integer $n \geq 0$ for which $f \in \Re^n(M,N)$ but $f \notin \Re^{n+1}(M,N)$. We denote the depth of f by $\operatorname{dp}(f)$.

Next, we state the definition of degree of an irreducible morphism given by S. Liu in [12].

1.1. Let $f: X \to Y$ an irreducible morphism in $\operatorname{mod} A$, with X or Y indecomposable. The *left degree* $d_l(f)$ of f is infinite, if for each integer $n \geq 1$, each module $Z \in \operatorname{mod} A$ and each morphism $g: Z \to X$ with $\operatorname{dp}(g) = n$ we have that $fg \notin \Re^{n+2}(Z,Y)$. Otherwise, the left degree of f is the least natural m such that there is an A-module Z and a morphism $g: Z \to Y$ with $\operatorname{dp}(g) = m$ such that $fg \in \Re^{m+2}(Z,Y)$.

The *right degree* $d_r(f)$ of an irreducible morphism f is dually defined.

The next result is an immediate consequence of the definition of degree.

Lemma 1.1. ([13, Lemma 1.3]). *Let $f: X \to Y$ be an irreducible morphism in* $\operatorname{mod} A$. *If Y' is a direct summand of Y and g is the co-restriction of f to Y', then $d_l(g) \leq d_l(f)$.*

1.2. By a path in Γ_A we mean a sequence of irreducible morphisms between indecomposable modules $Y_1 \to Y_2 \to \cdots \to Y_{n-1} \to Y_n$, and by a non-zero path (zero-path) we mean that the composition of the irreducible morphisms of the path does not vanish (vanishes).

In [11], Igusa and Todorov defined the notion of sectional paths. A path $Y_1 \to Y_2 \to \cdots \to Y_{n-1} \to Y_n$ in Γ_A is said to be sectional if for each $i = 1, \ldots, n-1$ we have that $Y_{i+1} \not\cong \tau Y_{i-1}$. In [12], Liu generalized such a concept defining what he called a pre-sectional path. A path $Y_1 \to Y_2 \to \cdots \to Y_{n-1} \to Y_n$ in Γ_A is said to be *pre-sectional* if for each i, $1 \leq i \leq n-1$, $Y_{i-1} = \tau Y_{i+1}$ implies that $Y_{i-1} \oplus \tau Y_{i+1}$ is a summand of the domain of a right almost split morphism for Y_i, or equivalently, $\tau^- Y_{i-1} = Y_{i+1}$ implies that $\tau^{-}Y_{i-1}\oplus Y_{i+1}$ is a summand of the codomain of a left almost split morphism for Y_i. Observe that any sectional path is a pre-sectional path.

Furthermore, in [11] Igusa and Todorov proved that if $X_0 \stackrel{f_1}{\to} X_1 \to \cdots \to X_{n-1} \stackrel{f_n}{\to} X_n$ is a sectional path then $f_n \dots f_1 : X_0 \to X_n$ is such that $\mathrm{dp}(f_n \dots f_1) = n$. In [12, Lemma 1.15], Liu extended the above result to pre-sectional paths and proved that if $X_0 \to X_1 \to \cdots \to X_{n-1} \to X_n$ is a pre-sectional path then there are irreducible morphisms $f_i : X_i \to X_{i+1}$ for $i = 0, \dots, n-1$ such that $\mathrm{dp}(f_{n-1} \dots f_0) = n$.

Let us recall that paths in Γ_A having the same starting vertex and the same ending vertex are called parallel paths.

Let Γ be a component of Γ_A. Following [9], we say that Γ is a component with length if parallel paths in Γ have the same length. Otherwise, we say that Γ is a component without length. We observe that a component of Γ_A with length has no oriented cycles.

If $X, Y \in \Gamma$, where Γ is a component of Γ_A with length, then we say that the length $\ell(X, Y)$ between X and Y is n if there is a path of irreducible morphisms from X to Y in Γ of length n.

1.3. Let Γ be a component of Γ_A. An arrow $\alpha : M \to N$ in Γ has valuation (a, b) if there is a minimal right almost split morphism $aM \oplus X \to N$ where M is not a summand of X, and a minimal left almost split morphism $M \to bN \oplus Y$ where N is not a summand of Y. If $a = b = 1$ then we say that the arrow α has trivial valuation.

A component Γ of Γ_A is said to satisfy the condition $\alpha(\Gamma) \le 2$ if $\alpha(X) \le 2$ for every X in Γ.

Following [16], a component Γ of Γ_A is called generalized standard if $\Re^{\infty}(X, Y) = 0$ for all $X, Y \in \Gamma$.

We recall the following result proven in [9], useful for our further purposes.

Proposition 1.2. ([9, Proposition 3.3]). *Let Γ in Γ_A be a generalized standard component with length. Let $X, Y \in \Gamma$ such that $\ell(X, Y) = n$. Then:*

(a) $\Re^{n+1}(X, Y) = 0$.
(b) If $g : X \to Y$ is a non-zero morphism then $g \in \Re^{n}(X, Y) \setminus \Re^{n+1}(X, Y)$.
(c) $\Re^{j}(X, Y) = \Re^{n}(X, Y)$, for each $j = 1, ..., n$.

In [9], the authors studied the degree of irreducible morphisms between indecomposable A-modules in generalized standard Auslander-Reiten components with length. Next, we shall generalize some of such results whenever f is an irreducible morphism with non-indecomposable codomain.

Proposition 1.3. *Let A be an artin algebra and $\Gamma \subset \Gamma_A$ be a generalized standard component with length. Let $f : X \to \oplus_{i=1}^{r} Y_i$ be an irreducible morphism with $Y_i \in \Gamma$, for $i = 1, \dots, r$. If $d_l(f) = n$ then there exists a morphism $\varphi \in \Re^{n}(M, X) \setminus \Re^{n+1}(M, X)$ for some indecomposable A-module M such that $f\varphi = 0$.*

Proof. Assume $d_l(f) = n$. Then, there exists a morphism $\varphi \in \Re^{n}(M, X) \setminus \Re^{n+1}(M, X)$ for some indecomposable A-module M such that $f\varphi \in \Re^{n+2}(M, Y)$. By hypothesis since Γ is a component with length and $\mathrm{dp}(\varphi) = n$ then $\ell(M, X) = n$. Moreover, since

$f : X \to Y$ is irreducible we get that $\ell(M, Y_i) = n+1$, for $i = 1, \ldots, r$. By Proposition 1.2 we have that $\Re^{n+2}(M, Y_i) = 0$ for $i = 1, \ldots, r$. Hence, $\Re^{n+2}(M, Y) = 0$. Therefore, $f\varphi = 0$. □

As an immediate consequence of the above result we get the following corollary.

Corollary 1.4. *Let A be an artin algebra and $\Gamma \subset \Gamma_A$ be a generalized standard component with length. Let $f : X \to \oplus_{i=1}^r Y_i$ be an irreducible morphism with $Y_i \in \Gamma$, for $i = 1, \ldots, r$. If $d_l(f) = n$ then f is an epimorphism. Moreover, an injective source is of infinite left degree.*

Theorem 1.5. *Let A be an artin algebra and $\Gamma \subset \Gamma_A$ be a generalized standard component with length. Let $f : X \to \oplus_{i=1}^r Y_i$ be an irreducible morphism with $Y_i \in \Gamma$, for $i = 1, \ldots, r$. Then, $d_l(f) = n$ if and only if the inclusion morphism $\iota : \ker f \hookrightarrow X$ is such that $\mathrm{dp}(\iota) = n$.*

Proof. Assume $d_l(f) = n$. Then, by Proposition 1.3 there exists a morphism $\varphi \in \Re^n(M, X) \backslash \Re^{n+1}(M, X)$ for some indecomposable A-module M such that $f\varphi = 0$. Then, $\varphi = \iota\delta$ where $\iota : kerf \hookrightarrow X$ is the inclusion morphism and $\delta : M \to kerf$. Furthermore, δ is an isomorphism and $\mathrm{dp}(\iota) = n$, otherwise $d_l(f) < n$.

Now, if $\mathrm{dp}(\iota) = n$ then $d_l(f) \leq n$. Assume that $d_l(f) = m$ with $m < n$. By the above implication we get to the contradiction that $\mathrm{dp}(\iota) = m$. Therefore, $d_l(f) = n$. □

For the convenience of the reader, we state some results proven in [12] needed throughout this paper.

Lemma 1.6. ([12, Lemma 1.2]). *Let $m \geq 1$ be an integer and let $p : X \to Y$ and $f : Y \to Z$ be morphisms in* mod A. *Suppose that f is irreducible and Z indecomposable. If $p \notin \Re^{m+1}$ and $fp \in \Re^{m+2}$, then*

(1) *Z is not projective, and*
(2) *if $0 \to \tau Z \stackrel{(g,g')^t}{\to} Y \oplus Y' \stackrel{(f,f')}{\to} Z \to 0$ is an almost split sequence, then there exist a morphism $q : X \to \tau Z$ in* mod A *such that $q \notin \Re^m$, $p + gq \in \Re^{m+1}$ and $g'q \in \Re^{m+1}$.*

Let $f : X \to Y$ and $g : Y \to Z$ be irreducible morphisms in mod A. Following [12], we say that the pair $\{f, g\}$ is a component of an almost split sequence if there are irreducible morphisms $f' : X \to Y'$ and $g' : Y' \to Z$ such that $0 \to X \stackrel{(f,f')^t}{\to} Y \oplus Y' \stackrel{(g,g')}{\to} Z \to 0$ is an almost split sequence.

If $(f_1, f_2) : \tau Y_1 \oplus \tau Y_2 \to X$ and $(g_1, g_2)^t : X \to Y_1 \oplus Y_2$ are irreducible morphisms, with Y_1, Y_2 indecomposable non-projective modules, $\{f_1, g_1\}$ and $\{f_2, g_2\}$ are components of $\epsilon(Y_1)$ and $\epsilon(Y_2)$ respectively, then (f_1, f_2) is called a left neighbor of $(g_1, g_2)^t$.

Lemma 1.7. ([12, Lemma 1.11]). *Let $f : X \to Y$ be an irreducible morphism in* mod A. *If f has finite left degree and $Y = Y_1 \oplus Y_2$ where Y_1 and Y_2 are indecomposable then f has a left neighbor $g : \tau Y_1 \oplus \tau Y_2 \to X$ with $d_l(g) < d_l(f)$.*

Proposition 1.8. ([12, Proposition 1.12]). *Let $f : X \to Y$ be an irreducible morphism in* mod A *with either X or Y indecomposable. Then*

(1) *f is a surjective sink if and only if $d_l(f) = 1$,*
(2) *f is an injective source if and only if $d_r(f) = 1$.*

Lemma 1.9. ([12, Lemma 1.5]). *Assume that $Y_0 \to Y_1 \to \cdots \to Y_{n-1} \to Y_n$ is a pre-sectional path in Γ_A. Then, for any integer m the path $\tau^m Y_0 \to \tau^m Y_1 \to \cdots \to \tau^m Y_{n-1} \to \tau^m Y_n$ is pre-sectional whenever it is defined.*

Proposition 1.10. ([12, Proposition 1.6]). *Let $f : X \to Y$ be an irreducible morphism of finite left degree in $\operatorname{mod} A$ with Y indecomposable. Assume that*

$$Y_n \to Y_{n-1} \to ... \to Y_1 \to Y_0 = Y$$

is a pre-sectional path in Γ_A with $n \geq 1$. If $X \oplus Y_1$ is a summand of the middle term of $\varepsilon(Y)$ then $d_l(f) > n$.

