On the eigenvalues of some nonhermitian oscillators

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Abstract. We consider a class of one-dimensional nonhermitian oscillators and discuss the relationship between the real eigenvalues of PT-symmetric oscillators and the resonances obtained by different authors. We also show the relationship between the strong-coupling expansions for the eigenvalues of those oscillators. Comparison of the results of the complex rotation and the Riccati-Padé methods reveals that the optimal rotation angle converts the oscillator into either a PT-symmetric or an Hermitian one. In addition to the real positive eigenvalues the PT-symmetric oscillators exhibit real positive resonances under different boundary conditions. They can be calculated by means of the straightforward diagonalization method. The Riccati-Padé method yields not only the resonances of the nonhermitian oscillators but also the eigenvalues of the PT-symmetric ones.

1. Introduction

In a recent paper Jentschura et al[1] discussed the resonances for the anharmonic oscillator $H = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{1}{2}q^2 + \sqrt{g}q^3$and their weak- and strong-coupling expansions. They showed analytical expressions for the coefficients of the former and numerical estimates for those of the latter. In particular, the leading coefficients of the strong-coupling expansions are the eigenvalues of $H = -\frac{1}{2}\frac{d^2}{dq^2} + q^3$.

Some time earlier Bender and Boettcher[2] had discussed the eigenvalues of PT-symmetric oscillators of the form $H = -\frac{d^2}{dx^2} - (ix)^N$ that exhibit a finite number of real positive eigenvalues for $1 < N < 2$ and an infinite number when $N \geq 2$.

Alvarez[3] discussed the analytical properties of the solutions of the Hamiltonian operator $H = \frac{1}{2}p^2 + \frac{1}{2}kx^2 + gx^3$ and showed that it supports real and complex resonances depending on the complex values of the coupling constant g. His results suggest that the resonances calculated by Jentschura et al[1] and the real eigenvalues obtained by Boettcher and Bender[2] (see also Bender[9]) may by related in a simple way by means of the Symanzik scaling[4] already invoked by Alvarez is his investigation[3]. In exactly the same way the strong-coupling expansion obtained by Jentschura et al[1] may be related to that obtained some time earlier by Fernández et al[5] for the PT-symmetric oscillator $H = p^2 + ix^3 + \lambda x^2$. The purpose of this paper is the exploration into such relationships as well as into other properties of a class of nonhermitian oscillators.

In section 2 we investigate the relationship among some of the earlier results on one-dimensional nonhermitian oscillators. In section 3 we discuss the application of the complex-rotation[6] and Riccati-Padé[7, 8] methods to those oscillators. Finally, in section 4 we summarize the main results and draw conclusions.

2. Real and complex eigenvalues

As outlined above, Jentschura et al[1] discussed several properties of the resonances for the anharmonic oscillator

$$H_c = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{1}{2}q^2 + \sqrt{g}q^3 \tag{1}$$

as well as their weak-coupling

$$E_n(g) = \sum_{k=0}^{\infty} E_{n,k} g^k \tag{2}$$

and strong-coupling expansions

$$E_n(g) = g^{1/5} \sum_{k=0}^{\infty} L_{n,k} g^{-2k/5} \tag{3}$$

The coefficients of the former can be obtained exactly by means of perturbation theory and those of the latter in a numerical way. In particular, the leading coefficients of the strong-coupling expansions $L_{n,0}$ are the eigenvalues of the pure cubic anharmonic oscillator

$$H_l = -\frac{1}{2}\frac{d^2}{dq^2} + q^3 \tag{4}$$

On the other hand, the closely related PT-symmetric oscillators

$$H_{PT} = -\frac{d^2}{dx^2} - (ix)^N \tag{5}$$

exhibit an infinite number of real positive eigenvalues when $N \geq 2$[2] and accurate results for $N = 3$ and $N = 4$ are available for comparison[9].

