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On Application of Theory of Distributions to Static and Dynamic Analysis of Cracked Beams

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This paper presents a rigorous study on the static and dynamic behavior of beams affected by cracks. The theory of distributions developed by Laurent Schwartz[1] is adopted as it is particularly suitable for the treatment of discontinuities in functions for the deflection and derivatives of the beam. Thus, this paper presents a contribution towards the understanding and application of the theory of distributions to the static and dynamic behavior of structural elements affected by cracks. A simple, computationally efficient and accurate algorithm is developed for the problems of concern. Numerical results are presented for beams with two cracks. The algorithms developed for beams with discontinuities are obtained in a rigorous framework for static and vibration problems.

Keywords: Cracked beam; distribution theory; generalized function; discontinuity.

1. Introduction

In the past years, the static and dynamic analysis of structural beams and frames with single or multiple concentrated damages has received considerable treatment. Such a topic is of great engineering interest since the presence of concentrated cracks may drastically change the behavior of the beams or frames, due to alternations in the continuity of the physical and geometrical properties. For this reason, the effect of concentrated cracks has been widely studied. Several models for describing the variation of flexural stiffness of the beams in the vicinity of damages have been implemented, classified as the direct and inverse problems. The direct problems have been addressed by formulating the mechanical models of the beam containing the damages. Several works belonging to this class will be cited below. The inverse problems have been solved with the detection of position and severity of the damage by using either static or dynamic tests. Various method for the identification of damage have been developed by Morassi,[2,3] Caddemi and Morassi,[4,5] Narkis[6] and Vestroni and Cappechi.[7,8]

The effect of concentrated damages on the flexural stiffness in the vicinity of a crack has been modeled as an equivalent rotational spring connecting two segments of the damaged beam. The model containing an elastically restrained hinge, with rotational stiffness dependent on the extent of the damage, proved to be accurate within certain limits, and has extensively been used. In this popular model, the stiffness of the rotational restraint r_c is determined by relating the local flexibility to the strain energy concentrated in the vicinity of the crack through the principle of fracture mechanics and is given by

$$r_c = C / f\left(\frac{a}{h}\right), \quad f\left(\frac{a}{h}\right) = \sum_{k=0}^{N} d_k \left(\frac{a}{h}\right)^k,$$

where h denotes the height of the cross-section of the beam and a the depth of the crack. The above expressions can be found in several papers with different values of the parameters C, d_k and N. It is not possible to give a detailed account because of the great amount of information, so without the claim of being exhaustive, in this paper only some references will be cited. Ostachowicz and Krawczuk[9] and Farghaly[10] modeled the crack as a continuous flexibility using the displacement field in the vicinity of the crack modeled by the fracture mechanics methods. Dimarogonas[11] presented a state of the art review with particular reference to dynamics. Chondros *et al.*[12] developed a continuous cracked beam vibration theory for the lateral vibration of cracked Euler–Bernoulli beams with single-edge or double-edge open cracks. The crack was also modeled as a continuous flexibility using the displacement field in the vicinity of the damage, found by the fracture mechanics methods. Shifrin and Ruotolo[13] presented a new method for evaluating the natural frequencies of a beam with an arbitrary number of cracks. The method was based on the use of massless rotational springs to represent the cracks and, as a main feature, it leads to a system of $N + 2$ linear equations for a beam with N cracks. Fernandez–Saez and Navarro[14] presented an analytical approach for analysing the fundamental frequency of cracked Euler–Bernoulli beams in bending vibration. The influence of the crack was represented by an elastic rotational spring connecting the two segments of the beam at the cracked section. Li[15,16] presented a model of massless rotational springs adopted for analysing the free vibrations of multi-step uniform and nonuniforms beams, with an arbitrary number of cracks and concentrated masses. Binici[17] proposed a method to determine the eigenfrequency changes of axially loaded beams where cracks were modeled as rotational springs. Loya *et al.*[18] presented exact and perturbative solutions for the natural frequencies of vibration of cracked Timoshenko beams. In this work the beam was modeled as two segments connected by two springs, one extensional and the other rotational. Lellep and Kraav[19] analysed the influence of cracks on the stability of beams by the method of distributed line spring. Batihan and Kadioglu[20] determined analytical solutions for the transverse vibrations of cracked beams attached to various types of elastic foundations.

Regarding the mathematical tools, the Dirac's delta function has been extensively used to obtain representations of point loads and singularities in a variety of structural problems. Nevertheless, analytical formulations currently available for these problems are not completely satisfactory in terms of rigorous formulation and computational efficiency or in terms of physical consistency. Engineers increasingly used techniques of mathematical analysis adopting heuristics procedures. As early as in 1919, the problem of deflection of beams with discontinuous loading conditions has been treated by a method introduced by Macaulay,[21] consisting of the use of a bracket notation to take into account the discontinuities. This method was generalized to two-dimensional problems. Later, Wittrick[22] analysed beams with lateral loads and circular plates with axisymmetric lateral loads. Mahig[23] applied the method for rectangular plates under a point load and for circular plates with axisymmetric loading. Conway[24] extended the use of the method to solving two-dimensional problems governed by partial differential equations considering long rectangular plates and concentric circular plates subjected to a normal concentrated load. Selek and Conway[25] applied Macaulay's method to obtain the exact solutions of three point loaded plate problems. Falsone[26] extended Macaulay's method to the cases in which discontinuous external loads are applied, giving discontinuities on displacements and rotations.

It is remarkable that all these works present heuristics formulations and these approaches require some caution due to the involved mathematical subtleties. Macaulay's method is, in essence, an exposure with meaning given over the bracket notation, typically of the form $[x - c]^n$, where if the argument within the bracket is negative, the term is ignored, while if the argument is positive, it is unaltered. All the generalized functions used in Macaulay's method are the Dirac's delta function and its generalized derivatives and integrals. The Dirac's delta function is commonly defined by means of properties, such as:

$$\delta(x-c) = 0, \quad \text{if } x \neq c,$$

$$\int_a^b \delta(x-c)dx = \begin{cases} 0, & \text{if } c \notin [a,b], \\ 1 & \text{if } c \in [a,b], \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x-c)u(x)dx = u(c),$$

where u is a sufficiently smooth function. Engineers and physicists have used this function with success, but it must be noted that the definition is not rigorous since the above properties are contradictory. In consequence, the use of Macaulay's method leads to analytical manipulations which are confusing and not mathematically precise. Nevertheless, the distribution theory developed by Laurent Schwartz,[1] provides rigorous justifications for these common formal mathematical manipulations published in the engineering literature.

