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Bounds for axially symmetric linear perturbations for the extreme Kerr black hole

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Abstract

We obtain remarkably simple integral bounds for axially symmetric linear perturbations for the extreme Kerr black hole in terms of conserved energies. From these estimates we deduce pointwise bounds for the perturbations outside the horizon.

Keywords: stability, black hole, extreme, Kerr

1. Introduction

In this article we continue the work initiated in [4] destined to study the linear stability of an extreme Kerr black hole under axially symmetric gravitational perturbations using conserved energies. For a general introduction to the subject and a list of relevant references, we refer to [4]. The main result of that article is that there exists a positive definite and conserved energy for the axially symmetric gravitational perturbations. In the present article, using this energy, we prove the existence of integral bounds for the first and second derivatives of the perturbation. In particular, these bounds imply pointwise estimates for the perturbation outside the black hole horizon. This result is presented in theorem 1.1 and corollary 1.2. In the following we introduce the basic definitions and notation needed to formulate the theorem, and then we discuss the meaning and scope of these estimates.

Axially symmetric perturbations are characterized by two functions σ_1 and ω_1, which represent the linear perturbation of the norm and the twist of the axial Killing vector. The coordinates system (t, ρ, z) is fixed by the maximal-isothermal gauge condition. Partial derivatives with respect to the space coordinates (ρ, z) is denoted by ∂_ρ and ∂_z respectively, and partial derivative with respect to t is denoted by a dot. We use the following notation to abbreviate the products of gradients in the spatial coordinates (ρ, z) for functions f and g

$$\partial f \partial g = \partial_\rho f \partial_\rho g + \partial_z f \partial_z g, \quad |\partial f|^2 = \left(\partial_\rho f\right)^2 + \left(\partial_z f\right)^2. \tag{1}$$

The two-dimensional Laplacian Δ is defined by

$$\Delta f = \partial_\rho^2 f + \partial_z^2 f, \tag{2}$$

and the operators ${}^{(3)}\Delta$ and ${}^{(7)}\Delta$ are defined by

$${}^{(3)}\Delta f = \Delta f + \frac{\partial_\rho f}{\rho}, \quad {}^{(7)}\Delta f = \Delta f + 5\frac{\partial_\rho f}{\rho}. \tag{3}$$

The operators ${}^{(3)}\Delta$ and ${}^{(7)}\Delta$ correspond to the flat Laplace operator in three-dimensions and seven-dimensions respectively written in cylindrical coordinates and acting on axially symmetric functions.

The domain for the space coordinates (ρ, z) is the half plane $\mathbb{R}^2_+$ defined by $0 \leqslant \rho < \infty$, $-\infty < z < \infty$. The axis of symmetry is give by $\rho = 0$. In these coordinates, the horizon is located at the origin $r = 0$, where $r = \sqrt{\rho^2 + z^2}$. We follow the same notation and conventions used in [4] and we refer to that article for further details.

The linear equations for axially symmetric gravitational perturbations for the extreme Kerr black hole in the maximal-isothermal gauge were obtained in [4]. In appendix A we briefly review the set of equations needed in the proof of theorem 1.1. The background quantities are denoted with a subindex 0, and the first order perturbation with a subindex 1. The square norm of the axial Killing vector of the background extreme Kerr metric is denoted by η_0 and the background function σ_0 is defined by

$$e^{\sigma_0} = \frac{\eta_0}{\rho^2}. \tag{4}$$

The other relevant background quantities are the twist ω_0 and the function q_0. In appendix B we review the behaviour of these explicit functions.

It is useful to define the following rescaling of ω_1

$$\bar{\omega}_1 = \frac{\omega_1}{\eta_0^2}. \tag{5}$$

The extreme Kerr solution depends on only one parameter m_0 which represents the total mass of the black hole. The first order linearization of the total mass of the spacetime m_1 vanished. The second order expansion of the total mass m_2 provides a positive definite and conserved quantity for the perturbation (see [4]) that is given explicitly in equation (A.5). Taking time derivatives of the linear equations we get an infinity number of conserved quantities that have the same form as m_2 but in terms of the time derivatives of the corresponding quantities. For the result presented bellow, we will make use of $\bar{m}_2$ which is obtained taking one time derivative of the equations. The explicit expression for $\bar{m}_2$ is given in equation (A.7). The conserved quantities m_2 and $\bar{m}_2$ depend only on the initial conditions for the perturbation.

We will assume in the following that the perturbations satisfy the fall off and boundary conditions discussed in detail in [4]. Physically, these conditions imply that the system is isolated (and hence it has finite total energy m_2) and that that the perturbations do not change the angular momentum of the background (i.e. ω_1 vanished at the axis).

Theorem 1.1. *Axially symmetric linear gravitational perturbations* (σ_1, ω_1) *for the extreme Kerr black hole satisfy the following bound*

$$\int_{\mathbb{R}^2_+}\left(\frac{1}{2}\eta_0^2\left|\partial\bar{\omega}_1\right|^2+\left|\partial\eta_0\right|^2\bar{\omega}_1^2+\left|\partial\sigma_1\right|^2+\frac{\sigma_1^2}{r^2}\right)\rho\mathrm{d}\rho\mathrm{d}z\leqslant Cm_2, \tag{6}$$

$$\int_{\mathbb{R}^2_+}\left(\left({}^{(3)}\Delta\sigma_1\right)^2+\eta_0^2\left({}^{(7)}\Delta\bar{\omega}_1\right)^2\right)\mathrm{e}^{-2\left(q_0+\sigma_0\right)}\rho\mathrm{d}\rho\mathrm{d}z\leqslant C\left(\bar{m}_2+m_2\right), \tag{7}$$

where C is a positive constant that depends only on the mass m_0 of the background extreme Kerr black hole.