It is not difficult to obtain a connection between the results outlined above by means of the Symanzik scaling

$$U^{\dagger}pU = \gamma^{-1}p,\ U^{\dagger}xU = \gamma x \tag{6}$$

where U is a well known unitary operator[4]. This transformation was already used by Alvarez in his investigation of the cubic anharmonic oscillator[3]. For example, if we take into account that $2\gamma^2U^{\dagger}H_lU = H_{PT}$ when $\gamma = (i/2)^{1/5}$ then we realize that the complex eigenvalues $L_{n,0}$ of H_l calculated by Jentschura et al[1] and the real positive eigenvalues E_n^{PT} of H_{PT} for $N = 3$ calculated by Boettcher and Bender[2] and Bender[9] are related by

$$L_{n,0} = 2^{-3/5}i^{-2/5}E_n^{PT} \tag{7}$$

Some time ago Fernández et al[5] obtained the perturbation expansion for the PT-symmetric oscillator

$$H_F = p^2 + ix^3 + \lambda x^2 \tag{8}$$

in the form

$$E_n(\lambda) = \sum_{j=0}^{\infty} W_{n,j}\lambda^j \tag{9}$$

Arguing as before, we can obtain the coefficients of the strong-coupling expansion (3) from those of the perturbation series (9) as follows:

$$L_{n,j} = 2^{-(4j+3)/5}i^{(4j-2)/5}W_{n,j} \tag{10}$$

The first coefficients are shown in Table 1 as an illustrative example.

3. Complex-rotation and Riccati-Padé methods

For concreteness we consider the family of anharmonic oscillators

$$\begin{aligned} H_K &= \frac{1}{2}p^2 - x^K, \\ H_K\psi &= E\psi \end{aligned} \tag{11}$$

The complex-rotation method (CRM) consists of the diagonalization of the rotated Hamiltonian operator

$$U^{\dagger}H_{K}U=\gamma^{-2}\left(\frac{1}{2}p^{2}-\gamma^{K+2}x^{K}\right) \tag{12}$$

where $\gamma=\eta e^{i\theta}$. The parameter $\eta>0$ produces a dilatation or contraction of the scale and θ a rotation of the coordinate in the complex x-plane. On tuning η we improve the rate of convergence of the diagonalization method as the matrix dimension increases and the value of θ enables us to uncover the resonances[6]. For the diagonalization method we choose the basis set of eigenfunctions of the harmonic oscillator $H=p^{2}+x^{2}$.

For comparison purposes we also apply the Riccati-Padé method (RPM) for asymmetric potentials[8]. It consists of the expansion of the logarithmic derivative of the eigenfunction $\psi(x)$

$$f(x)=-\frac{\psi'(x)}{\psi(x)} \tag{13}$$

in a Taylor series about the origin

$$f(x)=\sum_{j=0}^{\infty}f_{j}x^{j} \tag{14}$$

where the coefficients f_j depend on the two unknowns E and $f_0=-\psi'(0)/\psi(0)$. From the coefficients of the even and odd powers of the coordinate $f_{e,j}=f_{2j}$ and $f_{o,j}=f_{2j-1}$, $j=1,2,\ldots$, respectively, we construct the Hankel determinants $H_{D}^{ed}(E,f_0)=\left|f_{e,i+j+d-1}\right|_{i,j=1}^{D}$, $H_{D}^{od}(E,f_0)=\left|f_{o,i+j+d-1}\right|_{i,j=1}^{D}$ and obtain both E and f_0 from the roots of the system of nonlinear equations $\{H_{D}^{ed}(E,f_0)=0,\ H_{D}^{od}(E,f_0)=0\}$. For every fixed value of $d=0,1,\ldots$ we look for convergent sequences of roots $E^{[D,d]}$, $D=2,3,\ldots$. Commonly, we obtain reasonable results for $d=0$ but calculations with other values of d enable us to test the consistency of the method.

