Advances in Applied Mathematics 89 (2017) 67–75

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

# The automorphism group of the $s$-stable Kneser graphs ☆

Pablo Torres

*Universidad Nacional de Rosario and CONICET, Rosario, Argentina*

A R T I C L E   I N F O

*Article history:*
Received 15 September 2015
Received in revised form 20 March 2017
Accepted 5 April 2017
Available online xxxx

*MSC:*
05C99
05E99

*Keywords:*
Stable Kneser graph
Automorphism group

A B S T R A C T

For $k, s \geq 2$, the $s$-stable Kneser graphs are the graphs with vertex set the $k$-subsets $S$ of $\{1, \ldots, n\}$ such that the circular distance between any two elements in $S$ is at least $s$ and two vertices are adjacent if and only if the corresponding $k$-subsets are disjoint. Braun showed that for $n \geq 2k + 1$ the automorphism group of the 2-stable Kneser graphs (Schrijver graphs) is isomorphic to the dihedral group of order $2n$. In this paper we generalize this result by proving that for $s \geq 2$ and $n \geq sk+1$ the automorphism group of the $s$-stable Kneser graphs also is isomorphic to the dihedral group of order $2n$.



## 1. Introduction

Given a graph $G$, $V(G)$, $E(G)$ and $\mathrm{Aut}(G)$ denote its vertex set, edge set and automorphism group, respectively. Let $[n] := \{1, 2, 3, \ldots, n\}$. For positive integers $n$ and $k$ such that $n \geq 2k$, the *Kneser graph* $\mathrm{KG}(n, k)$ has as vertices the $k$-subsets of $[n]$ with edges defined by disjoint pairs of $k$-subsets. A subset $S \subseteq [n]$ is *$s$-stable* if any two of its

☆ Partially supported by MathAmSud Project 13MATH-07 (Argentina–Brazil–Chile–France), PID CONICET 2015-0277, PICT-2012-1324 and PICT-2015-2218.

*E-mail address:* ptorres@fceia.unr.edu.ar.

http://dx.doi.org/10.1016/j.aam.2017.04.001

elements are at least "at distance $s$ apart" on the $n$-cycle, i.e. $s \leq |i-j| \leq n-s$ for distinct $i, j \in S$. For $s, k \geq 2$, we denote $[n]^k_s$ the family of $s$-stable $k$-subsets of $[n]$. The *$s$-stable Kneser graph* $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$ [6,10] is the subgraph of $\mathrm{KG}(n,k)$ induced by $[n]^k_s$.

In a celebrated result, Lovász [5] proved that the chromatic number of $\mathrm{KG}(n,k)$, denoted $\chi(\mathrm{KG}(n,k))$, is equal to $n-2k+2$, verifying a conjecture due to M. Kneser [3]. After this result, Schrijver [7] proved that the chromatic number remains the same for $\mathrm{KG}(n,k)_{2-\mathrm{stab}}$. Moreover, this author showed that $\mathrm{KG}(n,k)_{2-\mathrm{stab}}$ is $\chi$-critical. Due to these facts, the 2-stable Kneser graphs have been named *Schrijver graphs*. These results were the base for several papers devoted to Kneser graphs and stable Kneser graphs (see e.g. [1,4,6,8–10]). In addition, it is well known that for $n \geq 2k+1$ the automorphism group of the Kneser graph $\mathrm{KG}(n,k)$ is isomorphic to $S_n$, the symmetric group of order $n$ (see [2] for a textbook account).

More recently, in 2010 Braun [1] proved that the automorphism group of the Schrijver graphs $\mathrm{KG}(n,k)_{2-\mathrm{stab}}$ is isomorphic to the dihedral group of order $2n$, denoted $D_{2n}$. In this paper we generalize this result by proving that the automorphism group of the $s$-stable Kneser graphs is isomorphic to $D_{2n}$, for $n \geq sk+1$.

Firstly, notice that if $n = sk$, the $s$-stable Kneser graph $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$ is isomorphic to the complete graph on $s$ vertices and the automorphism group of $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$ is isomorphic to $S_s$.

