

Existence, uniqueness and convergence of simultaneous  
distributed-boundary optimal control problems\*

by

Claudia M. Gariboldi<sup>1</sup>, Domingo A. Tarzia<sup>2</sup>

<sup>1</sup>Departamento de Matemática, FCEFQyN, Universidade Nacional de Río  
Cuarto, Ruta 36 Km 601, 5800 Río Cuarto, Argentina  
cgariboldi@exa.unrc.edu.ar

<sup>2</sup>Departamento de Matemática-CONICET, FCE, Universidade Austral,  
Paraguay 1950, S2000FZF Rosario, Argentina  
DTarzia@austral.edu.ar

**Abstract:** We consider a steady-state heat conduction problem  $P$  for the Poisson equation with mixed boundary conditions in a bounded multidimensional domain  $\Omega$ . We also consider a family of problems  $P_\alpha$  for the same Poisson equation with mixed boundary conditions,  $\alpha > 0$  being the heat transfer coefficient defined on a portion  $\Gamma_1$  of the boundary. We formulate simultaneous *distributed and Neumann boundary* optimal control problems on the internal energy  $g$  within  $\Omega$  and the heat flux  $q$ , defined on the complementary portion  $\Gamma_2$  of the boundary of  $\Omega$  for quadratic cost functional. Here, the control variable is the vector  $(g, q)$ . We prove existence and uniqueness of the optimal control  $(\bar{g}, \bar{q})$  for the system state of  $P$ , and  $(\bar{g}_\alpha, \bar{q}_\alpha)$  for the system state of  $P_\alpha$ , for each  $\alpha > 0$ , and we give the corresponding optimality conditions. We prove strong convergence, in suitable Sobolev spaces, of the vectorial optimal controls, system and adjoint states governed by the problems  $P_\alpha$  to the corresponding vectorial optimal control, system and adjoint states governed by the problem  $P$ , when the parameter  $\alpha$  goes to infinity. We also obtain estimations between the solutions of these vectorial optimal control problems and the solution of two scalar optimal control problems characterized by fixed  $g$  (with boundary optimal control  $\bar{q}$ ) and fixed  $q$  (with distributed optimal control  $\bar{g}$ ), respectively, for cases both of  $\alpha > 0$  and  $\alpha = \infty$ .

**Keywords:** simultaneous optimal control problems, mixed elliptic problems, optimality condition, elliptic variational equalities, vectorial optimal control problems

---

\*Submitted: February 2014; Accepted: March 2015

## 1. Introduction

We consider a bounded domain  $\Omega$  in  $\mathbb{R}^n$ , whose regular boundary  $\Gamma$  consist of the union of the two disjoint portions  $\Gamma_1$  and  $\Gamma_2$  with  $med(\Gamma_i) > 0$  for  $i = 1, 2$ . We consider the following steady-state heat conduction problems  $P$  and  $P_\alpha$  (for each parameter value  $\alpha > 0$ ) respectively, with mixed boundary conditions:

$$-\Delta u = g \text{ in } \Omega \quad u|_{\Gamma_1} = b \quad -\frac{\partial u}{\partial n}|_{\Gamma_2} = q \quad (1)$$

$$-\Delta u = g \text{ in } \Omega \quad -\frac{\partial u}{\partial n}|_{\Gamma_1} = \alpha(u - b) \quad -\frac{\partial u}{\partial n}|_{\Gamma_2} = q \quad (2)$$

where  $g$  is the internal energy in  $\Omega$ ,  $b > 0$  is the temperature on  $\Gamma_1$  for (1) and the temperature of the external neighborhood of  $\Gamma_1$  for (2),  $q$  is the heat flux on  $\Gamma_2$  and  $\alpha > 0$  is the heat transfer coefficient on  $\Gamma_1$  (Newton law or Robin condition on  $\Gamma_1$ ). The following hypothesis:  $g \in L^2(\Omega)$ ,  $q \in L^2(\Gamma_2)$ , and  $b \in H^{\frac{1}{2}}(\Gamma_1)$  is assumed to hold. Problems (1) and (2) can be considered as the steady-state Stefan problem for suitable data  $q$ ,  $g$  and  $b$ , see Tarzia (1979), or Tabacman and Tarzia (1989).

We denote by  $u_{(g,q)}$  and  $u_{(\alpha,g,q)}$  the unique solutions of the elliptic problems (1) and (2), respectively, whose variational formulations are given, as in Kinderlehrer and Stampacchia (1980), by:

$$a(u_{(g,q)}, v) = L_{(g,q)}(v), \quad \forall v \in V_0, \quad u_{(g,q)} \in K \quad (3)$$

$$a_\alpha(u_{(\alpha,g,q)}, v) = L_{(\alpha,g,q)}(v), \quad \forall v \in V, \quad u_{(\alpha,g,q)} \in V \quad (4)$$

where

$$V = H^1(\Omega), \quad V_0 = \{v \in V / v|_{\Gamma_1} = 0\}, \quad K = v_0 + V_0,$$

$$R = L^2(\Gamma_1), \quad H = L^2(\Omega), \quad Q = L^2(\Gamma_2)$$

for  $v_0 \in V$  given, with  $v_0|_{\Gamma_1} = b$  and

$$(g, h)_H = \int_{\Omega} gh \, dx; \quad (q, \eta)_Q = \int_{\Gamma_2} q\eta \, d\gamma, \quad (b, v)_R = \int_{\Gamma_1} bv \, d\gamma$$

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx; \quad a_\alpha(u, v) = a(u, v) + \alpha (u, v)_R$$

$$L_{(g,q)}(v) = (g, v)_H - (q, v)_Q; \quad L_{(\alpha,g,q)}(v) = L_{(g,q)}(v) + \alpha (b, v)_R.$$

The bilinear form  $a$  is coercive on  $V_0$ , with coerciveness constant  $\lambda > 0$  and the bilinear form  $a_\alpha$  is coercive on  $V$  with coerciveness constant  $\lambda_\alpha = \lambda_1 \min(1, \alpha)$ , where  $\lambda_1 > 0$  is the coerciveness constant for the bilinear form  $a_1$ , see Kinderlehrer and Stampacchia (1980), or Tabacman and Tarzia (1989).

