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Fernando Tohmé, Juan M.C. Larrosa

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Architectures engender Crises:
The Emergence of Power Laws in Social Networks

Fernando Tohmé
Instituto de Matemática de Bahia Blanca (INMABB)
Department of Economics, Universidad Nacional del Sur, San Juan y 12 de Octubre
Bahia Blanca, Buenos Aires 8000, Argentina
ftohme@criba.edu.ar

Juan M.C. Larrosa
Instituto de Investigaciones Económicas y Sociales del Sur (IIESS)
Department of Economics, Universidad Nacional del Sur, San Juan y 12 de Octubre
Bahia Blanca, Buenos Aires 8000, Argentina
jlarrosa@criba.edu.ar

Abstract

Recent financial crises posed a number of questions. The most salient were related to the cogency of derivatives and other sophisticated hedging instruments. One claim is that all those instruments rely heavily on the assumption that events in the world are guided by normal distributions while, instead, all the evidence shows that they actually follow fat-tailed power laws. Our conjecture is that it is the very financial architecture that engenders extreme events. Not on purpose but just because of its complexity. That is, the system has an internal connection structure that is able to propagate and enhance initially small disturbances. The final outcome ends up not being correlated with its triggering event. To support this claim, we appeal to the intuition drawn from the behavior of social networks. Most of the interesting cases constitute scale-free structures. In particular, we contend, those that arise from strategic decisions of the agents.

1 Introduction

The structure of the financial system, particularly the network of financial entities and agents as well as the ensuing structure of debts and obligations of all the parties involved in financial transactions are usually jointly deemed as constituting the *architecture* of the system (Eichengreen 2000).[1] Its main participants are banks, stock exchanges, private investors, governments, etc. The first financial structures, banks, arose as a way of making a more efficient use of resources in societies in which agents had different time schedules for the use of their money assets. But in the last two centuries the financial architecture became a fundamental component of the economy of modern nations, and more recently of the global economy (Burton and Brown 2009).

From bank deposits to credit default swaps, the financial instruments involve two parties, one lending some resources and the other borrowing them. Interestingly, the last decades showed a continuous growth of the length of the chains of liability, with different forms of insurance against negative events added as a protection for lenders. New and more complicated instruments arose to distribute both potential premiums as well as obligations among even unsuspecting individuals (like pensioners or owners of current accounts). While

[1]This characterization combines the institutional and regulatory infrastructure (which for some purists is the only component that should be called "architecture") with the intermediation structure.

in many cases a lender of last resort exists (usually governments bailing out institutions in trouble, but also supranational entities), in others cascades of defaulted debts can ensue, hitting all the firms and individuals in the complex chains of obligations (Kindleberger and Aliber 2011). So, for instance, in 2007, defaults on loan mortgages impacted on pension funds, insurance companies, mutual funds, etc. giving raise to the worst economic crisis since the 1930s.

The outbreak of this financial crisis posed a number of puzzling questions. Besides the search for whom to blame, some theoretical questions arose. The most salient were related to the cogency of *derivatives* and other sophisticated hedging instruments. A popular account of why they seem to be flawed appeared in Nassim Nicholas Taleb's "The Black Swan" (2007). There, Taleb claims that all those instruments rely heavily on the assumption that events in the world are guided by normal distributions while, instead, all the evidence shows that they actually follow fat-tailed power laws.

While this hypothesis explains why the financial instruments were not able to cover the title owners from a cascade of negative shocks and their consequences (Elliot et al. 2014), it requires further examination (Karimi and Raddant 2014). The underlying mechanism that generates events obeying power laws must be better understood, before engaging in the design of new regulation and prevention policies.

Our conjecture is that it is the very financial architecture that engenders extreme events. Not on purpose but just because of its complexity. That is, the system has an internal connection structure that is able to propagate and enhance initially small disturbances. The final outcome ends up not being correlated with its triggering event.

To support this claim, we appeal to the intuition drawn from the behavior of social networks. Most of the interesting cases constitute *scale-free* structures. In particular, we contend, those that arise from strategic decisions of the agents.

On the other hand, it has been shown that random disturbances in such networks start multiplicative branching processes that generate flows obeying to power laws. This is precisely the core of our argument.

In this paper we will abstract away all the specific details of the financial architecture and see it as a directed graph in which each node is able to borrow from the nodes to which it accesses and lend to the nodes from which it is acceded. Each node is identified with an agent, which is endowed with some amount of a good that can be transferred from and to other agents. By establishing links to other agents she can gain access to the goods held by them, but she has to pay a fee to establish those links. The strategic goal of each rational agent is to maximize the access to the resources held by the others, at the lowest cost possible. Nash equilibria yield the structures from which no individual agent can deviate to increase her benefit.

One of the most influential approaches in the literature has been to conceive the social structure as a network that may arise from random interactions among individuals. Changes in the probability of encounters may lead to drastic changes in the final structure that obtains. In mathematics this is a well-known result, that originates in Erdös and Renyi groundbreaking study on *random graphs* (1959). They showed that if the average number of links established by any agent is slightly increased in a small neighborhood of 1, a complete disconnected graph becomes a completely connected one. Newman et al. (2001) have generalized this result for generic probability distributions, showing that intermediate phase transitions exist, at which new components arise in the graph.

We reproduce a similar result, but here it arises from the intentional behavior of rational agents. Unlike in the case of random connections, "mavens", i.e. agents sought for connection, tend to appear. This gives these networks a distinctive scale-free structure.

Consider now a given Nash network. A random shock on a node will follow the already established connections, to fulfill this new request of resources. If this demand surpasses the amount of resources available in the node, it will request, in turn, this difference from its providers. Depending on the stored amounts and the connections, the initial shock might die out quite soon or it might propagate through large portions of the network.