From the definitions we have that $D_{2n}$ injects into $\mathrm{Aut}(\mathrm{KG}(n,k)_{s-\mathrm{stab}})$, as $D_{2n}$ acts on $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$ by acting on $[n]$. Then, we have the following fact.

**Remark 1.1.** $D_{2n} \subseteq \mathrm{Aut}(\mathrm{KG}(n,k)_{s-\mathrm{stab}})$.

In the sequel, the arithmetic operations are taken *modulo* $n$ on the set $[n]$ where $n$ represents the 0. Let us recall an important result due to Talbot.

**Theorem 1.2** *(Theorem 3 in [8]). Let $n, s, k$ be positive integers such that $n \geq sk$ and $s \geq 3$. Then, every maximum independent set in $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$ is of the form $\mathcal{I}_i = \{I \in [n]^k_s : i \in I\}$ for a fixed $i \in [n]$.*

For $n \geq sk+1$ we observe that $\{i, i+s, i+2s, \ldots, i+(k-1)s\}$ and $\{i, i+s+1, i+2s+1, \ldots, i+(k-1)s+1\}$ belong to $[n]^k_s$ for all $i \in [n]$. Then, we can easily obtain the following fact.

**Remark 1.3.** Let $n \geq sk+1$ and $i, j \in [n]$. If $i \neq j$, then $\mathcal{I}_i \neq \mathcal{I}_j$.

## 2. Automorphism group of $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$

This section is devoted to obtain the automorphism group of $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$. To this end, let us introduce the following graph family. Let $n, s, k$ be positive integers such that $n \geq sk+1$. We define the graph $G(n,k,s)$ with vertex set $[n]$ and two vertices $i, j \in [n]$

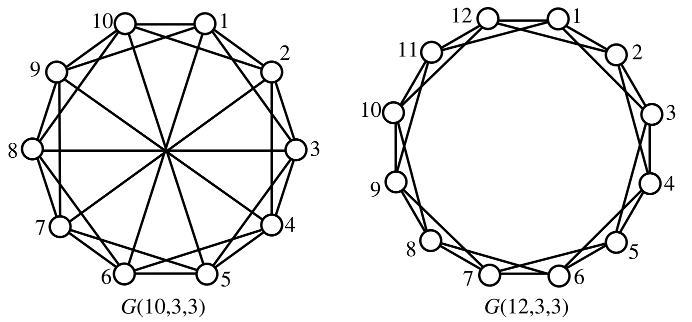


**Fig. 1.** Examples of graphs $G(n, k, s)$.

are adjacent if and only if it does not exist $S \in [n]^k_s$ such that $\{i, j\} \subseteq S$. See examples in Fig. 1.

Two vertices $i, j$ of $G(n, k, s)$ are *consecutive* if $i = j+1$. Let us see a direct result about consecutive vertices and dihedral groups, which we will use in the following theorem.

**Remark 2.1.** An injective function $f : [n] \mapsto [n]$ sends consecutive vertices of $G(n, k, s)$ to consecutive vertices of $G(n, k, s)$ if and only if $f$ belongs to the dihedral group $D_{2n}$.

Next, we obtain the main result of this section that states the link between the automorphism groups of $\mathrm{KG}(n, k)_{s-\mathrm{stab}}$ and $G(n, k, s)$.

**Theorem 2.2.** *Let $n, s, k$ be positive integers such that $n \geq sk + 1$ and $s \geq 3$. Then, the automorphism group of $\mathrm{KG}(n, k)_{s-\mathrm{stab}}$ is isomorphic to the automorphism group of $G(n, k, s)$.*