We formulate the following simultaneous *distributed and Neumann boundary* optimal control problems, see Lions (1968) or Tröltzsch (2010):

$$\text{Find } (\bar{g}, \bar{q}) \in H \times U_{ad} \quad \text{such that} \quad J(\bar{g}, \bar{q}) = \min_{g \in H, q \in U_{ad}} J(g, q) \quad (5)$$

$$\text{Find } (\bar{g}_\alpha, \bar{q}_\alpha) \in H \times U_{ad} \quad \text{such that} \quad J_\alpha(\bar{g}_\alpha, \bar{q}_\alpha) = \min_{g \in H, q \in U_{ad}} J_\alpha(g, q) \quad (6)$$

with  $U_{ad} = \{q \in Q : q \geq 0 \text{ on } \Gamma_2\}$  and the cost functionals  $J : H \times Q \rightarrow \mathbb{R}_0^+$  and  $J_\alpha : H \times Q \rightarrow \mathbb{R}_0^+$  being given by:

$$J(g, q) = \frac{1}{2} \|u_{(g,q)} - z_d\|_H^2 + \frac{M_1}{2} \|g\|_H^2 + \frac{M_2}{2} \|q\|_Q^2 \quad (7)$$

$$J_\alpha(g, q) = \frac{1}{2} \|u_{(\alpha,g,q)} - z_d\|_H^2 + \frac{M_1}{2} \|g\|_H^2 + \frac{M_2}{2} \|q\|_Q^2 \quad (8)$$

where  $z_d \in H$ ,  $u_{(g,q)}$  and  $u_{(\alpha,g,q)}$  are the unique solutions of the elliptic variational equalities (3) and (4) respectively, and the positive constants  $M_1$  and  $M_2$  are given. We remark that we denote the control variables by  $g$  and  $q$ , these two variables corresponding usually to the internal energy and the heat flux respectively, in heat transfer problems.

The use of the variational equality theory in connection with optimization and optimal control problems was done in Belgacem, El Fekih and Metoui (2003), Bensoussan (1974), Casas (1986), Casas and Raymond (2006), Kirchner, Meidner and Vexler (2011), Mignot and Puel (1984).

In Section 2, we obtain the existence and uniqueness of the vectorial optimal control  $(\bar{g}, \bar{q})$  of the problem (5) and of the vectorial optimal control  $(\bar{g}_\alpha, \bar{q}_\alpha)$  of the problem (6), for each  $\alpha > 0$ . We also give the optimality conditions in relation to the adjoint state  $p_{(\bar{g}, \bar{q})}$  for (5) and  $p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}$  for (6).

In Section 3, we obtain estimations between the first component of the simultaneous optimal control  $\bar{g}$  and the scalar optimal control  $\bar{g}$  studied in Gariboldi and Tarzia (2003) (see the optimization problem (18)), and the second component of the simultaneous optimal control  $\bar{q}$  and the scalar optimal control  $\bar{q}$  studied in Gariboldi and Tarzia (2008) (see the optimization problem (19)). In the optimal control problems (5) and (6) we have considered two control variables simultaneously, that is, the solution is a vectorial optimal control, while in the optimal control problems, given in Gariboldi and Tarzia (2003) and Gariboldi and Tarzia (2008), respectively, we have considered only one control variable, namely the solutions are scalar optimal controls. Moreover, we characterize the optimal control  $(\bar{g}, \bar{q})$  as a fixed point on  $H \times Q$  for a suitable operator  $W$ . In a similar way, we obtain estimations for the optimal controls of the problems  $P_\alpha$ , for each  $\alpha > 0$ , and we characterize the optimal control  $(\bar{g}_\alpha, \bar{q}_\alpha)$  as a fixed point on  $H \times Q$  for a suitable operator  $W_\alpha$ .

In Section 4, we prove the strong convergence, in suitable Sobolev spaces, of the optimal controls  $(\bar{g}_\alpha, \bar{q}_\alpha)$  of the problems (6) to the optimal control  $(\bar{g}, \bar{q})$  of the problem (5), of the system states  $u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}$  to the system state  $u_{(\bar{g}, \bar{q})}$ , and

of the adjoint states  $p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}$  to the adjoint state  $p_{(\bar{g}, \bar{q})}$ , when the parameter  $\alpha$  goes to infinity. We also prove the convergence of the corresponding cost functional when  $\alpha$  goes to infinity.