Since the rate of convergence of the RPM is considerably greater than the one for the CRM we choose the results of the former as exact or reference eigenvalues. Figure 1 shows $\log\left|\left(E_{n}^{RPM}-E_{n}^{CRM}\right)/E_{n}^{RPM}\right|$ as a function of θ for the first resonances of the cubic oscillator ($K=3$). Those results suggest that the minimum of the logarithmic deviation appears at $\theta=\pi/10$ (in all our calculations we have chosen

$\eta = e^{-1}$ that provides a reasonable rate of convergence). In order to understand this empirical result we resort to the scaling transformation (12) for $K = 3, 5, \ldots$. It is clear that $U^{\dagger}H_K U = \gamma_j^{-2}\left[\frac{1}{2}p^2 - \gamma_j^{K+2}x^K\right]$ is proportional to the PT-symmetric oscillator $\frac{1}{2}p^2 - (-1)^j ix^K$ when $\gamma_j = e^{(2j+1)i\pi/[2(K+2)]}$, $j = 0, 1, \ldots, K+1$. For $K = 3$ and $j = 0$ we obtain $\theta = \pi/10$ as suggested by Figure 1. The obvious conclusion is that the optimal rotation angle converts each of the anharmonic oscillators of this particular class into a PT-symmetric one.

It also follows from equation (12) that $\gamma_j^2 U^{\dagger}H_K U = H_K$ when $\gamma_j = e^{2\pi ij/(K+2)}$. Therefore, instead of just one eigenvalue E_n we expect $K+1$ replicas located at

$$E_{n,j} = e^{4\pi ij/(K+2)}E_n, \; j = 0, 1, \ldots, K+1 \tag{15}$$

The RPM yields all these eigenvalues simultaneously as limits of different sequences of roots of the same sequence of pairs of Hankel determinants. On the other hand, the CRM uncovers them at different values of θ. Figure 2 shows $\log\left|\left(E_{0,j}^{RPM} - E_{0,j}^{CRM}\right)/E_{0,j}^{RPM}\right|$ as a function of θ for the lowest resonance of the cubic oscillator. We appreciate that the closest agreement between both methods takes place exactly at the rotation angles $\theta_j = (2j+1)\pi/10$ derived above. Table 2 shows these results more precisely and Table 3 a similar calculation for the quintic oscillator.

The case $j = 0$ for the cubic oscillator agrees with the resonance calculated by Jentschura et al[1]. These authors claimed to have chosen the rotation angle $\theta = \pi/5$ for all their calculations on the cubic oscillator (in particular for the strong-coupling expansion). However, we could not obtain acceptable results for this rotation angle. In fact, our calculations for the cubic oscillator suggest that the multiples of $\theta = \pi/5$ are the worst choices. Figure 3 shows the real and imaginary parts of the first resonance as functions of θ. We appreciate that the regions of stability appear at $j\pi/5 < \theta < (j+1)\pi/5$, $j = 0, 1, 2, 3, 4$ (the boundaries are marked by vertical dashed lines). The optimal rotation angles discussed above (those that convert the anharmonic oscillator into a PT-symmetric one) bisect each of these regions and the rotation angle chosen by Jentschura et al corresponds to one of the boundaries. Present results agree with those of Alvarez[3] who proposed to integrate the differential equation along the rays $\arg(\pm x) = \pi/10 - \arg(g)/5$ in the case of a harmonic oscillator perturbed by the cubic term gx^3. More precisely, he also showed that the left and right boundary conditions for the resonances hold in the common sector $0 < \frac{1}{2}\arg(g) + \frac{5}{2}\arg(x) < \frac{\pi}{2}$ so that $0 < \arg(x) < \frac{\pi}{5}$ for $g = 1$ in agreement with the first region of stability shown in Figure 3. The appearance of the optimal rotation angle $\theta = \pi/5$ in the paper by Jentschura et al[1] is merely due to a misprint[12].

Table 4 shows the first resonances for the cubic and quintic oscillators calculated by means of the RPM. They may be useful as benchmark for testing other approximate methods. For example, the first three of them for $K = 3$ agree with those of Jentschura et al[1].