We will prove that if the network has a number of relatively important players and the others are able to pay the fee to attach to them, the network will show all possible behaviors in the propagation of shocks. In particular, the generation, albeit with a rather low probability, of extreme events.

The organization of the paper is as follows. In section 2 we briefly discuss the financial network of liabilities (the financial *architecture*) and how a crisis may ensue, triggering a chain of losses of wealth. In section 3 we formalize this in the framework of equilibria networks. In section 4 we characterize the architecture that may arise when resources are unevenly distributed and the richest nodes act as mavens. We will show that the structure of this network is scale-free. In Section 5, we will analyze the multiplicative branching processes engendered by random shocks and see that the theoretical distribution of perturbations follows a power law.

2 The Financial Architecture and Minskyan Crises

A schematic description of how a contemporary financial system works is as follows. Consider an individual or firm who intends to make an investment, say to buy a house in the former case or to incorporate new technology in the latter. Instead of using savings they borrow loans from banks (maybe from more than one). Banks, in turn use the savings from final lenders. Borrowers pay back their debts (with some premium) which increases the assets of the banks, which in turn yield a higher premium on the securities. Final lenders receive them, increasing their wealth. But that constitutes a virtuous circle in which more patient agents (lenders) are rewarded by more impatient ones (borrowers), who may use the loans to increase their own wealth. On the contrary, if a borrower fails to repay the loan, he goes bankrupt (eventually becoming subject to foreclosure). The bank in turn writes down the loan and accounts it as a loss. If enough borrowers default their debts, the bank itself goes bankrupt, transferring the losses to the final lenders that see their savings diminished. The circle of loss of wealth, maybe started by a worse market situation, leads to a chain of defaults.

The actual financial architecture is far more complicated. Banks, at least in the developed world, have created a number of financial instruments backed on the loans given to borrowers (Elton et al. 2014). They are sold to mutual funds, pension systems and banks abroad. The same scheme as above works here, only that it is much more complicated, since in turn these instruments are used as collateral for other ("derivative") instruments, that are in turn sold to other financial institutions, etc. On the other hand, governments and supranational financial institutions provide lending of last resort to some of those financial institutions (particularly banks). Again, if this works well, wealth is created at the borrowing side, which is used to repay the debts and the proceeds go all trough the chain of obligations to primary and secondary lenders (i.e. the ones that securitize the loans as well as those that get those loans as collateral).

Hyman Minsky (1986) described the value of stocks of this system in terms of the *balance sheet* of all the entities involved. Minsky analyzed the onset of crises in which credit finances the acquisition of assets, which are at first more valuable than the securities used to provide the funds. But then, those assets loss value (like when a housing bubble bursts) increasing the debt and increasing the number of defaults. A massive loss of wealth ensues.[2] The question that

[2]For a more complete presentation of the views of Minsky on this matter, see the first two essays in Fazzari and Papadimitriou (2015).

remains is what is the probability of such a Minskyan crisis.

To analyze this, we will formalize this idea of a financial architecture in which the balance sheet of every involved party is affected. Once a contract is established between two agents, one a lender and the other a borrower, if the latter defaults his obligation, triggers a variation in the balance sheet of the lender, in which the loss is seen as being "borrowed" by the other party. In turn, a variation in the balance sheet of the lender, is equivalent to her being able to lend more (think of a bank with an increased amount of deposits, looking after potential borrowers). In any case the simple representation in the next section is able to capture the main features of the financial architecture and the onset of financial crises.

The novelty of our approach is the mix between the strategic behavior displayed in the emergence of the network and the multiplicative process that ensues after a random shock affects one of its nodes. The literature on Econophysics has also analyzed similar phenomena but with other approaches. So for instance Hawkins (2011) has described a similar Minsky-like crisis (see section 2). Di Guilmi et al (2008) focus on similar borrowing relation among firms while Lee et al . (2011) center their attention on the topology of the network of relations as a source of economic crises. In many ways our approach complements, albeit covering a partially different ground, these important contributions.

3 Equilibrium Networks

We focus on a static model of a network formation with complete information where our solution concepts are Nash-type equilibria. Our theoretical model is based on a seminal paper in the field of network formation by Bala and Goyal (2000). They characterize and provide a constructive proof of the existence of Nash equilibrium under the assumption of homogeneous costs and benefits through the players. In their model players can link unilaterally (without requiring the consent of the partner). The distinctive feature of this approach it its emphasis on individual incentives in shaping the networking decisions (Goyal (2007: ch. 7)). The formation of a link is expensive and only the player who forms the link pays this cost. Players can form direct links, but can also be connected indirectly with other players. As linking evolve a fairly complex structure of connections may emerges. We conceive this network as a representation of the global financial architecture.

Let $N = (1, \ldots, n)$ be a set of agents. To avoid trivial results we will always assume that $n \geq 3$. If i and j are two typical members of N, a link among them, without intermediaries, originated by i and ending in j will be represented as ij. The interpretation of ij is that i establishes a contact with j that allows i to partner with j as well as connect to j's network of contacts. Each agent $i \in N$ has some resources of her own, $I_i \in \Re^+$, (i.e. represented as a nonnegative real number). i can have access to more wealth by forming links with other agents who can lend her extra resources. The formation of links is costly but we will assume that a link ij has a fixed cost c. For simplicity, we assume that $l : N \times N \to Z^+$, i.e. that the length of each link is a non-negative integer and that $l_{ij} = l_{ji}$ (each i is trivially connected to itself through a link of length 0).