This asymptotic behavior can be considered to be very important in the optimal control of heat transfer problems, because the Dirichlet boundary condition, given in (1), can be approximated by the relevant physical condition given by the Newton law or the Robin boundary condition given in (2), see Carslaw and Jaeger (1959). Therefore, the goal of this paper is to approximate a Dirichlet boundary condition in a vectorial optimal control problem, governed by an elliptic variational equality, by a Robin boundary condition in a family of vectorial optimal control problems, governed also by elliptic variational equalities, for a large positive coefficient  $\alpha$ . As the particular cases of our results can be considered the ones given in Gariboldi and Tarzia (2003) when the scalar control variable is the internal energy  $g$  for both state systems (1) and (2), and in Gariboldi and Tarzia (2008) when the scalar control variable is the heat flux  $q$  on the boundary  $\Gamma_2$  for both state systems (1) and (2). In Belgacem, El Fekih and Metoui (2003) the control variable is the temperature  $b$  on the boundary  $\Gamma_1$  for the state system (1), and the temperature of the external neighborhood  $b$  on the boundary  $\Gamma_1$  for the state systems (2), this being essentially different with respect to the present vectorial optimal control problems.

## 2. Existence and uniqueness of optimal controls

### 2.1. Problem $P$ and its optimal control problem

Let  $C : H \times Q \rightarrow V_0$  be the application defined by  $C(g, q) = u_{(g, q)} - u_{(0, 0)}$ , where  $u_{(0, 0)}$  is the solution of the problem (1) for  $g = 0$  and  $q = 0$ . We define, in the way similar to that in Gariboldi and Tarzia (2003, 2008), and Lions (1968), the applications  $\Pi : (H \times Q) \times (H \times Q) \rightarrow \mathbb{R}$ , and  $L : H \times Q \rightarrow \mathbb{R}$  as follows:

$$\Pi((g, q), (h, \eta)) = (C(g, q), C(h, \eta))_H + M_1(g, h)_H + M_2(q, \eta)_Q$$

$$L((g, q)) = (C(g, q), z_d - u_{(0, 0)})_H, \quad \forall (g, q), (h, \eta) \in H \times Q.$$

For each  $(g, q) \in H \times Q$ , we define the adjoint state  $p_{(g, q)}$  corresponding to the problem (1), as the unique solution of the mixed elliptic problem

$$-\Delta p = u - z_d \text{ in } \Omega, \quad p|_{\Gamma_1} = 0, \quad \frac{\partial p}{\partial n}|_{\Gamma_2} = 0 \quad (9)$$

whose variational formulation is given by

$$a(p_{(g, q)}, v) = (u_{(g, q)} - z_d, v)_H, \quad \forall v \in V_0, \quad p_{(g, q)} \in V_0. \quad (10)$$

and we have the following properties.

**THEOREM 1** *a)  $J$  is a coercive and strictly convex functional on  $H \times Q$ .*

b) The adjoint state  $p_{(g,q)}$  satisfy,  $\forall (h, \eta) \in H \times Q$ :

$$a(p_{(g,q)}, C(h, \eta)) = (C(h, \eta), u_{(g,q)} - z_d)_H = (h, p_{(g,q)})_H - (\eta, p_{(g,q)})_Q$$

c)  $J$  is Gâteaux differentiable and  $J'$  is given by,  $\forall (h, \eta) \in H \times Q$ :

$$J'(g, q)(h - g, \eta - q) = \Pi((g, q), (h - g, \eta - q)) - L(h - g, \eta - q) \quad (11)$$

d) There exists a unique solution  $(\bar{g}, \bar{q}) \in H \times U_{ad}$  of the vectorial optimal control problem (5) and its optimality condition is given by,  $\forall (h, \eta) \in H \times U_{ad}$ :

$$(h - \bar{g}, p_{(\bar{g}, \bar{q})} + M_1 \bar{g})_H + (\eta - \bar{q}, M_2 \bar{q} - p_{(\bar{g}, \bar{q})})_Q \geq 0.$$

PROOF (a) It is sufficient to prove that,  $\forall (g_2, q_2), (g_1, q_1) \in H \times Q$  and  $\forall t \in [0, 1]$ , we have, see Lions (1968), Boukrouche and Tarzia (2007), or Tröltzsch (2010):

$$u_{((1-t)g_2+tg_1, (1-t)q_2+tq_1)} = (1-t)u_{(g_2, q_2)} + tu_{(g_1, q_1)} \quad (12)$$

and

$$\begin{aligned} & (1-t)J(g_2, q_2) + tJ(g_1, q_1) - J((1-t)(g_2, q_2) + t(g_1, q_1)) = \\ &= \frac{t(1-t)}{2} [\|u_{(g_2, q_2)} - u_{(g_1, q_1)}\|_H^2 + M_1 \|g_2 - g_1\|_H^2 + M_2 \|q_2 - q_1\|_Q^2] \geq \\ &\geq \frac{Mt(1-t)}{2} \|(g_2 - g_1, q_2 - q_1)\|_{H \times Q}^2, \end{aligned} \quad (13)$$

and

$$\begin{aligned} a(p_{(g,q)}, C(h, \eta)) &= (-\Delta p_{(g,q)}, u_{(g,q)} - u_{(0,0)})_H = \\ &= (h, p_{(g,q)})_H - (\eta, p_{(g,q)})_Q \end{aligned} \quad (14)$$

where

$$\|(g, q)\|_{H \times Q}^2 = \|g\|_H^2 + \|q\|_Q^2, \quad M = \min(M_1, M_2) > 0. \quad \square$$

