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On first integrals of the geodesic flow on Heisenberg nilmanifolds

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In this paper we study the geodesic flow on nilmanifolds equipped with a left-invariant metric. We write the underlying definitions and find general formulas for the Poisson involution. As an application we develop the Heisenberg Lie group equipped with its canonical metric. We prove that a family of first integrals giving the complete integrability can be read off at the Lie algebra of the isometry group. We also explain the complete integrability for any invariant metric and on compact quotients.

1. Introduction

Given a smooth manifold M, any complete Riemannian structure $\langle \cdot, \cdot \rangle$ induces the geodesic flow $\Gamma\colon M \times \mathbb{R} \to M$ which can be defined as the Hamiltonian flow associated to the energy function $E(v) := 1/2\langle v, v \rangle$ on TM. Usually, this flow is not integrable in the sense of Liouville and it is generally expected that the integrability of the geodesic flow imposes important obstructions to the topology of the supporting manifold. However contrasting some results of Taimanov [19,27,28] on topological obstructions for real-analytic manifolds supporting real-analytic integrable geodesic flows with some smooth

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examples of smoothly integrable geodesic flows on manifolds that do not satisfy the above obstructions constructed by Butler [8], and Bolsinov and Taimanov [7], we observe that the regularity of first integrals plays a fundamental role that nowadays is not completely well understood. For that reason, when dealing with locally homogeneous manifolds to have the possibility of constructing real-analytic first integrals is very desirable.

In the advances reached in the theory of Hamiltonian systems in the 1980's one can recognize the role of Lie theory in the study of several examples. This is the case of the so known Adler–Kostant–Symes [2,18,26] scheme used for the study of some mechanical systems and of the so known Thimm's method for the study of the geodesic flow [29]. In both cases the main examples arise from semisimple Lie groups. These results appeared parallel to the many studies given by the Russian school which can be found for instance in [14]. Examples of integrable geodesic flows and other systems on Lie groups and quotients or bi-quotients can be found in [6,3–5,8,16,17,21,23,24].

For other Hamiltonian systems on non-semisimple Lie groups only few examples and generalizations are known in the case of the geodesic flow. This is the situation for nilpotent and solvable Lie groups or even their compact quotients which are locally homogeneous manifolds. For instance, Eberlein started a study of the geometry concerning the geodesic flow on Lie groups following his own and longer study in this topic, giving a good material and references in [12,13]. This study of Eberlein is much more general and is mixed with many other geometrical questions.

In the case of 2-step nilpotent Lie groups and their compact quotients there are some interesting results. On the one hand Butler [9] proved the Liouville integrability of the geodesic flow whenever the Lie algebra is Heisenberg–Reiter. On the other hand he also proved the non-commutative integrability – in the sense of Nekhoroshev [22] – whenever the Lie algebra is almost non-singular. Since Bolsinov and Jovanovic [4] proved that integrability in the non-commutative sense implies Liouville integrability, the previous results of Butler give the Liouville integrability for an important family of 2-step nilpotent Lie groups and their compact quotients.

In the present paper we concentrate in the geodesic flow of Lie groups endowed with a metric invariant by left-translations. In the first part we write the basic definitions and get general conditions and formulas for the involution of first integrals making use of the Lie theory tools, that is assuming some natural identifications. We put special emphasis on 2-step nilpotent Lie groups, we take as nilmanifold after [31], for which there exists a developed geometrical theory and several examples and applications see [11].

We apply the results we get to the case of the Heisenberg Lie group H_n of dimension $2n + 1$ equipped with the canonical left-invariant metric. Although this is a naturally reductive space the methods of Thimm do not apply directly in this case, since the full isometry group is not semi-simple. Compare also with the Mishchenko–Fomenko method [14,25].

Our main goal is to investigate the nature of the first integrals one can construct. We prove that all the first integrals we get can be visualized on the isometry group. In fact this is the case of quadratic polynomials which are invariant, so as first integrals arising from Killing vector fields. Recall that given a Killing vector field X^* on a Riemannian manifold M one has a first integral on the tangent Lie bundle TM defined by $f_{X^*}(v) = \langle X^*, v\rangle$. We proved that

(i) There is a bijection between the set of quadratic first integrals of the geodesic flow on H_n – with the canonical metric – and the Lie subalgebra of skew-symmetric derivations of the Heisenberg Lie algebra $\mathfrak{h}_n$, so that involution of quadratic first integrals would correspond to a torus of skew-symmetric derivations – Theorem 3.2. Actually a general formulation of quadratic polynomials on a 2-step nilpotent Lie algebras to be first integrals is found so as the pairwise commutativity condition.
(ii) The linear morphism $X^* \mapsto f_{X^*}$ builds a Lie algebra isomorphism onto the image. This is the first example we found of this situation.

Making use of all these results we exhibit families of first integrals in involution for the geodesic flow on H_n, see Theorem 3.4. After that we consider any lattice Λ and passing to the quotient we explain the integrability of the geodesic flow on the compact spaces $\Lambda\backslash H_n$. Finally we consider any left-invariant metric on the Heisenberg Lie group H_n and prove that the corresponding geodesic flow is also completely integrable.

2. Preliminaries and basic facts

In this section we present some basic results to study the geodesic flow on Lie groups with a left-invariant metric. See [1] for more details.

Let N be a Lie group endowed with a left-invariant metric $\langle\cdot,\cdot\rangle$ and let $\mathfrak{n}$ be the Lie algebra of N. We identify the tangent bundle TN of N with $N \times \mathfrak{n}$. So, TN is a Lie group regarded as the direct product of N and the abelian group $\mathfrak{n}$. We shall consider the left-invariant metric on TN given by the product of the left-invariant metric on N and the Euclidean metric on $\mathfrak{n}$. We also identify a tangent vector at $(p, Y) \in TN$ with a pair $(U, V) \in \mathfrak{n} \times \mathfrak{n}$ in the obvious way. With these identifications, the tautological 1-form on TN is given by

$$\Theta_{(p,Y)}(U, V) = \langle Y, U\rangle$$

and the canonical symplectic form $\Omega = -d\Theta$ on TN is given by

$$\Omega_{(p,Y)}((U, V), (U', V')) = \langle U, V'\rangle - \langle V, U'\rangle + \langle Y, [U, U']\rangle. \tag{2.1}$$

Given a smooth function $f : TN \to \mathbb{R}$, the Hamiltonian vector field of f is denoted by X_f and it is given implicitly by

$$df_{(p,Y)}(U, V) = \Omega_{(p,Y)}(X_f(p, Y), (U, V)).$$

The associated Poisson bracket on $C^\infty(TN)$ shall be denoted as

$$\{f, g\} = \Omega(X_f, X_g).$$

We will denote the gradient of f at the point (p, Y), with respect to the product metric on $TN = N \times \mathfrak{n}$, by $\operatorname{grad}_{(p,Y)} f = (U, V) \in \mathfrak{n} \times \mathfrak{n}$. The following result is straightforward.