A dual result holds for the right degree. Next, we state it since we shall use it very frequently all over this work.

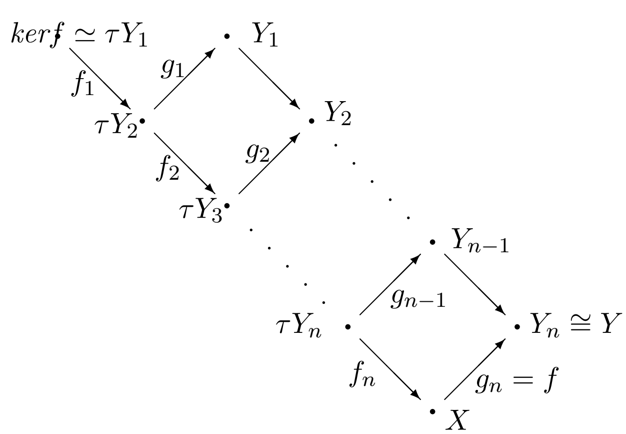
Proposition 1.11. ([12, Dual of Proposition 1.6]). *Let $f : X \to Y$ be an irreducible morphism of finite right degree in $\operatorname{mod} A$ with X indecomposable. Assume that*

$$X = X_0 \to X_1 \to ... \to X_{n-1} \to X_n$$

is a pre-sectional path in Γ_A with $n \geq 1$. If $Y \oplus X_1$ is a summand of the middle term of $\varepsilon(X)$ then $d_r(f) > n$.

In [5], the degree of irreducible morphisms between indecomposable A-modules in Auslander-Reiten components Γ of Γ_A with $\alpha(\Gamma) \leq 2$ were characterized. We state such a result below.

Proposition 1.12. ([5, Proposition 5.1]). *Let A be an artin algebra and Γ a component of Γ_A satisfying $\alpha(\Gamma) \leq 2$. Let $f : X \to Y$ be an irreducible morphism, with $X, Y \in \Gamma$. Then, $d_l(f) = n$ if and only if there exists a configuration of almost split sequences*



(1)

where $\delta : \tau Y_1 \xrightarrow{f_1} \tau Y_2 \xrightarrow{f_2} \cdots \to \tau Y_n \xrightarrow{f_n} X$ is a pre-sectional path of length n with $\operatorname{dp}(\delta) = n$, $f\delta = 0$ and $\alpha'(\tau Y_1) = 1$. Moreover, $d_l(g_i) = i$ for $i = 1, \ldots, n$.

A dual result holds for the right degree of an irreducible morphism between indecomposable A-modules.

As a consequence of Proposition 1.12 we get the following corollary.

Corollary 1.13. *Let A be an artin algebra and Γ be a component of Γ_A with $\alpha(\Gamma) \leq 2$. Let $f : X \to Y$ be an irreducible morphism, with $X, Y \in \Gamma$. Then, $d_l(f) = n$ if and only if the inclusion morphism $\iota : kerf \to X$ is such that $dp(\iota) = n$.*

Proof. If $d_l(f) = n$ then by Proposition 1.12, there is a pre-sectional path $\varphi : \ker f \to X$ of length n, with $\mathrm{dp}(\varphi) = n$ and such that $f\varphi = 0$. Hence, $\varphi = \iota\delta$ with $\delta : \ker f \to \ker f$. Therefore, $dp(\iota) = n$ otherwise the left degree of f is less than n.

Now, since $\mathrm{dp}(\iota) = n$ and $f\iota = 0$ then $d_l(f) \leq n$. Suppose that $d_l(f) = m$ with $m < n$. By Proposition 1.12, there is a pre-sectional path $\varphi : \ker f \to X$ of length m and with $\mathrm{dp}(\varphi) = m$ such that $f\varphi = 0$. Hence, $\varphi = \iota\delta$ with $\delta : \ker f \to \ker f$ contradicting that $dp(\iota) = n$. □

We end up this section recalling this useful result for our further considerations.

Lemma 1.14. ([5, Lemma 5.4]). *Let A be an artin algebra and Γ an Auslander-Reiten component with $\alpha(\Gamma) \leq 2$. Let $f = (f_1, f_2)^t : X \to X_1 \oplus X_2$ be an irreducible morphism with $X, X_i \in \mathrm{ind}\, A$ for $i = 1, 2$. If $d_l(f) = n$ then $d_l(f_i) < n$ for $i = 1, 2$.*

For unexplained notions on representation theory we refer the reader to [1, 2, 15] and for notions on degrees to [5, 7, 9, 12, 13].

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2. The results

The aim of this section is to complete the study of the degrees of irreducible morphisms between modules in Auslander-Reiten components Γ with $\alpha(\Gamma) \leq 2$ started in [5]. In particular, we would like to study the degree of an irreducible epimorphism $f : X \to Y_1 \oplus Y_2$ with $Y_i \in \Gamma$ for $i = 1, 2$. For such a purpose we start with some technical lemmas.

2.1. Let $f : X \to Y$ be an irreducible morphism in $\mathrm{mod}\, A$. We set

$$\mathrm{Irr}(X, Y) = \Re(X, Y)/\Re^2(X, Y)$$

and $k_X = \mathrm{End}(X)/\Re(X, X)$. We denote by $\overline{f}$ the residue class of f in $\mathrm{Irr}(X, Y)$. Recall that $\mathrm{Irr}(X, Y)$ is a $k_Y - k_X^{op}$−bimodule and that k_X is a division ring whenever X is an indecomposable A-module. We consider the composition of morphisms from the right to the left.

In [10], for regular components of Γ_A of type ZA_∞ or stable tubes, whenever A is a finite dimensional k algebra over an algebraically closed field k, the authors proved that if $(f, g)^t : X \to Y_1 \oplus Y_2$ is a left minimal almost split morphism with Y_1, Y_2 indecomposable non-isomorphic A-modules, $\alpha \in k^*$ and $\mu \in \Re^2(X, Y_2)$ then the irreducible morphism $(f, \alpha g + \mu)^t : X \to Y_1 \oplus Y_2$ is also a left minimal almost split morphism.

Next, we generalize such result for components Γ of Γ_A with $\alpha(\Gamma) \leq 2$ where A is any artin algebra.

Lemma 2.1. *Let Γ be an Auslander-Reiten component with $\alpha(\Gamma) \leq 2$ and $(f, g)^t : X \to Y_1 \oplus Y_2$ be a left minimal almost split morphism with $Y_1, Y_2 \in \Gamma$ non-isomorphic A-modules. Consider $\varphi_X \in k_X$ and $\mu \in \Re^2(X, Y_2)$. Then, the irreducible morphism $(f, g\varphi_X + \mu)^t : X \to Y_1 \oplus Y_2$ is also a left minimal almost split morphism.*

Proof. Since $(f, g)^t : X \to Y_1 \oplus Y_2$ is a left minimal almost split morphism and $g\varphi_X + \mu$ is not an isomorphism, there is a morphism $\delta_1 : Y_1 \to Y_2$ and a morphism $\delta_2 : Y_2 \to Y_2$ such that $g\varphi_X + \mu = \delta_1 f + \delta_2 g$. Furthermore, $\delta_1 : Y_1 \to Y_2$ is not an isomorphism since $Y_1 \not\simeq Y_2$. Hence, $\delta_1 f \in \Re^2(X, Y_2)$. Then, $g\varphi_X = \delta_2 g + \mu'$, with $\mu' \in \Re^2(X, Y_2)$.

Note that δ_2 is an isomorphism, otherwise $g \in \Re^2(X, Y_2)$ a contradiction to the fact that g is irreducible.

Let $t = \begin{pmatrix} id & 0 \\ \delta_1 & \delta_2 \end{pmatrix} : Y_1 \oplus Y_2 \to Y_1 \oplus Y_2$. One can verify that t is an isomorphism, with inverse

$$\begin{pmatrix} id & 0 \\ -\delta_2^{-1}\delta_1 & \delta_2^{-1} \end{pmatrix}$$

and that $(f, g\varphi_X + \mu)^t = t\,(f, g)^t$. Thus, $(f, g\varphi_X + \mu)^t$ is also a left minimal almost split morphism, as we want to prove. □

A dual result holds true for a right minimal almost split morphism. We state the result below.

Lemma 2.2. *Let Γ an Auslander-Reiten component with $\alpha(\Gamma) \leq 2$ and $(f, g) : X_1 \oplus X_2 \to Y$ be a right minimal almost split morphism with $X_1, X_2 \in \Gamma$ non-isomorphic A-modules. Consider $\varphi_Y \in k_Y$, and $\mu \in \Re^2(X_2, Y)$. Then, the irreducible morphism $(f, \varphi_Y g + \mu) : X_1 \oplus X_2 \to Y$ is also a right minimal almost split morphism.*

With a similar proof as in Lemma 2.1 we infer the next two results.

Lemma 2.3. *Let Γ be an Auslander-Reiten component with $\alpha(\Gamma) \leq 2$ and $(f, g)^t : X \to Y_1 \oplus Y_2$ be a left minimal almost split morphism with $Y_1, Y_2 \in \Gamma$ non-isomorphic A-modules. Consider $\varphi_{Y_i} \in k_{Y_i}$ and $\mu \in \Re^2(X, Y_i)$ for $i = 1, 2$. Then, for $i = 1, 2$ the irreducible morphism $(f, \varphi_{Y_i} g + \mu)^t : X \to Y_1 \oplus Y_2$ is also a left minimal almost split morphism.*

Lemma 2.4. *Let Γ an Auslander-Reiten component with $\alpha(\Gamma) \leq 2$ and $(f, g) : X_1 \oplus X_2 \to Y$ be a right minimal almost split morphism with $X_1, X_2 \in \Gamma$ non-isomorphic A-modules. Consider $\varphi_{X_i} \in k_{X_i}$, and $\mu \in \Re^2(X_i, Y)$ for $i = 1, 2$. Then, for $i = 1, 2$ the irreducible morphism $(f, \varphi_{X_i} g + \mu) : X_1 \oplus X_2 \to Y$ is also a right minimal almost split morphism.*

We recall some useful results given by R. Bautista in [3] and by S. Liu in [12].

Proposition 2.5. ([3]) *Let A be an artin algebra and $X, Y \in \text{ind}A$. Then, the morphism*

$$g = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} : X \to Y \sqcup Y ... \sqcup Y$$

is irreducible if and only if $\overline{g_1}, ..., \overline{g_n} \in \Re(X,Y)/\Re^2(X,Y)$ are linearly independent over $k_Y = \text{End}(Y)/\Re(Y,Y)$.