## 2.2. Problem $P_\alpha$ and its optimal control problem

Let  $C_\alpha : H \times Q \rightarrow V$  be the application defined by  $C_\alpha(g, q) = u_{(\alpha, g, q)} - u_{(\alpha, 0, 0)}$ , where  $u_{(\alpha, 0, 0)}$  is the solution of the problem (2) for  $g = 0$  and  $q = 0$ . We define the applications  $\Pi_\alpha : (H \times Q) \times (H \times Q) \rightarrow \mathbb{R}$  and  $L_\alpha : H \times Q \rightarrow \mathbb{R}$  by the following expressions,  $\forall (g, q), (h, \eta) \in H \times Q$ :

$$\Pi_\alpha((g, q), (h, \eta)) = (C_\alpha(g, q), C_\alpha(h, \eta))_H + M_1(g, h)_H + M_2(q, \eta)_Q,$$

$$L_\alpha(q) = (C_\alpha(g, q), z_d - u_{(\alpha, 0, 0)})_H.$$

For each  $(g, q) \in H \times Q$  and  $\alpha > 0$ , we define the adjoint state  $p_{(\alpha, g, q)}$ , corresponding to the problem (2), as the unique solution of the mixed elliptic problem

$$-\Delta p = u - z_d \text{ in } \Omega, \quad -\frac{\partial p}{\partial n}\Big|_{\Gamma_1} = \alpha p, \quad \frac{\partial p}{\partial n}\Big|_{\Gamma_2} = 0, \quad (15)$$

whose variational formulation is given by

$$a_\alpha(p_{(\alpha, g, q)}, v) = (u_{(\alpha, g, q)} - z_d, v)_H, \quad \forall v \in V, \quad p_{(\alpha, g, q)} \in V. \quad (16)$$

We can obtain similar properties to Theorem 2.1, following Boukrouche and Tarzia (2007), Kinderlehrer and Stampacchia (1980), Lions (1968), or Tröltzsch (2010).

**THEOREM 2** *We have, for each  $\alpha > 0$ , the following properties:*

- a)  $J_\alpha$  is a coercive and strictly convex functional on  $H \times Q$ .
- b) The adjoint state  $p_{(\alpha, g, q)}$  satisfies,  $\forall (h, \eta) \in H \times Q$ :

$$a_\alpha(p_{(\alpha, g, q)}, C_\alpha(h, \eta)) = (C_\alpha(h, \eta), u_{(\alpha, g, q)} - z_d)_H = (h, p_{(\alpha, g, q)})_H - (\eta, p_{(\alpha, g, q)})_Q.$$

- c)  $J_\alpha$  is Gâteaux differentiable and  $J'_\alpha$  is given by,  $\forall (h, \eta) \in H \times Q$ :

$$J'_\alpha(g, q)(h - g, \eta - q) = \Pi_\alpha((g, q), (h - g, \eta - q)) - L_\alpha(h - g, \eta - q). \quad (17)$$

- d) There exists a unique solution  $(\bar{g}_\alpha, \bar{q}_\alpha) \in H \times U_{ad}$  of the vectorial optimal control problem (6) and its optimality condition is given by,  $\forall (h, \eta) \in H \times U_{ad}$ :

$$(h - \bar{g}_\alpha, p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} + M_1 \bar{g}_\alpha)_H + (\eta - \bar{q}_\alpha, M_2 \bar{q}_\alpha - p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)})_Q \geq 0.$$

### 3. Estimations

#### 3.1. Estimations with respect to the problem P

We consider the scalar distributed optimal control problem:

$$\text{Find } \bar{g} \in H \quad \text{such that} \quad J_1(\bar{g}) = \min_{g \in H} J_1(g), \quad \text{for fixed } q \in Q, \quad (18)$$

and the scalar Neumann boundary optimal control problem:

$$\text{Find } \bar{q} \in U_{ad} \quad \text{such that} \quad J_2(\bar{q}) = \min_{q \in U_{ad}} J_2(q), \quad \text{for fixed } g \in H, \quad (19)$$

where  $J_1$  is the cost functional given in Gariboldi and Tarzia (2003) plus the constant  $\frac{M_2}{2} \|q\|_Q^2$ , and  $J_2$  is the functional given in Gariboldi and Tarzia (2008) plus the constant  $\frac{M_1}{2} \|g\|_H^2$ , that is,  $J_1 : H \rightarrow \mathbb{R}_0^+$  and  $J_2 : Q \rightarrow \mathbb{R}_0^+$ , are given by:

$$J_1(g) = \frac{1}{2} \|u_g - z_d\|_H^2 + \frac{M_1}{2} \|g\|_H^2 + \frac{M_2}{2} \|q\|_Q^2, \quad (\text{fixed } q \in Q) \quad (20)$$

$$J_2(q) = \frac{1}{2} \|u_q - z_d\|_H^2 + \frac{M_2}{2} \|q\|_Q^2 + \frac{M_1}{2} \|g\|_H^2, \quad (\text{fixed } g \in H) \quad (21)$$

where  $u_g$  and  $u_q$  are the unique solutions of the problem (1) for fixed  $q$  and  $g$  data, respectively.

REMARK 1 *The functionals  $J$ ,  $J_1$  and  $J_2$  satisfy the elemental estimations*

$$J(\bar{g}, \bar{q}) \leq J_1(\bar{g}), \quad \forall q \in Q \quad \text{and} \quad J(\bar{g}, \bar{q}) \leq J_2(\bar{q}), \quad \forall g \in H.$$

In the next theorem we will obtain estimations between the solution of the scalar distributed optimal control problem (18) with the first component of the solution of the vectorial distributed and Neumann boundary optimal control problem (5), and between the solution of the scalar Neumann boundary optimal control problem (19) with the second component of the solution of the vectorial distributed and Neumann boundary optimal control problem (5).