Belonging to the stream of rigorous mathematical research several important papers have been presented, which provide interesting results based on adoption of the theory of distributions. Yavari *et al.*[27–30] made use of the distribution theory in the integration procedure, providing a formulation of the governing differential equations over a unique integration domain. But these authors introduced the auxiliary beam method, to avoid finding the solution of a differential equation in the space of distributions, and instead solved another differential equation in the space of classical functions. This procedure also requires the enforcement of a single transition condition at each singularity. Wang and Qiao[31] analysed the vibration of beams in the presence of any type of discontinuity expressing the modal displacement function of the entire beam with N discontinuities by means of a single function through the use of distributions. The Laplace transform method was used to solve the corresponding differential equations.

Bernoulli beams under static loads in presence of discontinuities in the curvature and in the slope functions have been studied by Biondi and Caddemi.[32] They considered the theory of distributions to propose an integration procedure over a unique integration domain without enforcement of continuity conditions. A nontrivial generalization to multiple different singularities was proposed by these authors.[33]

Exact closed-form solutions, for both static and dynamic problems for beams affected by multiple damages, have been obtained by Caddemi and Calió.[34–36] The model adopted by these researchers implies a representation of the singularities by suitable Dirac's delta distributions in the beam's flexural rigidity. However, this approach implies the use of a product of two distributions operation not allowed by the classical distribution theory. In order to give some mathematical meaning to a differential equation that involves the product of two Dirac's delta distributions, a definition for this operation should be adopted. This question was treated by Bagarello[37,38] who introduced a definition of the product of two Dirac's delta distributions. This new product has been used by Caddemi and Calió in the above cited works.

Caddemi and Morassi[39] demonstrated that the formulation and solution of the bending problem for multi-cracked beams can be included in the classical formalism of the theory of distributions. Stankovik and Atanakovik[40] analysed linear discontinuous differential equations which correspond to important problems in mechanics and determined the corresponding weak solutions. Hormann and Oparnica[41] proved that a differential equation governing the transversal displacement function of beams with jump discontinuous coefficients cannot possess a distributional solution if the solution shows a jump at the same cross section. Palmieri and Cicirello[42] demonstrated that analytical formulations currently available for the use of Dirac's delta functions in the problem of cracked beams under static loads is not completely satisfactory in terms of physical consistency.

It must be noted that the classical method for solving problems in presence of concentrated cracks in beams, relies on the integration of the governing differential equations between singularities and on the enforcement of the transition conditions at those points where the singularities occur. If many discontinuities are present along the beam, this procedure turns out to be inadequate. The sub-division of the beam into sub-beams between two subsequent cracks, requires the enforcement of four continuity conditions at each point where a crack is located. In consequence, for a beam with N cracks, the statical problem requires the resolution of a system of $4(N+1)$ algebraic equations and in the case of free vibrations, the characteristic equation relies on the solution of a determinant of order $4(N+1)$. Clearly, this traditional procedure is analytically and computationally inefficient. Consequently, studies aimed at providing integration procedures able to treat singularities more efficiently have been proposed, as those based on the theory of distributions which have been described above.

In civil and mechanical engineering, there are many types of additional elements such as rotational and translational springs, elastically restrained hinges, spring roller supports, etc. Each of these devices implies different kinds of discontinuities, such as jump discontinuities in slope, deflection, bending moment, shear force, etc. The theory of distributions is particularly suited to analysing beams with an arbitrary type of discontinuities located at different points. The jump of a function located at a point c allows obtaining the distributional derivative as the sum of the classical derivative (in an interval excluding the point c) plus a measure of mass given by the jump, concentrated at c. In consequence, through successive differentiations the corresponding differential equations are obtained. In this procedure the rigorous definition of product of an infinitely differentiable function and a distribution is used.[43,44] This approach, which allows the direct application of the distributions theory and provides a better understanding of the mathematical manipulations, is applied in the present paper.

One feature of this work is to present an application of the Hamilton's principle for the derivation of the corresponding boundary value problem. A relevant product of this procedure is the determination of the transition conditions and the jumps introduced in the involved functions by intermediate additional elements. Boundary conditions are also included, but these are well known and are listed in several textbooks.[45,46] It is also the purpose of the present paper to provide the differential equations and the corresponding solutions in the distributional framework.

This paper is organized in the following way. In Sec. 2 the governing differential equations, the boundary conditions and the transition conditions, are obtained. In Sec. 3 the jump discontinuities involved in the transitions conditions are determined. In Sec. 4, a general differential equation in the framework of the theory of distributions is obtained. In Sec. 5 several particular cases and intermediate attachments are analysed. In Sec. 6 numerical results are presented. Verifications and numerical applications are also included. Finally, Sec. 7 contains the conclusions of this paper.

2. The Variation of the Energy Functional

Let us consider the beam of length l, which has elastically restrained ends and has an internal device which includes several elastic restraints. Particularly it allows a vertical displacement or the action of a hinge elastically restrained against rotation, as shown in Fig. 1. For convenience of formulation, only one discontinuity point at $x = c$ is considered on the beam. This formulation is easily extended to a beam with multiple discontinuity points as is demonstrated in Sec. 5.3.

The beam system is made up of two different spans, which correspond to the intervals $[0, c]$ and $[c, l]$ respectively, with the variable mass per unit length and variable flexural rigidity of the ith span denoted as $\rho_i A_i$ and $E_i I_i$, respectively. It is also assumed that the ends are elastically restrained against rotation and translation characterised respectively by the spring constants r_i and t_i, $i = 1, 2$. The device located at the intermediate point c has rotational restraints characterised by the spring constants r_{12} and r_c, and translational restraints characterised by the spring constants t_{12} and t_c. Adopting the values of the parameters r_i and t_i, $i = 1, 2$, all the possible combinations of classical end conditions, (i.e.: clamped, pinned, sliding and free) can be generated. On the other hand, adopting the appropriate values of the parameters r_c, r_{12}, t_c and t_{12}, different constraints on the point $x = c$ can be generated. It is supposed that the mentioned device has an internal shear guide to simulate a discontinuity in the deflection function.

In order to analyse the transverse planar displacement of the system under study, we suppose that the vertical position of the beam at any time t is described by the function:

$$u = u(x, t), \quad x \in [0, l], \; t \geq 0.$$

It is well known that at time t the kinetic energy of the beam can be expressed as

$$T_b = \frac{1}{2} \sum_{i=1}^{2} \int_{G_i} m_i(x) \left(\frac{\partial u}{\partial t}(x, t) \right)^2 dx, \tag{1}$$

where $G_1 = (0, c)$, $G_2 = (c, l)$, $m_i(x) = \rho_i(x) A_i(x)$, $i = 1, 2$.