Lemma 2.6. ([12, Lemma 1.10]) *Let*

$$0 \to X \overset{(e,f)}{\to} Y_1 \oplus Y_2 \overset{(g,h)^t}{\to} Z \to 0$$

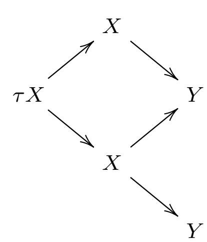
and

$$0 \to X \overset{(e',f')}{\to} Y_1 \oplus Y_2 \overset{(g',h')^t}{\to} Z \to 0$$

be two almost split sequences in mod A. *Assume that Y_1 and Y_2 have no common direct summands. Then, $\overline{e}$, $\overline{e'}$ in $I(X, Y_1)$ are linearly dependent over k_X if and only if $\overline{g}$, $\overline{g'}$ in $I(Y_1, Z)$ are linearly dependent over k_Z.*

2.2. There is an arrow $X \to Y$ in Γ_A if and only if there is an irreducible morphism $f : X \to Y$ with $X, Y \in \text{ind } A$. Such irreducible morphism is not unique. In [12], Liu proved that any irreducible morphism from X to Y have the same left (right) degree. Moreover, Liu showed that this allow us to define the left (right) degree of an arrow $X \to Y$ in Γ_A as the left (right) degree of any irreducible morphism from X to Y. Furthermore, by [12, Lemma 1.7], if $X \to Y$ an arrow in Γ_A of finite left or right degree with valuation (a, b) then $a = 1$ or $b = 1$.

As an immediate consequence of the above mentioned result we infer that if $f : X \to Y \oplus Y$ is an irreducible epimorphism of finite left degree then there is not a configuration of almost split sequences as follows

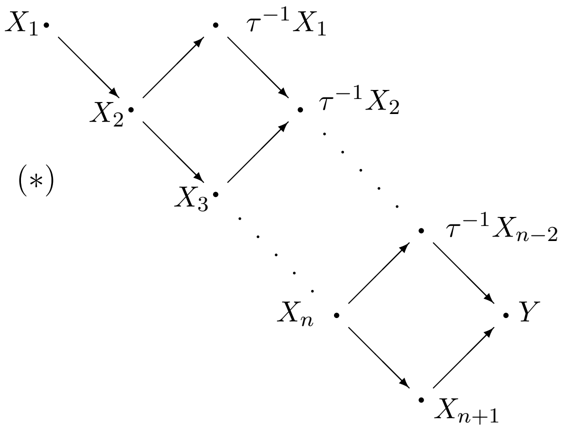


otherwise, the valuation of the arrow $X \to Y$ is $(2, 2)$ a contradiction to the fact that the arrow $X \to Y$ has finite left degree, see Lemma 1.1.

Next, we study the valuations of the arrows involved in the configuration of almost split sequences stated in Proposition 1.12.

Lemma 2.7. *Let A be an artin algebra and $\xi : X_1 \to X_2 \to \cdots \to X_n \to X_{n+1}$ ($n \geq 2$) be the pre-sectional path in the configuration of almost split sequences state below.*

(*)

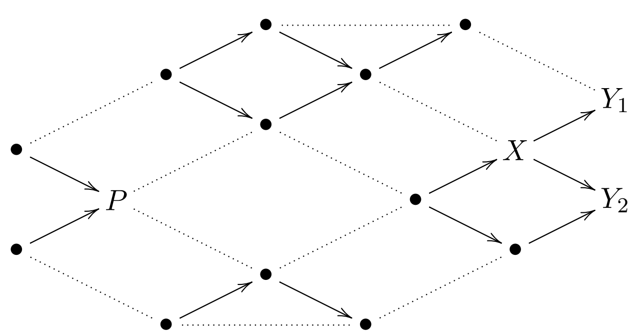


with $\mathrm{dp}(\xi) = n$ *and* $\alpha'(X_i) \leq 2$ *for* $i = 1, ..., n$. *Then, the valuations of the arrows in the path* $X_1 \to X_2 \to \cdots \to X_n \to X_{n+1}$ *are not* $(2, 2)$.

Proof. Note that the arrow $X_1 \to X_2$ does not have valuation $(2, 2)$. In fact, assume that the valuation is (a, b). Then, by [12, Lemma 1.7] since $d_r(X_1 \to X_2) = 1$ we have that $a = 1$ or $b = 1$.

Now, assume that there is an $i \in \{2, \ldots, n_1 - 1\}$ such that the arrow $X_i \to X_{i+1}$ has valuation $(a, b) = (2, 2)$. Then, there are a left and a right minimal almost split morphism of the form $X_i \to X_{i+1} \oplus X_{i+1}$ and $X_i \oplus X_i \to X_{i+1}$ for $i = 1, 2$, respectively. Since $\alpha(X_i) = 2$, then the arrow $X_{i+1} \to \tau^{-1}X_i$ has valuation $(2, 2)$, a contradiction to the fact that the arrow $X_{i+1} \to \tau^{-1}X_i$ is of finite left degree, see Proposition 1.12. Then, we prove that the arrows $X_i \to X_{i+1}$ have valuation different from $(2, 2)$, for $i = 1, \ldots, n$. More precisely, since $\alpha'(X_i) \leq 2$ for $i = 1, ..., n$ the valuations of the arrows are either one of the following $(1, 2)$, $(2, 1)$ or $(1, 1)$. □

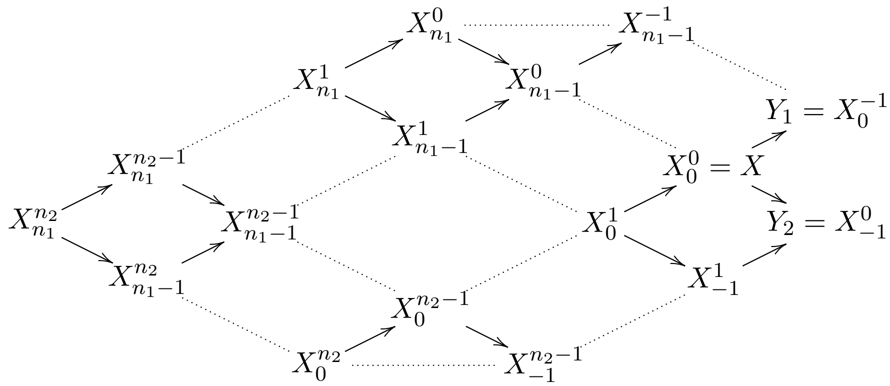
2.3. In [14, Lemma 4.9], by using length arguments, the authors proved that there is not a translation subquiver in Γ_A of the form



where P is either a projective module or a direct successor of a projective-injective module and X is either an injective module or a direct predecessor of a projective-injective module.

2.4. Let $f = (f_1, f_2)^t : X \to Y_1 \oplus Y_2$ be an irreducible epimorphism and $d_l(f_i) = n_i$ for $i = 1, 2$. Suppose there is a configuration of almost split sequences as follows,

(2)



involving only almost split sequences with at most two indecomposable middle terms, with $\alpha'(X_{n_1}^0) = \alpha'(X_0^{n_2}) = 1$ and where the paths $\beta_1 : X_{n_1}^0 \to X_{n_1-1}^0 \rightsquigarrow X$ and $\beta_2 : X_0^{n_2} \to X_0^{n_2-1} \rightsquigarrow X$ are pre-sectional of length n_i with $\mathrm{dp}(\beta_i) = n_i$ for $i = 1, 2$, respectively.

Notation 2.8. *In the configuration of almost split sequences stated above, for* $0 \leq i \leq n_2$ *we denote by* φ_i *the following paths*

$$\varphi_i : X_{n_1}^i \stackrel{t_{n_1}^i}{\to} X_{n_1-1}^i \stackrel{t_{n_1-1}^i}{\to} \cdots \to X_2^i \stackrel{t_2^i}{\to} X_1^i \stackrel{t_1^i}{\to} X_0^i$$

and by φ'_i *the paths*

$$\varphi'_i : X_{n_1}^i \stackrel{t_{n_1}^i}{\to} X_{n_1-1}^i \stackrel{t_{n_1-1}^i}{\to} \cdots \to X_1^i \stackrel{t_1^i}{\to} X_0^i \stackrel{t_0^i}{\to} X_{-1}^i.$$

For $0 \leq j \leq n_1$, *we denote by* δ_j *the paths*

$$\delta_j : X_j^{n_2} \stackrel{g_0^j}{\to} X_j^{n_2-1} \stackrel{g_1^j}{\to} \cdots \to X_j^1 \stackrel{g_{n_1}^j}{\to} X_j^0$$

and by δ'_j *the paths*

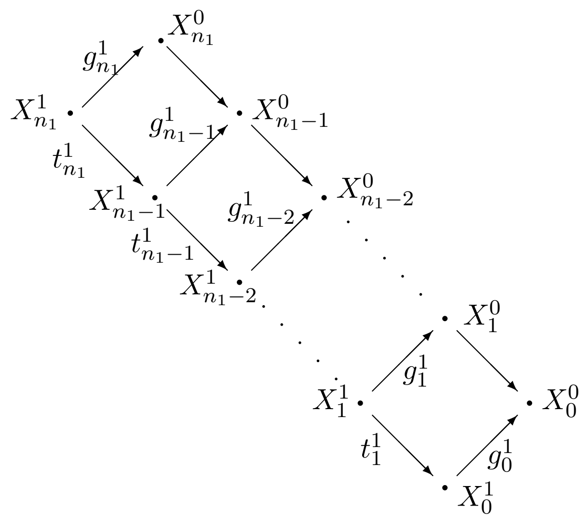
$$\delta'_j : X_j^{n_2} \stackrel{g_0^j}{\to} X_j^{n_2-1} \stackrel{g_1^j}{\to} \cdots \to X_j^1 \stackrel{g_{n_1}^j}{\to} X_j^0 \stackrel{g_{n_1+1}^j}{\to} X_j^{-1}.$$

With the above notations we prove the following lemma.

Lemma 2.9. *Consider* Γ *an Auslander-Reiten component with* $\alpha(\Gamma) \leq 2$. *Let* $(f_1, f_2)^t : X \to Y_1 \oplus Y_2$ *be an irreducible epimorphism with* $Y_i \in \Gamma$ *and* $d_l(f_i) = n_i$ *for* $i = 1, 2$. *Then,*

(a) φ_i *is a pre-sectional path with* $\mathrm{dp}(\varphi_i) = n_1$, *for* $i = 1, \dots, n_2$.
(b) δ_j *is a pre-sectional path with* $\mathrm{dp}(\delta_j) = n_2$, *for* $j = 1, \dots, n_1$.
(c) φ'_i *is a pre-sectional path with* $\mathrm{dp}(\varphi'_i) = n_1 + 1$, *for* $i = 1, \dots, n_2$.
(d) δ'_j *is a pre-sectional path with* $\mathrm{dp}(\delta'_j) = n_2 + 1$, *for* $j = 1, \dots, n_1$.