THEOREM 3 *If  $(\bar{g}, \bar{q}) \in H \times U_{ad}$  is the unique solution of the vectorial optimal control problem (5), and  $\bar{g}$  and  $\bar{q}$  are the unique solutions of the scalar optimal control problems (18) and (19) respectively, then:*

$$\|\bar{q} - \bar{q}\|_Q \leq \frac{\|\gamma_0\|}{\lambda M_2} \|u_{(\bar{g}, \bar{q})} - u_{(\bar{g}, \bar{q})}\|_H \quad (22)$$

$$\|\bar{g} - \bar{g}\|_H \leq \frac{1}{\lambda M_1} \|u_{(\bar{g}, \bar{q})} - u_{(\bar{g}, \bar{q})}\|_H, \quad (23)$$

where  $\gamma_0$  is the trace operator.

PROOF For  $g = \bar{g}$ , from the optimality condition for  $\bar{q}$ , see Gariboldi and Tarzia (2008), we have

$$(M_2 \bar{q} - p_{(\bar{g}, \bar{q})}, \eta - \bar{q})_Q \geq 0, \quad \forall \eta \in U_{ad}. \quad (24)$$

If we take  $h = \bar{g} \in H$  in the optimality condition for  $(\bar{g}, \bar{q})$ , we obtain

$$(M_2 \bar{q} - p_{(\bar{g}, \bar{q})}, \eta - \bar{q})_Q \geq 0, \quad \forall \eta \in U_{ad}. \quad (25)$$

Now, taking  $\eta = \bar{q} \in U_{ad}$  in (24) and  $\eta = \bar{q} \in U_{ad}$  in (25), we obtain

$$(M_2(\bar{q} - \bar{q}) + (p_{(\bar{g}, \bar{q})} - p_{(\bar{g}, \bar{q})}), \bar{q} - \bar{q})_Q \geq 0,$$

and by using  $\|p_{(\bar{g}, \bar{q})} - p_{(\bar{g}, \bar{q})}\|_V \leq \frac{1}{\lambda} \|u_{(\bar{g}, \bar{q})} - u_{(\bar{g}, \bar{q})}\|_H$  we deduce

$$\|\bar{q} - \bar{q}\|_Q \leq \frac{\|\gamma_0\|}{M_2} \|p_{(\bar{g}, \bar{q})} - p_{(\bar{g}, \bar{q})}\|_V \leq \frac{\|\gamma_0\|}{\lambda M_2} \|u_{(\bar{g}, \bar{q})} - u_{(\bar{g}, \bar{q})}\|_H;$$

therefore the estimation (22) holds. Similarly, the estimation (23) holds.  $\square$

When we consider the vectorial distributed and Neumann boundary optimal control problem (5) without restrictions, i.e.  $U_{ad} = Q$ , then we can characterize the solution of (5) by using the fixed point theory.

Let  $W : H \times Q \rightarrow H \times Q$  be the operator defined by

$$W(g, q) = \left(-\frac{1}{M_1} p_{(g, q)}, \frac{1}{M_2} p_{(g, q)}\right). \quad (26)$$

THEOREM 4 *There exists a positive constant  $C_0 = C_0(\lambda, \gamma_0, M_1, M_2)$  such that,  $\forall (g_1, q_1), (g_2, q_2) \in H \times Q$ :*

$$\|W(g_2, q_2) - W(g_1, q_1)\|_{H \times Q} \leq C_0 \|(g_2, q_2) - (g_1, q_1)\|_{H \times Q} \quad (27)$$

and  $W$  is a contraction operator if and only if data satisfy the following condition:

$$C_0 = \frac{\sqrt{2}}{\lambda^2} \sqrt{\frac{1}{M_1^2} + \frac{\|\gamma_0\|^2}{M_2^2}} (1 + \|\gamma_0\|) < 1. \quad (28)$$

PROOF By using the estimations,  $\forall (g_1, q_1), (g_2, q_2) \in H \times Q$ :

$$\|u_{(g_1, q_1)} - u_{(g_2, q_2)}\|_V \leq \frac{1}{\lambda} (\|g_2 - g_1\|_H + \|\gamma_0\| \|q_2 - q_1\|_Q), \quad (29)$$

$$\|p_{(g_1, q_1)} - p_{(g_2, q_2)}\|_V \leq \frac{1}{\lambda} \|u_{(g_1, q_1)} - u_{(g_2, q_2)}\|_H \quad (30)$$

we obtain

$$\|W(g_2, q_2) - W(g_1, q_1)\|_{H \times Q}^2 \leq \left( \frac{1}{M_1^2} + \frac{\|\gamma_0\|^2}{M_2^2} \right) \frac{1}{\lambda^4} [\|g_2 - g_1\|_H + \|\gamma_0\| \|q_2 - q_1\|_Q]^2.$$

Then (27) holds and the operator  $W$  is a contraction if and only if data satisfy inequality (28).  $\square$

COROLLARY 1 *If data satisfy inequality (28) then the unique solution  $(\bar{g}, \bar{q}) \in H \times Q$  of the vectorial optimal control problem (5) can be obtained as the unique fixed point of the operator  $W$ , that is:*

$$W(\bar{g}, \bar{q}) = \left( -\frac{1}{M_1} p_{(\bar{g}, \bar{q})}, \frac{1}{M_2} p_{(\bar{g}, \bar{q})} \right) = (\bar{g}, \bar{q}).$$