The conserved quantity m_2 involves first spatial derivatives of σ_1 and ω_1, the quantity $\bar{m}_2$ also involves second spatial derivatives of σ_1 and ω_1. Note however, that these terms appear in a rather complicated way and hence it is by no means obvious that m_2 and $\bar{m}_2$ satisfy the bound (6) and (7).

Besides σ_1 and ω_1, gravitational perturbations involve other quantities (the shift vector β_1, the metric function q_1, the second fundamental form χ_1^{AB}, see [4] for details). These other functions are, in principle, calculated in terms σ_1 and ω_1 using the coupled system of equations. It is remarkable that the estimates (6) and (7) can be written purely in terms of the geometric functions σ_1 and ω_1 (which precisely encode the dynamical degree of freedom of the system) without involving the other functions.

The functions (σ_1, ω_1) satisfy the linear evolution equations (A.1)–(A.2). These equations have the well known structure of a wave map coupled with a non-trivial background metric. Recently, a model problem for an analogous wave map (but without the coupling) was studied in [6]. Remarkably enough the estimates proved in theorem 1.1 make use only of the wave map structure of the equations, but they hold for the complete coupled system. Also, these estimates are robust in the sense that they make use of energies that are also available for the nonlinear equations.

Since the estimates (6) and (7) essentially control up to second derivatives of the functions (σ_1, ω_1), using Sobolev embeddings we can obtain pointwise bounds. This is, of course, one of the main motivations to obtain these kind of estimates. Note however, that the different terms are multiplied by the background functions. Particularly relevant is the factor $\mathrm{e}^{-2(\sigma_0+q_0)}$ that appears in (7). This explicit function is positive, goes to 1 at infinity but vanished like r^2 at the origin, where the black hole horizon is located. This means that the estimate (7) degenerates at the origin (but not at infinity) and we can not expect to control pointwise the functions (σ_1, ω_1) at the origin using only this estimate. In the following corollary we prove pointwise bounds outside the horizon.

Let $\delta > 0$ be an arbitrary, small, number. We define the following two domains

$$\Omega_\delta=\left\{(\rho, z)\in\mathbb{R}^2_+,\ \text{such that}\ 0<\delta\leqslant r\right\} \tag{8}$$

$$\Gamma_\delta=\left\{(\rho, z)\in\mathbb{R}^2_+,\ \text{such that}\ 0<\delta\leqslant \rho\right\}. \tag{9}$$

We have the following result.

Corollary 1.2. *Under the same assumptions of theorem 1.1, the following pointwise bounds hold*

$$\sup_{\Omega_\delta}\left|\sigma_1\right|\leqslant C_\delta\left(\bar{m}_2+m_2\right), \tag{10}$$

$$\sup\nolimits_{\Gamma_\delta}\left|\bar{\omega}_1\right|\leqslant C_\delta\left(\bar{m}_2+m_2\right), \tag{11}$$

where the constant C_δ depends on m_0 and δ.

The bounds (10) and (11) are not intended to be sharp, they are meant as examples of possible pointwise bounds that can be deduced from (6) and (7). It is certainly conceivable that sharped weighted bounds can be proved using the estimates (6) and (7). But it is also clear that no pointwise bound at the horizon can be proved using these estimates, because the factor $\mathrm{e}^{-2(\sigma_0+q_0)}$ vanishes there. The situation strongly resembles the problem studied in [5]. In that article the wave equation on the extreme Reissner–Nordström black hole was analyzed using conserved energies. In order to prove a pointwise bound at the horizon it was not enough with the first two energies. An extra energy which involves 'integration in time' was needed. It is a relevant open question whether the same strategy can be applied to the present case, which is certainly much more complicated.

2. Proof of theorem 1.1

In this section we prove theorem 1.1. We begin with the estimate (6), which represents the most important part of the theorem. In the integrand on the left-hand side of (6) the terms involve up to first derivatives of σ_1 and ω_1. The integral is bounded only with the energy m_2, the higher order energy $\bar{m}_2$ is not needed for this estimate. Moreover, to prove the bound (6) we will make use only of the last three terms in the energy density ϵ_2 given by (A.6). Note that in ϵ_2 appears the same terms as in the integrand on the left-hand side of (6). However they appear arranged in different form (i.e. there are many cross products) and it not obvious how to deduce the bound (6).