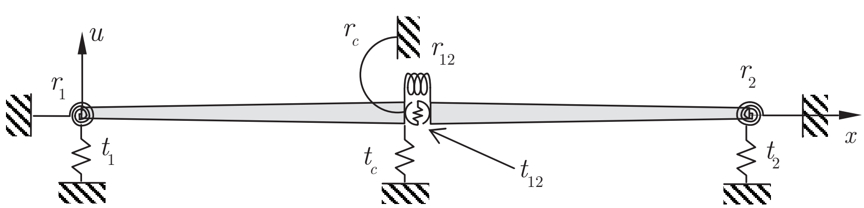


Fig. 1. Mechanical system under study.

The total potential energy due to the elastic deformation of the mechanical system is given by

$$E_p = \frac{1}{2}\left\{\sum_{i=1}^{2}\int_{G_i} e_i(x)\left(\frac{\partial^2 u}{\partial x^2}(x,t)\right)^2 dx + r_1\left(\frac{\partial u}{\partial x}(0^+,t)\right)^2 + r_c\left(\frac{\partial u}{\partial x}(c^-,t)\right)^2 \right.$$
$$+ t_1 u^2(0^+,t) + t_c u^2(c^-,t) + r_{12}\left(\frac{\partial u}{\partial x}(c^+,t) - \frac{\partial u}{\partial x}(c^-,t)\right)^2$$
$$\left. + t_{12}(u(c^+,t) - u(c^-,t))^2 + r_2\left(\frac{\partial u}{\partial x}(l^-,t)\right)^2 + t_2 u^2(l^-,t)\right\}, \tag{2}$$

where $e_i(x) = E_i(x)I_i(x)$, $i = 1, 2$. The notations $0^+, c^-, c^+$ and l^- imply the use of lateral limits and lateral derivatives. It can be observed that the strain energy due to the rotational restraint of coefficient r_c, is computed by means of the expression $0.5r_c(\partial u/\partial x)^2$ evaluated at (c^-, t), which implies that the spring is connected at the right end of the span which corresponds to the interval $[0, c]$ and is connected to a fixed wall. On the other hand, the strain energy corresponding to the rotational restraint of the internal hinge is computed by $0.5r_{12}((\partial u/\partial x)(c^+,t) - (\partial u/\partial x)(c^-,t))^2$, which implies that the spring is connected at right end of the first span and at the left end of the second span. The same situation is valid for the translational restraints respectively characterized by the coefficients t_c and t_{12}.

The potential energy of an external load $q = q(x,t)$ is given by

$$E_q = \int_G q(x,t)u(x,t)dx, \quad G = (0,l). \tag{3}$$

Hamilton's principle requires that between times t_a and t_b, at which the positions are known, the motion will make stationary the action integral $F(u) = \int_{t_a}^{t_b} L dt$ on the space of admissible functions,[47] where the Lagrangian L is given by $L = T_b - U$, and $U = E_p + E_q$. In consequence by using Eqs. (1) to (3), the energy functional to be considered is given by

$$F(u) = \frac{1}{2}\int_{t_a}^{t_b}\left[\sum_{i=1}^{2}\int_{G_i}\left(m_i(x)\left(\frac{\partial u}{\partial t}(x,t)\right)^2 - e_i(x)\left(\frac{\partial^2 u}{\partial x^2}(x,t)\right)^2\right)\right.$$
$$\left. - 2\int_G q(x,t)u(x,t)dx\right]dt - \frac{1}{2}\left[r_1\left(\frac{\partial u}{\partial x}(0^+,t)\right)^2 + r_c\left(\frac{\partial u}{\partial x}(c^-,t)\right)^2\right.$$
$$+ t_1u^2(0^+,t) + t_cu^2(c^-,t) + r_{12}\left(\frac{\partial u}{\partial x}(c^+,t) - \frac{\partial u}{\partial x}(c^-,t)\right)^2$$
$$\left. + t_{12}(u(c^+,t) - u(c^-,t))^2 + r_2\left(\frac{\partial u}{\partial x}(l^-,t)\right)^2 + t_2u^2(l^-,t)\right]. \tag{4}$$

The variation δI of a functional is a straightforward generalization of the definition of the directional derivative of a real valued function defined on a subset of $\mathbb{R}^n$. The stationary condition for the functional given by Eq. (4) requires that

$$\delta F(u; v) = 0, \quad \forall\, v \in D_a, \tag{5}$$

where $\delta F(u;v)$ is the first variation of F at u in the direction v and D_a is the space of admissible directions at u for the space D of admissible functions.[48,49] Unfortunately, the presence of a jump in the deflection function u at the point c leads to an integrand which includes the square of the derivative of a distribution. For this reason it is not possible to specify regularity assumptions of the functions involved in the functional (4) for the application of the techniques of calculus of variations. Nevertheless, it is possible to obtain the corresponding boundary value problem without discussing the regularity of the functions involved in the variational procedure by applying the techniques of calculus of variations with condition (5) in heuristic form. In this manner the following boundary value problem is obtained

$$\frac{\partial^2}{\partial x^2}\left(e_i(x)\frac{\partial^2 u}{\partial x^2}(x,t)\right)+m_i(x)\frac{\partial^2 u}{\partial t^2}(x,t)=q_i(x,t),\quad \forall\, x\in G_i,\ i=1,2, \tag{6}$$

$$r_1\frac{\partial u}{\partial x}(0^+,t)=e(0^+)\frac{\partial^2 u}{\partial x^2}(0^+,t), \tag{7}$$

$$t_1 u(0^+,t)=-\frac{\partial}{\partial x}\left(e(0^+)\frac{\partial^2 u}{\partial x^2}(0^+,t)\right), \tag{8}$$

$$r_{12}\left(\frac{\partial u}{\partial x}(c^+,t)-\frac{\partial u}{\partial x}(c^-,t)\right)-r_c\frac{\partial u}{\partial x}(c^-,t)=e(c^-)\frac{\partial^2 u}{\partial x^2}(c^-,t), \tag{9}$$

$$r_{12}\left(\frac{\partial u}{\partial x}(c^+,t)-\frac{\partial u}{\partial x}(c^-,t)\right)=e(c^+)\frac{\partial^2 u}{\partial x^2}(c^+,t), \tag{10}$$

$$t_{12}(u(c^+,t)-u(c^-,t))=-\frac{\partial}{\partial x}\left(e(c^-)\frac{\partial^2 u}{\partial x^2}(c^-,t)\right)+t_c u(c^-,t), \tag{11}$$

$$t_{12}(u(c^+,t)-u(c^-,t))=-\frac{\partial}{\partial x}\left(e(c^+)\frac{\partial^2 u}{\partial x^2}(c^+,t)\right), \tag{12}$$

$$r_2\frac{\partial u}{\partial x}(l^-,t)=-e(l^-)\frac{\partial^2 u}{\partial x^2}(l^-,t), \tag{13}$$

$$t_2 u(l^-,t)=\frac{\partial}{\partial x}\left(e(l^-)\frac{\partial^2 u}{\partial x^2}(l^-,t)\right),\quad t\geq 0. \tag{14}$$