### 3.2. Estimations with respect to the problem $P_\alpha$

For each  $\alpha > 0$ , we consider the scalar optimal control problems:

$$\text{Find } \bar{g}_\alpha \in H \quad \text{such that} \quad J_{1\alpha}(\bar{g}_\alpha) = \min_{g \in H} J_{1\alpha}(g), \quad (31)$$

$$\text{Find } \bar{q}_\alpha \in U_{ad} \quad \text{such that} \quad J_{2\alpha}(\bar{q}_\alpha) = \min_{q \in U_{ad}} J_{2\alpha}(q), \quad (32)$$

where  $J_{1\alpha} : H \rightarrow \mathbb{R}_0^+$  and  $J_{2\alpha} : Q \rightarrow \mathbb{R}_0^+$  are given by:

$$J_{1\alpha}(g) = \frac{1}{2} \|u_{\alpha g} - z_d\|_H^2 + \frac{M_1}{2} \|g\|_H^2 + \frac{M_2}{2} \|q\|_Q^2, \quad (\text{fixed } q \in Q) \quad (33)$$

$$J_{2\alpha}(q) = \frac{1}{2} \|u_{\alpha q} - z_d\|_H^2 + \frac{M_2}{2} \|q\|_Q^2 + \frac{M_1}{2} \|g\|_H^2, \quad (\text{fixed } g \in H) \quad (34)$$

where  $J_{1\alpha}$  is the functional studied in Gariboldi and Tarzia (2003) plus the constant  $\frac{M_2}{2} \|q\|_Q^2$ ,  $J_{2\alpha}$  is the functional studied in Gariboldi and Tarzia (2008) plus the constant  $\frac{M_1}{2} \|g\|_H^2$ , and the system states  $u_{\alpha g}$  and  $u_{\alpha q}$  are the unique solutions of the problem (2) for fixed data  $q$  and  $g$ , respectively.



REMARK 2 *The functionals  $J_\alpha$ ,  $J_{1\alpha}$  and  $J_{2\alpha}$  satisfy the estimations*

$$J_\alpha(\bar{g}_\alpha, \bar{q}_\alpha) \leq J_{1\alpha}(\bar{g}_\alpha), \forall q \in Q \text{ and } J_\alpha(\bar{g}_\alpha, \bar{q}_\alpha) \leq J_{2\alpha}(\bar{q}_\alpha), \forall g \in H.$$

Estimations between the solution of the scalar distributed optimal control problem (31) with respect to the first component of the solution of the vectorial distributed and Neumann boundary optimal control problem (6), and estimations between the solution of the scalar Neumann boundary optimal control problem (32) with respect to the second component of the solution of the vectorial distributed and Neumann boundary optimal control problem (6) are given in the next theorem whose proof is omitted.

THEOREM 5 *If  $(\bar{g}_\alpha, \bar{q}_\alpha) \in H \times U_{ad}$  is the unique solution of the vectorial optimal control problem (6), and  $\bar{g}_\alpha$  and  $\bar{q}_\alpha$  are the unique solutions of the scalar optimal control problems (31) and (32), respectively, then we have the following estimations*

$$\begin{aligned} \|\bar{q}_\alpha - \bar{q}_\alpha\|_Q &\leq \frac{\|\gamma_0\|}{\lambda M_2} \|u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}\|_H \\ \|\bar{g}_\alpha - \bar{g}_\alpha\|_H &\leq \frac{1}{\lambda M_1} \|u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}\|_H. \end{aligned}$$

In the way similar to Theorem 4, we can now characterize the solution of the vectorial distributed and Neumann boundary optimal control problem (6), without restrictions, proving that a suitable operator  $W_\alpha$  is a contraction. It is presented in the next theorem and the proof is omitted. We define the operator  $W_\alpha : H \times Q \rightarrow H \times Q$ , for each  $\alpha > 0$ , by the expression

$$W_\alpha(g, q) = \left(-\frac{1}{M_1} p(\alpha, g, q), \frac{1}{M_2} p(\alpha, g, q)\right). \quad (35)$$

THEOREM 6  *$W_\alpha$  is a Lipschitz operator over  $H \times Q$ , that is, there exists a positive constant  $C_{0\alpha} = C_{0\alpha}(\lambda_\alpha, \gamma_0, M_1, M_2)$ , such that:*

$$\|W_\alpha(g_2, q_2) - W_\alpha(g_1, q_1)\|_{H \times Q} \leq C_{0\alpha} \|(g_2 - g_1, q_2 - q_1)\|_{H \times Q} \quad (36)$$

and  $W_\alpha$  is a contraction operator if and only if data satisfy the following inequality:

$$C_{0\alpha} = \frac{\sqrt{2}}{\lambda_\alpha^2} \sqrt{\frac{1}{M_1^2} + \frac{\|\gamma_0\|^2}{M_2^2}} (1 + \|\gamma_0\|) < 1. \quad (37)$$

COROLLARY 2 *If data satisfy inequality  $C_{0\alpha} < 1$ , then the unique solution  $(\bar{g}_\alpha, \bar{q}_\alpha) \in H \times Q$  of the vectorial optimal control problem (6) can be obtained as the unique fixed point of the operator  $W_\alpha$ , that is:*

$$W_\alpha(\bar{g}_\alpha, \bar{q}_\alpha) = \left(-\frac{1}{M_1} p(\alpha, \bar{g}_\alpha, \bar{q}_\alpha), \frac{1}{M_2} p(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)\right) = (\bar{g}_\alpha, \bar{q}_\alpha).$$

#### 4. Convergence when $\alpha \rightarrow +\infty$

LEMMA 1 *For each  $\alpha > 0$ ,  $(g, q) \in H \times Q$ ,  $b \in H^{1/2}(\Gamma_1)$ , we have the following limits:*

$$i) \lim_{\alpha \rightarrow \infty} \|u_{(\alpha, g, q)} - u_{(g, q)}\|_V = 0 \quad ii) \lim_{\alpha \rightarrow \infty} \|p_{(\alpha, g, q)} - p_{(g, q)}\|_V = 0. \quad (38)$$

PROOF We proceed in a similar way to the one given in Gariboldi and Tarzia (2003) and in Gariboldi and Tarzia (2008).