The agents will try to maximize the utility of the resources available to them as well as to minimize the cost of connecting to other agents. In order to do this, they will be endowed with a set of *strategies*. Each strategy for $i \in N$ is a $(n-1)$-dimensional vector $g_i = \langle g_{i,1}, \ldots, g_{i,i-1}, g_{i,i+1} \cdots g_{i,n} \rangle$ where each $g_{i,j}$, for $j \neq i$, is either 0 or 1. This is interpreted as meaning that i establishes a direct link with j if $g_{i,j} = 1$ while if $g_{i,j} = 0$ there is no such direct link. The set of all i's strategies is denoted as G_i. We restrict our analysis to only pure strategies, and so $|G_i| = 2^{n-1}$. Finally, $G = G_1 \times \ldots \times G_n$ denotes the set of strategy profiles in the interaction among the agents in N.

The existence of a direct link ij indicates an asymmetric partnership between i and j. That is, $g_{i,j} = 1$ indicates that i establishes a link with j that permits i to access to j's resources but no viceversa (the symmetry between i and j is restored if also $g_{j,i} = 1$). Structures with this feature are called **one-way flow** networks.

In one-way flow networks a strategy profile can be represented as a directed graph $g = (g_1 \ldots g_n)$ over N. That is, in the directed graph the elements of N are the *nodes* while any established link like $g_{i,j} = 1$ is represented by an arrow beginning in j with its head pointing to i. That is, arrowheads always point toward the agent who establishes the link.

We define $N^{g_i} = \{k \in N | g_{i,k} = 1\}$ as the set of agents to whom i establishes a direct link according to her strategy g_i. We say that there exists a *path* from j to i according to $g \in G$ if there exists a sequence of different agents $j_0 \ldots j_m$ (with $i = j_0$ and $j = j_m$) such that $g_{j_0,j_1} = \ldots = g_{j_{m-1},j_m} = 1$. In other words, given the joint strategy g, we have that $j_1 \in N^{g_{j_0}}$, $j_2 \in N^{g_{j_1}}$, …, $j_m \in N^{g_{j_{m-1}}}$. The length of the path from $j = j_m$ to $i = j_0$, denoted as $j \to_g i$, is the sum of the lengths of links in the path, $\sum_{j=0}^{m-1} l_{j(j+1)}$. Notice that a direct link is a path of length 1.

We denote the set of agents accessed (directly and otherwise) by i as $N^{i;g} =$

$\{k \in N | k \to_g i\} \cup \{i\}$. We include i in $N^{i;g}$ to indicate that i knows her own valuation through a link, of length 0, from i to herself. Let $\mu_i : G \to \{0, \ldots, n \times (n-1)\}$ be the number of links in all paths that end in i, originated by agents in $N^{i;g}$ under any given joint strategy: $\mu_i(g) = \sum_{(j,k) \in \mathcal{L}^{i;g}} l_{jk}$, where $\mathcal{L}^{i,g}$=$\{(j,k) \in N \times N : g_{j,k} = 1,$ and $\exists l \in N^{i;g}$ and $l \to_g i$ with $j, k \in l \to_g i\}$.

Let $\Pi_i : G \to R$ be i's payoff function:

$$\Pi_i(g) \equiv \sum_{j \in N^{i;g}} I_j - c\mu_i(g)$$

where c is the cost of establishing each link. That is, i's payoff is just the sum of all the resources that can be accessed by her, less the total cost of the paths reaching her that are established according to g. The intuition here is that i gets a payoff from accessing to more resources but at the same time she has to pay a "fee" for each of the links on the paths to the sources of information.

For each $g \in G$, agent i obtains a structure $N^{i;g}$ and her payoff depends critically on the type of directed graph that corresponds to $N^{i;g}$. The goal of a rational agent is to get as much information as possible traversing the shortest possible paths.

Given a network $g \in G$, let g_{-i} be the directed graph obtained by removing all of agent i's direct links. Then, g can be written as $g = g_i \oplus g_{-i}$ where $\oplus$ indicates that g is formed by the union of the links of g_i and those in g_{-i}. A strategy g_i is said the *best response* of agent i to g_{-i} if

$$\Pi_i(g_i \oplus g_{-i}) \geq \Pi_i(g_i^{'} \oplus g_{-i})$$

for all $g_i^{'} \in G_i$.

The set of best responses to g_{-i} is $BR_i(g_{-i})$. A network $g = \langle g_1, \ldots, g_n \rangle$ is said to be a *Nash network* if for each i, $g_i \in BR_i(g_{-i})$ i.e. if g (as a joint strategy) is a Nash equilibrium.

Given a network g, a set $C \subset N$ is called a *component* of g if for every pair of agents i and j in C ($i \neq j$) we have that $j \in N^{i;g}$ and $i \in N^{j;g}$. Furthermore, there does not exist $C^{'}$, $C \subset C^{'}$ for which this is true. A component C it said to be *minimal* if C is not a component anymore once a link $g_{i,j} = 1$ between two agents i and j in C is cut off, i.e. if $g_{i,j} = 0$.

A network g is said to be *connected* if it supports a single component. If that component is minimal, g is *minimally connected*. A network that is not connected is *disconnected*.

Then, we have:

Lemma 1 *A strict Nash network is empty if for each i, $I_i < c$. If, instead, for each i, I_i is larger than the cost of accessing i from every $j \neq i$, the strict Nash network is minimally connected.*

Proof: See Appendix.

Let us note that the topology of the minimally connected Nash network is called the "wheel". That is, each node is connected just to one node. But this case arises only if the costs are almost zero, or if the amount of resources of each agent are large.

For intermediary cases we have:

Proposition 1 *If, for not every i, $I_i > c$, the strict Nash network has n directed edges.*

Proof: See Appendix.