3. Derivation of Jump Functions

The presence of the considered internal elastic restraints generates discontinuities in the deflection function u and some or all of its derivatives $\partial^k u/\partial x^k$, $k=1,2,3$, and for this reason, in the application of the distribution theory it is essential to determine all the jumps occurring in the involved functions. So, it is convenient to define a function $s(f)$ that determines the jump of another function f, at a certain point c and is given by

$$s(f):G\to\mathbb{R},\quad s(f)(c)=f(c^+)-f(c^-),\ G\subset\mathbb{R},\ c\in G. \tag{15}$$

It is well known that for a differential equation of order $2m$, the boundary conditions containing the function u and derivatives of this function of orders not greater than $m-1$, are called *stable* or *geometric* and those containing derivatives of orders higher than $m-1$, are called *unstable* or *natural*.[50,51]

Since the domain of work of the problem of the beam described above is given by $G=(0,l)$ and this is an open interval in the space $\mathbb{R}$, the boundary ∂G is given by the points 0 and l. Consequently, the point c is an interior point of G and the equations formulated at $x=c$ can be called *transition conditions*. In what follows we extend the above classification of boundary conditions to the transition conditions that are given by Eqs. (9) to (12).

The function given by Eq. (15) allows expressing the jumps involved in the transition conditions in Eqs. (9) to (12) as follows:

$$s(u)(c,t) = -\frac{1}{t_{12}}\frac{\partial}{\partial x}\left(e(c^-)\frac{\partial^2 u}{\partial x^2}(c^-,t)\right) + \frac{t_c}{t_{12}}u(c^-,t), \tag{16}$$

$$s\left(\frac{\partial u}{\partial x}\right)(c,t) = \frac{e(c^-)}{r_{12}}\frac{\partial^2 u}{\partial x^2}(c^-,t) + \frac{r_c}{r_{12}}\frac{\partial u}{\partial x}(c^-,t), \tag{17}$$

$$s(u)(c,t) = -\frac{1}{t_{12}}\frac{\partial}{\partial x}\left(e(c^+)\frac{\partial^2 u}{\partial x^2}(c^+,t)\right), \tag{18}$$

$$s\left(\frac{\partial u}{\partial x}\right)(c,t) = \frac{e(c^+)}{r_{12}}\frac{\partial^2 u}{\partial x^2}(c^+,t). \tag{19}$$

Let us consider the bending moment and the shear force, given respectively by the functions:

$$M = e\frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad V = \frac{\partial}{\partial x}\left(e\frac{\partial^2 u}{\partial x^2}\right).$$

If Eq. (9) is subtracted from Eq. (10), we have

$$s(M)(c,t) = r_c\frac{\partial u}{\partial x}(c^-,t). \tag{20}$$

Finally, if Eq. (11) is subtracted from Eq. (12), we have

$$s(V)(c,t) = -t_c u(c^-,t). \tag{21}$$

4. The Differential Equation in the Framework of Distributions

Now it is possible to reformulate the boundary value problem (6)–(14) in the distributional framework. The purpose is to introduce the jump conditions at $x=c$ within the differential equation which governs the deformation of the beam. Before presenting the differential equations derivation, we recall briefly some preliminary results.[43,44]

Let G be an open set in R. A distribution $\tilde{f}$ is a continuous linear functional

$$\tilde{f} : \mathscr{D}(G) \to \mathbb{R},$$

where $\mathscr{D}(G)$ denotes the space of test functions. The space of all distributions is denoted by $\mathscr{D}'(G)$. The Dirac's delta "function" δ can now be rigorously defined as the distribution $\delta_c \in \mathscr{D}'(G)$, given by

$$\delta_c(\varphi) = \varphi(c), \quad \forall\, \varphi \in \mathscr{D}(G).$$

Let the function $f \in C^1(\Omega)$ where $\Omega = G - \{c\}$, $c \in G$ and suppose that f has a jump discontinuity at $x = c$, where $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist and are finite. Suppose also that the classical derivative of f in Ω is a bounded function in Ω. Then, the derivative of f *in the sense of distributions* is denoted by $\tilde{f}'$, and is given by

$$\tilde{f}'(x) = f'(x) + s(f)(c)\delta(x - c) \tag{22}$$

where f' denotes the classical derivative of f in Ω, $\delta(x - c)$ denotes the Dirac's delta distribution and $s(f)(c)$ is given by Eq. (15).

Finally, we recall the definition of the product of a distribution and a function. Let the distribution be $\tilde{f} \in \mathscr{D}'(G)$ and the infinitely differentiable function $\alpha \in C^\infty(G)$. Then the product of α and $\tilde{f}$ is a distribution $\alpha\tilde{f}$ given by

$$\langle \alpha\tilde{f}, \varphi\rangle = \langle \tilde{f}, \alpha\varphi\rangle, \quad \forall\, \varphi \in \mathscr{D}(G),$$

where $\langle \tilde{f}, \varphi\rangle$ denotes the application of the distribution $\tilde{f}$ to the test function φ.

In the following, since functions of one or two variables will be used, a mixed notation of derivatives is adopted. If the function u, which describes the vertical position of the beam, and its derivatives $\partial^k u/\partial x^k$, $k = 1, 2, 3$ have a jump discontinuity at $x = c$, the first distributional derivative can be obtained by using Eq. (22), so we have

$$\frac{\tilde{\partial} u}{\partial x}(x,t) = \frac{\partial u}{\partial x}(x,t) + s(u)(c,t)\delta(x - c), \tag{23}$$

where $\partial u/\partial x$ is defined $\forall\,(x,t) \in \Omega \times [0,T]$ for some fixed time $T > 0$ and

$$\Omega = G - \{c\} = (0,c) \cup (c,l), \quad G = (0,l).$$

It can be observed that the derivative $\partial u/\partial x$ is defined $\forall x \in (0,c)$ and $\forall\, x \in (c,l)$, but is undefined at $x = c$. Taking the derivative with respect to x of Eq. (23), multiplying by $e \in C^\infty(G)$ and using the distributional properties of δ and δ', we have[43,44]

$$\begin{aligned} e(x)\frac{\tilde{\partial}^2 u}{\partial x^2}(x,t) = e(x)\frac{\partial^2 u}{\partial x^2}(x,t) &+ s(u)(c,t)e(c)\delta'(x - c) \\ &- s(u)(c,t)e'(c)\delta(x - c) + s(\partial u/\partial x)(c,t)e(c)\delta(x - c). \end{aligned}$$