**Proof.** As we have mentioned, given $i \in [n]$, Theorem 1.2 guarantees that the sets $\mathcal{I}_i$ are the maximum independent sets in $\mathrm{KG}(n, k)_{s-\mathrm{stab}}$. Besides, any automorphism of $\mathrm{KG}(n, k)_{s-\mathrm{stab}}$ send maximum independent sets into maximum independent sets, i.e. for each $\alpha \in \mathrm{Aut}(\mathrm{KG}(n, k)_{s-\mathrm{stab}})$ and $i \in [n]$, $\alpha(\mathcal{I}_i) = \mathcal{I}_j$ for some $j \in [n]$. From Remark 1.3, if $i \neq j$ then $\alpha(\mathcal{I}_i) \neq \alpha(\mathcal{I}_j)$ and so $\alpha$ permutes these independent sets. Hence we define the homomorphism $\phi$ from $\mathrm{Aut}(\mathrm{KG}(n, k)_{s-\mathrm{stab}})$ to $S_n$ such that

$$\phi(\alpha)(i) = j \Leftrightarrow \alpha(\mathcal{I}_i) = \mathcal{I}_j.$$

We will show that $\phi$ is injective and its image is $\mathrm{Aut}(G(n, k, s))$.

Given a non-trivial element $\alpha \in \mathrm{Aut}(\mathrm{KG}(n, k)_{s-\mathrm{stab}})$, there exists $S \in [n]^k_s$ such that $\alpha(S) \neq S$, i.e. there exists $j \in S$ such that $j \notin \alpha(S)$. It follows that $\alpha(S) \in \mathcal{I}_{\phi(\alpha)(j)}$, but $\alpha(S) \notin \mathcal{I}_j$, hence $\phi(\alpha)(j) \neq j$ and $\phi(\alpha)$ is non-trivial. Then, $\phi$ is injective.

Now, we first prove that $\mathrm{Aut}(G(n, k, s)) \subseteq \phi(\mathrm{Aut}(\mathrm{KG}(n, k)_{s-\mathrm{stab}}))$. For each $\beta \in \mathrm{Aut}(G(n, k, s))$ we define the function $\gamma : V(\mathrm{KG}(n, k)_{s-\mathrm{stab}}) \mapsto V(\mathrm{KG}(n, k)_{s-\mathrm{stab}})$ such that for each $S = \{s_1, \ldots, s_k\} \in V(\mathrm{KG}(n, k)_{s-\mathrm{stab}})$, $\gamma(S) = \{\beta(s_1), \ldots, \beta(s_k)\}$.

Since $S$ is a stable set of $G(n,k,s)$, $\gamma(S)$ is also a stable set of $G(n,k,s)$ and $\gamma$ is well defined. It is not hard to see that $\gamma$ is bijective. Furthermore, $S$ and $S'$ are adjacent in $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$ if and only if $\gamma(S)$ and $\gamma(S')$ are adjacent in $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$. Therefore $\gamma \in \mathrm{Aut}(\mathrm{KG}(n,k)_{s-\mathrm{stab}})$ and from definition $\phi(\gamma) = \beta$.

Let us prove that $\phi(\mathrm{Aut}(\mathrm{KG}(n,k)_{s-\mathrm{stab}})) \subseteq \mathrm{Aut}(G(n,k,s))$, i.e. $\phi(\alpha)$ is an automorphism of $G(n,k,s)$ for each $\alpha \in \mathrm{Aut}(\mathrm{KG}(n,k)_{s-\mathrm{stab}})$. Let $i,j \in [n]$, $i' = \phi(\alpha)(i)$ and $j' = \phi(\alpha)(j)$. If $ij \in E(G(n,k,s))$, since $\mathcal{I}_i \cap \mathcal{I}_j = \emptyset$ and $\alpha$ is injective, $\mathcal{I}_{i'} \cap \mathcal{I}_{j'} = \emptyset$, i.e. $\nexists\ S \in V(\mathrm{KG}(n,k)_{s-\mathrm{stab}})$ such that $\{i',j'\} \subseteq S$. Thus $i'j' \in E(G(n,k,s))$. Since $\phi(\alpha)$ is bijective, we conclude that $\phi(\alpha) \in \mathrm{Aut}(G(n,k,s))$.