Lemma 2.1. *Let $(N, \langle\cdot,\cdot\rangle)$ be a Lie group with a left-invariant metric and let $f \in C^\infty(TN)$. By $\operatorname{grad}_{(p,Y)} f = (U, V)$ we denote the gradient of f with respect to the left-invariant metric on TN. Then*

1. *the Hamiltonian vector field of f is given by*

$$X_f(p, Y) = (V, \operatorname{ad}^t(V)(Y) - U), \tag{2.2}$$

where $\operatorname{ad}^t(V)$ denotes the transpose of $\operatorname{ad}(V)$ with respect to the metric on $\mathfrak{n}$.

2. *If $g \in C^\infty(TN)$, the Poisson bracket is given by*

$$\{f, g\}(p, Y) = -\Omega_{(p,Y)}(\sigma \operatorname{grad}_{(p,Y)} f, \sigma \operatorname{grad}_{(p,Y)} g), \tag{2.3}$$

where $\sigma(U, V) = (V, U)$.

The geodesic field on TN is the vector field associated with the geodesic flow

$$\Phi_t(p, Y) = \gamma'(t)$$

where $\gamma(t)$ is the geodesic on N with initial conditions $\gamma(0) = p$, $\gamma'(0) = (p, Y)$. It is well known that the geodesic field arises as the Hamiltonian vector field of the energy function $E : TN \to \mathbb{R}$, $E(p, Y) = \frac{1}{2}\langle Y, Y\rangle$. One can also define the geodesic flow on the Lie algebra $\mathfrak{n}$ by using the so-called Gauss map $G : TN \to \mathfrak{n}$, given by $G(p, Y) = Y$. See [12,13] for more details.

We say that a smooth function $f : TN \to \mathbb{R}$ is a *first integral* of the geodesic flow if f is constant along the integral curves of the geodesic field, or equivalently if $\{f, E\} = 0$. We say that N has *completely integrable* geodesic flow (in the sense of Liouville) if there exist first integrals of the geodesic flow $f_1, \dots, f_n$, where $n = \dim N$, such that $\{f_i, f_j\} = 0$ for all i, j and the gradients of $f_1, \dots, f_n$ are linear independent on an open dense subset of TN.

Remark 2.2. (First integrals from Killing fields). If M is a Riemannian manifold and X^* is a Killing field on M, then the function $f_{X^*} : TM \to \mathbb{R}$ defined as $f_{X^*}(v) = \langle X^*(\pi(v)), v\rangle$ is a first integral of the geodesic flow.

Lemma 2.3. *Let $(N, \langle\cdot,\cdot\rangle)$ be a Lie group equipped with a left-invariant metric. A smooth function $f : TN \to \mathbb{R}$, with gradient $\operatorname{grad}_{(p,Y)} f = (U, V)$, is a first integral of the geodesic flow if and only if*

$$\langle Y, U\rangle = \langle Y, [V, Y]\rangle \tag{2.4}$$

for all $(p, Y) \in TN$.

Proof. It follows from Lemma 2.1 and the fact that $X_E(f) = 0$ if and only if f is a first integral of the geodesic flow. □

A function $f : TN \to \mathbb{R}$ is said to be *invariant* if $f(p, Y) = f(qp, Y)$ for all $p, q \in N$, $Y \in \mathfrak{n}$. That is, f is invariant under the left-action of N on TN. One can use Lemma 2.3 in order to find some invariant first integrals.

Proposition 2.4. *Let $(N, \langle\cdot,\cdot\rangle)$ be a Lie group with a left-invariant metric.*

1. *The function $f_{Z_0} : TN \to \mathbb{R}$, defined by $f_{Z_0}(p, Y) = \langle Y, Z_0\rangle$, is a first integral of the geodesic flow for all $Z_0 \in \mathfrak{z}$. Moreover, the family $\{f_{Z_0}\}_{Z_0\in\mathfrak{z}}$ is a Poisson-commutative family of first integrals.*
2. *Let $A : \mathfrak{n} \to \mathfrak{n}$ be a symmetric endomorphism of $\mathfrak{n}$ and let $g_A : TN \to \mathbb{R}$ denote the quadratic polynomial given by $g_A(p, Y) = \frac{1}{2}\langle Y, AY\rangle$. Then g_A is a first integral of the geodesic flow if and only if $\langle Y, [AY, Y]\rangle = 0$ for all $Y \in \mathfrak{n}$.*

Proof. It easily follows from Lemma 2.3 using that $\operatorname{grad}_{(p,Y)} f_{Z_0} = (0, Z_0)$ and $\operatorname{grad}_{(p,Y)} g_A = (0, AY)$. □

2.1. The case of 2-step nilpotent Lie groups

When N is a 2-step nilpotent Lie group, its Lie algebra $\mathfrak{n}$ can be decomposed as the orthogonal sum

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z},$$

where $\mathfrak{z}$ is the center of $\mathfrak{n}$. In such a case, for each $Z \in \mathfrak{z}$, we have a skew-symmetric linear map $j(Z) : \mathfrak{v} \to \mathfrak{v}$ given by

$$\langle j(Z)U, V\rangle = \langle [U, V], Z\rangle \tag{2.5}$$

for all $U, V \in \mathfrak{v}$. We say that the Lie algebra $\mathfrak{n}$ is *non-singular* if $\mathrm{ad}(X) : \mathfrak{n} \to \mathfrak{z}$ is surjective for all $X \notin \mathfrak{z}$. It is a well known fact that $\mathfrak{n}$ is non-singular if and only if $j(Z)$ is non-singular for any $Z \in \mathfrak{z} - \{0\}$ (which does not depend on the choice of the left-invariant metric, see for instance [11]). We can restate Lemma 2.1 for a 2-step nilpotent Lie group as follows. If $f, g \in C^\infty(TN)$ are functions with $\mathrm{grad}_{(p,Y)} f = (U, V)$ and $\mathrm{grad}_{(p,Y)} g = (U', V')$ then

$$X_f(p, Y) = (V, j(Y_{\mathfrak{z}})V_{\mathfrak{v}} - U) \tag{2.6}$$

and

$$\{f, g\}(p, Y) = \langle V', U\rangle - \langle V, U'\rangle + \langle j(Y_{\mathfrak{z}})V'_{\mathfrak{v}}, V_{\mathfrak{v}}\rangle, \tag{2.7}$$

where the subindexes $\mathfrak{v}$ and $\mathfrak{z}$ denote the respective components according to the decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$. In particular, since $\mathrm{grad}_{(p,Y)} E = (0, Y)$, we get

$$X_E(p, Y) = (Y, j(Y_{\mathfrak{z}})Y_{\mathfrak{v}})$$

and from Lemma 2.3, we get that f is a first integral of the geodesic flow on TN if and only if

$$\langle Y, U\rangle = \langle j(Y_{\mathfrak{z}})V_{\mathfrak{v}}, Y_{\mathfrak{v}}\rangle.$$