Example 1 *Consider a group of five agents, $N = \{A, B, C, D, E\}$, such that the information owned by them is: $I_A = 2$, $I_B = 1.8$, $I_C = 2.1$, $I_D = 0.5$ and $I_E = 0.5$. Under a connection cost $c = 1.8$, Figure 1 shows one of the possible equilibrium structures that can arise.*

4 A Scale-free Architecture

A directed graph, corresponding to a Nash equilibrium g^*, can be fully described by means of its *Laplacian matrix* $L(g^*) = D(g^*) - A(g^*)$. It obtains as the difference between the degree matrix $D(g^*)$ and the adjacency matrix $A(g^*)$. $D(g^*)$ is a diagonal $n \times n$ matrix, in which for each i, the ii entry is $\sum_{\{j:g^*_{i,j}\neq 0\}} l_{ij}$. $A(g^*)$ is a $n \times n$ matrix in which each entry ij is l_{ij} if $g^*_{ij} = 1$ and 0 otherwise.

The main properties of $L(g^*)$ are (Mohar 1991),(Wu 2005):

- The eigenvalues of $L(g^*)$, $\lambda_1, \lambda_2, \ldots, \lambda_n$ have all non-negative real parts.

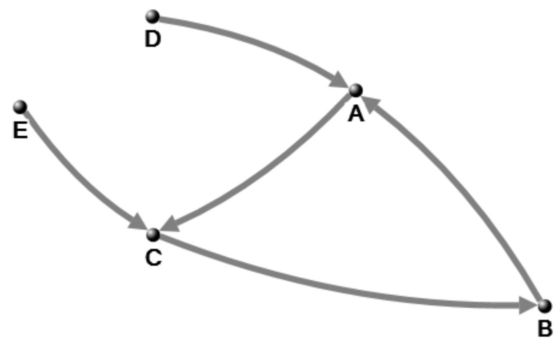

Figure 1: Example 1: Equilibrium network of 5 nodes with the different endowments

- Furthermore, $Re(\lambda_1) \leq Re(\lambda_2) \leq \ldots \leq Re(\lambda_n)$ are such $|\{k : Re(\lambda_k) = 0\}|$ is the number of undirected[3] components of g^*.

- In the case that g^* is an empty network, $Re(\lambda_k) = 0$ for every $k = 1, \ldots, n$, while if g^* is connected, $Re(\lambda_k) > 0$ for $k = 2, \ldots, n$.

Now consider the space $\mathcal{H}^n$ of the Laplacian matrices of directed graphs with n nodes and distances given by $\{l_{ij}\}_{i,j=1,\ldots,n}$. It can be endowed with a partial order $\preceq$, such that $L^i, L^{ii} \in \mathcal{H}^n$ are such that $L^i \preceq L^{ii}$ iff all the eigenvalues of $L^{ii} - L^i$ have non-negative real parts. That is, $L^{ii} - L^i$ corresponds to a direct graph (not necessarily to an equilibrium). This, in turn means that the graph corresponding to L^{ii} includes the undirected edges of L^i as a subgraph.

We can define a transformation $t : N \cup \{0\} \times \mathcal{I} \to \mathcal{I}$, where N is the class of natural numbers, defined as follows:

- For any distribution of resources I, $I = t(0, I)$.

- For $k \geq 1$, $t(k+1, I) = I'$ such that I' is a distribution of resources with a Nash graph, $g^*(I')$ that includes all the links in the Nash graph of $t(k, I)$. Furthermore, $t(k + 1, I)$ yields a Nash graph that *strictly* includes the graph of $t(k, I)$.[4] Otherwise, if no such graph exists, $t(k + 1, I) \in t(k, I)$.

We have that:

[3] That is, the components that obtain by disregarding the direction of edges.

[4] It is immediate that t is a correspondence, since there might exist several minimal graphs in this condition.

Theorem 1 *Consider distributions I^1 and I^2 in the conditions of Lemma 1 and the transformation t. Then, there exist two parameters $\bar{k}, \tilde{k} \in N$ such that $\bar{k} < \tilde{k}$ and for $k \leq \bar{k}$, $t(k, I^1)$ yields an empty network, while for $k \geq \tilde{k}$, $t(k, I^1)$ is a minimally connected network.*

Proof: See Appendix.

An important feature of each of the chains determined by t is that each one will be determined by the nodes that receive heavier demands of connection. This is expressed in the concept of *eigenvector centrality*, which should not be confused with the mere number of links pointed towards the central nodes. More broadly, it indicates their relative importance. To determine the centrality of each node, we have to consider the lowest non-zero eigenvalue of the Laplacian matrix and compute the corresponding eigenvector. Formally, given an equilibrium g^*, and the family of eigenvalues of $L(g^*)$, $Re(\lambda_1) \leq Re(\lambda_2) \leq \ldots \leq Re(\lambda_n)$, let k be the least such that $Re(\lambda_k) > 0$. Then let $\{\bar{x}^j\}_{j \geq k}$ be the eigenvectors corresponding to positive eigenvalues (Newman 2008). Then, the centrality of the i node is $\frac{1}{\sum_{j \geq k} \bar{x}_i^j}$.

Formally, we have the following result:

Proposition 2 *For each chain of resource distributions between I^1 and I^2 induced by t, the critical parameters $\{k_m\}_{m=1}^n$ that determine the transition from Nash graphs with $m+1$ components to Nash graphs with m components, are associated with different centrality values for different values of m. These values will be such that the centrality of nodes decreases in each transition.*

Proof: See Appendix.

This analysis is of static nature. We assume that changes in the distribution of resources do not trigger flows over the links. For each transition the central nodes will play the role of "mavens", i.e. highly connected individuals that facilitate the connection between agents that were separated under the previous distribution. Since the undirected subgraphs underlying the components are all minimal spanning trees, a transition implies that one node (a maven) receives a large number of links while the others receive only a few.