Taking two more derivatives we get

$$\begin{aligned}\frac{\tilde{\partial}^2}{\partial x^2}\left(e(x)\frac{\tilde{\partial}^2 u}{\partial x^2}(x,t)\right) = {} & \frac{\partial^2}{\partial x^2}\left(e(x)\frac{\partial^2 u}{\partial x^2}(x,t)\right) + e(c)s(u)(c,t)\delta'''(x-c) \\ & - s(u)(c,t)e'(c)\delta''(x-c) + e(c)s\left(\frac{\partial u}{\partial x}\right)(c,t)\delta''(x-c) \\ & + s(M)(c,t)\delta'(x-c) + s(V)(c,t)\delta(x-c), \qquad (24)\end{aligned}$$

where the derivative $\frac{\partial^2}{\partial x^2}(e\frac{\partial^2 u}{\partial x^2})$ is defined $\forall\,(x,t) \in \Omega \times [0,T]$. Finally, by substituting Eq. (24) into Eq. (6) we have

$$\begin{aligned}&\frac{\tilde{\partial}^2}{\partial x^2}\left(e(x)\frac{\tilde{\partial}^2 u}{\partial x^2}(x,t)\right) + m(x)\frac{\partial^2 u}{\partial t^2}(x,t) \\ &\quad = q(x,t) + e(c)s(u)(c,t)\delta'''(x-c) - s(u)(c,t)e'(c)\delta''(x-c) \\ &\qquad + e(c)s\left(\frac{\partial u}{\partial x}\right)(c,t)\delta''(x-c) + s(M)(c,t)\delta'(x-c) + s(V)(c,t)\delta(x-c). \qquad (25)\end{aligned}$$

The preceding equation coincides in a particular case with that obtained by Wang and Quiao.[31] It should be noted that according to Eq. (6), the functions e, m and q are respectively given by

$$e(x) = \begin{cases} e_1(x), & \forall\, x \in (0,c), \\ e_2(x), & \forall\, x \in (c,l), \end{cases} \qquad m(x) = \begin{cases} m_1(x), & \forall\, x \in (0,c), \\ m_2(x), & \forall\, x \in (c,l), \end{cases}$$

and

$$q(x,t) = \begin{cases} q_1(x,t), & \forall\, x \in (0,c), \\ q_2(x,t), & \forall\, x \in (c,l). \end{cases}$$

The functions m and q must be continuous, but according to the above definition of product of a function and a distribution, the function e must be in the space $C^\infty(0,l)$. Since Eq. (25) involves distributional derivatives, it must be included in the space of distributions $\mathscr{D}'(0,l)$.

5. Analysis of Particular Cases

5.1. *Free vibration*

The differential equation that describes the free vibration of the beam is derived by assuming $q \equiv 0$, in Eq. (25) and in the case:

$$e(x) = EI, \quad m(x) = \rho A, \quad \forall\, x \in [0,l],$$

the differential equation reduces to

$$\frac{\tilde{\partial}^4 u}{\partial x^4}(x,t) + \frac{\rho A}{EI}\frac{\partial^2 u}{\partial t^2}(x,t)$$
$$= s(u)(c,t)\delta'''(x-c) + s\left(\frac{\partial u}{\partial x}\right)(c,t)\delta''(x-c)$$
$$+ s\left(\frac{\partial^2 u}{\partial x^2}\right)(c,t)\delta'(x-c) + s\left(\frac{\partial^3 u}{\partial x^3}\right)(c,t)\delta(x-c) \quad \text{in } \mathscr{D}'(0,l). \quad (26)$$

It should be noted that the preceding equation coincides with the one obtained by Wang and Quiao.[31]

For the beam under free vibration, the variable separation method can be used. In accordance we adopt

$$u(x,t) = w(x)\cos\omega t,$$

where $w = w(x)$ denotes the modal displacement of the beam and ω the natural circular frequency. It is easy to demonstrate that for $k = 0, 1, 2, 3$ we have

$$s\left(\frac{\partial^k u}{\partial x^k}\right)(c,t) = s(w^{(k)})(c)\cos\omega t,$$

so Eq. (26) reduces to

$$\frac{\tilde{d}^4 w}{dx^4}(x) - \frac{\rho A\omega^2}{EI}w(x) = s(w)(c)\delta'''(x-c) + s(w')(c)\delta''(x-c)$$
$$+ s(w'')(c)\delta'(x-c) + s(w''')(c)\delta(x-c) \quad \text{in } \mathscr{D}'(0,l). \quad (27)$$

5.2. *Statical behavior*

Let us consider the statical behavior of the mechanical system described, when a load $q = q(x)$, which causes a transverse deflection $u = u(x)$, is applied. The governing differential equation is given by

$$\frac{\tilde{d}^2}{dx^2}\left(e(x)\frac{\tilde{d}^2 w}{dx^2}u(x)\right) = q(x) + e(c)s(u)(c)\delta'''(x-c)$$
$$- s(u)(c)e'(c)\delta''(x-c) + e(c)s(u')(c)\delta''(x-c)$$
$$+ s(eu'')(c)\delta'(x-c) + s((eu'')')(c)\delta(x-c) \quad \text{in } \mathscr{D}'(0,l). \quad (28)$$

In the case $e(x) = EI$, $\forall\, x \in [0,l]$ the above equation reduces to

$$\frac{\tilde{d}^4 w}{dx^4}(x) = \frac{q(x)}{EI} + s(u)(c)\delta'''(x-c) + s(u')(c)\delta''(x-c)$$
$$+ s(u'')(c)\delta'(x-c) + s(u''')(c)\delta(x-c) \quad \text{in } \mathscr{D}'(0,l). \quad (29)$$

5.3. *Generalization*

The above procedure can be generalized to beams with a finite number N_c of cracks located at the points c_i. In this case the differential Eq. (27) can be expressed as

$$\frac{\tilde{d}^4 w}{dx^4}(x) - \frac{\rho A \omega^2}{EI} w(x) = \sum_{i=1}^{N_c} \sum_{k=0}^{3} s(w^{(k)})(c_i) \delta^{(3-k)}(x - c_i) \quad \text{in } \mathscr{D}'(0, l).$$

In the same manner the differential Eq. (29) can be expressed as

$$\frac{\tilde{d}^4 w}{dx^4}(x) = \frac{q(x)}{EI} + \sum_{i=1}^{N_c} \sum_{k=0}^{3} s(w^{(k)})(c_i) \delta^{(3-k)}(x - c_i) \quad \text{in } \mathscr{D}'(0, l).$$

5.4. *Intermediate attachments*

Equation (25) was obtained assuming that the function u, which describes the vertical position of the beam, and its derivatives $\partial^k u/\partial x^k$, $k = 1, 2, 3$ have a jump discontinuity at $x = c$. These discontinuities are induced by different attachments. It is possible to generate several particular cases by adopting the appropriate parameter values of the general restraint device described in Sec. 2. Thus, Eqs. (16) to (21) and other situations can be generated by substituting values or limiting values of the restraint parameters r_c, r_{12}, t_c and t_{12}, into the transition conditions in Eqs. (9)–(12).