Proof. We only prove (a) since statements $(b), (c)$ and (d) follow similarly. By Proposition 1.12 we know that $\varphi_0 : X_{n_1}^0 \stackrel{t_{n_1}^0}{\to} X_{n_1-1}^0 \stackrel{t_{n_1-1}^0}{\to} \cdots \to X_1^0 \stackrel{t_1^0}{\to} X_0^0$ is a pre-sectional path of length n_1 with $\mathrm{dp}(\varphi_0) = n_1$. Let φ_1 be the path $\varphi_1 : X^1_{n_1} \overset{t^1_{n_1}}{\to} X^1_{n_1-1} \overset{t^1_{n_1-1}}{\to} \cdots \to X^1_2 \overset{t^1_2}{\to} X^1_1 \overset{t^1_1}{\to} X^1_0$ of length n_1. In order to prove that $\mathrm{dp}(\varphi_1) = n_1$, we illustrate the situation with the following diagram:



First, we observe that $\varphi_1 \in \Re^{n_1}$. If $g^1_{n_1} : X^1_{n_1} \to X^0_{n_1}$ is an irreducible epimorphism then $\mathrm{dp}(\varphi_0 g^1_{n_1}) = n_1 + 1$, because $d_r(g^1_{n_1}) = \infty$. Since $\alpha(\Gamma) \leq 2$ then $g^1_0 : X^1_0 \to X^0_0$ an irreducible epimorphism. Moreover, $\varphi_1 g^1_{n_1} = g^1_1 \varphi_1$. Then, $\mathrm{dp}(g^1_0 \varphi_1) = n_1 + 1$. Hence, $\mathrm{dp}(\varphi_1) = n_1$ and we are done.

If $g^1_{n_0}$ is an irreducible monomorphism then we prove that $\varphi_1 \notin \Re^{n_1+1}$. In fact, since $g^1_0 \varphi_1 = \varphi_0 g^1_{n_0}$ we get that $\varphi_0 g^1_{n_1} \in \Re^{n_1+2}$. Then, $d_l(g^1_{n_1}) \leq n_1$.

We claim that the path φ_1 is a pre-sectional path. First, note that $X^1_i \simeq \tau X^0_{i-1}$ for $i = 1, \dots, n_1$. By Lemma 1.9 we have that $\tau X^0_{n_1-1} \to \tau X^0_{n_1-2} \to \cdots \to \tau X^0_0$ is a pre-sectional path. It remains to prove that $\tau X^0_{n_1-1} \to \tau X^0_{n_1-2} \to \cdots \to \tau X^0_0 \to \tau Y_2$ is also a pre-sectional path. If $\tau X^0_2 \simeq \tau^2 Y_2$ then $X^0_1 \simeq \tau^1 Y_2$. Therefore, $\tau^1 Y_2 \oplus \tau^1 Y_2$ is the domain of the sink for X^0_0 proving that φ_1 is a pre-sectional path.

On the other hand, by Proposition 1.11 we get that $d_r(g^1_{n_1}) \geq n_1 + 1$ a contradiction to the fact that $d_l(g^1_{n_1}) \leq n_1$. Therefore, $\mathrm{dp}(\varphi_1) = n_1$.

With a similar argument as we used above since $\mathrm{dp}(\varphi_1) = n_1$ then we infer that $\mathrm{dp}(\varphi_2) = n_1$. Iterating successively this argument for each path φ_i for $i = 0, \dots, n_1 - 1$, we get the result. □

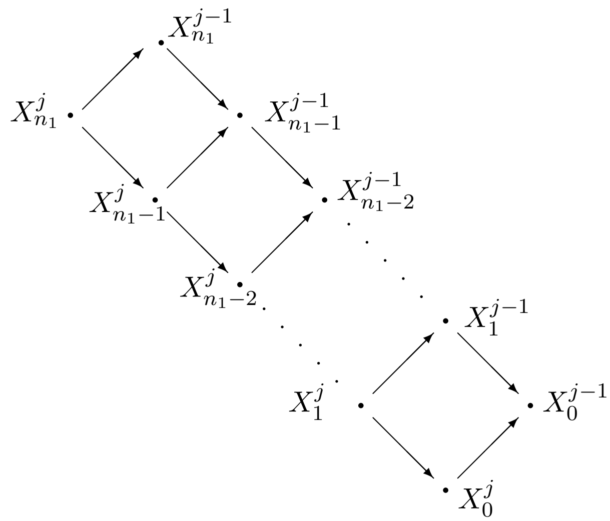
Lemma 2.10. *Let $f = (f_1, f_2)^t : X \to Y_1 \oplus Y_2$ be an irreducible epimorphism with $d_l(f) < \infty$ and X, Y_1, Y_2 indecomposable A-modules. Suppose there is a configuration of almost split sequences as in*

Then, the valuations of the arrows which are not in the paths $X^{n_2}_{n_1} \to X^{n_2-1}_{n_1} \rightsquigarrow X^0_{n_1}$, $X^{n_2}_{n_1} \to X^{n_2}_{n_1-1} \rightsquigarrow X^{n_2}_0$, $X^{n_2-1}_{-1} \to X^{n_2-2}_{-1} \rightsquigarrow Y_2$ and $X^{-1}_{n_1-1} \to X^{-1}_{n_1-2} \rightsquigarrow Y_1$ are either $(1,2), (2,1)$ or $(1,1)$.

Proof. By Proposition 1.12 we know that the arrows $X_i^0 \to X_i^{-1}$ have finite left degree for $i = 0, \ldots, n_1 - 2$. Then, by [12, Lemma 1.7] the valuations of such arrows are $(1,1), (2,1)$ or $(1,2)$. Moreover, by Lemma 2.7 the arrows $X_i^0 \to X_{i-1}^0$ also have such valuations.

On the other hand, the arrows of the form $X_i^{n_2} \to X_{i+1}^{n_2}$ for $i = 0, \ldots, n_1$ do not have valuation $(2,2)$ since such arrows are of finite right degree. In fact, the path $X_{n_1}^{n_2-1} \to X_{n_1-1}^{n_2-1} \rightsquigarrow X_1^{n_2-1} \to X_0^{n_2-1}$ is a pre-sectional path of length n_1 and such that $\mathrm{dp}() = n_1$. then, we get the result by the dual of Proposition 1.12.

Now, we analyze the valuations of the arrows which are in the configuration state below for $j = 1, \ldots, n_2 - 1$.



We start with the arrow $X_0^1 \to X_0^0$ and we assume that has valuation $(2,2)$. Then, there is a left minimal almost split morphism of the form $X_0^1 \to X_0^0 \oplus X_0^0$ and also a right minimal almost split morphism of the form $X_0^1 \oplus X_0^1 \to X_0^0$. Hence, $X_0^1 \simeq X_1^0$ and $X_0^0 \simeq X_{-1}^1$. Since $\epsilon'(X_0^1) = 2$ then we get that the arrow $X_0^0 \to X_0^{-1}$ has valuation $(2,2)$, an absurdly. Now, consider the arrow $X_1^1 \to X_0^1$ and assume that such an arrow has valuation $(2,2)$. Then again, there is a left minimal almost split morphism $X_1^1 \to X_0^1 \oplus X_0^1$ and a right minimal almost split morphism $X_1^1 \oplus X_1^1 \to X_0^1$. Hence, $X_0^1 \simeq X_1^0$ and $X_1^1 \simeq X_0^2$. Since $\epsilon'(X_0^1) = 2$ then we get that the arrow $X_0^1 \to X_0^0$ has valuation $(2,2)$ contradicting the step above. Repeating this argument in each arrow of such a configuration, that is, we fix an i for $i = 1, \ldots, n_1 - 1$ and we analyze that the valuation of the arrow $X_i^1 \to X_i^0$ is not $(2,2)$ and then we proceed to analyze the valuation of the arrow $X_i^1 \to X_{i+1}^1$.

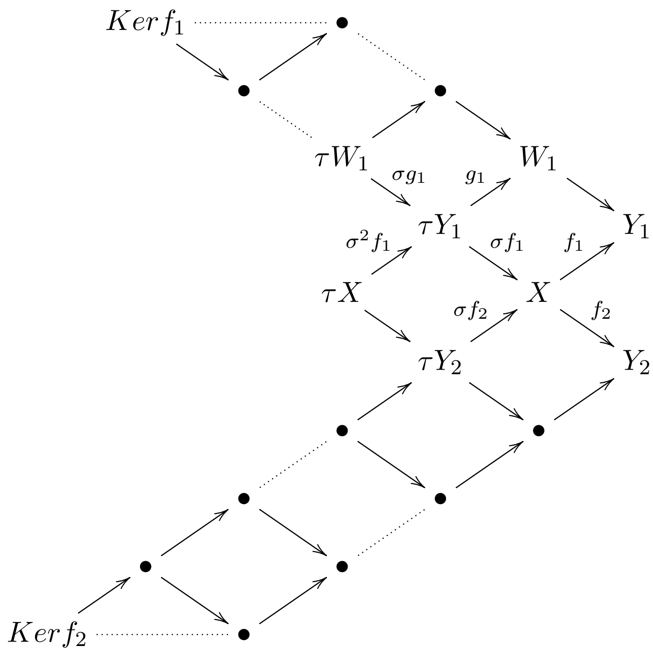
Iterating the above argument in each possible configuration of almost split sequences state in the lemma for $j = 1, \ldots, n_2 - 1$ we get the desired result. □

Our next result is essential for our further purposes. The aim is to show that we can built a configuration of almost split sequences as in Lemma 2.10.

Lemma 2.11. *Consider Γ an Auslander-Reiten component with $\alpha(\Gamma) \leq 2$. Let $f = (f_1, f_2)^t : X \to Y_1 \oplus Y_2$ be an irreducible epimorphism with $Y_i \in \Gamma$, for $i = 1, 2$. If $d_l(f) < \infty$ then there is a configuration of almost split sequences as in Lemma 2.10.*

Proof. We prove that it is possible to built such a configuration. First, since $d_l(f) < \infty$ then each irreducible morphism f_i satisfies that $d_l(f_i) < \infty$ (see Lemma 1.1). Furthermore, by Proposition 1.12 for each morphism $f_i : X \to Y_i$ there is a configuration of almost split sequences as in Equation (1).

On the other hand, by Lemma 1.7, the morphism $(\sigma f_1, \sigma f_2) : \tau Y_1 \oplus \tau Y_2 \to X$ is an irreducible morphism of finite left degree. Therefore, X is not projective and there exists an almost split sequence with two indecomposable middle terms starting in τX. Hence, we have a configuration of almost split sequences as follows



By Proposition 1.12, τW_1 is defined. Now, we prove that the morphism $(\sigma g_1, \sigma^2 f_1) : \tau W_1 \oplus \tau X \to \tau Y_1$ determine in the above configuration is a right minimal almost split morphism.