Therefore the image of $\phi$ is $\mathrm{Aut}(G(n,k,s))$ and the proof is complete. □

This result allows us to obtain $\mathrm{Aut}(\mathrm{KG}(n,k)_{s-\mathrm{stab}})$ from $\mathrm{Aut}(G(n,k,s))$. Next section is devoted to analyze the structure and the automorphism group of the graphs $G(n,k,s)$.

### *2.1. The automorphism group of $G(n,k,s)$*

Let $G$ be a simple graph. For a vertex $v \in V(G)$, the *open neighborhood* of $v$ in $G$ is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$. Then, the *closed neighborhood* of $v$ in $G$ is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is $\deg(v) = |N(v)|$. For any positive integer $d$, we denote by $G^d$ the *$d$-th power* of $G$, i.e. the graph with the same vertex set $V(G)$ and such that two vertices $u$, $v$ are adjacent if and only if $\mathrm{dist}_G(u,v) \leq d$, where $\mathrm{dist}_G(u,v)$ is the *distance* between $u$ and $v$ in $G$, i.e. the length of the shortest path in $G$ from $u$ to $v$. We denote by $C_n$ the *$n$-cycle graph* with vertex set $[n]$ and edge set $\{ij :\ i,j \in [n],\ j = i+1\}$.

**Theorem 2.3.** *Let $n, s, k$ be positive integers such that $n \geq sk+1$ and $s \geq 3$. Then,*

1. *if $s(k+1) - 1 \leq n$, then $G(n,k,s)$ is isomorphic to $C_n^{s-1}$, and*
2. *if $sk+1 \leq n \leq s(k+1)-2$. Then $G(n,k,s)$ is the graph on $[n]$ and edges defined as follows:*

$$ij \in E(G(n,k,s)) \Leftrightarrow i \neq j,\ |j-i| \notin \bigcup_{d=1}^{k-1} \{ds, ds+1, \ldots, ds+r\},$$

   *where $r = n - sk$.*

**Proof.** From the symmetry of $G(n,k,s)$ (see Remark 1.1), to prove this result it is enough to obtain the open/closed neighborhood of vertex 1 in $G(n,k,s)$ for each case.

1. **Case** $s(k+1) - 1 \leq n$: We have to prove that $[n] \setminus N[1] = \{s+1, \ldots, n-s+1\}$. By definitions, $i \in N(1)$ for every $i \in \{2, \ldots, s\} \cup \{n-s+2, \ldots, n\}$. We only need to show that for all $i \in \{s+1, \ldots, n-s+1\}$ there exists $S_i \in [n]^k_s$ such that $\{1, i\} \subseteq S_i$. So, let $i \in \{s+1, \ldots, n-s+1\}$ and $t = \left\lfloor \frac{i-1}{s} \right\rfloor$.