Let us consider the distribution of *degrees* of the nodes. That is, for each i, let $d_i^{k_m}$ be the number of agents to which i is connected in the Nash network corresponding to parameter k_m. It is easy to see that mavens will have a large degree, while the rest of the nodes will have a low degree. If $n \to \infty$, the degree of mavens would grow without bound. That is, the Nash network would be *scale-free*.

If we fix a parameter $k > \tilde{k}$, such that the Nash network is not a wheel, we have an architecture in which there are powerful players (the mavens) and the rest, who try to connect to them. In what follows, the architecture will be denoted G^k while the number of cases in which a generic node i has a degree j ($d_i^k = j$) is $d(j)$. We have that:

Proposition 3 *For* $k > \tilde{k}$, $d(j) = \binom{n-n_c}{j-1} VR^{n-n_c+1-j}_{n_c-1}$.[5]

Proof: See Appendix.

Example 2 *Consider the same nodes, information values and connection cost as in Example1. The degree of node* C *can be either* 1, 2 *or* 3. *In Figure 2 all the cases in which these degrees obtain are exhibited. It can be seen that for* $d(1)$ *the number of cases is* $\binom{2}{0}VR^2_2 = 4$, *while* $d(2) = \binom{2}{1}VR^1_2 = 4$ *and* $d(3) = \binom{2}{2}VR^0_2 = 1$.

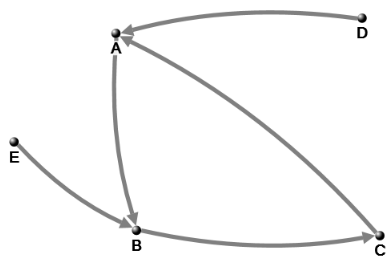

Figure 2: Example 2

For any given node, $d(j)$ is distributed according a power law:

Theorem 2 *For a large* n, *in* G^k *we have that*

$$d(j) \;\sim\; j^{-\tau}.$$

where τ *are the odds*[6] *that any node* i *is such that* $I_i < c$.

[5] Where $\binom{m}{p}$ indicates the number of p-combinations out of a set of m elements, while VR^m_p is the number of variations with repetition of p elements chosen from a set of m elements.

[6] That is, the probability of the event over the probability of the complementary event.

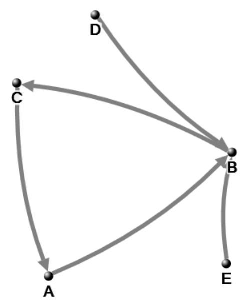

Figure 3: Example 2: Degree(C)= 1

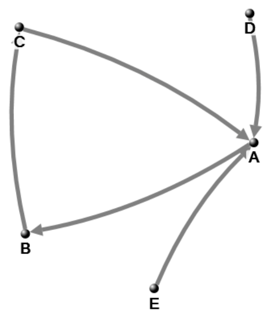

Figure 4: Example 2: Degree(C)= 1

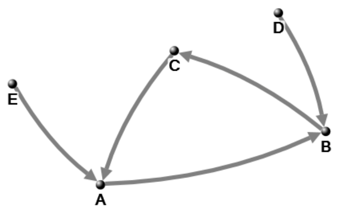

Figure 5: Example 2: Degree(C)= 1

Proof: See Appendix.

Two extreme cases are worth considering. One is when $\theta^* = 0$, i.e. when no node has information enough to make it worth a connection. Since then $\tau \to \infty$ this means that only the empty network will be obtained. On the contrary, when $\theta^* = 1$, $\tau = 0$, indicating that all the nodes will be part of the Nash

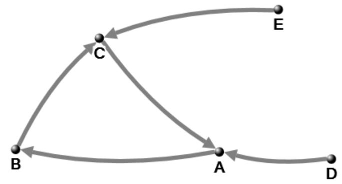

Figure 6: Example 2: Degree(C)= 2

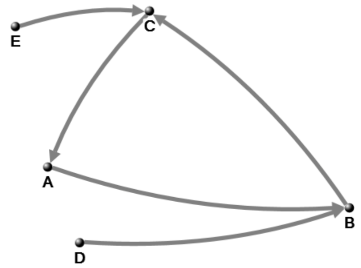

Figure 7: Example 2: Degree(C)= 2

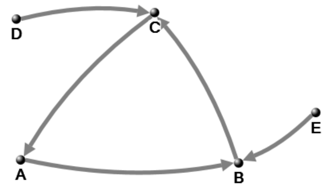

Figure 8: Example 2: Degree(C)= 2

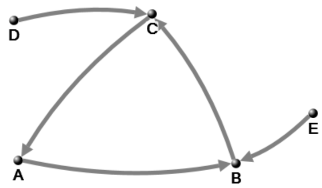

Figure 9: Example 2: Degree(C)= 2

network. Both results recast Lemma 1.