5.4.1. *Elastic rotational restraint connected to a fixed wall*

The presence of a rotational restraint at $x = c$ is generated by adopting $r_{12} \to \infty$, $t_{12} \to \infty$, $t_c = 0$ and $0 < r_c < \infty$, in Eqs. (9)–(12). Substituting the condition $t_{12} \to \infty$ in Eq. (12) and $r_{12} \to \infty$ in Eq. (10) leads to the continuity conditions

$$u(c^-, t) = u(c^+, t), \tag{30}$$

$$\frac{\partial u}{\partial x}(c^-, t) = \frac{\partial u}{\partial x}(c^+, t). \tag{31}$$

These equations imply that in this case, there is no internal hinge and the articulation is perfectly rigid. Finally, the condition $0 < r_c < \infty$ implies the existence of a rotational spring connected to the beam and to a fixed wall.

Substituting Eq. (10) into Eq. (9) leads to

$$r_c \frac{\partial u}{\partial x}(c^-, t) = e(c^+) \frac{\partial^2 u}{\partial x^2}(c^+, t) - e(c^-) \frac{\partial^2 u}{\partial x^2}(c^-, t), \tag{32}$$

which coincides with Eq (20). If we let $r_c \to \infty$ in Eq. (32) and Eq. (31) is used, we have the *geometric* transition condition

$$\frac{\partial u}{\partial x}(c^-, t) = \frac{\partial u}{\partial x}(c^+, t) = 0.$$

5.4.2. *Elastic traslational restraint connected to a fixed wall*

The presence of a traslational restraint al $x = c$ is generated by adopting $r_{12} \to \infty$, $t_{12} \to \infty$, $r_c = 0$, $0 < t_c < \infty$ in Eqs. (9)–(12). An analogous analysis to that used in the previous case, leads to

$$t_c u(c^-, t) = \frac{\partial}{\partial x}\left(e(c^-)\frac{\partial^2 u}{\partial x^2}(c^-, t)\right) - \frac{\partial}{\partial x}\left(e(c^+)\frac{\partial^2 u}{\partial x^2}(c^+, t)\right), \tag{33}$$

which coincides with Eq. (21). If we let $t_c \to \infty$ in Eq. (33) and Eq. (30) is used, we have the *geometric transition* condition

$$u(c^-, t) = u(c^+, t) = 0.$$

5.4.3. *Elastic rotational restraint connected at two points of the beam*

The presence of this kind of restraint at $x = c$ is generated by adopting $r_c = 0, t_{12} \to \infty, t_c = 0, 0 < r_{12} < \infty$ in Eqs. (9)–(12). In this case we obtain

$$r_{12}\left(\frac{\partial u}{\partial x}(c^+, t) - \frac{\partial u}{\partial x}(c^-, t)\right) = e(c^-)\frac{\partial^2 u}{\partial x^2}(c^-, t) = e(c^+)\frac{\partial^2 u}{\partial x^2}(c^+, t), \tag{34}$$

which coincides with Eq. (19). This case can be modeled by an internal hinge elastically restrained against rotation. If we let $r_{12} \to 0$ in Eq. (34), the *natural* transition condition is obtained

$$\frac{\partial^2 u}{\partial x^2}(c^-, t) = \frac{\partial^2 u}{\partial x^2}(c^+, t) = 0,$$

which corresponds to a free internal hinge and the articulation is perfect.

5.4.4. *Elastic translational restraint fixed at two points of the beam*

The presence of this kind of restraint at $x = c$ is generated by adopting $r_c = 0$, $r_{12} = 0$, $t_c = 0$, $0 < t_{12} < \infty$ in Eqs. (9)–(12). This elastic restraint introduces a discontinuity of deflection at $x = c$ and from Eqs. (11) and (12) we have

$$t_{12}(u(c^+, t) - u(c^-, t)) = -\frac{\partial}{\partial x}\left(e(c^-)\frac{\partial^2 u}{\partial x^2}(c^-, t)\right) = -\frac{\partial}{\partial x}\left(e(c^+)\frac{\partial^2 u}{\partial x^2}(c^+, t)\right), \tag{35}$$

which coincides with Eq. (18). Now if we let $t_{12} \to 0$ in Eq. (35) we have a case of a free internal shear guide and the *natural* transition condition

$$\frac{\partial}{\partial x}\left(e(c^-)\frac{\partial^2 u}{\partial x^2}(c^-, t)\right) = \frac{\partial}{\partial x}\left(e(c^+)\frac{\partial^2 u}{\partial x^2}(c^+, t)\right) = 0.$$

Following this procedure, all the transition conditions for the most relevant situations which arise from combination of the above considered cases, can be obtained.

6. Numerical Results

For the beam in free transverse vibrations, we consider Eq. (27), i.e.

$$\frac{\tilde{d}^4 w}{dx^4}(x) - \frac{\rho A\omega^2}{EI}w(x) = s(w)(c)\delta'''(x-c) + s(w')(c)\delta''(x-c) \\ + s(w'')(c)\delta'(x-c) + s(w''')(c)\delta(x-c) \quad \text{in } \mathscr{D}'(0,l). \quad (36)$$

In the case of the generally restrained beam described previously, the corresponding boundary conditions are given by Eqs. (7), (8), (13) and (14) with $u(x,t)$ replaced by $w(x)$. The equation in Eq. (36) can be solved by representing the solution as $w = w_h + w_p$, where w_h denotes the general solution of the homogeneous equation and w_p denotes a particular solution of Eq. (36). The solution of the homogeneous equation for Eq. (36) is the well-known linear combination

$$w_h(x) = c_1\cosh(\lambda x) + c_2\sinh(\lambda x) + c_3\cos(\lambda x) + c_4\sin(\lambda x),$$

where $\lambda^4 = \rho A\omega^2/EI$ and the constants c_i are determined by the boundary conditions at $x = 0$ and $x = l$. The particular solution of Eq. (36) can be obtained by assuming

$$w_p(x) = v(x)H(x-c), \quad (37)$$

where $H(x-c)$ is the well-known Heaviside function and v is unknown. This function can be obtained by substituting Eq. (37) and its derivatives into Eq. (36) and equating the coefficients of $H, \delta, \delta', \delta''$ and δ''' on both sides of the equation. This procedure leads to an initial value problem for v which can be solved by the method commonly known in the classical differential equations theory. For instance, let us consider the differential equation

$$\frac{\tilde{d}^4 w}{dx^4}(x) - \lambda^4 w(x) = s(w')(c)\delta''(x-c) \quad \text{in } \mathscr{D}'(0,l). \quad (38)$$