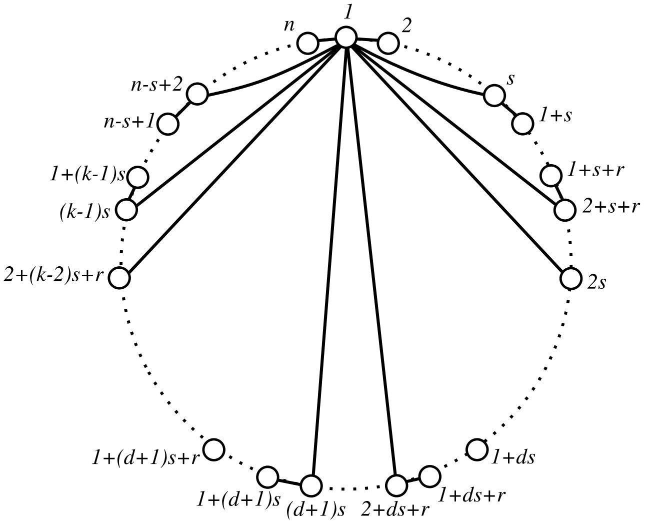


**Fig. 2.** Neighborhood of vertex 1 in $G(n,k,s)$.

If $t \geq k-1$, let $S_i = \{1, 1+s, \ldots, 1+(k-2)s, i\}$. Then $S_i \in [n]^k_s$, since $s+1 \leq i \leq n-s+1$ and $i-(1+(k-2)s) \geq i-(1+(t-1)s) \geq i-(1+(\frac{i-1}{s}-1)s) = s$.
If $t \leq k-2$, let $S_i = \{1, 1+s, \ldots, 1+(t-1)s, i, i+s, \ldots, i+(k-t-1)s\}$. To prove that $S_i \in [n]^k_s$ it is enough to show that $i-(1+(t-1)s) \geq s$ and $n-(i+(k-t-1)s) \geq s-1$. The first inequality trivially holds. To see the second inequality, notice that

$$i-(1+(t-1)s) = i-1+s-s\left\lfloor\frac{i-1}{s}\right\rfloor < i-1+s-s\left(\frac{i-1}{s}-1\right) = 2s.$$

Then, $i-(1+(t-1)s) \leq 2s-1$. Therefore $n-(i+(k-t-1)s) = n-(i-1-(t-1)s+(k-2)s+1) = n-(i-(1+(t-1)s))-((k-2)s+1) \geq n-(2s-1)-((k-2)s+1) = n-2s+1-(k-2)s-1 = n-sk \geq s-1$.

2. **Case** $sk+1 \leq n \leq s(k+1)-2$: Let $F_d = \{1+ds, 1+ds+1, \ldots, 1+ds+r\}$ for $d \in [k-1]$ and $F = \bigcup_{d=1}^{k-1} F_d$. We will prove that $N[1] = [n]-F$, which implies that 1 and $j$ are adjacent in $G(n,k,s)$ if and only if $j-1 \notin \bigcup_{d=1}^{k-1}\{ds, ds+1, \ldots, ds+r\}$, as required.
Firstly, since $n-(r+1+(k-1)s) = s-1$, the set $S_p = \{1, p+s, p+2s, \ldots, p+(k-1)s\} \in [n]^k_s$ for all $p \in [r+1]$. Furthermore, $\{1\} \cup F = \bigcup_{p=1}^{r+1} S_p$ and then $N[1] \subseteq [n]-F$.
To see the converse inclusion, observe first that if $h \in [s] \cup \{n-s+2, \ldots, n\}$ then $h \in N[1]$ from definition of $G(n,k,s)$.
Hence, if $k=2$ we have finished.
Now, let $k \geq 3$ (see Fig. 2). We have that

$$[n] \setminus (F \cup [s] \cup \{n-s+2, \ldots, n\}) = \bigcup_{m=1}^{k-2} \{ms+2+r, \ldots, (m+1)s\}.$$

Let $h \in \bigcup_{m=1}^{k-2}\{ms+2+r, \dots, (m+1)s\}$. We will prove that it does not exist $S \in [n]_s^k$ such that $\{1, h\} \subseteq S$. Let $W$ be an $s$-stable set of $[n]$ such that $\{1, h\} \subseteq W$. Notice that $|W \cap [h-1]| \leq \lfloor \frac{h-1}{s} \rfloor$ and

$$|W \cap \{h, \dots, n\}| \leq \left\lfloor \frac{n-h+1}{s} \right\rfloor.$$

Consider $m' \in [k-2]$ such that $h \in \{m's+2+r, \dots, (m'+1)s\}$. Then,

- $\lfloor \frac{h-1}{s} \rfloor \leq \left\lfloor \frac{(m'+1)s-1}{s} \right\rfloor = m'$.
- $\lfloor \frac{n-h+1}{s} \rfloor \leq \left\lfloor \frac{n-(m's+2+r)+1}{s} \right\rfloor \leq \lfloor \frac{n-r-1}{s} \rfloor - m' = \lfloor \frac{n-n+sk-1}{s} \rfloor - m' = k-1-m'$.