The proof of (6) can be divided in two parts. The first part consists of integral estimates, this is the subtle part of the proof. The second part consists of pointwise estimates. In the arguments, we make repeated use of different forms of the standard Cauchy inequality, and for readability we summarize them below. Let $a_1 \cdots a_n$ be arbitrary real numbers, then we have

$$a_1^2 + a_2^2 \geqslant \frac{1}{2}(a_1 + a_2)^2, \tag{12}$$

and, in general,

$$a_1^2 + a_2^2 + \ldots + a_n^2 \geqslant \frac{1}{n}(a_1 + a_2 + \ldots + a_n)^2. \tag{13}$$

Let $\lambda > 0$, then

$$a_1 a_2 \leqslant \lambda a_1^2 + \frac{a_2^2}{4\lambda}. \tag{14}$$

In the following two lemmas we prove the relevant integral estimates. The relevance of lemma 2.1 in the proof of the estimate (6) is clear: in this lemma the bound for the fourth term in (6) is proved. This integral bound is the key to prove the bounds for the second and the third term in (6). We will see in the following, that in order to prove these bounds we will need the integral estimate proved in lemma 2.2 with $v = \bar{\omega}_1$.

Lemma 2.1. *Consider the mass m_2 given by (A.5) and (A.6). Then, the following inequality holds*

$$m_2 \geqslant \int_{\mathbb{R}^2_+} \frac{\sigma_1^2}{r^2} \rho \mathrm{d}\rho \mathrm{d}z. \tag{15}$$

Proof. The mass m_2 is the second variation of the total ADM mass (see [4]). The first three terms in (A.6) correspond to the dynamical part of the mass (these terms vanished for stationary solutions), the last three terms correspond to the stationary part of the mass. These terms are precisely the second variation of the mass functional extensively studied in connection with the mass angular momentum inequality (see [2, 3] and reference therein). In a recent article [7], an important estimate has been proved for the second variation of this functional in terms of the distance function in the hyperbolic plane. From lemma 2.3 in [7] we deduce the following inequality

$$m_2 \geqslant 2 \int_{\mathbb{R}^2_+} \left| \partial \mathbf{d}\left((\eta_1, \omega_1), (\eta_0, \omega_0) \right) \right|^2 \rho \mathrm{d}\rho \mathrm{d}z, \tag{16}$$

where $\mathbf{d}$ is the distance function in the hyperbolic plane between the two points (η_1, ω_1) and (η_0, ω_0), where $\eta_1 = \rho^2 \mathrm{e}^{\sigma_1}$ (see, for example, [2] for the explicit expression of $\mathbf{d}$).

To obtain the desired lower bound for the right-hand side of the inequality (16) we first use the following weighted Poincare inequality proved in [1] (equation (31) in [1] with $\delta = -1/2$)

$$2 \int_{\mathbb{R}^2_+} |\partial \mathbf{d}|^2 \rho \mathrm{d}\rho \mathrm{d}z \geqslant \int_{\mathbb{R}^2_+} \frac{\mathbf{d}^2}{r^2} \rho \mathrm{d}\rho \mathrm{d}z, \tag{17}$$

and then we use the following bound for the distance function $\mathbf{d}$ proved in [2] (see equation (138) in that reference)

$$|\mathbf{d}| \geqslant |\sigma_1|. \tag{18}$$

□

Lemma 2.2. *Let η_0 and ω_0 be the norm and the twist function for the extreme Kerr black hole, and let v be an arbitrary smooth function with compact support outside the axis. Then, the following inequality holds*

$$\int_{\mathbb{R}^2_+} |\partial \omega_0|^2 v^2 \rho \mathrm{d}\rho \mathrm{d}z \leqslant 3 \int_{\mathbb{R}^2_+} |\partial \eta_0|^2 v^2 \rho \mathrm{d}\rho \mathrm{d}z + \int_{\mathbb{R}^2_+} \eta_0^2 |\partial v|^2 \rho \mathrm{d}\rho \mathrm{d}z. \tag{19}$$

Proof. We will use that the background function η_0 and ω_0 satisfy equation (B.3). Let v be an arbitrary function with compact support outside of the axis. We multiply (B.3) by $\eta_0^{-2\delta} v^2$ (where δ is an arbitrary number) and integrate, we obtain

$$\int_{\mathbb{R}^2_+} \eta_0^{-2\delta} v^{2\,(3)}\Delta\left(\ln \eta_0\right) \rho \mathrm{d}\rho \mathrm{d}z = -\int_{\mathbb{R}^2_+} \frac{|\partial \omega_0|^2}{\eta_0^2} \eta_0^{-2\delta} v^2 \rho \mathrm{d}\rho \mathrm{d}z. \tag{20}$$

Integrating by parts the left-hand side of (20) we obtain the following useful identity

$$\int_{\mathbb{R}^2_+} |\partial \omega_0|^2 \eta_0^{-2\delta-2} v^2 \rho \mathrm{d}\rho \mathrm{d}z = -2\delta \int_{\mathbb{R}^2_+} \eta_0^{-2\delta-2} |\partial \eta_0|^2 v^2 \rho \mathrm{d}\rho \mathrm{d}z + 2 \int_{\mathbb{R}^2_+} \eta_0^{-2\delta-1} v \partial v \partial \eta_0 \rho \mathrm{d}\rho \mathrm{d}z. \tag{21}$$