We may assume that $\tau W_1 \simeq \tau X$, otherwise depending on the valuation of the arrow $\tau X \to \tau Y_1$ by Lemma 2.2 or Lemma 2.4 we get the result. By our assumption, we have that $W_1 \simeq X$. Then, by (2.2), $Y_1 \not\simeq Y_2$ because the valuation of the arrow $X \to Y_1$ is different from $(2, 2)$. Hence, $\tau Y_1 \not\simeq \tau Y_2$. Since $(\sigma f_1, g_1) : \tau Y_1 \to X \oplus X$ is such that σf_1 and g_1 are k_X-linearly independent then by Proposition 2.5 the morphism $(\sigma g_1, \sigma^2 f_1) : \tau X \oplus \tau X \to \tau Y_1$ is irreducible. Since by hypothesis $\alpha(\Gamma) \leq 2$ then the morphism $(\sigma g_1, \sigma^2 f_1) : \tau X \oplus \tau X \to \tau Y_1$ is a right minimal almost split morphism. We observe that in such a case τY_1 is not projective. In fact, otherwise $(\sigma g_1, \sigma^2 f_1) : \tau X \oplus \tau X \to \tau Y_1$ is a monomorphism and since $f : X \to Y_1 \oplus Y_2$ is an epimorphism then using length arguments over the modules of the almost split sequences of such configuration we get to a contradiction. Then, τY_1 is not projective and there is an almost split sequence with two indecomposable middle terms ending in τY_1. Now again such an almost split sequence determines a sink morphism in $\tau^2 Y_1$ and with a similar argument as above we shall analyze that it is a minimal right almost split morphism.

Summarizing, we proceed as follows. If in the configuration it is determined a sink morphism of the form $(g_1, g_2)^t : N_1 \oplus N_2 \to M$ with $N_1 \not\simeq N_2$ then we apply Lemma 2.2 or Lemma 2.4, depending on the valuation of the arrow to prove that it is a minimal right almost split morphism. In case we deal with a sink morphism $(g_1, g_2)^t$ where $N_1 \simeq N_2$, since by Lemma 2.10 the valuation of the arrow $N_1 \to M$ is not $(2, 2)$, more precisely it is $(2, 1)$, then we can apply Proposition 2.6 in order to prove that g_1 and g_2 are k_N linearly independent. Then, by Proposition 2.5 we infer that such a morphism is irreducible. Moreover, it is a minimal right almost split morphism. Finally, using length arguments we can prove that M is not projective and we arrive to our next step since we can built an almost split sequence with two indecomposable middle terms ending in M. We observe that this procedure is finite since $d_l(f_1)$ and $d_l(f_2)$ are finite. □

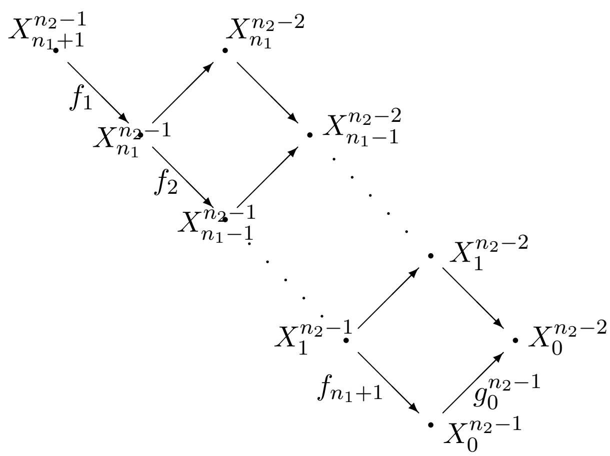
Remark 2.12. *Consider a configuration of almost split sequences as in Lemma 2.10 and assume that $n_2 \leq n_1$. Note that the length of any path of irreducible morphisms from $X^{n_2}_{n_1}$ to a module M, where $\epsilon'(M)$ is such that $\alpha'(M) = 1$, is greater or equal to n_2. Indeed, this fact is a consequence of the shape of such a configuration.*

With the notations of the configuration of almost split sequences given in Lemma 2.10, we state the following result.

Proposition 2.13. *Consider Γ an Auslander-Reiten component with $\alpha(\Gamma) \leq 2$. Let $f = (f_1, f_2)^t : X \to Y_1 \oplus Y_2$ be an irreducible epimorphism and $d_l(f_i) = n_i$ for $i = 1, 2$. Then, $\delta = \delta_0 \varphi_{n_2} = \varphi_0 \delta_{n_1} \in \Re^{n_1+n_2}(X^{n_2}_{n_1}, X) \backslash \Re^{n_1+n_2+1}(X^{n_2}_{n_1}, X)$ and $f\delta = 0$.*

Proof. Without loss of generality, we may assume that $n_2 \leq n_1$. By Lemma 2.9, we know that $\mathrm{dp}(\varphi_{n_2}) = n_1$. Moreover, since $g^{n_2}_0$ is a monomorphism then $\mathrm{dp}(g^{n_2}_0 \varphi_{n_2}) = n_1 + 1$. Furthermore, $d_l(g^{n_2}_0) = \infty$.

If $g^{n_2-1}_0 g^{n_2}_0 \varphi_{n_2} \in \Re^{n_1+3}(X^{n_2}_{n_1}, X^{n_2-2}_0)$ then $d_l(g^{n_2-1}_0) \leq n_1 + 1$ and $g^{n_2-1}_0 : X^{n_2-1}_0 \to X^{n_2-2}_0$ is an epimorphism. Then, applying Proposition 1.10 we have that $d_l(g^{n_2-1}_0) \geq n_1 + 1$. Therefore, $d_l(g^{n_2-1}_0) = n_1 + 1$. By Proposition 1.12 we have a configuration of almost split sequences as follows



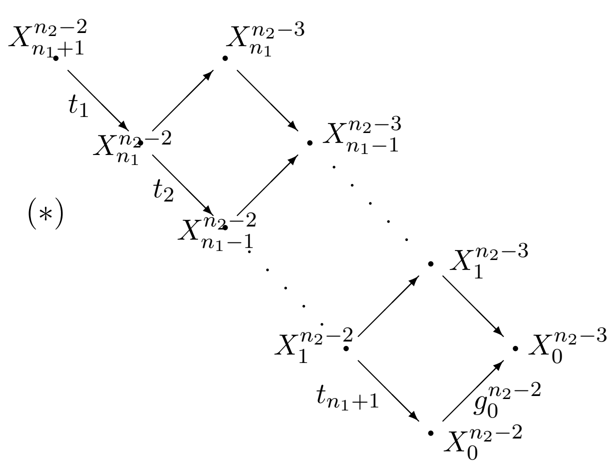
where $\delta_1 = f_{n_1+1} \ldots f_2 f_1 : X_{n_1+1}^{n_2-1} \rightsquigarrow X_0^{n_2-1}$ is a pre-sectional path of length n_1+1 and $\mathrm{dp}(\delta_1) = n_1 + 1$.

On the other hand, since $\mathrm{dp}(g_0^{n_2}\varphi_{n_2}) = n_1 + 1$ applying Lemma 1.6, we infer that $X_{n_1}^{n_2} \simeq X_{n_1+1}^{n_2-1}$ a contradiction to the fact that $\alpha(X_{n_1+1}^{n_2-1}) = 1$ and $\alpha(X_{n_1}^{n_2}) = 2$. Therefore, $\mathrm{dp}(g_0^{n_2-1}g_0^{n_2}\varphi_{n_2}) = n_1 + 2$.

Now, if $\mathrm{dp}(g_0^{n_2-2}g_0^{n_2-1}g_0^{n_2}\varphi_{n_2}) = n_1+4$ then again we get that $d_l(g_0^{n_2-2}) \leq n_1+2$, but by Proposition 1.10 we have that $d_l(g_0^{n_2-2}) \geq n_1 + 1$. Hence, either $d_l(g_0^{n_2-1}) = n_1 + 1$ or $d_l(g_0^{n_2-1}) = n_1 + 2$.

If $d_l(g_0^{n_2-1}) = n_1 + 2$ then using the same argument as above we get to the same contradiction that $\alpha(X_{n_1+1}^{n_2-2}) = 1$ and $\alpha(X_{n_1}^{n_2}) = 2$ with $X_{n_1}^{n_2} \simeq X_{n_1+1}^{n_2-2}$.

If $d_l(g_0^{n_2-1}) = n_1 + 1$ then, by Proposition 1.12 we have a configuration of almost split sequences as follows,



Since $g_0^{n_2-1}g_0^{n_2}\varphi_{n_2} \in \Re^{n_1+4}$ and by the above step we know that $(g_0^{n_2}\varphi_{n_2}) = n_1 + 2$ then applying successively Proposition 1.10 to the configuration of almost split sequences stated above we get that $g_0^{n_2-1}g_0^{n_2}\varphi_{n_2} = \delta'\theta + \mu$ where $\delta' = t_{n_1+1} \ldots t_2 t_1$, $\mu \in$

$\Re^{n_1+2}(X^{n_2-2}_{n_1+1}, X^{n_2-2}_0)$ and $\theta : X^{n_2}_{n_1} \to X^{n_2-2}_{n_1+1}$ is a non-zero morphism with $\mathrm{dp}(\theta) = 1$. Hence, θ is an irreducible morphism. We observe that $\alpha(X^{n_2-2}_{n_1+1}) = 1$ and by Remark 2.12 the length of any path from $X^{n_2}_{n_1}$ to any module M where $\epsilon'(M)$ is such that $\alpha'(M) = 1$ is greater or equal to n_2. Therefore, we get to the contradiction that there is a path from $X^{n_2}_{n_1}$ to M of length less than n_2.

Iterating this procedure, considering the composition of the previous morphisms with the next irreducible morphism of the path δ_0 and applying Proposition 1.10 as we explained above, we get that for each possible degree there is a path from $X^{n_2}_{n_1}$ to a module M of length less than n_2 or either we get the absurdly that there is a module N where $\epsilon'(N)$ is such that $\alpha'(N) = 1$ and $\alpha'(N) = 2$. Therefore, we prove that $\mathrm{dp}(\delta_0\varphi_{n_2}) = n_1 + n_2$. Moreover, by the mesh relations of the configuration of almost split sequences we get that $\delta = \delta_0\varphi_{n_2} = \varphi_0\delta_{n_1}$. We observe that this procedure is finite since $d_l(f_i) = n_i$ for $i = 1, 2$. It is not hard to verify that $f\delta = 0$, proving the result. □

As an immediate consequence of the above result we get the following corollary.

Corollary 2.14. *Let A be an artin algebra and $\Gamma \subset \Gamma_A$ satisfying $\alpha(\Gamma) \leq 2$. Let $f : X \to Y_1 \oplus Y_2$ be an irreducible epimorphism of finite left degree with $Y_1, Y_2 \in \Gamma$. Then, $d_l(f) \leq d_l(f_1) + d_l(f_2)$.*

We are in position to state one of the main results of this paper.