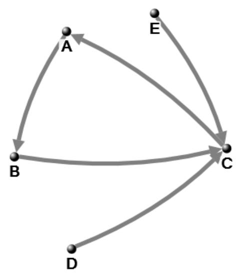

Figure 10: Example 2: Degree(C)= 3

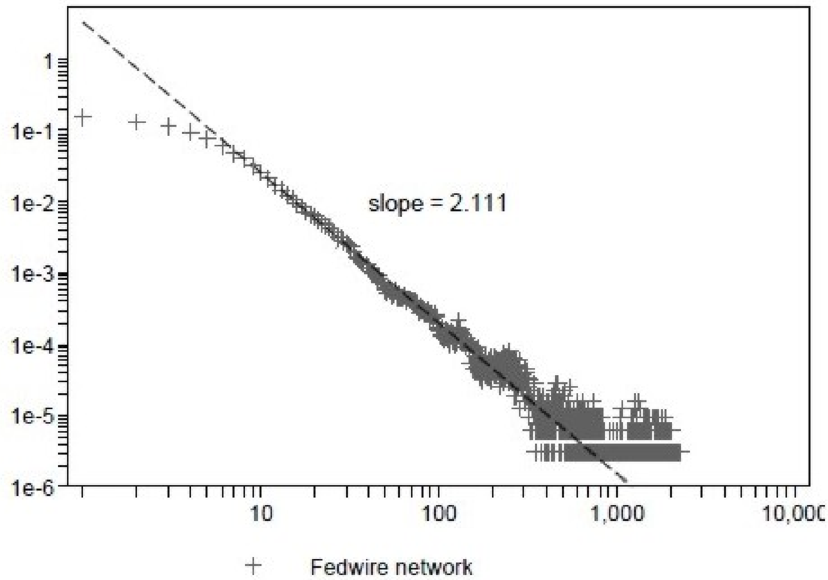

Figure 11: Structure of the Federal Reserve Payment Systems (Sorämaki et al. 2007). Reproduced with permission.

5 The Origin of Extreme Events

Suppose that in G^k one node i receives a shock, meaning a variation of the value of I_i. Given the links held by i with other agents, if the shock is negative and now $I_i < c$, she might borrow from its potential lender the resources needed to recover the loss. Alternatively, if the shock is positive and previously $I_i < c$, i

may get enough resources to attract new potential borrowers. In the former case, any agent j that became a lender of i, can borrow the extra units transferred to i from its own lender, etc. The only constraint on this process it that no loops are allowed, since they would imply that nodes previously 'bailed out' will have in turn to bail out other nodes.[7] The multiplicative effect of this process may either die out immediately (if the node can satisfy the request with its own resources) or propagate through the network. Notice that no matter the sign of the shock, a new structure $G^{k'}$ may appear. If the shock is negative the disturbance propagates through the links in G^k leaving some of them with resources below c and leading to a new network structure, while if the shock is positive, the node may become attractive, yielding $G^{k'}$ as a result.

Example 3 *Consider the network in Figure 3, with the same nodes and values as in Example1. Suppose that node D receives a negative shock of -1. Then, it has to borrow 1 from its lender, B, in order to return to its previous state. Now I_B becomes 0.8. B, in turn, will request 1 from C, which leaves C with 0.1. Then C requests 1 from A. And the process ends there (otherwise A would ask 1 from the previously bailed out B), with all the nodes with original values, except A that has now only 1, which implies that it will be left out of the wheel and the new topology represented in Figure 4 arises (A remains being a* potential *borrower from B). In the process four nodes have been affected: D, B, C, A.*

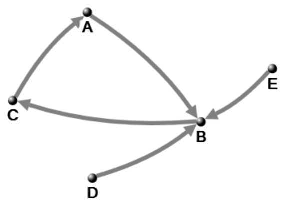

Figure 12: Example 3

On the other hand, suppose a positive shock of 1.3, again to D. Then, D will become part of the wheel (a possible resulting structure is shown in Figure 5).

Consider s the number of resources displaced due to a shock σ to node i. So, if the shock is negative, there will exist a path $k_1, k_2, \ldots, k_s$, where $k_1 = i$, such that for $j = 1$ to $j = s - 1$, $\sigma + I_{k_j} < c$ and either k_s is such that $\sigma + I_{k_s} \geq c$

[7]In the real world a bankrupt company is not allowed to lend to another one.

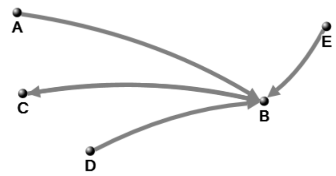

Figure 13: Example 3

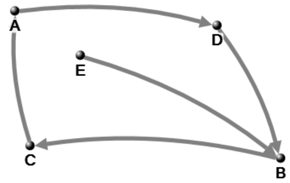

Figure 14: Example 3

or k_s has a direct link to a k_j in the path ($j < s$). Of course, the length of the path (i.e. the size of the disturbance generated by σ) is s.

If the shock of σ to i is positive and $\sigma + I_i \geq c$, $s = n_\sigma + 1$, where n_σ is the is the number of links that are established to i.[8]. Otherwise, if $\sigma > 0$ but $\sigma + I_i < c$, $s = 1$.

Let $p(s)$ be the probability that a shock generates a displacement of size s. Our main claim is that

$$p(s) \sim s^{-\gamma}.$$

Consider q_m the probability that a shock to a node generates a tree with m branches. This probability is given by:

$$q_m = \frac{m\ d(m)}{\sum_j j\ d(j)} \frac{1}{m}$$

i.e. the proportion of cases in which the disturbance may affect a node through one of m links, multiplied by the probability that the node will get a value below than c (if the shock is negative) or the probability that any nodes attracted to it has a value below than c (if the shock is positive).

From Theorem 2 we have:

$$q_m \sim \frac{m^{-\tau}}{\sum_j j^{1-\tau}} \sim \frac{m^{-\tau}}{\zeta(1-\tau)}$$

[8]Since N_c decreases now to $N_c \setminus \{i\}$, n_σ ranges between 1 and $n - n_c$

where $\zeta(\cdot)$ is Riemann's zeta function.

To derive an expression for $p(s)$ we consider, given complex numbers x and y, its generating function $\mathcal{P}(x) = \sum_{s=1}^{\infty} p(s)x^s$ and a generating function for q_m as $\mathcal{Q}(y) = \sum_{m=1}^{\infty} q_m y^m$. Then, we have that (Otter, 1949):

$$\mathcal{P}(\omega) = \omega\mathcal{Q}(\mathcal{P}(\omega))$$

Then we have:

Proposition 4 *For $\tau \in (2, 3)$, $\gamma = \frac{1-\theta}{2\theta-1}$ such that $p(s) \sim s^{-\gamma}$.*

Proof: See Appendix.