We take $\delta = -1$ in (21), and obtain

$$\int_{\mathbb{R}^2_+} |\partial\omega_0|^2 v^2 \rho \mathrm{d}\rho \mathrm{d}z = 2\int_{\mathbb{R}^2_+} |\partial\eta_0|^2 v^2 \rho \mathrm{d}\rho \mathrm{d}z + 2\int_{\mathbb{R}^2_+} \eta_0 v \partial v \partial\eta_0 \rho \mathrm{d}\rho \mathrm{d}z, \tag{22}$$

$$\leqslant 3\int_{\mathbb{R}^2_+} |\partial\eta_0|^2 v^2 \rho \mathrm{d}\rho \mathrm{d}z + \int_{\mathbb{R}^2_+} \eta_0^2 \, |\partial v|^2 \rho \mathrm{d}\rho \mathrm{d}z, \tag{23}$$

where, to obtain line (23) we have used in the second term of the right-hand side of (22) the inequality (14) with $a_1 = \eta_0 \partial\bar{\omega}_1$, $a_2 = \partial\eta_0 \bar{\omega}_1$ and $\lambda = 1/2$. □

We prove now the pointwise bounds in terms of the energy density ε_2. In the following we denote by C a generic positive constant that depends only on the background parameter m_0.

We begin with the first term in the integrand in (6). From the explicit expression for ε_2 given in (A.6), keeping only the fifth and sixth terms, we obtain

$$\frac{\varepsilon_2}{\rho} \geqslant \left(\partial\left(\omega_1\eta_0^{-1}\right) - \eta_0^{-1}\sigma_1\partial\omega_0\right)^2 + \left(\eta_0^{-1}\sigma_1\partial\omega_0 - \omega_1\eta_0^{-2}\partial\eta_0\right)^2 \tag{24}$$

$$= \left(\eta_0\partial\bar{\omega}_1 + \bar{\omega}_1\partial\eta_0 - \eta_0^{-1}\sigma_1\partial\omega_0\right)^2 + \left(\eta_0^{-1}\sigma_1\partial\omega_0 - \bar{\omega}_1\partial\eta_0\right)^2, \tag{25}$$

where in (25) we have used the definition of $\bar{\omega}_1$ given in (5). We use the Cauchy inequality (12) in (25) to finally obtain

$$\frac{\varepsilon_2}{\rho} \geqslant \frac{1}{2}\eta_0^2\left(\partial\bar{\omega}_1\right)^2. \tag{26}$$

From the second term in (6) we take ε_2 given in (A.6) and keep only the last term, we obtain

$$\frac{\varepsilon_2}{\rho} \geqslant \left(\frac{\partial\omega_0}{\eta_0}\sigma_1 - \partial\eta_0\bar{\omega}_1\right)^2, \tag{27}$$

$$= \frac{|\partial\omega_0|^2}{\eta_0^2}\sigma_1^2 + |\partial\eta_0|^2\bar{\omega}_1^2 - 2\sigma_1\bar{\omega}_1\frac{\partial\omega_0}{\eta_0}\partial\eta_0, \tag{28}$$

$$\geqslant -\frac{|\partial\omega_0|^2}{\eta_0^2}\sigma_1^2 + \frac{1}{2}\,|\partial\eta_0|^2\bar{\omega}_1^2, \tag{29}$$

$$\geqslant -\frac{C}{r^2}\sigma_1^2 + \frac{1}{2}\,|\partial\eta_0|^2\bar{\omega}_1^2, \tag{30}$$

where in the inequality (29) we have used the Cauchy inequality (14) with $\lambda = 1$ and in line (30) we have used the bound (B.5) for the background quantities. We have obtained

$$\frac{\varepsilon_2}{\rho} + \frac{C}{r^2}\sigma_1^2 \geqslant \frac{1}{2}\,|\partial\eta_0|^2\bar{\omega}_1^2. \tag{31}$$

We integrate the bounds (26) and (31), and use the integral bound (15) to obtain

$$\int_{\mathbb{R}^2_+}\left(\frac{1}{2}\eta_0^2\left|\partial\bar{\omega}_1\right|^2+\left|\partial\eta_0\right|^2\bar{\omega}_1^{\,2}+\frac{\sigma_1^{\,2}}{r^2}\right)\rho\mathrm{d}\rho\mathrm{d}z\leqslant Cm_2. \tag{32}$$

To prove (6) it only remains to bound the term $|\partial\sigma_1|^2$. For that term, we use the fourth term in (A.6) to obtain

$$\frac{\varepsilon_2}{\rho}\geqslant\left(\partial\sigma_1+\omega_1\eta_0^{-2}\partial\omega_0\right)^2 \tag{33}$$

$$=\left(\partial\sigma_1+\bar{\omega}_1\partial\omega_0\right)^2, \tag{34}$$

$$=\left|\partial\sigma_1\right|^2+\bar{\omega}_1^{\,2}\left|\partial\omega_0\right|^2+2\bar{\omega}_1\partial\sigma_1\partial\omega_0, \tag{35}$$

$$\geqslant\frac{1}{2}\left|\partial\sigma_1\right|^2-\left|\partial\omega_0\right|^2\bar{\omega}_1^{\,2}, \tag{36}$$

where in line (34) we have just used the definition of $\bar{\omega}_1$ and in line (36) we have used inequality (14) with $\lambda=\frac{1}{4}$. Then, we have obtained

$$\frac{\varepsilon_2}{\rho}+\left|\partial\omega_0\right|^2\bar{\omega}_1^{\,2}\geqslant\frac{1}{2}\left|\partial\sigma_1\right|^2. \tag{37}$$

We integrate the pointwise estimate (37), to handle second term on the left-hand side of (37) we use the integral bound (15) $v=\bar{\omega}_1$ and the bound (32). Hence we have obtained the desired estimate (6).