From the results just discussed one may be tempted to conclude that the CMR with $\theta = 0$ should yield the eigenvalues of the PT-symmetric oscillators. This conjecture is supported by the convergence of this method towards the accurate RPM eigenvalues shown in Table 5. However, such conclusion is wrong. Although the eigenvalues produced by two quite different methods like the RPM and CRM agree accurately for all $N = 3, 5, 7 \ldots$ only in the case $N = 3$ they are those of the PT-symmetric oscillators. For $N = 5, 7, \ldots$ both methods yield the resonances discussed above rotated in the complex plane. In fact, Bender and Boettcher[10] clearly stated that the diagonalization method is useful only for $1 < N < 4$ because in the other cases the wedges in which the eigenfunction vanishes as $|x| \to \infty$ do not contain the real x axis. More precisely, since those wedges are not symmetric about the origin the complex rotation outlined above is insufficient to take into account both the left and right PT boundary conditions[2].

Table 6 shows the first eigenvalues for the PT-symmetric oscillators (5) with $N = 5$ and $N = 7$ calculated by means of the RPM, CRM ($\theta = 0$, $\eta = 0.4$) and WKB method. The first two approaches agree between them but not with the WKB method that provides estimates to the actual eigenvalues of the PT-symmetric oscillators[2]. Note that the discrepancy increases with the quantum number that makes the WKB increasingly accurate. On the other hand, it is well known that the eigenvalues of the Hamiltonian matrix agree with the WKB ones for the $N = 3$ case[11]. As an additional confirmation that the eigenvalues of the CRM are not those of the PT-symmetric oscillators compare the results of Table 6 with the accurate upper and lower bounds derived by Yan and Handy[13]. Although the functional form of the operators is the same the boundary conditions are different[2]. For simplicity, from now we will refer to resonance[3] and PT-symmetric boundary conditions[2]. Although the CRM takes into account only the former, it is interesting that it yields real positive eigenvalues for the PT-symmetric oscillators (5) with $N = 5, 7, \ldots$. The reason is discussed below.

In order to understand the results just outlined we inspect the form of the eigenfunctions provided by the CRM for the PT-symmetric oscillators. We calculated the eigenfunctions $\psi_n(x)$, $n = 0, 1, 2$ and their absolute squares are shown in Figure 4 for $N = 3, 5, 7$. We appreciate that they all look similar and satisfy $|\psi_n(-x)|^2 = |\psi_n(x)|^2$as expected from the fact that $\psi_n(-x)^* = \lambda\psi_n(x)$, where $|\lambda| = 1$[9]. More precisely, our numerical calculations suggest that in these particular cases $\psi_n(-x)^* = (-1)^n\psi_n(x)$. Even though the resonance boundary conditions are different from the PT-symmetric ones for $N = 5, 7, \ldots$ there appears to be an unbroken symmetry that produces real eigenvalues. Besides, all those eigenfunctions are strongly localized about $x = 0$ as expected for a resonance. It is interesting that both the RPM and the CRM yield real and positive eigenvalues with localized eigenfunctions for the PT-symmetric oscillators although they are not the true eigenvalues and eigenfunctions of the PT-symmetric oscillators for $N > 3$.

In addition to the resonances just discussed the RPM also yields the true eigenvalues of the PT-symmetric oscillators for all $N = 3, 5, \ldots$. For example, for $N = 5$ we estimated $E_0 = 1.9082646$ from determinants of dimension $D = 10, \ldots, 20$. Note that this eigenvalue is considerably greater than that in Table 6 obtained from the resonance boundary condition. We will discuss this issue again below.

The situation is remarkably different for $K = 4, 6, \ldots$. The PT-symmetric oscillators require asymmetric boundary conditions[2] and, consequently, one should apply the RPM for asymmetric potentials outlined above. However, in the case of resonances the boundary conditions are symmetric (for example, outgoing waves to the right and left) and the oscillator exhibits true even parity. In this case the appropriate logarithmic derivative of the wavefunction is of the form

$$f(x) = \frac{s}{x} - \frac{\psi'(x)}{\psi(x)} \tag{16}$$

where $s = 0$ or $s = 1$ for even or odd eigenfunctions, respectively. From the coefficients of the Taylor expansion

$$f(x) = \sum_{j=0}^{\infty} f_j x^{2j+1} \tag{17}$$

we construct the Hankel determinants $H_D^d(E) = |f_{i+j-1+d}|_{i,j=1}^D$ that depend on the only unknown E and obtain the eigenvalues from sequences of roots of $H_D^d(E) = 0$[7].