Theorem 2.15. *Let A be an artin algebra and $\Gamma \subset \Gamma_A$ satisfying $\alpha(\Gamma) \leq 2$. Let $f : X \to Y$ be an irreducible epimorphism with $X \in \Gamma$ or $Y \in \Gamma$. If $d_l(f) < \infty$ then there is a positive integer n, a module $Z \in \Gamma$ and a morphism $\varphi : Z \to X$ with* $\mathrm{dp}(\varphi) = n$ *such that $f\varphi = 0$.*

Proof. If $f : X \to Y$ is an irreducible morphism with X and Y indecomposable then the result follows by Proposition 1.12. In particular if Y is indecomposable but $X = X_1 \oplus X_2$ decomposes into two indecomposable summands then since $d_l(f) < \infty$ by Lemma 1.6 we have that Y is not projective. Moreover, f is a surjective sink and we get the result by Proposition 1.8.

Finally, if X is indecomposable but $Y = Y_1 \oplus Y_2$ with Y_i indecomposable for each $i = 1, 2$ then Proposition 2.13 gives the result. □

As an immediate consequence of the above theorem we get our next result.

Corollary 2.16. *Let A be an artin algebra and Γ a component of Γ_A satisfying $\alpha(\Gamma) \leq 2$. Let $f : X \to Y_1 \oplus Y_2$ be an irreducible epimorphism of finite left degree with $X, Y_1, Y_2 \in \Gamma$. Then, $\iota : kerf \to X$ the inclusion morphism is such that $pd(\iota) = n$, for some positive integer n. Moreover,* $\ker f \in \Gamma$.

Proof. Since $\alpha(\Gamma) \leq 2$ and $f : X \to Y_1 \oplus Y_2$ is an irreducible epimorphism then X is injective. Hence, $\ker(f)$ is a simple A-module. Moreover, since $d_l(f) = \infty$ by Theorem 2.15 there is a positive integer m, a module $Z \in \Gamma$ and a morphism $\varphi : Z \to X$ with $dp(\varphi) = m$ such that $f\varphi = 0$. Hence, $\varphi = \phi\iota$ with $\phi : Z \to \ker(f)$. Therefore, ι is such that $pd(\iota) = n$, for some positive integer $n \leq m$. Moreover, $kerf \in \Gamma$. □

Since any irreducible epimorphism from an indecomposable module X to $Y_1 \oplus Y_2$ is a right minimal almost split morphism, we get the following result.

Proposition 2.17. *Let A be an artin algebra and Γ a component of Γ_A satisfying $\alpha(\Gamma) \leq 2$. Let $f : X \to Y_1 \oplus Y_2$ and $g : X \to Y_1 \oplus Y_2$ be irreducible epimorphisms, with $X, Y_1, Y_2 \in \Gamma$. Then, $d_l(f) = d_l(g)$.*

Proof. Since both morphisms are right minimal almost split morphisms, there is an automorphism $t \in \mathrm{Aut}(Y_1 \oplus Y_2)$ such that $tf = g$. Clearly, if $d_l(f) = \infty$ then $d_l(f) = d_l(g)$. In case $d_l(f) = n$ then there is a module $M \in \Gamma$ and a morphism $\varphi : M \to X$ with $dp(\varphi) = n$ such that $f\varphi = 0$. Since $tf = g$ then $g\varphi = 0$. Therefore $d_l(g) \leq d_l(f)$. Similarly, we get that $d_l(f) \leq d_l(g)$, proving the result. □

A dual result holds for the right degree.

By Lemma 1.1 we know that any co-restriction of an irreducible morphism of finite left degree is of finite left degree. For these particular components, if f is an irreducible epimorphism then the converse follows by Proposition 2.11 and Proposition 1.12. We state the result below.

Proposition 2.18. *Consider Γ an Auslander-Reiten component with $\alpha(\Gamma) \leq 2$. Let $(f_1, f_2)^t : X \to Y_1 \oplus Y_2$ be an irreducible epimorphism with $Y_i \in \Gamma$. Then, $d_l(f) < \infty$ if and only if $d_l(f_i) < \infty$ for each i.*

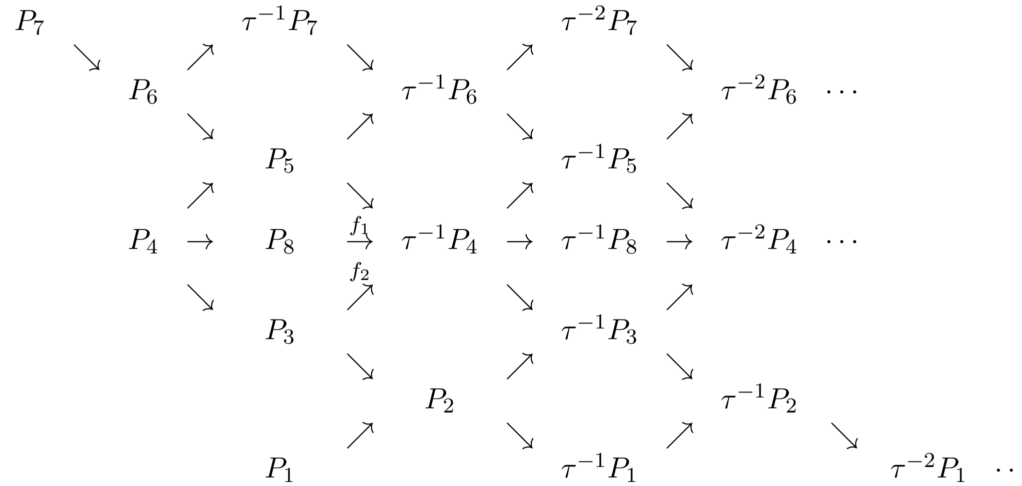
A dual result holds true for the right degree of an irreducible monomorphism $(f_1, f_2) : X_1 \oplus X_2 \to Y$ with $Y_i \in \Gamma$.

In our next example, we show that the above corollary does not hold if we drop the hypothesis that Γ is an Auslander-Reiten component with $\alpha(\Gamma) \leq 2$.

Example 2.19. *Let A be the hereditary k-algebra of type $\widetilde{E_7}$, given by the quiver:*

$$\begin{array}{ccccccccccccc} & & & & & & 8 & & & & & & \\ & & & & & & \downarrow & & & & & & \\ 1 & \leftarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \leftarrow & 5 & \rightarrow & 6 & \rightarrow & 7 \end{array}$$

The preprojective component Γ of the Auslander-Reiten quiver is:



We observe that Γ *is a generalized standard convex component of* Γ_A *with length, see* [9, Proposition 2.6]. *Consider the irreducible monomorphism* $f : P_8 \oplus P_3 \to \tau^{-1}P_4$ *where* $f_1 : P_8 \to \tau^{-1}P_4$ *and* $f_2 : P_3 \to \tau^{-1}P_4$ *are irreducible monomorphisms.*

Since f_1 *is an injective source then by Proposition 1.8 we have that* $d_r(f_1) = 1$. *By Theorem 1.5 we infer that* $d_r(f_2) = 3$, *since ker*$(f_2) = \tau^{-2}P_1$ *and* $\tau^{-1}P_4 \to \tau^{-1}P_3 \to \tau^{-1}P_2 \to \tau^{-2}P_1$ *is a sectional path of length three. Furthermore, dp*$(g_3g_2g_1) = 3$, *see* [11]. *Note that* $d_r(f) = \infty$ *since coker*$f = I_5$ *and* I_5 *belongs to another component.*

2.5. Next, we present some applications to finite dimensional algebras over an algebraically closed field. We start with a result concerning the kernel of an irreducible epimorphism $f : X \to Y_1 \oplus Y_2$ with Y_1 and Y_2 indecomposable.

Proposition 2.20. *Let* A *be a finite dimensional* k*-algebra over an algebraically closed field and* Γ *be a component of* Γ_A *satisfying* $\alpha(\Gamma) \leq 2$. *Let* $f = (f_1, f_2) : X \to Y_1 \oplus Y_2$ *be an irreducible epimorphism, with* $d_l(f_i) = n_i$ *for* $i = 1, 2$ *and* $X, Y_1, Y_2 \in \Gamma$. *Consider the configuration of almost split sequences stated in Lemma 2.10. Then the following statements hold.*

(a) *The module* $X_{n_1}^{n_2} \simeq \ker f$.
(b) $\alpha(\ker f) = 2$ *and* $\epsilon'(\ker f) : 0 \to \ker f \stackrel{(t_1,t_2)^t}{\to} M_1 \oplus M_2 \to \tau^{-1}\ker f \to 0$ *is such that* $d_r(t_1) = n_1$ *and* $d_r(t_2) = n_2$.
(c) *The inclusion morphism* $\iota : ker f \to X$ *is such that* $pd(\iota) = d_l(f_1) + d_l(f_2)$.

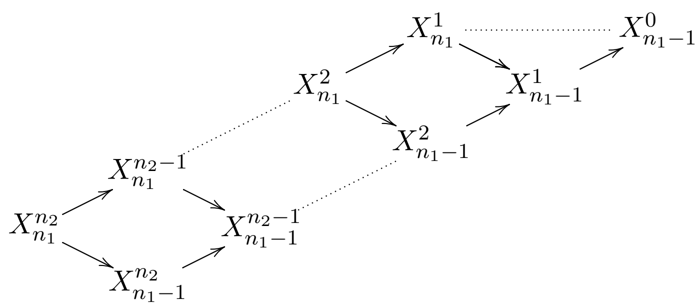
Proof. Since k is an algebraically closed field then all the division rings $T_X = \mathrm{End}(X)/\Re(X, X)$ with X indecomposable are isomorphic to k. Given an arrow $X \to Y$, its valuation is given by the dim $\mathrm{Irr}(X, Y)$ as vector spaces over T_X^{op} and T_Y, respectively. Moreover, they are all of the form (n, n). By [12, Lemma 1.7], we know that if an arrow $X \to Y$ in Γ_A of finite left (or right) degree with valuation (a, b) we have that $a = 1$ or $b = 1$. Hence, the arrow $X \to Y$ has trivial valuation.

Now, we prove Statement (a). Assume that $X_{n_1}^{n_2} \not\simeq \ker(f)$. Since $d_l(f) < \infty$, by Theorem 2.15 there is a morphism $\varphi : X_{n_1}^{n_2} \to X$ with $\mathrm{dp}(\varphi) = n$ such that $f\varphi = 0$.

Then, $f_i\varphi = 0$, for $i = 1, 2$. Therefore, Ker$f$ factors through Kerf_i for $i = 1, 2$ and $X^{n_2}_{n_1}$ factors through Ker$f$. Then S is a module that belongs to the path $\varphi_{n_1} : X^{n_2}_{n_1} \to X^1_{n_1}$ and also to the path $\varphi_{n_2} : X^{n_2}_{n_1} \to X^{n_2}_1$ in the configuration stated in Lemma 2.2. Moreover, since $\dim_k(\mathrm{Hom}(S, I_S)) = 1$ then $\ell(S, I_S)$ is unique. Hence, $n_1 = n_2$ since the morphisms in the mentioned paths behaves well respect of the powers of the radical. Then, $Y_1 \simeq Y_2$ and we get that the valuation of the arrow $X \to Y_1$ is not trivial a contradiction to the fact that the arrow $X \to Y_1$ has finite left degree. Therefore, $Z \simeq kerf$ getting the result.