THEOREM 7 *i) If  $u_{(\bar{g}, \bar{q})}$  and  $u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}$  are the unique system states, corresponding of the vectorial optimal control problems (5) and (6), respectively, then:*

$$\lim_{\alpha \rightarrow \infty} \|u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\bar{g}, \bar{q})}\|_V = 0. \quad (39)$$

*ii) If  $p_{(\bar{g}, \bar{q})}$  and  $p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}$  are the unique adjoint states, corresponding to the vectorial optimal control problems (5) and (6), respectively, then:*

$$\lim_{\alpha \rightarrow \infty} \|p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})}\|_V = 0. \quad (40)$$

*iii) If  $(\bar{g}, \bar{q})$  and  $(\bar{g}_\alpha, \bar{q}_\alpha)$  are the unique solutions of the simultaneous distributed and Neumann boundary optimal control problems (5) and (6), respectively, then:*

$$\lim_{\alpha \rightarrow \infty} \|(\bar{g}_\alpha, \bar{q}_\alpha) - (\bar{g}, \bar{q})\|_{H \times Q} = 0. \quad (41)$$

PROOF The proof is given in two steps:

**Step 1.** From the optimal control problem (6) we deduce that there exist positive constants  $C_1$ ,  $C_2$ , and  $C_3$ , independent of  $\alpha$ , such that

$$\|u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - z_d\|_H \leq C_1, \quad \|\bar{g}_\alpha\|_H \leq C_2, \quad \|\bar{q}_\alpha\|_Q \leq C_3. \quad (42)$$

Now, if we take  $v = u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\bar{g}, \bar{q})} \in V$  in the variational equality (4), following Gariboldi and Tarzia (2003) or Gariboldi and Tarzia (2008), we obtain, for  $\alpha > 1$ ,  $\|u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}\|_V \leq C_4$  where  $C_4 = C_4(C_2, C_3, \gamma_0, u_{(\bar{g}, \bar{q})}, \lambda_1)$  is independent of  $\alpha$ . Therefore,

$$\exists \mu \in K \text{ such that } u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} \rightharpoonup \mu \text{ weakly in } V, \text{ when } \alpha \rightarrow +\infty. \quad (43)$$

Taking  $v = p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})} \in V$  in the variational equality (16), we obtain that there exists a positive constant  $C_5 = C_5(C_1, p_{(\bar{g}, \bar{q})}, \lambda_1)$ , such that  $\|p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)}\|_V \leq C_5$  and next

$$\exists \xi \in V_0 \text{ such that } p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} \rightharpoonup \xi \text{ weakly in } V, \text{ when } \alpha \rightarrow +\infty. \quad (44)$$

Moreover, from (42), we deduce that there exist  $f \in Q$  and  $h \in H$  such that

$$\bar{q}_\alpha \rightharpoonup f \text{ weakly in } Q, \text{ when } \alpha \rightarrow +\infty \quad (45)$$

$$\bar{g}_\alpha \rightharpoonup h \text{ weakly in } H, \text{ when } \alpha \rightarrow +\infty. \quad (46)$$

For  $v \in V_0$ , taking into account (43), (45), (46) and taking the limit as  $\alpha$  goes to infinity, we have that

$$a(\mu, v) = (h, v)_H - (f, v)_Q, \quad \forall v \in V_0, \mu \in K \quad (47)$$

and by the uniqueness of the solution of (3), we get  $\mu = u_{hf}$ .

Now, for  $v \in V_0$ , taking into account (44), with the parameter  $\alpha$  going to infinity in the variational equality (16), we have that

$$a(\xi, v) = (u_{hf} - z_d, v)_H, \quad \forall v \in V_0, \xi \in V_0 \quad (48)$$

and from the uniqueness of the solution of (10), we get  $\xi = p_{hf}$ . Next,

$$\begin{aligned} J(h, f) &\leq \liminf_{\alpha \rightarrow \infty} J_\alpha(\bar{g}_\alpha, \bar{q}_\alpha) \leq \liminf_{\alpha \rightarrow \infty} J_\alpha(h', f') = \\ &= \lim_{\alpha \rightarrow \infty} J_\alpha(h', f') = J(h', f'), \quad \forall (h', f') \in H \times Q, \end{aligned}$$

and from the uniqueness of the solution to the problem (5), we have that  $h = \bar{g}$  and  $f = \bar{q}$ . Therefore, we have proved that

$$u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} \rightharpoonup u_{(\bar{g}, \bar{q})} \text{ weakly in } V, \text{ when } \alpha \rightarrow +\infty \quad (49)$$

$$p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} \rightharpoonup p_{(\bar{g}, \bar{q})} \text{ weakly in } V, \text{ when } \alpha \rightarrow +\infty. \quad (50)$$

**Step 2.** Taking  $h = 0$  and  $\eta = \bar{q}$  in the optimality condition for the vectorial optimal control problem (6),  $h = 0$  and  $\eta = \bar{q}_\alpha$  in the optimality condition for the vectorial optimal control problem (5), we have

$$(\bar{q}_\alpha - \bar{q}, M_2(\bar{q} - \bar{q}_\alpha) + (p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})}))_Q \geq 0;$$

then, we deduce that

$$\|\bar{q} - \bar{q}_\alpha\|_Q \leq \frac{\|\gamma_0\|}{M_2} \|p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})}\|_V. \quad (51)$$