Figure 11 shows the log-log representation of the degree distribution of the Federal Reserve Wire Network, which follows a power law. In log-log scale the power law function is a straight line with scope 2.111. The degree distribution resulting from this estimation is scale free, with a power law analogous to the form proposed by Proposition 4.

6 Discussion

The abstract features of our presentation should not hide the initial purpose we stated in the Introduction. The financial architecture (G^k in our formalism) reflects the basic feature of financial systems: it consists of an organized scheme in which richer entities lend money to less well endowed ones. The links, in turn, reflect the contracts that may arise between borrowers and lenders, including the payment of fees for the use of the financial resources.

Shocks represent sudden increases or decreases of wealth that have impact on other entities in the system. Agents connected to the one who received the shock may either borrow or lend to her. The propagation of the shock proceeds through the network, yielding either small overall effects or (with much less probability), a high impact on the entire system. These effects are Taleb's *black swans*. In particular, huge financial crises may arise in this way, starting from innocent-looking changes in wealth in some sectors of the economy.

The calls for more regulation or state intervention should take into account that all depends on the architecture of the system. If it is scale-free, shocks will propagate following a power law, which in turn may engender extreme events. Thus, if prudential regulation must be enacted to avoid this kind of crisis (in Minsky's sense), the key resides in restricting the scale-free structure and not so much in separating roles of banks (commerce and investment).[9] The key for doing this consists in detecting the "mavens" and then limiting their degree in the network. This will reduce their participation in the chains of propagation of defaults and thus, cascades of bankruptcies may be contained, instead of affecting the entire architecture.

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[9]Like int the frequent calls to reinstate the 1933 Glass-Steagall Act (Ugeux 2014).

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8 Appendix

Proof of Lemma 1: *Let us consider* $\Pi_i : G \to \Re^+$*, for each* $i \in N$*. If we show that there exists a unique (up to isomorphism)* $g^* \in G$ *that maximizes* Π_i *for each* i*, given that the others choose* g^*_{-i} *we would establish that there exists only one Nash equilibrium in the game.*

On the other hand, since given two joint strategies g *and* $g^{'}$, $\Pi_i(g) \geq \Pi_i(g^{'})$ *iff the corresponding graphs* $N^{i;g}$ *and* $N^{i;g^{'}}$ *are such that:*

$$\sum_{j\in N^{i;g}} I_j - \sum_{j\in N^{i;g^{'}}} I_j \geq c(\mu_i(g) - \mu_i(g^{'}))$$

it follows that for every i *the corresponding payoff should be:*

$$\Pi_i(g^*) = \sum_{j\in N^{i;g^*_{-i}}_{>c}} I_j - c\bar{\mu}_i(g^*)$$

where $N^{i;g^*_{-i}}_{>c} = \{j : I_j > c\bar{\mu}_i(i \to_{g_i;g^*_{-i}} j)\} \cup \{i\}$*. Here* $\bar{\mu}_i(j \to_{g_i;g^*_{-i}})i$ *is the shortest length of a path from* j *to* i*, for any* g_i *while keeping fixed* g^*_{-i}*. In turn,* $\bar{\mu}_i(g^*)$ *is* $\sum_{(j,k)\in\mathcal{G}L^{i;g^*}} l_{jk}$*, for* $\mathcal{G}L^{i;g^*}$*, the class of* shortest paths *between* i *and any other node in* $N^{i;g^*}$*.*

Proof of Proposition 1: *Let us consider the class of nodes with information at least as valuable as the cost of connection:* $N_c = \{i \in N : I_i \geq c\}$*. By a straightforward application of Lemma 1, restricted to* N_c*, we have that* N_c *constitutes a wheel (i.e. with* $|N_c|$ *directed edges). On the other hand, all nodes* *in $N \setminus N_c$ will connect to one of the nodes in N_c getting access to all of them ($|N \setminus N_c|$ directed edges). So, in total, we have $|N| + |N \setminus N_c| = n$ directed edges in the graph.*

Proof of Theorem 1: *First of all, notice that t determines an increasing sequence in $\mathcal{H}^n$. This is immediate from the fact that $t(k, I)$ yields a class of information distributions such that the corresponding Nash graphs all have the undirected Nash graph of I as a subgraph. This in turn means that the Laplacian matrixes will verify that $L^I \preceq L^{t(k,I)}$.*

Since the number of eigenvalues with zero real part of a Laplacian is the number of components of its graph, and $L^I \preceq L^{t(k,I)}$ it follows that, since a graph corresponding to $t(k, I)$ keeps the links of the graph for I, the number of components in $t(k, I)$ has to be greater or equal than the amount of components for I. Then, the number of eigenvalues with zero real part must either remain the same or decrease from L^I to $L^{t(k,I)}$.

Therefore, $t(0, I^1)$ yields the empty network, with the maximal number of components. So, trivially, $\bar{k} = 0$ since for any $k > 0$, $t(k, I^1)$ has already at least one link among two nodes. On the other hand, considering the ordering among Laplacian matrixes, there must exist L^{I^2}, corresponding to I^2. It is clear that $I^2 \in t(k, I^1)$ for some k, since the minimally connected graph for I^2 includes the empty graph as subgraph. Take $\tilde{k}$ as the minimal k that verifies that the information distribution supports a minimally connected graph. It follows that for every $k > \tilde{k}$, $t(k, I^1)$ will also be an information distribution that corresponds to a minimally connected graph.