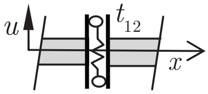
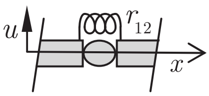
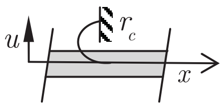
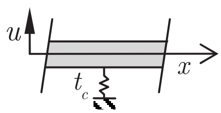
The substitution of (37) and its derivatives into Eq. (38) and equating the coefficients of $H, \delta, \delta', \delta''$ and δ''' on both sides of the equation leads to

$$w_p(x) = v(x)H(x-c),$$
$$v(x) = \frac{s(w')(c)}{2\lambda}(\sinh(\lambda(x-c)) + \sin(\lambda(x-c))).$$

Table 1 also includes the particular solutions corresponding to different cases analysed in Sec. 4. In this table the following notation is used:

$$v_1(x) = \frac{1}{2}(\cosh(\lambda x) + \cos(\lambda x)), \quad v_2(x) = \frac{1}{2\lambda}(\sinh(\lambda x) + \sin(\lambda x)),$$
$$v_3(x) = \frac{1}{2\lambda^2}(\cosh(\lambda x) - \cos(\lambda x)), \quad v_4(x) = \frac{1}{2\lambda^3}(\sinh(\lambda x) - \sin(\lambda x)),$$

Table 1. Jump functions, differential equations and particular solutions.

Device	Jump	Differential equation	Particular solution
	$-\frac{EI}{t_{12}}\frac{d^3w}{dx^3}(c^-)$	$L(w)=s(w)(c)\delta'''(x-c)$	$w_p(x)=s(w)(c)v_1(x-c)H(x-c)$
	$\frac{EI}{r_{12}}\frac{d^2w}{dx^2}(c^-)$	$L(w)=s(w')(c)\delta''(x-c)$	$w_p(x)=s(w')(c)v_2(x-c)H(x-c)$
	$\frac{r_c}{EI}\frac{dw}{dx}(c^-)$	$L(w)=s(w'')(c)\delta'(x-c)$	$w_p(x)=s(w'')(c)v_3(x-c)H(x-c)$
	$-\frac{t_c}{EI}w(c^-)$	$L(w)=s(w''')(c)\delta(x-c)$	$w_p(x)=s(w''')(c)v_4(x-c)H(x-c)$

and

$$L(w)=\frac{\tilde{d}^4 w}{dx^4}(x)-\lambda^4 w(x).$$

In order to establish the accuracy and applicability of the approach developed, numerical results were computed for a cantilevered beam for which the reference values are available in the literature. The results were obtained by introducing two cracks at two locations of the cantilevered beam, as shown in Fig. 2, and compared with the uncracked condition.

The values of the frequency parameters

$$\lambda_i=\left(\frac{\rho A}{EI}\omega_i^2\right)^{1/4},$$

were obtained with the classical bisection method. Then, the ratios between the first three natural frequencies of the cracked and uncracked beam were calculated, i.e. $\bar{\lambda}_i/\lambda_i,\ i=1,2,3$, where $\bar{\lambda}_i$ and λ_i are the frequency coefficients of the cracked and uncracked beam, respectively. The first crack location is given by $c_1=0.15$, whereas the second crack position c_2 varies in the interval $[0,1]$. The cracks severities are given by the ratios $\eta_i=a_i/h$, where a_i is the depth of the ith crack and h is the height of the rectangular cross-section of the beam. In all cases the following values have been adopted:

$$\eta_1=a_1/h=0.1 \quad \text{and} \quad \eta_2=a_2/h=0.1, \quad 0.2 \text{ and } 0.3.$$

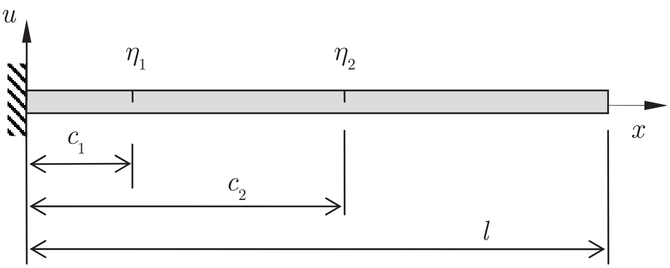


Fig. 2. Cantilevered beam with two cracks. $\eta_i = a_i/h$, $i = 1, 2$.

By using the method developed in the present paper, numerical computations have been carried out and results were compared with those given by Shifrin and Ruotolo.[13] The graphics in Figs. 3, 4 and 5, where continuous and discontinuous lines denote results of this study, show quite close agreement with those obtained by Shifrin and Ruotolo.[13]

Additionally, new numerical results were generated for a beam with one end and an intermediate point elastically restrained. Table 2 lists the values of the first six frequency parameters λ_i^* for a beam with one end elastically restrained against rotation, the other end simply supported, and the intermediate point $c^* = 0.4$ restrained by a spring with elastic constant r_{12}.

In Table 2 different values of the rotational restriction $R_1 = r_1 l/(EI)$ are considered. The remaining parameters are given by

$$R_c = r_c l/(EI), \quad R_{12} = r_{12} l/(EI), \quad R_2 = r_2 l/(EI), \quad T_1 = t_1 l^3/(EI),$$

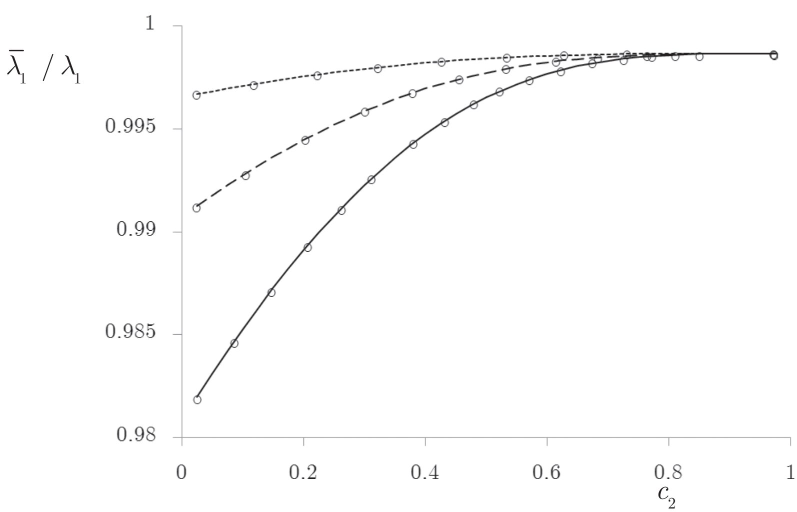


Fig. 3. Comparison of first frequency coefficients $\bar{\lambda}_1/\lambda_1$ of a cantilevered beam with two cracks, $c_1 = 0.15$, $\eta_1 = 0.1$ and ---- $\eta_2 = 0.1$, – – $\eta_2 = 0.2$, —— $\eta_2 = 0.3$; ∘ Shifrin and Ruotolo.[13]