Next, in a similar way, taking  $h = \bar{g}$  and  $\eta = 0$  in the optimality condition for the problem (6) and  $h = \bar{g}_\alpha$  and  $\eta = 0$  in the optimality condition for the problem (5), we prove that

$$\|\bar{g} - \bar{g}_\alpha\|_H \leq \frac{1}{M_2} \|p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})}\|_V. \quad (52)$$

Now, from (49) and the following inequalities, for  $\alpha > 1$ ,

$$\begin{aligned}
& \lambda_1 \|u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\bar{g}, \bar{q})}\|_V^2 + (\alpha - 1) \|u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\bar{g}, \bar{q})}\|_R^2 \leq \\
& \leq (g, u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\bar{g}, \bar{q})})_H - (q, u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\bar{g}, \bar{q})})_Q \\
& \quad - a(u_{(\bar{g}, \bar{q})}, u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - u_{(\bar{g}, \bar{q})})
\end{aligned}$$

we get that the result (39) holds. In a similar way, from (50) and the inequality

$$\begin{aligned}
& \lambda_1 \|p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})}\|_V^2 \leq \\
& \leq (u_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - z_d, p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})})_H - \\
& \quad - a(p_{(\bar{g}, \bar{q})}, p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})}) - \alpha(p_{(\bar{g}, \bar{q})}, p_{(\alpha, \bar{g}_\alpha, \bar{q}_\alpha)} - p_{(\bar{g}, \bar{q})})_R
\end{aligned}$$

we obtain the limit (40). Finally, from (39), (40), and the estimations (51) and (52), we deduce that the limit (41) holds.  $\square$

**COROLLARY 3** *If  $(\bar{g}, \bar{q})$  and  $(\bar{g}_\alpha, \bar{q}_\alpha)$  are the unique solutions of the simultaneous distributed and Neumann boundary optimal control problems (5) and (6), respectively, then we have:*

$$\lim_{\alpha \rightarrow \infty} |J_\alpha(\bar{g}_\alpha, \bar{q}_\alpha) - J(\bar{g}, \bar{q})| = 0.$$

**PROOF** The conclusion follows from the definition of  $J$ ,  $J_\alpha$  and the last theorem.  $\square$

## Acknowledgements

This paper has been partially sponsored by the Project PIP No.0534 from CONICET-Universidad Austral (Rosario, Argentina) and AFOSR-SOARD Grant FA 9550-14-1-0122. The authors would like to thank two anonymous referees for their constructive comments, which improved the readability of the manuscript.

## References

- BEN BELGACEM, F., EL FEKIH, H. and METOUI, H. (2003) Singular perturbation for the Dirichlet boundary control of elliptic problems. *ESAIM: M2AN* **37**, 833-850.
- BENSOUSSAN, A. (1974) Teoría moderna de control óptimo. *Cuadern. Inst. Mat. Beppo Levi* # **7**, Rosario.
- BOUKROUCHE, M. and TARZIA, D.A. (2007) On a convex combination of solutions to elliptic variational inequalities. *Electronic Journal of Differential Equations* **31**, 1-10.
- CARSLAW, H.S. and JAEGER, J.C. (1959) *Conduction of Heat in Solids*. Clarendon Press, Oxford.

- CASAS, E. (1986) Control of an elliptic problem with pointwise state constraints. *SIAM J. Control Optim.* **24**, 1309-1318.
- CASAS, E. and RAYMOND, J.P. (2006) Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations. *SIAM J. Control Optim.* **45** (5), 1586-1611.
- GARIBOLDI, C.M. and TARZIA, D. A. (2003) Convergence of distributed optimal controls on the internal energy in mixed elliptic problems when the heat transfer coefficient goes to infinity. *Appl. Math. Optim.* **47**, 213-230.
- GARIBOLDI, C.M. and TARZIA, D.A. (2008) Convergence of boundary optimal control problems with restrictions in mixed elliptic Stefan-like problems. *Adv. in Diff. Eq. and Control Processes* **1**(2), 113-132.
- KINDERLEHRER, D. and STAMPACCHIA, G. (1980) *An Introduction to Variational Inequalities and Their Applications*. Academic Press, New York.
- KIRCHNER, A., MEIDNER, D. and VEXLER, B. (2011) Trust region methods with hierarchical finite element models for PDE-constrained optimization. *Control and Cybernetics* **40** (4), 213-230.
- LIONS, J.L. (1968) *Contrôle optimal des systèmes gouvernés par des équations aux dérivées partielles*. Dunod-Gauthier Villars, Paris.
- MIGNOT, F. and PUEL, J.P. (1984) Optimal control in some variational inequalities. *SIAM J. Control Optim.* **22**, 466-476.
- TABACMAN, E.D. and TARZIA, D.A. (1989) Sufficient and/or necessary condition for the heat transfer coefficient on  $\Gamma_1$  and the heat flux on  $\Gamma_2$  to obtain a steady-state two-phase Stefan Problem. *J. Diff. Eq.* **77**, 16-37.
- TARZIA, D.A. (1979) Sur le problème de Stefan à deux phases. *C. R. Acad. Sc. Paris* **288A**, 941-944.
- TRÖLTZSCH, F. (2010) *Optimal Control of Partial Differential Equations. Theory, Methods and Applications*. American Mathematical Society, Providence, Rhode Island.