We turn to the bound (7) which involves second derivatives of the functions σ_1 and $\bar{\omega}_1$ and hence we need the higher order mass $\bar{m}_2$.

We begin with the term with σ_1. We use the evolution equation (A.1) to obtain

$$\left({}^{(3)}\Delta\sigma_1\right)^2=\left(\mathrm{e}^{2\left(\sigma_0+q_0\right)}\dot{p}+\frac{2}{\eta_0^2}\left(\sigma_1\left|\partial\omega_0\right|^2-\partial\omega_1\partial\omega_0\right)\right)^2 \tag{38}$$

$$=\left(\mathrm{e}^{2\left(\sigma_0+q_0\right)}\dot{p}+\frac{2}{\eta_0^2}\left(\sigma_1\left|\partial\omega_0\right|^2-2\eta_0\bar{\omega}_1\partial\eta_0\partial\omega_0+\eta_0^2\partial\omega_0\partial\bar{\omega}_1\right)\right)^2 \tag{39}$$

$$\leqslant 4\mathrm{e}^{4\left(\sigma_0+q_0\right)}\dot{p}^2+\frac{16}{\eta_0^4}\sigma_1^2\left|\partial\omega_0\right|^4+\frac{64}{\eta_0^2}\left(\partial\eta_0\partial\omega_0\right)^2\bar{\omega}_1^{\,2}+16\left(\partial\omega_0\partial\bar{\omega}_1\right)^2 \tag{40}$$

$$\leqslant 4\mathrm{e}^{4\left(\sigma_0+q_0\right)}\dot{p}^2+C\frac{\left|\partial\omega_0\right|^2}{\eta_0^2}\frac{\sigma_1^2}{r^2}+\frac{64}{\eta_0^2}\left|\partial\eta_0\right|^2\left|\partial\omega_0\right|^2\bar{\omega}_1^{\,2}+16\left|\partial\bar{\omega}_1\right|^2\left|\partial\omega_0\right|^2, \tag{41}$$

where in line (39) we have used the definition of $\bar{\omega}_1$ (5), in line (40) we have used the inequality (13) and line (41) follows from the bound (B.5) and the Cauchy–Schwartz inequality.

Multiplying by $e^{-2(\sigma_0+q_0)}$ the inequality (41) we obtain

$$e^{-2(\sigma_0+q_0)}\left({}^{(3)}\Delta\sigma_1\right)^2 \leqslant 4e^{2(\sigma_0+q_0)}\dot{p}^2 + e^{-2(\sigma_0+q_0)}\frac{|\partial\omega_0|^2}{\eta_0^2} \times \left(C\frac{\sigma_1^2}{r^2} + 64\,|\partial\eta_0|^2\bar{\omega}_1^2 + 16\,|\partial\bar{\omega}_1|^2\eta_0^2\right) \tag{42}$$

$$\leqslant 2\frac{\bar{\varepsilon}_2}{\rho} + C\left(\frac{\sigma_1^2}{r^2} + |\partial\eta_0|^2\bar{\omega}_1^2 + |\partial\bar{\omega}_1|^2\eta_0^2\right), \tag{43}$$

where in line (43) we have used (A.8) and the bound (B.7).

Integrating (43) and using the previous bounds we finally obtain

$$\int_{\mathbb{R}^2_+} e^{-2(\sigma_0+q_0)}\left({}^{(3)}\Delta\sigma_1\right)^2\rho\,d\rho\,dz \leqslant C(\bar{m}_2 + m_2). \tag{44}$$

To estimate the second derivatives of $\bar{\omega}_1$ we proceed in a similar way. We will make use of the evolution equation (A.2). First, it is useful to write this equation in terms of $\bar{\omega}_1$ instead of ω_1. To do that we first obtain the following relation

$$\eta_0^2\ {}^{(7)}\Delta\bar{\omega}_1 = {}^{(3)}\Delta\omega_1 - \frac{4}{\rho}\partial_\rho\omega_1 - 4\partial\omega_1\partial\sigma_0 - 2\omega_1{}^{(3)}\Delta\sigma_0 + 8\frac{\omega_1}{\rho}\partial_\rho\sigma_0 + 4\omega_1\,|\partial\sigma_0|^2, \tag{45}$$

where we have used the definition of $\bar{\omega}_1$ given in (5), the expression of η_0 in terms of σ_0 given in (4), the definitions of the operators ${}^{(3)}\Delta$ and ${}^{(7)}\Delta$ given in (3) and the identity (B.4). Using the evolution equations (A.2) and (45) we obtain

$$\eta_0^2\ {}^{(7)}\Delta\bar{\omega}_1 = e^{2(\sigma_0+q_0)}\dot{d} - 2\eta_0^2\bar{\omega}_1{}^{(3)}\Delta\sigma_0 - 2\eta_0^2\partial\sigma_0\partial\bar{\omega}_1 + 2\partial\omega_0\partial\sigma_1. \tag{46}$$