The second item in Proposition 2.4 says that in the 2-step nilpotent case, the commutativity of the quadratic first integrals depends on their restriction to the orthogonal complement of the center.

Theorem 2.5. *Let $(N, \langle\cdot,\cdot\rangle)$ be a 2-step nilpotent Lie group with a left-invariant metric, let $A : \mathfrak{v} \to \mathfrak{v}$ be a symmetric endomorphism and let g_A defined as in Proposition 2.4. Assume that $\{Z_1, \ldots, Z_m\}$ is a basis of $\mathfrak{z}$. Then*

1. *g_A is a first integral of the geodesic flow if and only if $[j(Z_i), A] = 0$ for all $i = 1, \ldots, m$, where the skew-symmetric endomorphism $j(Z_i) \in \mathfrak{so}(\mathfrak{v})$ is extended to $\mathfrak{n}$ by $j(Z_i)|_{\mathfrak{z}} = 0$;*
2. *two quadratic first integrals g_A, g_B Poisson commute if and only if $j(Z_i)AB = j(Z_i)BA$ for all $i = 1, \ldots, m$. In particular, if $\mathfrak{n}$ is non-singular, then $\{g_A, g_B\} = 0$ if and only if $[A, B] = 0$.*

Proof. Since $j(Y_{\mathfrak{z}}) = \langle Y, Z_1\rangle Z_1 + \cdots + \langle Y, Z_m\rangle Z_m$, it follows from Proposition 2.4 that g_A is a first integral if and only if $\langle j(Z_i)AY_{\mathfrak{v}}, Y_{\mathfrak{v}}\rangle = 0$ for all $Y_{\mathfrak{v}}$. Since A is symmetric and $j(Z_i)$ is skew-symmetric, this is equivalent to $j(Z_i)A = Aj(Z_i)$. This proves the assertion 1.

For the second part assume that g_A, g_B are first integrals (and so they satisfy 1). From Lemma 2.1 we get that $\{g_A, g_B\} = 0$ if and only if

$$0 = \langle Y, [AY, BY]\rangle = \langle j(Y_{\mathfrak{z}})AY_v, BY_{\mathfrak{v}}\rangle = \langle Bj(Y_{\mathfrak{z}})AY_{\mathfrak{v}}, Y_{\mathfrak{v}}\rangle,$$

or equivalently, $Bj(Z_i)A$ must be skew-symmetric for all $i = 1, \ldots, m$. By taking the transpose, one has that $Aj(Z_i)B = Bj(Z_i)A$, and since A, B both commute with $j(Z_i)$, we obtain $j(Z_i)AB = j(Z_i)BA$ for all $i = 1, \ldots, m$. □

2.1.1. Non-integrable Lie algebras

We shortly review a particular family of 2-step nilpotent Lie algebras discussed by Butler in [10]. Given a 2-step nilpotent Lie algebra $\mathfrak{n}$ and $\lambda \in \mathfrak{n}^*$, we put $\mathfrak{n}_\lambda = \{X \in \mathfrak{n} : \mathrm{ad}^*(X)\lambda = 0\}$. An element $\lambda \in \mathfrak{n}^*$ is called regular if $\mathfrak{n}_\lambda$ has minimal dimension. The Lie algebra $\mathfrak{n}$ is called *non-integrable* if there exists a dense open subset $\mathcal{W}$ of $\mathfrak{n}^* \times \mathfrak{n}^*$ such that for each $(\lambda, \mu) \in \mathcal{W}$, both λ and μ are regular and $[\mathfrak{n}_\lambda, \mathfrak{n}_\mu]$ has positive dimension.

It is clear that $\mathfrak{z} \subset \mathfrak{n}_\lambda$ for all $\lambda \in \mathfrak{n}^*$. Moreover if the Lie algebra $\mathfrak{n}$ is equipped with a metric any $\lambda \in \mathfrak{n}^*$ has the form $\lambda = \langle V + Z, \cdot\rangle$ for certain $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$. Now, $X \in \mathfrak{n}_\lambda$ if and only if $\langle j(Z)X_\mathfrak{v}, \cdot\rangle = 0$. So if $\mathfrak{n}$ is non-singular, it must be $X_\mathfrak{v} = 0$. In this situation, $\mathfrak{n}_\lambda$ has minimal dimension for $Z \neq 0$ and $[\mathfrak{n}_\lambda, \mathfrak{n}_\mu] = 0$. This proves the following result.

Lemma 2.6 *(See [9]). Let $\mathfrak{n}$ be a non-singular 2-step nilpotent Lie algebra, then $\mathfrak{n}$ is not a non-integrable Lie algebra.*

The above lemma says that non-singular Lie algebras form a suitable family where one can look for nilmanifolds with completely integrable geodesic flow. The following theorem by Butler relates non-integrable Lie algebras with the integrability of the geodesic flow.