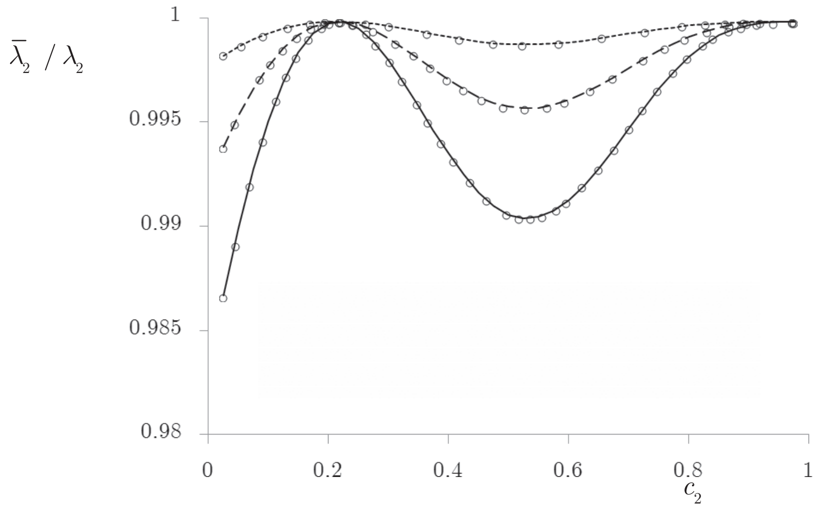


Fig. 4. Comparison of the second frequency coefficients $\bar{\lambda}_2/\lambda_2$ of a cantilevered beam with two cracks, $c_1 = 0.15$, $\eta_1 = 0.1$ and ---- $\eta_2 = 0.1$, -- $\eta_2 = 0.2$, —— $\eta_2 = 0.3$; ○ Shifrin and Ruotolo.[13]

$T_c = t_c l^3/(EI)$, $T_{12} = t_{12} l^3/(EI)$ and $T_2 = t_2 l^3/(EI)$. The frequency parameters are defined as

$$\lambda_i^* = \left(\frac{\rho A}{EI}\omega_i^2\right)^{1/4} l,$$

with the following values adopted:

$$R_c = R_2 = T_c = 0, \quad T_1 = T_2 = T_{12} = \infty, \quad R_{12} = 10 \text{ and } c^* = c/l = 0.4.$$

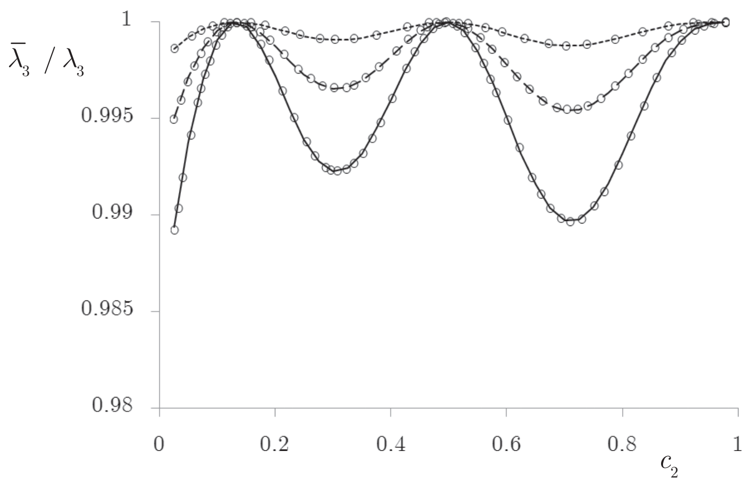


Fig. 5. Comparison of third frequency coefficients $\bar{\lambda}_3/\lambda_3$ of a cantilevered beam with two cracks, $c_1 = 0.15$, $\eta_1 = 0.1$ and ---- $\eta_2 = 0.1$, -- $\eta_2 = 0.2$, —— $\eta_2 = 0.3$; ○ Shifrin and Ruotolo.[13]

Table 2. First six frequency parameters λ_i^* for a beam with one end elastically restrained against rotation, the other end simply supported, and the intermediate point $c^* = 0.4$, elastically restrained.
$R_c = R_2 = T_c = 0, T_1 = T_2 = T_{12} = \infty$ and $R_{12} = 10$.

	Modal sequence					
R_1	1	2	3	4	5	6
0	3.013049	6.193163	9.290986	12.164007	15.707963	18.282046
0.001	3.013230	6.193233	9.291050	12.164040	15.707995	18.282078
0.01	3.014851	6.193861	9.291629	12.164336	15.708281	18.282365
0.1	3.030712	6.200089	9.297378	12.167284	15.711134	18.285227
1	3.161077	6.256958	9.351570	12.195671	15.738551	18.313171
10	3.589833	6.543086	9.680792	12.396958	15.931375	18.534957
100	3.831665	6.809479	10.098501	12.761689	16.275231	19.082064
1000	3.867617	6.858169	10.187538	12.862797	16.368142	19.273845
∞	3.871807	6.864013	10.198452	12.875891	16.380044	19.299903

7. Conclusions

In this work, the application of the theory of distributions to the statical and dynamical behavior of beams affected by cracks has been presented. The damage has been implemented in the form of a crack modeled by a constrained intermediate point and an internal device which allows a vertical displacement affected by a translational restraint and a hinge elastically restrained against rotation.

Hamilton's principle has been applied to obtaining the transition conditions and particularly the analytical expressions of the jumps of the involved functions. The governing differential equations of an Euler–Bernoulli beam with jump discontinuities in the deflection function u and its derivatives $\partial^k u/\partial x^k, k = 1, 2, 3$, have been derived in the space of distributions $\mathscr{D}'(0, l)$.

One of the contributions of this work is that if offers a more clear understanding of the mathematical description of the beam problem considered. Thus, a formulation and solution of the statical problem and the free vibration problem of a beam with multiple cracks has been treated in the realm of the theory of distributions. In consequence, a simple, computationally efficient and accurate algorithm has been developed for the mentioned problems. Although numerical results are presented for beams with two cracks, the algorithm developed is applicable for any number of cracks.

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