On the other hand, we can apply the CRM exactly in the same way discussed above. In this case the optimal rotation angles are given by $\gamma_j = e^{(2j+1)i\pi/(K+2)}$, $j = 0, 1, \ldots, K+1$ that make $U^{\dagger}HU = \gamma_j^{-2}\left(\frac{p^2}{2} + x^K\right)$ proportional to the Hermitian operator $\frac{p^2}{2} + x^K$. Table 7 compares the CRM and RPM results for the first resonance of the quartic oscillator ($K = 4$). There are $K/2 + 1$ replicas of every resonance given by

$$E_{n,j} = e^{-2\pi ij/(K/2+1)} E_n,\ j = 0, 1, \ldots, \frac{K}{2} \tag{18}$$

In this case the RPM for asymmetric potentials yields the eigenvalues of the corresponding PT-symmetric oscillator. Table 8 shows the first three eigenvalues of the PT-symmetric oscillators (5) with $N = 4$ and $N = 5$. The results in the first column agree with those obtained earlier by means of numerical integration[2, 9] and those in the second column lie within the upper and lower bounds derived by Yan and Handy[13]. The rate of convergence of the RPM is considerably greater for $N = 4$; in addition to it, we experienced considerable numerical difficulties in obtaining the roots of the pair of Hankel determinants for $N = 5$ by means of the Newton-Raphson method.

4. Conclusions

In section 2 we have shown that a simple scaling argument enables one to connect the results obtained earlier by several authors for a class of nonhermitian oscillators. Although this relationship is contained in Alvarez's work[3] the actual connection formulas have not been made explicit as far as we know. For example, the resonances of the cubic oscillator are straightforwardly related to the eigenvalues of the corresponding PT-symmetric oscillator. Such connection is not possible for other oscillators because the boundary conditions that give rise to the resonances and PT-symmetric eigenvalues are different.

The comparison of the RPM and CRM results enabled us to obtain the optimal rotation angle for the latter approach. We have shown that the effect of the optimal coordinate rotation is to convert the nonhermitian oscillators (11) into either a PT-symmetric or Hermitian one, for K odd or even, respectively. Such results are consistent with Alvarez's analysis of the harmonic oscillator with a cubic perturbation[3].

We have also shown that both the RPM and CRM yield real positive eigenvalues for the PT-symmetric oscillators (5) with $N = 3, 5, 7, \ldots$ but those results are not the actual eigenvalues of the PT-symmetric oscillators when $N > 3$ because the resonance and PT-symmetric boundary conditions are different. The CRM eigenfunctions are strongly localized and their absolute squares exhibit the symmetry coming from unbroken symmetry. It is also interesting that the eigenfunctions for the case $N = 3$ (where the CRM yields the actual eigenvalues of the PT-symmetric oscillator) are similar to those of $N = 5, 7$ (where the boundary conditions are those for the resonances).

On the other hand, the RPM yields both the real positive resonances mentioned above for odd N as well as the actual PT-symmetric eigenvalues obtained by Bender and Boettcher[2] and Bender[9]. In the case of $N = 3, 5, 7, \ldots$ all the eigenvalues are limits of sequences of roots of the same Hankel determinants given by the RPM for nonsymmetric potentials. In the case of even-parity potentials $N = 4, 6, \ldots$ the RPM for even parity potentials yields the resonances and the approach for nonsymmetric potentials provides the eigenvalues of the corresponding PT-symmetric oscillators. The main disadvantage of this approach as a practical tool is that it provides results for so many different problems that it is sometimes difficult to pick up the correct sequence of roots of the system of two Hankel determinants necessary for the treatment of nonsymmetric problems. On the other hand, from a mathematical point of view, this property of the RPM is most intriguing and interesting.