Thus, $|W| \leq \lfloor \frac{h-1}{s} \rfloor + \lfloor \frac{n-h+1}{s} \rfloor \leq k-1$. Therefore, any $s$-stable set of $[n]$ containing the set $\{1, h\}$ has cardinality at most $k-1$, i.e. it does not exist $S \in [n]_s^k$ such that $\{1, h\} \subseteq S$. Hence $h \in N[1]$ and the result follows. □

In order to obtain $\mathrm{Aut}(G(n, k, s))$, let us recall a well known result on automorphism group (see, e.g. [9]).

**Remark 2.4.** Let $m$ and $q$ be positive integers such that $m \geq 2q+3$. Then, the automorphism group of $C_m^q$ is the dihedral group $D_{2m}$.

Let $x$ be the degree of the vertices in $G(n, k, s)$ (which is a regular graph). Then, we have the following result.

**Theorem 2.5.** *Let $n, s, k$ be positive integers such that $n \geq sk+1$ and $s \geq 3$. Then, the automorphism group of $G(n, k, s)$ is the dihedral group $D_{2n}$.*

**Proof.** Firstly, observe that if $s(k+1)-1 \leq n$ the result immediately follows from Case 1 of Theorem 2.3 and Remark 2.4. Let us consider $sk+1 \leq n \leq s(k+1)-2$. From Remark 2.1 we only need to prove that every $\alpha \in \mathrm{Aut}(G(n, k, s))$ sends consecutive vertices to consecutive vertices. Moreover, by Remark 1.1 and Theorem 2.2 it is enough to show that $\alpha(1)$ and $\alpha(2)$ are consecutive vertices. Without loss of generality we consider $\alpha(1) = 1$.

Let $r = n - sk$. From Theorem 2.3,

$$N[2] \cap \{1+ds, \dots, 1+ds+r\} = \{1+ds\}$$

for $d = 1, \dots, k-1$, and

$$[n] \setminus N[1] = \bigcup_{d=1}^{k-1}\{1+ds, 1+ds+1, \dots, 1+ds+r\}.$$

So,

$$|N[1] \cap N[2]| = x + 1 - (k-1) = x - k + 2.$$

Analogously, $|N[1] \cap N[n]| = x - k + 2$.

Let $i \in N(1)$. Recall that

$$N(1) = \{2, \dots, s\} \cup \{n - s + 2, \dots, n\} \cup \left( \bigcup_{m=1}^{k-2} \{ms + 2 + r, \dots, (m+1)s\} \right).$$

If $i \in \{3, \dots, s\}$ we have that $\{1 + s, 2 + s\} \subseteq N[i]$. Besides, if $k \geq 3$ observe that $\{i + 1 + (d-1)s + r, \dots, i + ds - 1\} \subseteq N[i]$ for $d = 2, \dots, k-1$. Hence, since $3 \leq i \leq s$, $1 + ds \leq i + ds - 1$ and $i + 1 + (d-1)s + r \leq 1 + ds + r$. Therefore, $\{i + 1 + (d-1)s + r, \dots, i + ds - 1\} \cap \{1 + ds, \dots, 1 + ds + r\} \neq \emptyset$ for $d = 2, \dots, k-1$. Then, $|N[i] \cap \{1 + s, \dots, 1 + s + r\}| \geq 2$ and $|N[i] \cap \{1 + ds, \dots, 1 + ds + r\}| \geq 1$ for $d = 2, \dots, k-1$. So, if $k \geq 2$, $|N[1] \cap N[i]| \leq x + 1 - k \leq x - 1$ and thus $\alpha(2) \neq i$. Similarly if $i \in \{n - s + 2, \dots, n - 1\}$, we have $\alpha(2) \neq i$. So, if $k = 2$ the result follows.

Now, let $k \geq 3$. Consider $i \in \bigcup_{m=1}^{k-2} \{ms+2+r, \dots, (m+1)s\}$ and let $m_i \in \{1, \dots, k-2\}$ such that $i \in \{m_i s + 2 + r, \dots, (m_i + 1)s\}$.