Proof of Proposition 2: *Consider two critical values k_m and k_{m-1}. They will be associated to eigenvalues $\{\lambda_j^m\}_{j=1}^n$ and $\{\lambda_j^{m-1}\}_{j=1}^n$, respectively. Notice that since the real parts of the eigenvalues are ordered, $Re(\lambda_{m-1}^m) = 0$ and $Re(\lambda_m^m) > 0$. In turn, $Re(\lambda_{m-2}^{m-1}) = 0$ while $Re(\lambda_{m-1}^{m-1}) > 0$. Therefore, the corresponding least non-zero eigenvalues are $Re(\lambda_m^m) > 0$ and $Re(\lambda_{m-1}^{m-1}) > 0$ and the corresponding eigenvectors are different. The centrality of nodes will differ acordingly. But even so, notice that, since each Nash graph contains the undirected Nash graph of the previous stage in the chain, if the number of components becomes reduced from m to $m - 1$ is because two components become linked. That is because some nodes acquire enough information to makes them central for the two previous components.*

Proof of Proposition 3: *Consider the class of nodes with information at least as valuable as the cost of connection:* $N_c = \{i \in N : I_i \geq c\}$. *Recall that the cardinality of* N *is* n *and let* $|N_c| = n_c$. *Then, the degree of each* $i \in N_c$ *can range from* 1 *to* $n - n_c + 1$. *This is a straightforward consequence of Proposition 1: since the members of* N_c *will form a wheel, one of them will connect to* i *while any subset of* $N \setminus N_c$ *might connect also to* i.

Proof of Theorem 2: *Let us now consider a fixed node* $i^* \in N_c$ *(nodes in* $N \setminus N_c$ *have degree* 0*). For any given degree* j *of* i^*, $n - n_c + 1 - j$ *elements of* $N \setminus N_c$ *will be assigned to the rest of* $n_c - 1$ *nodes in the wheel (*$VR_{n_c-1}^{n-n_c+1-j}$ *possibilities), while* $j - 1$ *(one connection among the* j *edges comes from* N_c*) elements from among* $n - n_c$ *may be assigned to* i^* *(a number* $\binom{n-n_c}{j-i}$ *of cases). That is,* $d(j) = \binom{n-n_c}{j-1} VR_{n_c-1}^{n-n_c+1-j}$.

Consider a fixed node $i^* \in N_c$ *(nodes in* $N \setminus N_c$ *have degree* 0*). The number of cases in which* i^* *can have a degree* j*, according to Proposition 3, is* $\binom{n-n_c}{j-1} VR_{n_c-1}^{n-n_c+1-j}$. *For a large* n *this expression can be approximated by:*

$$\frac{\Gamma(M-1+j)}{\Gamma(M)\Gamma(j-1)}(n-j-M)^M$$

where $M = n - n_c + 1 - j$ *and therefore* $n - n_c = M - 1 + j$ *and* $n_c - 1 = n - M - j$. *Taking logarithmic derivatives (where* ψ^0 *is Gauss'* digamma *function) we obtain:*

$$\psi^0(M-1+j) - \psi^0(j-1) - \frac{M}{n-j-M} = \sum_{k=1}^{M-1+j-1} \frac{1}{k} - \sum_{k=1}^{j-1-1} \frac{1}{k} - \frac{M}{n-j-M} =$$

$$= \sum_{k=j-1}^{M-1+j-1} \frac{1}{k} - \frac{M}{n-j-M}$$

To obtain τ *we multiply this expression by* $-j$. *That is:*

$$\tau \sim -\sum_{k=j-1}^{M-1+j-1} \frac{j}{k} + \frac{jM}{n-j-M}$$

To assess the value of τ *we find a lower and an upper bound, by taking the lowest and highest terms in* $\sum_{k=j-1}^{M-1+j-1} \frac{j}{k}$ *and eliminating* M*:*

$$\tau_- \sim -\frac{j(n-n_c-j)}{j-1} + \frac{j(n-n_c-j)}{n_c-1} + \frac{j}{n_c-1} =$$

$$= \frac{j}{j-1}(j-n_c)\frac{n-n_c-j}{n_c-1} + \frac{j}{n_c-1}$$

and

$$\tau^- \;\sim\; -\frac{j(n-n_c-j)}{n-n_c-1} \;+\; \frac{j(n-n_c-j)}{n_c-1} \;+\; \frac{j}{n_c-1} \;=$$

$$=\; \frac{j}{n_c-1}(n-2n_c)\frac{n-n_c-j}{n+n_c-1} \;+\; \frac{j}{n_c-1}$$

For large values of n *and* n_c, $n_c \sim n_c$. *Furthermore, large values of* j *imply that* $j \sim n - n_c$. *Therefore,* $\tau_-, \tau^- \to \frac{n-n_c}{n_c}$. *That is,* $\tau \sim \frac{n-n_c}{n_c}$.

By taking, for large n, $n_c = \theta^* n$, *where* θ^* *is the probability that a given node* $i \in N_c$, *we have that* $\tau \sim \frac{1-\theta^*}{\theta^*}$, *i.e. the odds of the event '*$i \notin N_c$*'.*

Proof of Proposition 4: *For* s *large and* $\omega \sim 1$, *we find an asymptotic behavior for* $p(s)$ *(Goh et al., 2003, eq.* **(7)***):*

$$p(s) \sim a(\tau)s^{\frac{-\tau}{\tau-1}}$$

where $a(\tau) = \frac{-(\frac{\Gamma(\tau-1)}{\zeta(1-\tau)})^{\frac{1}{\tau-1}}}{\Gamma(\frac{1}{\tau-1})}$. *Then, since* $\tau = \frac{1-\theta}{\theta}$ *we have that* $s^{\frac{-\tau}{\tau-1}} = s^{\frac{-(1-\theta)}{2-\theta}}$.

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Highlights

- A model of network formation that engenders scale-free structures is proposed.
- Nodes are rational lenders or borrowers and links are financial transactions.
- Shocks represent sudden increases or decreases of wealth.
- They propagate on the system yielding small overall effects.
- However, with smaller probability, they impact the entire system.