[1] Jentschura U D, Surzhykov A, Lubasch M, and Zinn-Justin J 2008 *J. Phys. A* **41** 095302.

[2] Bender C M and Boeftcher S 1998 *Phys. Rev. Lett.* **80** 5243.

[3] Alvarez G 1995 *J. Phys. A* **27** 4589.

[4] Alvarez G 1988 *Phys. Rev. A* **37** 4079.

[5] Fernández F M, Guardiola R, Ros J, and Znojil M 1998 *J. Phys. A* **31** 10105.

[6] Yaris R, Bendler J, Lovett R A, Bender C M, and Fedders P A 1978 *Phys. Rev. A* **18** 1816.

[7] Fernández F M, Ma Q, and Tipping R H 1989 *Phys. Rev. A* **39** 1605.

[8] Fernández F M and Tipping R H 1996 *Can. J. Phys.* **74** 697.

[9] Bender C M 2007 *Rep. Prog. Phys.* **70** 947

[10] Bender C M and Boettcher S 1997 Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry arXiv:physics/9712001v1 [math-ph].

[11] Amore P and Fernández F M 2012 Comment on: ‘Numerical estimates of the spectrum for anharmonic PT symmetric potentials’ [Phys. Scr. **85** (2012) 065005].

[12] U D Jentschura, personal communication.

[13] Yan Z and Handy C R 2001 *J. Phys. A* **34** 9907.

Table 1. Coefficients $L_{0,j}$ of the strong-coupling expansion

(3)

j	Ref.[1]	From Ref.[5]
0	$0.617160050 - 0.448393023i$	$0.617160049536 - 0.448393022571i$
1	$-0.013228193 + 0.040712191i$	$-0.0132281928671 + 0.0407121914135i$
2	$0.009259259 + 0.000000000i$	0.0925925925868
3	$-0.000294361 - 0.000905951i$	$-0.000294361224639 - 0.000905950695052i$

Table 2. Lowest resonance for the cubic oscillator ($K = 3$). The first column shows the value of j that determines the optimal rotation angle $\theta_j = \frac{2j+1}{10}\pi$ for the CRM. The first and second entries in the second and third columns correspond to the CRM and RPM eigenvalues, respectively).

j	$\Re(E)$	$\Im(E)$
0	0.6171600495373	−0.448393022575
	0.61716004953893673754	−0.44839302257593285633
1	−0.2357341624247	−0.725515150844
	−0.23573416242530496269	−0.72551515084615828994
2	−0.762851774225	0.00000000000000
	−0.76285177422726354970	0.00000000000000000000
3	−0.2357341624247	0.725515150844
	−0.23573416242530496269	0.72551515084615828994
4	0.617160049537	0.448393022575
	0.61716004953893673754	0.44839302257593285633

Table 3. Idem Table 2 for the quintic oscillator ($K = 5$); in this case the optimal rotation angle is $\theta_j = \frac{2j+1}{14}\pi$.

j	$\Re(E)$	$\Im(E)$
0	0.639629797817 0.63962979781725182920	−0.308029476062 −0.30802947606177966696
1	0.1579755139908 0.15797551399078843452	−0.692135950055 −0.69213595005459668245
2	−0.442637553984 −0.44263755398395529372	−0.555049936656 −0.55504993665591387838
3	−0.709935515648 −0.70993551564816994002	0.00000000000000 0.00000000000000000000
4	−0.442637553984 −0.44263755398395529372	0.555049936656 0.55504993665591387838
5	0.1579755139909 0.15797551399078843452	0.692135950055 0.69213595005459668245
6	0.639629797817 0.63962979781725183008	0.308029476062 0.30802947606177966739