To obtain the estimate, we take the square of each side of equation (46) and use the Cauchy inequality (13) to obtain

$$\eta_0^4\left({}^{(7)}\Delta\bar{\omega}_1\right)^2 \leqslant 4e^{4(\sigma_0+q_0)}\dot{d}^2 + 16\eta_0^4\left({}^{(3)}\Delta\sigma_0\right)^2\bar{\omega}_1^2 + 16\eta_0^4\left(\partial\sigma_0\partial\bar{\omega}_1\right)^2 + 16\left(\partial\omega_0\partial\sigma_1\right)^2 \tag{47}$$

$$\leqslant 4e^{4(\sigma_0+q_0)}\dot{d}^2 + 16\,|\partial\omega_0|^2\left(|\partial\omega_0|^2\bar{\omega}_1^2 + |\partial\sigma_1|^2\right) + 16\eta_0^4\,|\partial\sigma_0|^2|\partial\bar{\omega}_1|^2, \tag{48}$$

where in line (48) we have used the Cauchy–Schwartz inequality and equation (B.1) to substitute the factor ${}^{(3)}\Delta\sigma_0$. We multiply by $e^{-2(\sigma_0+q_0)}\eta_0^{-2}$ each side of inequality (48)

$$\eta_0^2 e^{-2(\sigma_0+q_0)}\left({}^{(7)}\Delta\bar{\omega}_1\right)^2 \leqslant 4\frac{e^{2(\sigma_0+q_0)}}{\eta_0^2}\dot{d}^2 + 16e^{-2(\sigma_0+q_0)}\frac{|\partial\omega_0|^2}{\eta_0^2}\left(|\partial\omega_0|^2\bar{\omega}_1^2 + |\partial\sigma_1|^2\right) + 16e^{-2(\sigma_0+q_0)}\,|\partial\sigma_0|^2\eta_0^2\,|\partial\bar{\omega}_1|^2. \tag{49}$$

Then, we bound the first term on the right-hand side of inequality (49) with the energy density $\bar{\varepsilon}_2$(A.8), for the other terms we use the inequalities (B.7) and (B.8) to bound the background functions by a constant C. We obtain

$$\eta_0^2 e^{-2(\sigma_0+q_0)}\left({}^{(7)}\Delta\bar{\omega}_1\right)^2 \leqslant 2\frac{\bar{\varepsilon}_2}{\rho} + C\left(|\partial\omega_0|^2\bar{\omega}_1^2 + |\partial\sigma_1|^2 + \eta_0^2\,|\partial\bar{\omega}_1|^2\right). \tag{50}$$

Integrating (50) and using (6) we finally have

$$\int_{\mathbb{R}^2_+} \eta_0^2 e^{-2(\sigma_0+q_0)} \left({}^{(7)}\Delta \bar{\omega}_1 \right)^2 \rho d\rho dz \leqslant C(\bar{m}_2 + m_2). \tag{51}$$

3. Proof of corollary 1.2

In the proof of corollary 1.2 we essentially use an appropriated variant of the Sobolev embedding and standard cut off function arguments.

Let χ: $\mathbb{R} \to \mathbb{R}$ be a smooth cut off function such that $\chi \in C^\infty(\mathbb{R})$, $0 \leqslant \chi \leqslant 1$, $\chi(r) = 1$ for $0 \leqslant r \leqslant 1$, $\chi(r) = 0$ for $2 \leqslant r$. Define $\chi_\delta(r) = \chi(r/\delta)$.

Consider the following function

$$\bar{\sigma}_1 = \left(1 - \chi_\delta\right)\sigma_1. \tag{52}$$

Note that $\bar{\sigma}_1 = 0$ in B_δ and $\bar{\sigma}_1 = \sigma_1$ in $\Omega_{2\delta}$, where B_δ denotes the ball of radius δ, and $\Omega_{2\delta} = \mathbb{R}^2_+ \backslash B_{2\delta}$.

The function $\bar{\sigma}_1$ is smooth and decay at infinity, and then it satisfies the hypothesis of lemma B.1 in [4]. Hence, the following bounds holds

$$\int_{\mathbb{R}^2_+} \left(\left({}^{(3)}\Delta \bar{\sigma}_1 \right)^2 + \left| \partial \bar{\sigma}_1 \right|^2 \right) \rho d\rho dz \geqslant C \sup_{\mathbb{R}^2_+} \left| \bar{\sigma}_1 \right|, \tag{53}$$

where C is a numerical constant independent of $\bar{\sigma}_1$ (see also equations (121) and (122) in [4] to handle the term with the Laplacian).

Since $\sigma_1 = \bar{\sigma}_1$ in $\Omega_{2\delta}$, we have

$$\sup_{\mathbb{R}^2_+} \left| \bar{\sigma}_1 \right| \geqslant \sup_{\Omega_{2\delta}} \left| \bar{\sigma}_1 \right| = \sup_{\Omega_{2\delta}} \left| \sigma_1 \right|. \tag{54}$$

That is, if we can bound the integral in the left-hand side of the inequality (53) by the energies m_2 and $\bar{m}_2$ then the desired estimate (10) follows. To bound this integral we proceed as follows.

We decompose the domain of integration $\mathbb{R}^2_+$ in (53) in three region $\mathbb{R}^2_+ = \Omega_{2\delta} + A_{2\delta} + B_\delta$, where $A_{2\delta} = B_{2\delta} \backslash B_\delta$.