Theorem 2.7 *(See [10, Theorem 1.3]). Let $\mathfrak{n}$ be a non-integrable 2-step nilpotent Lie algebra with associated simply connected Lie group N and let Λ be a discrete co-compact subgroup of N. Then for any left-invariant metric g on N, the geodesic flow on $(\Lambda\backslash N, g)$ is not completely integrable.*

3. First integrals of the geodesic flow on Heisenberg manifolds

We consider the presentation of the $(2n+1)$-dimensional Heisenberg Lie group H_n given by the underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$ endowed with the multiplication map

$$(v, z)(v', z') = \left(v + v', z + z' - \tfrac{1}{2} v^t J v'\right)$$

where J is the linear map given by the multiplication for $\sqrt{-1}$ on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ (where one identifies $(x_1, \ldots, x_{2n})$ with $(x_1 + ix_2, \ldots, x_{2n-1} + ix_{2n})$). If one denotes by $(x_1, \ldots, x_{2n}, z)$ the standard coordinate system on $\mathbb{R}^{2n} \times \mathbb{R}$, then the basis of the Lie algebra $\mathfrak{h}_n$ of H_n consisting of the left-invariant vector fields which coincide with the canonical basis of $\mathbb{R}^{2n+1}$ at the origin, is given by

$$X_{2i-1} = \partial_{x_{2i-1}} - \frac{x_{2i}}{2}\partial_z, \qquad X_{2i} = \partial_{x_{2i}} + \frac{x_{2i-1}}{2}\partial_z, \qquad Z_1 = \partial_z.$$

Recall that for this basis, the only non-vanishing brackets are $[X_{2i-1}, X_{2i}] = Z_1$ and the exponential map $\exp : \mathfrak{h}_n \to H_n$ is identified with the identity map on $\mathbb{R}^{2n+1}$. Let $\langle\cdot,\cdot\rangle$ be the left-invariant metric on H_n that makes $X_1, \ldots, X_{2n}, Z_1$ an orthonormal basis. The map J defined above is nothing but the element in $\mathfrak{so}(\mathfrak{v})$ given by $J = j(Z_1)$, and so, for each $Z \in \mathfrak{z}$ we have that $j(Z) = \langle Z, Z_1\rangle J$. Therefore $[X, Y] = \langle JX, Y\rangle Z_1$.

First of all, we make use of Theorem 2.5 in order to find first integrals of the geodesic flow on TH_n that are invariant under the action of H_n. On the one hand since the center of $\mathfrak{h}_n$ is one dimensional, we only have one independent linear first integral $f_{Z_1}(p, Y) = \langle Y, Z_1\rangle$. On the other hand we can relate the quadratic first integrals from Theorem 2.5 to the isotropy subgroup of the full isometry group of H_n.

Remark 3.1. It is well known that the isometry group of a simply connected 2-step nilpotent Lie group, endowed with a left-invariant metric, is the semidirect product $\mathrm{Iso}(N) = K \ltimes N$, where the isotropy group K consists of the isometric automorphisms. See [31] for more details.

In our case (see [20] for instance), the Lie algebra of the isotropy group of $\mathrm{Iso}(H_n)$ is given by

$$\mathfrak{k} = \{B \in \mathfrak{so}(\mathfrak{v}) : [J, B] = 0\}. \tag{3.1}$$

Now according to Theorem 2.5 a quadratic first integrals has the form $g_A(p, Y) = \frac{1}{2}\langle AY, Y\rangle$ where A is a symmetric endomorphism of $\mathfrak{v}$ (extended to all $\mathfrak{h}_n$ in the trivial way) such that $[J, A] = 0$. If we call $B = JA$, then B belongs to $\mathfrak{so}(\mathfrak{v})$ and satisfies $[J, B] = 0$. Conversely, given $B \in \mathfrak{k}$, one can define a symmetric endomorphism $A = JB$ of $\mathfrak{v}$ such that g_A is a first integral for the geodesic flow on TH_n.

Theorem 3.2. *The map $\psi(g_A) = JA$ gives a bijection between the set of quadratic first integrals of the geodesic flow and the Lie subalgebra of skew-symmetric derivations of $\mathfrak{h}_n$. Moreover, $\{g_{A_1}, g_{A_2}\} = 0$ if and only if $[\psi(g_{A_1}), \psi(g_{A_2})] = 0$ in $\mathfrak{so}(\mathfrak{v})$.*

Notice that ψ is not a Lie algebra morphism.

Let us consider, for $i = 1, \ldots, n$, the symmetric endomorphism A_i of $\mathfrak{v}$ given by

$$A_i Y = \langle Y, X_{2i-1}\rangle X_{2i-1} + \langle Y, X_{2i}\rangle X_{2i}. \tag{3.2}$$

It is not hard to see that $[J, A_i] = 0$ and $[A_i, A_j] = 0$ for all $i, j = 1, \ldots, n$, which gives $\{g_{A_i}, g_{A_j}\} = 0$ by Theorem 2.5. Moreover, the family $A_1, \ldots, A_n$ is maximal such that the gradients of $g_{A_1}, \ldots, g_{A_n}$ are linearly independent, since $A_1, \ldots, A_n$ is a family of commuting symmetric endomorphisms that also commute with the multiplication by $\sqrt{-1}$ on $\mathbb{R}^{2n}$.

Observe that from Remark 2.2 we have another way of constructing first integrals from isotropic Killing fields. In fact, let $T \in \mathfrak{k}$, that is T satisfies (3.1), and let ρ_{sT} be the corresponding 1-parameter subgroup of automorphisms of H_n. Thus $\rho_{sT}(v, z) = (e^{sT}v, z)$, where $e^{sT} = \sum_{k=0}^{\infty} \frac{s^k}{k!} T^k$. Thus, the associated Killing field is given by

$$X_T^*(v, z) = \left.\frac{d}{ds}\right|_0 \rho_{sT}(v, z) = TW_{\mathfrak{v}} - \frac{1}{2}\langle TW_{\mathfrak{v}}, JW_{\mathfrak{v}}\rangle Z_1 \tag{3.3}$$

where $\exp W = (v, z)$. Now the first integral associated with X_T^* given by Remark 2.2 is

$$F_T(p, Y) = \langle TW_{\mathfrak{v}}, Y_{\mathfrak{v}}\rangle - \frac{1}{2}\langle AW_{\mathfrak{v}}, W_{\mathfrak{v}}\rangle\langle Y, Z_1\rangle \tag{3.4}$$

where $T = JA$ as in Theorem 3.2.

Recall that any right-invariant vector field on H_n may be regarded as a Killing vector field (associated with a 1-parameter subgroup of left-translations). In particular, we have the following basis of right-invariant vector fields:

$$X_{2i-1}^* = \partial_{x_{2i-1}} + \frac{x_{2i}}{2}\partial_z, \qquad X_{2i}^* = \partial_{x_{2i}} - \frac{x_{2i-1}}{2}\partial_z, \qquad Z_1^* = \partial_z.$$

Since $X_{2i-1}^* = X_{2i-1} + x_{2i}Z_1$ and $X_{2i}^* = X_{2i} - x_{2i-1}Z_1$, the first integrals given by Remark 2.2 and associated to the Killing fields $X_1^*, \ldots, X_{2n}^*$ are given by

$$F_{2i-1}(p, Y) = \langle X_{2i-1} + \langle W_{\mathfrak{v}}, X_{2i}\rangle Z_1, Y\rangle$$
$$F_{2i}(p, Y) = \langle X_{2i} - \langle W_{\mathfrak{v}}, X_{2i-1}\rangle Z_1, Y\rangle$$

which can be written, for $k = 1, \ldots, 2n$, as

$$F_k(p,Y) = \langle Y - j(Y_{\mathfrak{z}})W_{\mathfrak{v}}, X_k\rangle = \langle Y, X_k\rangle - \langle j(Y_{\mathfrak{z}})W_{\mathfrak{v}}, X_k\rangle. \tag{3.5}$$

Note that the first integral associated to $Z_1^* = Z_1$ is just f_{Z_1}.