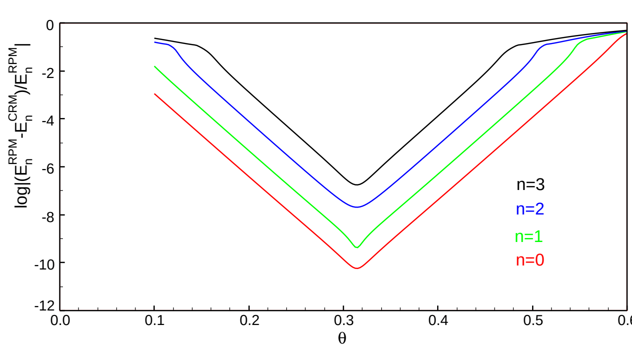

Figure 1. $\log\left|\left(E_n^{RPM} - E_n^{CRM}\right)/E_n^{RPM}\right|$ for the first resonances of the cubic oscillator.

Table 4. First resonances for the cubic and quintic oscillators calculated by means of the RPM.

n	$\Re E_n$	$\Im E_n$
	$K = 3$	
0	0.617160049538936737543	−0.4483930225759328563
1	2.1933097310211208676	−1.5935327966748432597
2	4.0363800198348283252	−2.9326017436011248866
3	6.03909710846479453	−4.38766088005387693
4	8.16189987482112373	−5.92996736837917780
5	10.3822957279796942	−7.54317938470562561
	$K = 5$	
0	0.6396297978172518292	−0.3080294760617796669
1	2.396357680279750382	−1.154025036407214392
2	4.9177001900469598	−2.3682395944315758
3	7.91746214032848	−3.812848812151728
4	11.31798850540404	−5.450456000157942
5	15.062218927774	−7.253582338538

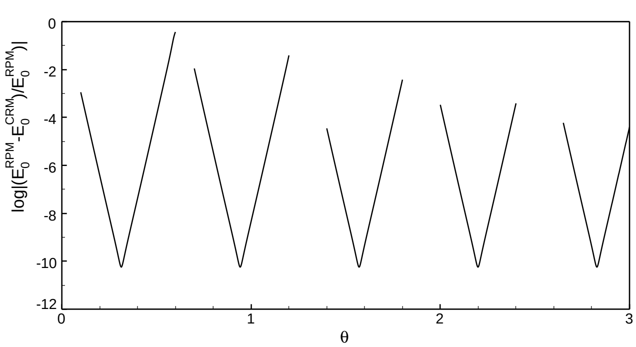

Figure 2. Logarithmic error $\log\left|\left(E_{0,j}^{RPM}-E_{0,j}^{CRM}\right)/E_{0,j}^{RPM}\right|$ for the first set of eigenvalues of the cubic oscillator ($K=3$) as functions of θ.

Table 5. Rayleigh-Ritz method for $H = p^2 + ix^N$ with basis sets of M harmonic-oscillator eigenfunction

M	$N = 5$	$N = 7$
10	1.13770276661976	1.29785656512558
20	1.16571028907153	1.22599499851804
30	1.16477239347223	1.22470989807491
40	1.16477042677832	1.22471162741409
50	1.16477040815780	1.22471168644715
60	1.16477040794314	1.22471168904864
70	1.16477040794343	1.22471168965977
80	1.16477040794342	1.22471168936849
RPM	1.16477040794341499419	1.2247116893311451

Table 6. First eigenvalues of the PT-symmetric oscillators $H = p^2 + ix^N$ calculated by means of the RPM, CRM and WKB method

RPM	CRM	WKB
	$N = 5$	
1.16477040794341499419	1.1647704079434150203	1.771244715
4.3637843677121091602	4.3637843677121073149	8.509035978
8.9551669982406716852	8.955166998240678966	17.65253759
	$N = 7$	
1.2247116893311451	1.2247116896597694535	2.855548625
4.72146253539246	4.72144769127068	15.77168804
10.0754495630818	10.0757623417291	34.91212093

Table 7. First resonance for the quartic oscillator ($K = 4$); in this case the optimal rotation angle is $\theta_j = \frac{2j+1}{6}\pi$.