(b). By Lemma 2.9 the path $\delta'_{n_1-1} : X^{n_2}_{n_1-1} \to X^{n_2-1}_{n_1-1} \to \cdots \to X^0_{n_1-1} \to X^{-1}_{n_1-1}$ is such that $\mathrm{dp}(\delta'_{n_1-1}) = n_2 + 1$.

On the other hand by the dual of Proposition 1.12 since there is a configuration of almost split sequences as follows



with δ'_{n_1-1} a path of length n_2. We observe that since $\alpha(\Gamma) \leq 2$ then the path δ'_{n_1-1} is pre-sectional. Moreover, $\delta'_{n_1-1}g_0^{n_1} = 0$. By Proposition 1.12, we infer that $d_r(g_0)^{n_1} = n_2 + 1$.

(c). It is a consequence of Proposition 2.13 and Statement (a). □

Let $f : X \to Y_1 \oplus Y_2$ be an irreducible epimorphism with $Y_1, Y_2 \in \Gamma$, where $\alpha(\Gamma) \leq 2$. If $d_l(f) = \infty$ then $d_l(f_i) = \infty$ for some $i = 1, 2$. Therefore, $d_l(f) = d_l(f_1) + d_l(f_2)$. Next, we study the case where $d_l(f) < \infty$.

Theorem 2.21. *Let A be a finite dimensional k-algebra over an algebraically closed field and $\Gamma \subset \Gamma_A$ be a component with $\alpha(\Gamma) \leq 2$. Let $f : X \to Y_1 \oplus Y_2$ be an irreducible morphism, with $X, Y_1, Y_2 \in \Gamma$. Then, $d_l(f) = d_l(f_1) + d_l(f_2)$.*

Proof. By Proposition 1.5 and [7, Proposition] we have that $d_l(f) = n$ if and only if $\iota : kerf \hookrightarrow X$ the inclusion morphism is such that $\mathrm{dp}(\iota) = n$. Therefore, we get the equality. □

We observe that the hypothesis that the Auslander-Reiten component Γ is such that $\alpha(\Gamma) \leq 2$ is a necessary condition for the result to be true.

Example 2.22. *Consider the hereditary algebra given by the quiver*

$$\begin{array}{ccc} 1 & & \\ & \searrow & \\ 2 & \rightarrow & 4 \\ & \nearrow & \\ 3 & & \end{array}$$

The Auslander-Reiten quiver is the following,

$$\begin{array}{ccccccccccc}
 & & \begin{matrix}1\\4\end{matrix} & & \cdots & & \begin{matrix}23\\4\end{matrix} & & \cdots & & 1 \\
 & \nearrow & & \searrow & & \overset{h_1}{\nearrow} & & \overset{g_1}{\searrow} & & \overset{f_1}{\nearrow} & \\
4 & \rightarrow & \begin{matrix}2\\4\end{matrix} & \rightarrow & \begin{matrix}123\\44\end{matrix} & \overset{h_2}{\rightarrow} & \begin{matrix}13\\4\end{matrix} & \overset{g_2}{\rightarrow} & \begin{matrix}123\\4\end{matrix} & \overset{f_2}{\rightarrow} & 2 \\
 & \searrow & & \overset{j}{\nearrow} & & \searrow & & \nearrow & & \searrow & \\
 & & \begin{matrix}3\\4\end{matrix} & & \cdots & & \begin{matrix}12\\4\end{matrix} & & \cdots & & 3
\end{array}$$

where the modules are given by their composition factors.

Since both irreducible morphisms f_1, f_2 are surjective sinks then by Proposition 1.8, $d_l(f_1) = d_l(f_2) = 1$. By Proposition 1.5 since Γ is a generalized standard component with length then the left degree of $f = (f_1, f_2)$ is given by the inclusion morphism from Kerf to X, that is, from $\iota : P_3 \to I_4$. Hence, since $dp(\iota) = 3$ then $d_l(f) = 3$.

Let A be an artin algebra and $f : X \to Y_1 \oplus Y_2$ be an irreducible epimorphism of finite left degree with $Y_1, Y_2 \in \Gamma$ and $\alpha(\Gamma) \leq 2$. We want to prove that $d_l(f) = d_l(f_1) + d_l(f_2)$. Though we do not know the answer in general, we can prove that this is the case when $d_l(f) < c\infty$ and $d_l(f_2) = 1$ or 2. In fact, we prove the following result.

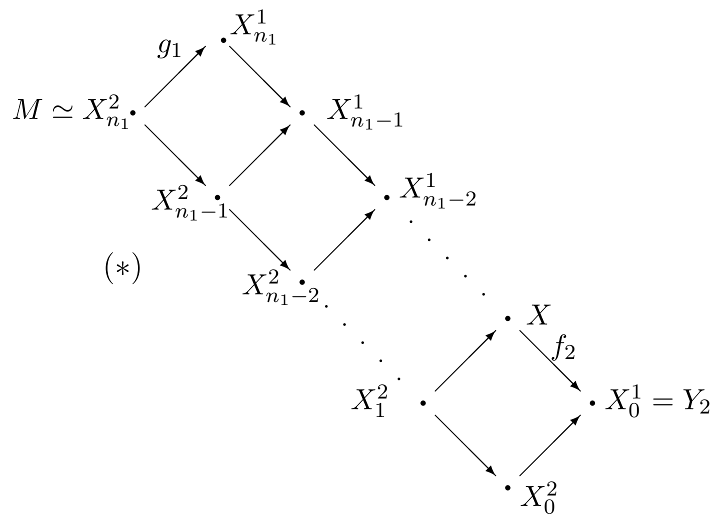
Proposition 2.23. *Let A be an artin algebra and $\Gamma \subset \Gamma_A$ satisfying $\alpha(\Gamma) \leq 2$. Let $f : X \to Y_1 \oplus Y_2$ be an irreducible epimorphism of finite left degree with $Y_1, Y_2 \in \Gamma$. If $d_l(f_2) = 1$ or $d_l(f_2) = 2$ then $d_l(f) = d_l(f_1) + d_l(f_2)$.*

Proof. Consider $d_l(f_1) = n_1$. By Lemma 1.14 and Proposition 2.21, $d_l(f_1) < d_l(f) \leq d_l(f_1) + d_l(f_2)$. Hence, $d_l(f_1) + 1 \leq d_l(f) \leq d_l(f_1) + d_l(f_2)$. If $d_l(f_2) = 1$ then we have the equality.

Now, assume that $d_l(f_2) = 2$ and that $d_l(f) < d_l(f_1) + d_l(f_2)$. Then, $d_l(f) \leq d_l(f_1) + d_l(f_2) - 1$. Since $d_l(f_2) = 2$ then $n_1 + 1 \leq d_l(f) \leq n_1 + 1$. Hence, $d_l(f) = n_1 + 1$. Then, there exists a module $M \in \Gamma$ and a morphism $\varphi \in \Re^{n_1+1}(M, X) \backslash \Re^{n_1+2}(M, X)$ such that $f\varphi \in \Re^{n_1+3}(M, Y_1 \oplus Y_2)$. Moreover, $\varphi = \delta_1\varphi_1 + \mu$ with $\mu \in \Re^{n_1+2}(X, Y_1)$ and $\delta_1\varphi_1 \in \Re^{n_1+1}(M, X) \backslash \Re^{n_1+2}(M, X)$. Note that $f_1\delta_1\varphi_1 = 0$.

Since $\delta_1 \in \Re^{n_1}(\mathrm{Ker} f_1, X) \backslash \Re^{n_1+1}(\mathrm{Ker} f_1, X)$ then $\varphi_1 \in \Re(M, \mathrm{Ker} f_1) \backslash \Re^2(M, \mathrm{Ker} f_1)$. Furthermore, since M and $\mathrm{Ker} f_1$ are indecomposable then φ_1 is an irreducible morphism and $f_2\varphi = f_2\delta_1\varphi_1 + f_2\mu$. Then, $f_2\delta_1\varphi_1 \in \Re^{n_1+3}(M, Y_2)$ with $\mathrm{dp}(f_2\delta_1) = n_1 + 1$. Therefore, $d_r(\varphi_1) \leq n_1 + 1$. Hence, by Proposition 1.12 we infer that φ_1 is a monomorphism and that $\tau^{-1}M$ is defined.

Observe that since φ_1 is an irreducible morphism from M to $\mathrm{Ker} f_1$ then $\alpha(M) = 2$ and the module $M \simeq X^2_{n_1}$. We illustrate the situation with the following diagram:



Since $d_r(\varphi_1) = d_r(g_1)$ then $d_r(g_1) \leq n_1 + 1$. By Proposition 1.11, $d_r(g_1) > n_1 + 1$ a contradiction proving that $d_l(f) = d_l(f_1) + d_l(f_2)$ if $d_l(f_2) = 2$. □

It would be of interest to extend Theorem 2.15 for any component $\Gamma \subset \Gamma_A$ where A is any artin algebra. Next, we study the situation where the component Γ has only one almost split sequence with three indecomposable middle terms, where one is projective-injective, and all the others almost split sequences with at least two middle terms.

Proposition 2.24. *Let A be an artin algebra and Γ be an Auslander-Reiten component with only one almost split sequence with three indecomposable middle terms $0 \to \tau Y \to X_1 \oplus X_2 \oplus X_3 \to Y \to 0$ with X_3 projective-injective and all the others almost split sequences in Γ with at most two indecomposable middle terms. Let $f : X \to Y$ be an irreducible morphism with $X \in \Gamma$ or $Y \in \Gamma$. Assume that $d_l(f) < \infty$. Then, there is a positive integer n, a module $Z \in \Gamma$ and a path φ of irreducible morphisms with $\mathrm{dp}(\varphi) = n$ such that $f\varphi = 0$.*

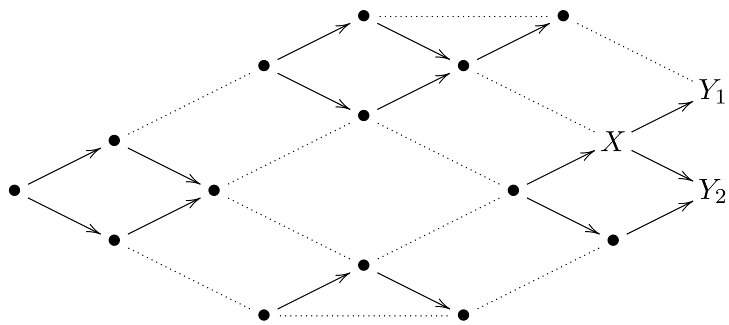
Proof. By the above lemma it remains to analyze the irreducible morphisms in the almost split sequence with three middle terms. Consider the almost split sequence

$$0 \to \tau Y \stackrel{(g_1,g_2,g_3)}{\to} X_1 \oplus X_2 \oplus X_3 \stackrel{(f_1,f_2,f_3)^t}{\to} Y \to 0$$

with X_3 injective-projective.