For the first integrals f_{Z_1} and the ones given in (3.4) and (3.5) we have that

$$\begin{aligned}
\operatorname{grad}_{(p,Y)} F_T &= \left(-\langle Z_1, Y\rangle AW_{\mathfrak{v}} - TY_{\mathfrak{v}}, TW_{\mathfrak{v}} - \tfrac{1}{2}\langle AW_{\mathfrak{v}}, W_{\mathfrak{v}}\rangle Z_1\right)\\
\operatorname{grad}_{(p,Y)} F_k &= (\langle Y, Z_1\rangle JX_k, X_k + \langle W, JX_k\rangle Z_1)\\
\operatorname{grad}_{(p,Y)} f_{Z_1} &= (0, Z_1)
\end{aligned} \tag{3.6}$$

which shows that the gradients of $F_1, \dots, F_{2n}, f_{Z_1}$ are linear independent on an open dense subset. The formula (2.7) gives $\{F_j, F_k\}(p,Y) = \langle Y, Z_1\rangle\langle X_k, JX_j\rangle$, so the only non-trivial Poisson brackets between the $F_1, \dots, F_{2n}$ are

$$\{F_{2i-1}, F_{2i}\} = f_{Z_1} \tag{3.7}$$

(see also [5,29]).

Let us study now the commutativity of first integrals associated with isotropic Killing fields. In fact let $T_A = JA$, $T_B = JB$ be elements of $\mathfrak{k}$ as in (3.1). Assume that $\{F_{T_A}, F_{T_B}\} = 0$, then by Lemma 2.1 we get

$$\begin{aligned}
0 = \{F_{T_A}, F_{T_B}\}(p,Z) &= -\langle T_B W_{\mathfrak{v}}, AW_{\mathfrak{v}}\rangle + \langle T_A W_{\mathfrak{v}}, BW_{\mathfrak{v}}\rangle + \langle JT_B W_{\mathfrak{v}}, T_A W_{\mathfrak{v}}\rangle\\
&= -\langle JABW_{\mathfrak{v}}, W_{\mathfrak{v}}\rangle + \langle JBAW_{\mathfrak{v}}, W_{\mathfrak{v}}\rangle + \langle J^2BW_{\mathfrak{v}}, JAW_{\mathfrak{v}}\rangle\\
&= -\langle JABW_{\mathfrak{v}}, W_{\mathfrak{v}}\rangle + \langle JBAW_{\mathfrak{v}}, W_{\mathfrak{v}}\rangle + \langle JABW_{\mathfrak{v}}, W_{\mathfrak{v}}\rangle,
\end{aligned}$$

and this implies, as in the proof of Theorem 2.5 that $[A, B] = 0$. Conversely, it is not hard to see that $\{F_{T_A}, F_{T_B}\} = 0$ if $[A, B] = 0$.

Remark 3.3. Making use of (3.6) we can compute all the Poisson brackets between our first integrals. In fact if A, B are symmetric maps on $\mathfrak{h}_n$ commuting with J then

- $\{F_{T_A}, g_B\}(p,Y) = \langle JABY_{\mathfrak{v}}, Y_{\mathfrak{v}}\rangle$. In particular $\{F_{T_A}, g_B\}(p,Y) = 0$ if and only if $[A, B] = 0$.
- $\{F_{T_A}, F_{T_B}\} = F_{[T_A, T_B]}$ since

$$\begin{aligned}
\{F_{T_A}, F_{T_B}\}(p,Y) &= \langle [T_A, T_B]W_{\mathfrak{v}}, Y_{\mathfrak{v}}\rangle + \langle Y, Z_1\rangle\langle JBAW_{\mathfrak{v}}, W_{\mathfrak{v}}\rangle\\
&= \langle [T_A, T_B]W_{\mathfrak{v}}, Y_{\mathfrak{v}}\rangle + \frac{1}{2}\langle Y, Z_1\rangle\langle (JBA - JAB)W_{\mathfrak{v}}, W_{\mathfrak{v}}\rangle
\end{aligned}$$

  and $\frac{1}{2}(JAB - JBA)$ is the symmetric part of JAB. Thus $\{F_{T_A}, F_{T_B}\} = 0$ if and only if $[A, B] = 0$.
- $\{F_T, F_k\}(p,Y) = F_k(p, TY_{\mathfrak{v}} + Y_{\mathfrak{z}})$, for all $k = 1, \dots, 2n$.
- $\{f_{Z_1}, F_k\} = \{f_{Z_1}, F_T\} = 0$ for all $k = 1, \dots, 2n$.

Let A_i be the symmetric map defined in (3.2) and let $T_i = JA_i$ denote the corresponding skew-symmetric derivation of $\mathfrak{h}_n$. From above calculations one can also see that the gradients of $F_{T_1}, \dots, F_{T_n}, g_{A_1}, \dots, g_{A_n}$ are independent on an open dense subset of TH_n. Note that $\{JA_i\}_{i=1}^n$ corresponds to a Cartan subalgebra of $\mathfrak{so}(\mathfrak{v})$, which in particular gives an abelian subalgebra of skew-symmetric derivations. The proof of the next theorem follows from canonical computations and the above remark.

Theorem 3.4. *The geodesic flow on TH_n is completely integrable in the sense of Liouville. In fact, the sets*

1. $\mathcal{G} = \{E\} \cup \{g_{A_i}\}_{i=1}^n \cup \{F_{T_k}\}_{k=1}^n$,
2. $\mathcal{F} = \{f_{Z_1}\} \cup \{g_{A_i}\}_{i=1}^n \cup \{F_{2k-1}\}_{k=1}^n$ *and*
3. $\mathcal{F}' = \{f_{Z_1}\} \cup \{g_{A_i}\}_{i=1}^n \cup \{F_{2k}\}_{k=1}^n$

give three independent commuting families of first integrals for the geodesic flow.