Define the constant C_δ by

$$C_\delta = \min_{\Omega_\delta} \left\{ e^{-2\left(\sigma_0 + q_0\right)} \right\}. \tag{55}$$

For the region B_δ we have that, by construction, $\bar{\sigma}_1 = 0$ and hence the integral (53) is trivial in B_δ. For the region $\Omega_{2\delta}$ we have $\bar{\sigma}_1 = \sigma_1$. For the term with first derivatives we obtain

$$Cm_2 \geqslant \int_{\mathbb{R}^2_+} \left| \partial \sigma_1 \right|^2 \rho d\rho dz, \tag{56}$$

$$\geqslant \int_{\Omega_{2\delta}} \left| \partial \sigma_1 \right|^2 \rho d\rho dz, \tag{57}$$

$$= \int_{\Omega_{2\delta}} \left| \partial \bar{\sigma}_1 \right|^2 \rho d\rho dz. \tag{58}$$

Where in (56) we have used the bound (6). For the terms with the Laplacian we have

$$Cm_2 \geqslant \int_{\mathbb{R}^2_+} \mathrm{e}^{-2\left(q_0+\sigma_0\right)}\left({}^{(3)}\Delta\sigma_1\right)^2 \rho \mathrm{d}\rho \mathrm{d}z, \tag{59}$$

$$\geqslant \int_{\Omega_{2\delta}} \mathrm{e}^{-2\left(q_0+\sigma_0\right)}\left({}^{(3)}\Delta\sigma_1\right)^2 \rho \mathrm{d}\rho \mathrm{d}z, \tag{60}$$

$$\geqslant C_\delta \int_{\Omega_{2\delta}} \left({}^{(3)}\Delta\sigma_1\right)^2 \rho \mathrm{d}\rho \mathrm{d}z, \tag{61}$$

$$= C_\delta \int_{\Omega_{2\delta}} \left({}^{(3)}\Delta\bar{\sigma}_1\right)^2 \rho \mathrm{d}\rho \mathrm{d}z, \tag{62}$$

where in line (61) we have used the definition (55).

It remains only to bound the integral in the transition region $A_{2\delta}$. For the first derivatives we have

$$\partial\bar{\sigma}_1 = \left(1 - \chi_\delta\right)\partial\sigma_1 - \sigma_1\partial\chi_\delta, \tag{63}$$

and then we obtain the pointwise estimate

$$\left|\partial\bar{\sigma}_1\right|^2 \leqslant 2\left(1 - \chi_\delta\right)^2\left|\partial\sigma_1\right|^2 + 2\sigma_1^2\left|\partial\chi_\delta\right|^2, \tag{64}$$

$$\leqslant C_\delta\left(\left|\partial\sigma_1\right|^2 + \sigma_1^2\right), \tag{65}$$

where we have used that the derivatives of χ_δ are bounded by a constant C_δ that depends only on δ. Integrating (64) on $A_{2\delta}$ and using the bound (6) we obtain

$$C_\delta m_2 \geqslant \int_{A_{2\delta}} \left|\partial\bar{\sigma}_1\right|^2 \rho \mathrm{d}\rho \mathrm{d}z. \tag{66}$$

For the term with the Laplacian we proceed in a similar way, we have

$${}^{(3)}\Delta\bar{\sigma}_1 = \left(1 - \chi_\delta\right){}^{(3)}\Delta\sigma_1 - 2\partial\sigma_1\partial\chi_\delta - \sigma_1{}^{(3)}\Delta\chi_\delta. \tag{67}$$

Then we obtain

$$\left({}^{(3)}\Delta\bar{\sigma}_1\right)^2 \leqslant C_\delta\left(\left({}^{(3)}\Delta\sigma_1\right)^2 + \left|\partial\sigma_1\right|^2 + \sigma_1^2\right), \tag{68}$$

where we have used again that all derivatives of χ_δ are bounded by a constant C_δ . Integrating (68) on $A_{2\delta}$, using the definition (55) and the bound we finally obtain

$$C_\delta \int_{A_{2\delta}} \left({}^{(3)}\Delta\bar{\sigma}_1\right)^2 \rho \mathrm{d}\rho \mathrm{d}z \leqslant m_2, \tag{69}$$

and hence, collecting all the bounds, we have proved

$$C_\delta \int_{\mathbb{R}^2_+} \left(\left({}^{(3)}\Delta\bar{\sigma}_1\right)^2 + \left|\partial\bar{\sigma}_1\right|^2\right)\rho \mathrm{d}\rho \mathrm{d}z \leqslant m_2. \tag{70}$$

To obtain the bound (11) for $\bar{\omega}_1$ the argument is similar.