Notice that

$$\begin{aligned} &\{1 + m_i s, \dots, 1 + m_i s + r\} \cup \{1 + (m_i + 1)s, \dots, 1 + (m_i + 1)s + r\} \\ &\quad \subseteq \{i - (s-1), \dots, i + s - 1\} \subseteq N[i]. \end{aligned} \tag{1}$$

Therefore,

$$\{1 + m_i s, \dots, 1 + m_i s + r\} \cup \{1 + (m_i + 1)s, \dots, 1 + (m_i + 1)s + r\} \subseteq N[i] \setminus N[1].$$

Now, let $m \in [k-2]$. If $m < m_i$ then $1+(m_i-m)s+r \leq i-(1+ms) \leq (m_i+1-m)s-1$. From Theorem 2.3, we have that $1 + ms \in N[i]$ if $m < m_i$. By a similar reasoning we have that $1 + ms + r \in N[i]$ if $m > m_i + 1$. Then,

$$\{1 + ms : \ m < m_i, \ m \in [k]\} \cup \{1 + ms + r : \ m > m_i + 1, \ m \in [k]\} \subseteq N[i] \setminus N[1].$$

These facts together with (1) imply that

$$|N[1] \cap N[i]| = x + 1 - (N[i] \setminus N[1]) \leq x + 1 - (2(r+1) + (k-4)) \leq x - k + 1.$$

Thus $\alpha(2) \neq i$. Therefore $\alpha(2) \in \{2, n\}$ and the thesis holds. □

Finally, we have the main result of this work.

**Theorem 2.6.** *Let $n, s, k$ be positive integers such that $n \geq sk + 1$ and $s \geq 2$. Then, the automorphism group of $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$ is isomorphic to the dihedral group $D_{2n}$.*

**Proof.** The result for the case $s = 2$ follows from [1] and for the remaining cases can be obtained from Theorems 2.2 and 2.5. $\square$

## 3. Further results

In this section we will obtain some properties of $s$-stable Kneser graphs as a consequence of the results in the previous sections. Firstly, as a consequence of Theorem 2.6, we have the following result.

**Theorem 3.1.** *Let $n, k, s \geq 2$ with $n \geq sk+1$. Then,* $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$ *is vertex transitive if and only if $n = sk+1$.*

**Proof.** Without loss of generality, we assume that every vertex $S = \{s_1, s_2, \ldots, s_k\}$ of the $s$-stable Kneser graph $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$ verifies that $s_1 < s_2 < \ldots < s_k$. Then, $S$ is described unequivocally by $s_1$ and the gaps $l_1(S), \ldots, l_k(S)$ such that for $i \in [k-1]$, $l_i(S) = s_{i+1} - s_i$ and $l_k(S) = s_1 + n - s_k$. Observe that every automorphism of $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$ "preserves" the gaps $l_i$, i.e. if $\phi \in \mathrm{Aut}(\mathrm{KG}(n,k)_{s-\mathrm{stab}})$ there exists $\alpha \in D_{2k}$ such that $l_i(\phi(S)) = l_{\alpha(i)}(S)$ for all $i \in [k]$.

If $n \geq sk+2$, then $S_1 = \{1, 1+s, 1+2s, \ldots, 1+(k-1)s\} \in [n]^k_s$ and $S_2 = \{1, 2+s, 2+2s, \ldots, 2+(k-1)s\} \in [n]^k_s$. Therefore, from Theorem 2.6, we have that no automorphism of $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$ maps $S_1$ to $S_2$, since $l_1(S_2) = s+1$ but $l_i(S_1) = s$ for $i \in [k-1]$ and $l_k(S_1) \geq s+2$.

Besides, in [9] it is proved that if $S \in [ks+1]^k_s$ then exactly one gap $l_m(S)$ is equal to $s+1$ and the remaining gaps are equal to $s$. From this fact we have that $\mathrm{KG}(sk+1,k)_{s-\mathrm{stab}}$ is vertex transitive. $\square$

Next, we will analyze some aspects related to colorings of $s$-stable Kneser graphs. Let $\alpha(G)$ and $\chi^*(G)$ the *independence number* and *fractional chromatic number* of a graph $G$, respectively.