Remark 3.5. It turns out that the families given in [9] are of the same type as the families in $\mathcal{F}$ and $\mathcal{F}'$, while the first integrals F_{T_k}, obtained from the Lie algebra of the isotropy subgroup $K \subset \mathrm{Iso}(H_n)$ are particular cases of the Noether integrals given by the momentum mapping. Recall that in the simply connected case all the first integrals introduced in Theorem 3.4 are analytic. As we shall see later this cannot be achieved for compact quotients of H_n.

The explicit calculation of the Poisson brackets between F_T, F_k, f_{Z_1}, where $T \in \mathfrak{k}$, $k = 1, \dots, 2n$, gives evidence of the following result.

Theorem 3.6. *The linear morphism between the Lie algebra of Killing vector fields on H_n and its image in $C^\infty(TH_n)$ given by $X^* \mapsto f_{X^*}$ as in Remark 2.2 builds a Lie algebra isomorphism between the isometry Lie algebra of H_n and its image equipped with the Poisson bracket.*

Proof. A Killing vector field X^* can be written as $X^* = X_T^* + X_U^*$ where $T + U \in \mathfrak{k} \ltimes \mathfrak{h}_n$. Moreover, $U(p) = V(p) + zZ_1^*$ and $V(p) = \sum_{k=1}^{2n} s_k X_k^*(p)$. Thus,

$$f_{X^*}(p, Y) = F_T(p, Y) + \sum_{k=1}^{2n} s_k F_k(p, Y) + z f_{Z_1}(p, Y)$$

and so $X^* \mapsto f_{X^*}$ is a Lie algebra homomorphism which is also injective. In fact, assume that

$$0 = F_T(p, Y) + \sum_{k=1}^{2n} s_k F_k(p, Y) + z f_{Z_1}(p, Y) \tag{3.8}$$

for all (p, Y). By taking $p = \exp Z_1$ and $Y = Z_1$ one gets $z = 0$. Now, if we let p be arbitrary and take $Y = Z_1$, $F_T(p, Y) = -\frac{1}{2}\langle AW_\mathfrak{v}, W_\mathfrak{v}\rangle$ and $F_k(p, Z_1) = -\langle JW_\mathfrak{v}, X_k\rangle$. Thus the equality (3.8) becomes

$$0 = -\frac{1}{2}\langle AW_\mathfrak{v}, W_\mathfrak{v}\rangle - \sum_{k=1}^{2n} s_k \langle JW_\mathfrak{v}, X_k\rangle.$$

If $\langle AX_j, X_j\rangle = 0$ for all $j = 1, \dots, 2n$, then $A = 0$ and so $F_T = 0$. Choosing $W = X_l$ one gets $0 = \sum_{k=1}^{2n} s_k \langle JX_l, X_k\rangle = \pm s_l$, which says $s_l = 0$ for every l. If $\langle AX_j, X_j\rangle \neq 0$, then there is a j such that $a_{jj} = \langle AX_j, X_j\rangle \neq 0$. Take $W = tX_j$ for $t \in \mathbb{R}$. Then Equation (3.8) becomes $0 = -\frac{1}{2}a_{jj}t^2 \mp s_j t = t(-\frac{1}{2}ta_{jj} \mp s_j)$ which should holds for every t. Note that the sign $\pm$ depends on the parity of j. Thus $a_{jj} = 0$, $s_j = 0$ for all j and $X^* \mapsto f_{X^*}$ is injective onto its image. □

The first integrals of the previous theorem follow a known property of integrals for a G-action on $Q = G/K$: they correspond to the homomorphism of the Poisson brackets induced by the momentum map $T^*Q \to \mathfrak{g}^*$. Up to our knowledge, this is the first example among nilpotent Lie groups where this homomorphism is bijective, that is, an isomorphism. It could be interesting to know if there are more examples of this situation among nilpotent Lie groups, and if there is a relationship with the complete integrability of the geodesic flow.

Remark 3.7. The metric considered in this section makes of H_n a naturally reductive Riemannian space. That means that there exists a Lie group of isometries G acting transitively on H_n such that its Lie algebra $\mathfrak{g}$ splits as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{h}$ is the Lie algebra of the isotropy subgroup and $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant subspace such that

$$\langle X, [Y, Z]_{\mathfrak{m}} \rangle + \langle Y, [X, Z]_{\mathfrak{m}} \rangle = 0$$

for all $X, Y, Z \in \mathfrak{m}$. However, the Heisenberg Lie group does not correspond to the examples in [29].

3.1. Riemannian Heisenberg manifolds

Recall that a Riemannian Heisenberg manifold is given as a quotient $\Lambda \backslash H_n$ where Λ is a discrete co-compact subgroup of H_n. Note that $\Lambda \backslash H_n$ becomes a Riemannian manifold with the metric that makes the projection $\pi : H_n \to \Lambda \backslash H_n$ a Riemannian submersion. Now, for each n-tuple $r = (r_1, \ldots, r_n) \in (\mathbb{Z}^+)^n$ such that $r_1 \mid r_2 \mid \cdots \mid r_n$, we define

$$\Lambda_r = \{(v, z) : v = (x, y) \text{ with } x \in r\mathbb{Z}^n,\ y \in 2\mathbb{Z}^n,\ z \in \mathbb{Z}\}, \tag{3.9}$$

where $x = (x_1, \ldots, x_n) \in r\mathbb{Z}^n$ means that $x_i \in r_i\mathbb{Z}$ for all $i = 1, \ldots, n$. It follows from [15] that this family classifies the co-compact discrete subgroups of H_n up to isomorphism. This is explained in the following remark.

Remark 3.8. In several works the Heisenberg Lie group is defined as $(2n+2) \times (2n+2)$ real matrices of the form

$$\gamma(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & \mathbf{1}_n & y^t \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y \in \mathbb{R}^n$ and $\mathbf{1}_n$ is the $n \times n$ identity matrix. The map Φ defined as $\Phi(\gamma(x, y, z)) = (x, y, z - \frac{1}{2}xy^t)$ gives an isomorphism with the Heisenberg Lie group defined as in the first paragraphs of this section. In [15] it is proved that any lattice on the Heisenberg group, with this matrix presentation, is isomorphic to one of the form

$$\Gamma_r = \{\gamma(x, y, z) : x \in r\mathbb{Z}^n,\ y \in \mathbb{Z}^n,\ z \in \mathbb{Z}\},$$

where $r = (r_1, \ldots, r_n) \in (\mathbb{Z}^+)^n$ is such that $r_1 \mid r_2 \mid \cdots \mid r_n$. Moreover, Γ_r is isomorphic to Γ_s if and only if $r = s$. Recall that, $\Phi^{-1}(\Lambda_r) \subset \Gamma_r$. This inclusion is strict, but one can still prove that Λ_r is isomorphic to Γ_r. For the sake of completeness we include the proof of this fact in the following remark.