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Appendix A. Axially symmetric linear perturbations of the extreme Kerr black hole

In this appendix we summarize the some of the equations obtained in [4] for axially symmetric perturbations for the extreme Kerr black hole in the maximal-isothermal gauge. In this gauge, Einstein equations are naturally divided into three groups: evolution equations, constraint equations and gauge equations. The evolution equations are further divided into two groups, evolution equations for the dynamical degree of freedom (σ_1, ω_1) and evolution equations for the metric (which is determined by the function q_1) and second fundamental form χ_{AB}. In the proof of theorem 1.1 we use the evolution equations for (σ_1, ω_1) given by

$$-\mathrm{e}^{2\left(\sigma_0+q_0\right)}\dot{p} + {}^{(3)}\Delta\sigma_1 = \frac{2}{\eta_0^2}\left(\sigma_1\left|\partial\omega_0\right|^2 - \partial\omega_1\partial\omega_0\right), \tag{A.1}$$

$$-\mathrm{e}^{2\left(\sigma_0+q_0\right)}\dot{d} + {}^{(3)}\Delta\omega_1 = 4\frac{\partial_\rho\omega_1}{\rho} + 2\partial\omega_1\partial\sigma_0 + 2\partial\omega_0\partial\sigma_1, \tag{A.2}$$

with

$$p = \dot{\sigma}_1 - 2\frac{\beta_1^\rho}{\rho} - \beta_1^A\partial_A\sigma_0, \tag{A.3}$$

$$d = \dot{\omega}_1 - \beta_1^A\partial_A\omega_0. \tag{A.4}$$

In these equations the indices $A, B\cdots$ are two-dimensional, they have the values ρ, z and β_1^A represents the shift vector of the foliation. The crucial property of these equations is that there exists a conserved mass given by

$$m_2 = \frac{1}{16}\int_{\mathbb{R}_+^2} \epsilon_2 \, d\rho dz, \tag{A.5}$$

where the positive definite energy density ϵ_2 is given by

$$\frac{\epsilon_2}{\rho} = 2\mathrm{e}^{2\left(\sigma_0+q_0\right)}p^2 + 2\frac{\mathrm{e}^{2\left(\sigma_0+q_0\right)}}{\eta_0^2}d^2 + 4\mathrm{e}^{-2u_0}\chi_1^{\mathrm{AB}}\chi_{1\mathrm{AB}} + \left(\partial\sigma_1 + \omega_1\eta_0^{-2}\partial\omega_0\right)^2$$
$$+ \left(\partial\left(\omega_1\eta_0^{-1}\right) - \eta_0^{-1}\sigma_1\partial\omega_0\right)^2 + \left(\eta_0^{-1}\sigma_1\partial\omega_0 - \omega_1\eta_0^{-2}\partial\eta_0\right)^2. \tag{A.6}$$

See [4] for the proof. For completeness, we have written the explicit expressions (A.3)–(A.4) for the functions p and d which involve the shift vector. However, we not make use of them in this article. The important point is that the functions p and d appear in the energy density (A.6).

The higher order mass is given by

$$\bar{m}_2 = \frac{1}{16}\int_{\mathbb{R}_+^2} \bar{\epsilon}_2 \, d\rho dz, \tag{A.7}$$

with energy density $\bar{\varepsilon}_2$ is given by

$$\frac{\bar{\varepsilon}_2}{\rho} = 2e^{2\left(\sigma_0+q_0\right)}\dot{p}^2 + 2\frac{e^{2\left(\sigma_0+q_0\right)}}{\eta_0^2}\dot{d}^2 + 4e^{-2u_0}\dot{\chi}_1^{AB}\dot{\chi}_{1AB} + \left(\partial\dot{\sigma}_1 + \dot{\omega}_1\eta_0^{-2}\partial\omega_0\right)^2$$
$$+ \left(\partial\left(\dot{\omega}_1\eta_0^{-1}\right) - \eta_0^{-1}\dot{\sigma}_1\partial\omega_0\right)^2 + \left(\eta_0^{-1}\dot{\sigma}_1\partial\omega_0 - \dot{\omega}_1\eta_0^{-2}\partial\eta_0\right)^2. \tag{A.8}$$

Appendix B. Extreme Kerr black hole

The extreme Kerr black hole solution depends only on one parameter m_0, which represents the total mass of the black hole. In the maximal-isothermal gauge, the relevant functions used in this article associated with this solution are: the square norm and the twist of the axial Killing vector denoted by η_0 and ω_0 respectively and the function q_0 which determines the intrinsic metric of the t = constant slices of the foliation. The function σ_0 is calculated from η_0 by equation (4). For the explicit expression for these functions and further details see appendix A in [4]. In this article we will only use the following properties of these functions.

They satisfy the stationary equations

$$^{(3)}\Delta\sigma_0 = -\frac{|\partial\omega_0|^2}{\eta_0^2}, \tag{B.1}$$

$$\partial^A\left(\frac{\rho\partial_A\omega_0}{\eta_0^2}\right) = 0. \tag{B.2}$$

Note that equation (B.1) is equivalent to

$$^{(3)}\Delta\left(\ln\eta_0\right) = -\frac{\left|\partial\omega_0\right|^2}{\eta_0^2}, \tag{B.3}$$

where we have used equation (4) and

$$^{(3)}\Delta(\ln\rho) = 0. \tag{B.4}$$

They satisfies the following elementary inequalities in $\mathbb{R}^2_+$

$$\frac{\left|\partial\omega_0\right|^2}{\eta_0^2} \leqslant \frac{C}{r^2}, \tag{B.5}$$

$$\left|\partial\sigma_0\right|^2 \leqslant \frac{C}{r^2}, \tag{B.6}$$

$$e^{-2\left(\sigma_0+q_0\right)}\frac{\left|\partial\omega_0\right|^2}{\eta_0^2} \leqslant C, \tag{B.7}$$

$$e^{-2\left(\sigma_0+q_0\right)}\left|\partial\sigma_0\right|^2 \leqslant C, \tag{B.8}$$

where the positive constant C depends only on m_0.

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