**Proposition 3.2.** *Let $n, k, s \geq 2$ with $n \geq sk+1$. Then,* $\chi^*(\mathrm{KG}(n,k)_{s-\mathrm{stab}}) = \frac{n}{k}$.

**Proof.** It is immediate to observe that $\chi^*(\mathrm{KG}(n,k)_{s-\mathrm{stab}}) \leq \frac{n}{k}$ (see, e.g. Theorem 7.4.5 in [2]). To see the converse inequality, we use the fact that for all graph $G$, $\chi^*(G) \geq \frac{|V(G)|}{\alpha(G)}$.

So, let us compute $|[n]^k_s| = |V(\mathrm{KG}(n,k)_{s-\mathrm{stab}})|$. From [8], since the sets $\mathcal{I}_i$ are maximum independent sets for $i \in [n]$, $\alpha(\mathrm{KG}(n,k)_{s-\mathrm{stab}}) = |\mathcal{I}_i| = \binom{n-(s-1)k-1}{k-1}$.

Then, to compute $|[n]^k_s|$, let us observe that $\bigcup_{i=1}^n \mathcal{I}_i = [n]^k_s$ and $\sum_{i=1}^n |\mathcal{I}_i| = n\binom{n-(s-1)k-1}{k-1}$, where each vertex of $\mathrm{KG}(n,k)_{s-\mathrm{stab}}$ is computed $k$ times. Then,

$$|[n]_s^k| = n\binom{n-(s-1)k-1}{k-1} - (k-1)|[n]_s^k|.$$

Hence $|[n]_s^k| = \frac{n}{k}\binom{n-(s-1)k-1}{k-1}$ and the result follows. □

As we have mentioned before, Schrijver [7] proved that the graphs $\mathrm{KG}(n,k)_{2-\mathrm{stab}}$ are $\chi$-critical subgraphs of $\mathrm{KG}(n,k)$ but it is an open problem to compute the chromatic number of $s$-stable Kneser graphs. From the last result and Proposition 2 in [6] we have

$$\frac{n}{k} \leq \chi(\mathrm{KG}(n,k)_{s-\mathrm{stab}}) \leq n-(k-1)s.$$

In particular, if $n = ks+1$ we obtain that $\chi(\mathrm{KG}(ks+1,k)_{s-\mathrm{stab}}) = s+1$, which is an alternative proof to compute the exact value of $\chi(\mathrm{KG}(ks+1,k)_{s-\mathrm{stab}})$ already studied in [6] and [9].

## References

[1] B. Braun, Symmetries of the stable Kneser graphs, Adv. in Appl. Math. 45 (2010) 12–14.
[2] C.D. Godsil, G. Royle, Algebraic Graph Theory, Graduate Texts in Mathematics, Springer, 2001.
[3] M. Kneser, Aufgabe 360, in: Jahresbericht der Deutschen Mathematiker-Vereinigung, 2. Abteilung, vol. 50, 1955, 27 pp.
[4] B. Larose, F. Laviolette, C. Tardif, On normal Cayley graphs and hom-idempotent graphs, European J. Combin. 19 (1998) 867–881.
[5] L. Lovász, Kneser's conjecture, chromatic number and homotopy, J. Combin. Theory Ser. A 25 (1978) 319–324.
[6] F. Meunier, The chromatic number of almost stable Kneser hypergraphs, J. Combin. Theory Ser. A 118 (2011) 1820–1828.
[7] A. Schrijver, Vertex-critical subgraphs of Kneser graphs, Nieuw Arch. Wiskd. (3) 26 (3) (1978) 454–461.
[8] J. Talbot, Intersecting families of separated sets, J. Lond. Math. Soc. (2) 68 (1) (2003) 37–51.
[9] P. Torres, M. Valencia-Pabon, Shifts of the stable Kneser graphs and hom-idempotence, European J. Combin. 62 (2017) 50–57.
[10] G. Ziegler, Generalized Kneser coloring theorems with combinatorial proofs, Invent. Math. 147 (2002) 671–691.