Remark 3.9. Recall that the center $Z(\Gamma_r)$ of Γ_r is $\{\gamma(0, 0, z) : z \in \mathbb{Z}\}$ for all r. So $Z(\Gamma_r)$ is cyclic with two generators $\gamma(0, 0, 1)$ and $\gamma(0, 0, -1)$. As it follows from [15], for each $i = 1, \ldots, n$ we have that

$$\gamma(r_i e_i, 0, 0)\gamma(0, e_i, 0)\gamma(r_i e_i, 0, 0)^{-1}\gamma(0, e_i, 0)^{-1} = \gamma(0, 0, 1)^{r_i},$$

and for all $s \in (\mathbb{Z}^+)^n$ such that $s_1 \mid s_2 \mid \cdots \mid s_n$, $s \neq r$, there are no elements $\gamma_1, \gamma_2, \ldots, \gamma_{2n} \in \Gamma_s$ such that

$$\gamma_{2i-1}\gamma_{2i}\gamma_{2i-1}^{-1}\gamma_{2i}^{-1} = \gamma(0,0,\pm 1)^{r_i}.$$

This shows, in particular, that Γ_r is not isomorphic to Γ_s if $r \neq s$. Now, from the classification theorem in [15], we have that Λ_r is isomorphic to Γ_s for some s. But

$$(r_i e_i, 0, 0)(0, 2e_i, 0)(-r_i e_i, 0, 0)(0, -2e_i, 0) = (0, 0, r_i) = (0, 0, 1)^{r_i},$$

where $(0, 0, 1)$ is one of the two generators of $Z(\Lambda_r)$. Therefore, Λ_r is isomorphic to Γ_r.

Since the quotient projection $\pi : H_n \to \Lambda_r \backslash H_n$ is a Riemannian submersion and furthermore a local isometry, we can identify the tangent bundle of $\Lambda \backslash H_n$ with $(\Lambda_r \backslash H_n) \times \mathfrak{h}_n$. Since π maps geodesics into geodesics, the energy function $\tilde{E}$ on $T(\Lambda_r \backslash H_n)$ is related to the energy function E on TH_n by

$$\tilde{E}(\Lambda_r p, Y) = E(p, Y) = \frac{1}{2}\langle Y, Y\rangle$$

and this does not depend on the given representative.

Since the integrals f_{Z_1}, g_{A_i} of the geodesic flow of TH_n, as given in Theorem 3.4 do not depend on the first coordinate, they descend to first integrals

$$\tilde{f}_{Z_1}(\Lambda_r p, Y) = f_{Z_1}(p, Y), \qquad \tilde{g}_{A_i}(\Lambda_r p, Y) = g_{A_i}(p, Y)$$

of the geodesic flow of $T(\Lambda_r \backslash H_n)$. Moreover, such first integrals are in involution, since for all $f, g \in C^\infty(T(\Lambda_r \backslash H_n))$ we have

$$\{f \circ \pi, g \circ \pi\} = \{f, g\} \circ \pi.$$

Note that the integrals F_k, $k = 1, \dots, 2n$ from Theorem 3.4 do not descend to the quotient. However one can construct first integrals on the quotient with the following argument. Let $(p, Y) \in TH_n$ and $q \in \Lambda_r$, say $p = (x, y, z)$ and $q = (x', y', z')$, with $x, y \in \mathbb{R}^n$, $z \in \mathbb{R}$, $x' \in r\mathbb{Z}^n$, $y' \in 2\mathbb{Z}^n$ and $z' \in \mathbb{Z}$. Take $W, W', W'' \in \mathfrak{h}_n$ such that $\exp W = p$, $\exp W' = q$ and $\exp W'' = qp$. Observe that $W''_{\mathfrak{v}} = W_{\mathfrak{v}} + W'_{\mathfrak{v}}$. So we get

$$\begin{aligned} F_k(qp, Y) &= \langle Y, X_k\rangle - \langle Y, Z_1\rangle\langle J(W'_{\mathfrak{v}} + W_{\mathfrak{v}}), X_k\rangle \\ &= F_k(p, Y) - f_{Z_1}(p, Y)\langle JW'_{\mathfrak{v}}, X_k\rangle. \end{aligned}$$

Since $\langle JW'_{\mathfrak{v}}, X_k\rangle \in \mathbb{Z}$ we have that

$$F_k(qp, Y) = F_k(p, Y) \mod f_{Z_1}(p, Y)\mathbb{Z}$$

and since f_{Z_1} is a first integral of the geodesic flow, we have that the function

$$\hat{F}_k(p, Y) = \sin\left(2\pi \frac{F_k(p, Y)}{f_{Z_1}(p, Y)}\right)$$

descends to $\Lambda_r \backslash H_n$ and is constant along the integral curves of the geodesic vector field in $T(\Lambda_r \backslash H_n)$. In order to get a smooth first integral let

$$\bar{F}_k(p, Y) = e^{-1/f_{Z_1}(p,Y)^2}\hat{F}_k(p, Y)$$

and let us define

$$\tilde{F}_k(\Lambda_r p, Y) = \bar{F}_k(p, Y).$$

So the functions $\tilde{F}_k$ are smooth (non-analytic) first integrals for the geodesic flow on $T(\Lambda_r\backslash H_n)$. It follows from a direct calculation that the families $f_{Z_1}, g_{A_i}, \bar{F}_{2k-1}$ and $f_{Z_1}, g_{A_i}, \bar{F}_{2k}$, $i, k = 1, \ldots, n$ are in involution. So the geodesic flow in $T(\Lambda_r\backslash H_n)$ is completely integrable in the sense of Liouville.