We only prove the result for the left degree since for the right degree follows by duality. By Liu, we know that if $d_l(f_1) < \infty$ then $d_l((g_2, g_3)) < \infty$. Since X_3 is projective then $d_l(f_1) = \infty$. With a similar argument we can prove that the left degree of the morphisms f_2 and $(f_1, f_2)^t$ are infinite. Let analyze first the morphism f_3. Note that f_3 is an epimorphism and therefore also (g_1, g_2). If $d_l(f_3) < \infty$ then $d_l((g_1, g_2)) < \infty$. By Lemma 2.11 we have a subquiver as follows

(3)



We observe that $d_l(g_3) = \infty$ since X_3 is projective. Hence $g_3\delta \in \Re^{n_1+n_2} \backslash \Re^{n_1+n_2+1}$. Moreover, $f_3 g_3 \delta = 0$. Conversely, if there is a subquiver as in (3) then $d_l(f_3) < \infty$.

Finally, the morphisms g_3, $(g_1, g_3)^t$ and $(g_2, g_3)^t$ are always of infinite left degree since X_3 is projective. □

As an immediate consequence we obtain the following corollary.

Corollary 2.25. *Let A be a finite dimensional k-algebra over an algebraically closed field k, and Γ be an Auslander-Reiten component with only one almost split sequence with three indecomposable middle terms $0 \to \tau Y \stackrel{(g_1, g_2, g_3)^t}{\to} X_1 \oplus X_2 \oplus X_3 \stackrel{(f_1, f_2, f_3)}{\to} Y \to 0$ with X_3 projective-injective. Let $f_3 : X_3 \to Y$ be an irreducible morphism with $X_3 \in \Gamma$ and $d_l(f_3) < \infty$. Then, $d_l(f_3) = d_l(g_1) + d_l(g_2) + 1$.*

We illustrate the above corollary by the following example.

Example 2.26. *Consider the algebra given by the quiver*

$$\begin{array}{ccccc} & & 2 & & \\ & \stackrel{\alpha}{\nearrow} & & \stackrel{\beta}{\searrow} & \\ 1 & & & & 4 \\ & \stackrel{\gamma}{\searrow} & & \stackrel{\delta}{\nearrow} & \\ & & 3 & & \end{array}$$

with $\beta\alpha = \delta\gamma$.

The Auslander-Reiten quiver is the following,

$$\begin{array}{ccccccccccccc} & & P_2 & & & & S_3 & & & & I_2 & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\ S_4 & & & & \tau^{-1}S_4 & \to & P_1 = I_4 & \stackrel{f}{\to} & \tau^{-2}S_4 & & & & I_1 \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & P_3 & & & & S_2 & & & & I_3 & & \end{array}$$

By the above result we claim that $d_l(f) = 3$.

3. Applications

3.1. On pre-sectional paths over directed components. As an application of the above results we study in this section the composition of two and three irreducible morphisms in pre-sectional paths, over directed Auslander-Reiten components of Γ_A satisfying that $\alpha(\Gamma) \leq 2$.

Proposition 3.1. *Let A be an artin algebra and Γ a directed component of Γ_A satisfying that $\alpha(\Gamma) \leq 2$.*

(a) *If $f_2 f_1$ is a pre-sectional path then $\mathrm{dp}(f_2 f_1) = \infty$ or $\mathrm{dp}(f_2 f_1) = 2$.*
(b) *If $f_3 f_2 f_1$ is a pre-sectional path then $\mathrm{dp}(f_3 f_2 f_1) = \infty$ or $\mathrm{dp}(f_3 f_2 f_1) = 3$.*

Proof. Assume that $X_1 \overset{f_1}{\to} X_2 \overset{f_2}{\to} X_3$ is a non-zero pre-sectional path in $\Re^3(X_1, X_3)$. By Lemma [6] we have that $d_l(f_2) = 1$ and $d_r(f_1) = 1$. Hence $X_1 \simeq \tau X_3$ and since $f_2 f_1$ is pre-sectional then we have irreducible morphisms as follows:

$$\begin{array}{ccccc} \tau X_3 & & \cdots & & X_3 \\ & \overset{f_1}{\searrow} & & \overset{g_2}{\nearrow} & \\ & & X_2 & & \\ & \overset{g_1}{\nearrow} & & \overset{h_2}{\searrow} & \\ \tau X_3 & & \cdots & & X_3 \end{array}$$

By length arguments it is not difficult to see that X_2 can not be either a projective and an injective A-module. Without loss of generality, assume that X_2 is not projective. Then since $\alpha(\Gamma) \leq 2$ then we have a sub-quiver in Γ as follows

$$\begin{array}{ccccccc} & & \tau X_3 & & \cdots & & X_3 \\ & \nearrow & & \overset{f_1}{\searrow} & & \overset{g_2}{\nearrow} & \\ \tau X_2 & & \cdots & & X_2 & & \\ & \searrow & & \overset{g_1}{\nearrow} & & \overset{h_2}{\searrow} & \\ & & \tau X_3 & & \cdots & & X_3 \end{array}$$

with $\alpha(X_3) = 1$ and $\alpha(X_2) = 2$. Moreover, since $g_2 f_1 = 0$ then $\mathrm{dp}(h_2 f_1) = 2$. In fact, otherwise $d_r(g_2, h_2)^t = 1$ a contradiction to Proposition 1.8. With a similar argument we claim that $h_2 g_1 = 0$ and $\mathrm{dp}(g_2 g_1) = 2$.

On the other hand, since $(g_2, h_2)^t$ is a left minimal almost split morphism then $f_2 = \varphi_2 h_2 + \gamma_2 g_2$ with $\varphi_2, \gamma_2 \in \mathrm{End}(X_3)$. Then, $f_2 f_1 = \varphi_2 h_2 f_1 + \gamma_2 g_2 f_1$. Hence, $f_2 f_1 = \varphi_2 h_2 f_1$ because $g_2 f_1 = 0$. By hypothesis Γ is a directed component and $0 \neq f_2 f_1 \in \Re^3$ then, since $f_2 f_1 = \varphi_2 h_2 f_1$ we infer that $\varphi_2 \in \Re^\infty$, otherwise there is a cycle in Γ from X_3 to X_3. Therefore, we prove that $f_2 f_1 \in \Re^\infty$.

(b) Let $f_3 f_2 f_1$ be a non-zero pre-sectional path. By the above result since $f_2 f_1$ is a non-zero pre-sectional path then $\mathrm{dp}(f_2 f_1) = 2$ or $dp(f_2 f_1) = \infty$.

Assume that $f_3f_2f_1 \in \Re^4$, $dp(f_2f_1) = 2$ and $dp(f_3f_2) = 2$, otherwise we get the result. Then, by [5, Theorem 3.9] $d_l(f_3) = 2$ and there is a configuration of almost split sequences as follows

$$\begin{array}{ccccccc} \tau X' & & \cdots & & X' & & \\ & \searrow & & \nearrow & & \searrow & \\ & & \tau Y & & \cdots & & Y \\ & & & & & \overset{f}{\nearrow} & \\ & & & \searrow & & & \\ & & & & X & & \end{array}$$

with $Y, X' \in ind\,A$, $\alpha(X') = 1$ and where $\varphi : \tau X' \to \tau Y \to X$ is a pre-sectional path with $\varphi \in \Re^2(\tau X', X) \backslash \Re^3(\tau X', X)$.

Therefore, by Proposition 1.12 there is a configuration of almost split sequences as in $(*)$ with $\tau X_4 \simeq X_2$. Since $f_3f_2f_1$ is a pre-sectional path then there is an irreducible morphism $X_3 \overset{(f_3,t_3)}{\to} X_4 \oplus X_4$. By [5, Theorem 5.3] there is a cycle in X_3, a contradiction that there is a cycle in Γ. Hence, in this case we have the result. □

Clearly, if Γ is a generalized standard component then we get the following corollary.

Corollary 3.2. *Let A be an artin algebra and Γ a directed component of Γ_A satisfying that $\alpha(\Gamma) \leq 2$. In addition, if Γ is a generalized standard component of Γ_A then*

(a) *If f_2f_1 is a non-zero pre-sectional path then* $\mathrm{dp}(f_2f_1) = 2$.
(b) *If $f_3f_2f_1$ is a non-zero pre-sectional path then* $\mathrm{dp}(f_3f_2f_1) = 3$.

3.2. Degrees and finite representation type of an algebra. The aim of this subsection is to establish a connection between degrees of irreducible morphisms and the representation type of the algebra.

We start recalling the following characterization given in [8], where ι_S and π_S are the injective hull and the projective cover of a simple S.

Theorem 3.3. *Let A be an artin algebra. The following statements are equivalent.*

(*a*) *The representation type of A is finite.*
(*b*) *The depth of ι_S is finite, for every simple module S.*
(*c*) *The depth of π_S is finite, for every simple module S.*
(*d*) *The map θ_S does not lie in* $(\mathrm{rad}^\infty(\mathrm{mod}A))^2$, *for every simple module S.*

Moreover, in this case, the nilpotency of $\mathrm{rad}(\mathrm{mod}A)$ *is $m+1$, where m is the maximal depth of the θ_S with S ranging over the simple modules.*

By the above theorem we get that the characterization given in [7, Theorem A] for a finite dimensional k-algebra over an algebraically closed field, still holds true for artin algebras with Auslander-Reiten components Γ satisfying $\alpha(\Gamma) \leq 2$.

Theorem 3.4. *Let A be an artin algebra where all the Auslander-Reiten components Γ of Γ_A are such that $\alpha(\Gamma) \leq 2$. Then, the following conditions are equivalent.*

(a) *A is finite representation type.*
(b) *For every non-simple indecomposable injective A-module I, the irreducible morphism $I \to I/socI$ has finite left degree.*

(c) *For every non-simple indecomposable projective A-module P, the irreducible morphism $radP \to P$ has finite right degree.*
(d) *For every irreducible epimorphism $f : X \to Y$ with X or Y indecomposable, the left degree of f is finite.*
(e) *For every irreducible monomorphism $f : X \to Y$ with X or Y indecomposable, the right degree of f is finite.*

Proof. Suppose first that A is of finite representation type. Then $\mathrm{rad}^{\infty}(\mathrm{mod}A) = 0$; see [**?**, (1.1)]. In particular, Statements (2), (3), (4) and (5) hold true.

If Statement (2) holds then, for every non-simple indecomposable injective A-module I_S, the irreducible morphism $I \to I_S/socI_S$ has finite left degree. By Lemma 2.20 we have that the inclusion morphism $inc : S \to I_S$ is such that $\mathrm{dp}(inc) = n$ for some positive integer n. Therefore, by Theorem 3.3 the depth of ι_S is finite, for every simple module S. Hence A is of finite representation type. Therefore, we prove that (1) is equivalent to (2).

Dually, Statement (3) implies Statement (1).

Finally, since Statement (4) ((5)) implies Statement (2) ((3), respectively) and therefore we get the result. □

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