j	$\Re(E)$	$\Im(E)$
0	0.33399312957789	−0.57849306980780
	0.33399312957788855414	−0.57849306980783854716
1	−0.66798625915576	0.00000000000000
	−0.66798625915577710827	0.00000000000000000000
2	0.3339931295779	0.57849306980787
	0.33399312957788855414	0.57849306980783854716

Table 8. Eigenvalues of the PT-symmetric oscillators (5) for $N = 4$ and $N = 5$

n	$N = 4$	$N = 5$
0	1.47714975357799945721	1.9082645782
1	6.00338608330082771515	8.58722083623
2	11.80243359513478158 0	17.7108090118

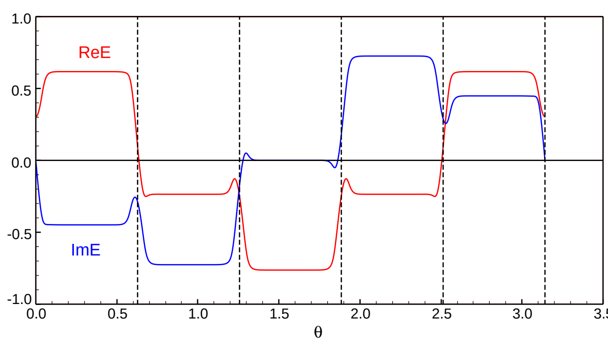

Figure 3. Real and imaginary parts of the first resonance $E(\theta)$ for the cubic oscillator ($K = 3$). The vertical lines mark multiples of $\pi/5$.

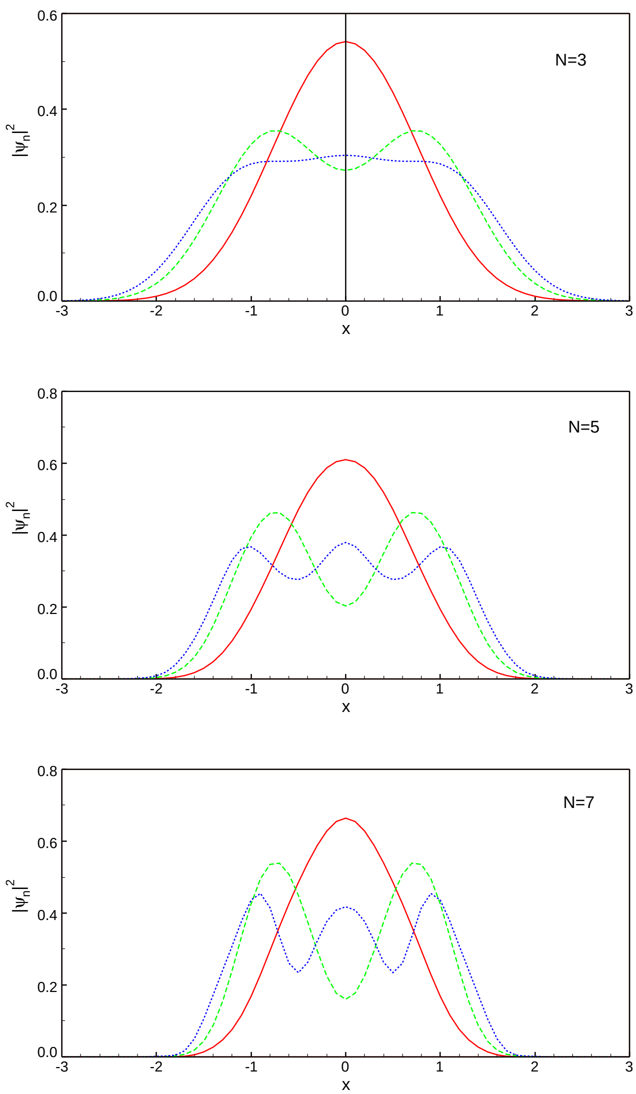

Figure 4. $|\psi_n|^2$, $n = 0$ (solid line, red), $n = 1$ (dashed line, green), $n = 2$ (dotted line, blue) for the PT-symmetric oscillators (5) with $N = 3, 5, 7$