Theorem 3.10. *Let H_n be the Heisenberg Lie group endowed with the standard metric and let Λ_r defined as in (3.9). If $\Lambda_r\backslash H_n$ is the corresponding Heisenberg manifold, then the geodesic flow in $T(\Lambda_r\backslash H_n)$ is completely integrable with smooth first integrals.*

4. The case of a general left-invariant metric on H_n

In this section we show how to construct first integrals of the geodesic flow on TH_n for arbitrary left-invariant metrics on H_n. Recall that any left-invariant metric g on H_n is isometric to one of the form $\langle\cdot,\cdot\rangle_P$ defined as follows. Let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathfrak{h}_n$ and let P be a symmetric positive definite operator on $\mathfrak{h}_n$, with respect to the standard inner product, which has the matrix form

$$P = \begin{pmatrix} \tilde{P} & \\ & \lambda \end{pmatrix},$$

where $\tilde{P} : \mathfrak{v} \to \mathfrak{v}$ is symmetric and positive definite and $\lambda > 0$. We can think of $\tilde{P}$ as a symmetric matrix with positive eigenvalues. The metric $\langle\cdot,\cdot\rangle_P$ is defined in $\mathfrak{h}_n$ by $\langle X, Y\rangle_P = \langle PX, Y\rangle$. Moreover, we can assume that P is diagonal, see [30, Theorem 3.1].

Let $j_P(Z)$ be the $\langle\cdot,\cdot\rangle_P$-skew-symmetric operator such that

$$\langle j_P(Z)X, Y\rangle_P = \langle [X, Y], Z\rangle_P.$$

Note that

$$\begin{aligned}\langle [X, Y], Z\rangle_P &= \langle [X, Y], PZ\rangle = \langle [X, Y], \lambda Z\rangle \\ &= \lambda\langle j(Z)X, Y\rangle = \lambda\langle PP^{-1}j(Z)X, Y\rangle \\ &= \lambda\langle P^{-1}j(Z)X, Y\rangle_P.\end{aligned}$$

This proves the following lemma.

Lemma 4.1. *With the assumptions and notation of this section we have that $\lambda j(Z) = \tilde{P} j_P(Z)$ or, equivalently $j_P(Z) = \lambda\tilde{P}^{-1}j(Z)$, for all $Z \in \mathfrak{z}$.*

We also need the next result.

Lemma 4.2. *If $f : TH_n \to \mathbb{R}$ is a smooth function and $\operatorname{grad}_{(p,Y)} f$, $\operatorname{grad}^P_{(p,Y)} f$ denote the gradients of f with respect to the metrics on TH_n induced by the standard metric and by $\langle\cdot,\cdot\rangle_P$ respectively, then*

$$\operatorname{grad}_{(p,Y)} f = P \operatorname{grad}^P_{(p,Y)} f,$$

where P acts diagonally on $T_{(p,Y)}(TH_n)$.

Proof. By a direct calculation,

$$\langle \operatorname{grad}^P_{(p,Y)} f, S\rangle_P = \langle P \operatorname{grad}^P_{(p,Y)} f, S\rangle = df_{(p,Y)}(S) = \langle \operatorname{grad}_{(p,Y)} f, S\rangle$$

which proves the lemma. □

So if $\operatorname{grad}_{(p,Y)} f = (U, V)$ then $\operatorname{grad}^P_{(p,Y)} f = (P^{-1}U, P^{-1}V)$. It follows from Proposition 2.4 that f_{Z_1} is also a first integral for the geodesic flow on $(TH_n, \langle\cdot,\cdot\rangle_P)$. In order to find quadratic first integrals, we use the following lemma.

Lemma 4.3. *An endomorphism A of $\mathfrak{v}$ is symmetric with respect to $\langle\cdot,\cdot\rangle_P$ if and only if $PA = A^tP$, where A^t is the transpose of A with respect to the standard metric.*

Proof. Let $X, Y \in \mathfrak{h}_n$ be arbitrary elements. Then $\langle AX, Y\rangle_P = \langle X, AY\rangle_P$ if and only if $\langle PAX, Y\rangle = \langle PX, AY\rangle$, and this holds if and only if $\langle PAX, Y\rangle = \langle A^tPX, Y\rangle$. □

Since $\tilde{P}$ is symmetric with respect to the standard metric, there exist a basis $U_1, \ldots, U_{2n}$ of $\mathfrak{v}$ such that $\tilde{P}U_i = \lambda_i U_i$ (with $\lambda_i > 0$, since $\tilde{P}$ is positive definite). This basis can be chosen orthonormal with respect to the standard metric, and moreover, since $j(Z)$ acts on $\mathfrak{v}$ as a multiple of the multiplication by $\sqrt{-1}$ on $\mathbb{C}^n$, we can assume that $U_{2i} = j(Z_1)U_{2i-1}$. So we can define the operators $\tilde{A}_i$ on $\mathfrak{v}$ such that

$$\tilde{A}_i U_{2i-1} = U_{2i-1} \quad \tilde{A}_i U_{2i} = U_{2i} \text{ and } \tilde{A}_i U_k = 0 \ \text{ if } k \neq 2i-1, 2i.$$

It follows that $\tilde{A}_i$ is symmetric with respect to $\langle\cdot,\cdot\rangle_P$, $\tilde{A}_i$ commutes with $j_P(Z)$ and $[\tilde{A}_i, \tilde{A}_j] = 0$ for all i, j. So, Theorem 2.5 shows that we get a commuting independent family of first integrals for the geodesic flow on $(TH_n, \langle\cdot,\cdot\rangle_P)$. Namely

$$g_{\tilde{A}_i}(p, Y) = \frac{1}{2}\langle \tilde{A}_i Y, Y\rangle_P,$$

where $\tilde{A}_i$ is extended so that $\tilde{A}_i Z_1 = 0$. Finally, in order to obtain the remaining integrals we use the Killing fields U_k^*, $k = 1, \ldots, 2n$. In fact, we have, in the same manner as in the previous section, that

$$\tilde{F}_k(p, Y) = \langle Y, U_k\rangle_P - \langle j_P(Y_{\mathfrak{z}})W_{\mathfrak{v}}, U_k\rangle_P$$

forms a family of first integrals for the geodesic flow such that

$$\{\tilde{F}_{2i-1}, \tilde{F}_{2j}\}^P = \delta_{ij} f_{Z_1}.$$

Theorem 4.4. *The geodesic flow on $(TH_n, \langle\cdot,\cdot\rangle_P)$ is completely integrable in the sense of Liouville. Moreover the sets*

1. $\mathcal{F} = \{f_{Z_1}\} \cup \{g_{\tilde{A}_i}\}_{i=1}^n \cup \{\tilde{F}_{2k-1}\}_{k=1}^n$ *and*
2. $\mathcal{F}' = \{f_{Z_1}\} \cup \{g_{\tilde{A}_i}\}_{i=1}^n \cup \{\tilde{F}_{2k}\}_{k=1}^n$

give two independent commuting families of first integrals of the geodesic flow.

Remark 4.5. Note that the case of a general Riemannian left-invariant metric on H_n does not give a naturally reductive space. Also the compact quotients considered here are not globally homogeneous but locally.

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