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Use of discrete Wigner functions in the study of a hybrid dissipative system

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Abstract

The persistence of coherence in a hybrid dissipative system, which is composed of superconducting flux-qubits (SFQs) and an electron ensemble, is analyzed. Both the interactions between the electrons and the SFQs are taken into account. The time evolution of the hybrid system is discussed in terms of the discrete Wigner function of each subsystem and in terms of the entropic uncertainty relations. The inclusion of a linewidth, both for the electrons and the SFQs, influences the coherence of each subsystem and the pattern of spin squeezing of the corresponding steady state.

Keywords: discrete Wigner functions, entropy squeezing, decoherence, quantum hybrid systems

1. Introduction

The study of macroscopic quantum coherence is a central issue in the field of quantum information processing and of quantum computing [1–3]. Quantum technologies allow the construction of quantum processors with the potential of carrying out specific tasks at exponentially reduced computation time. One of the greatest challenges in the development of a quantum computer is related to the possibility of generating the superposition and entanglement of quantum states with controlled decoherence [4–7]. In this direction, a variety of novel experimental approaches have enabled tests of fundamental quantum physics, such as superpositions, entanglement, tunneling, and quantum phase transitions [8–10].

In recent years, a great deal of effort has been devoted to the study of hybrid systems. Hybrid quantum circuits [11] are built up from two or more physical systems, so as to combine the best features of each. Typical elements employed in the design of hybrid quantum circuits are atoms, spins, superconducting qubits, cavities, and resonators. As an example, we can mention the experimental realization of hybrid systems composed of spins and superconducting circuits, i.e. spins coupled to superconducting resonators [12], or indirect-coupling hybrid circuits with nitrogen-vacancy (NV) centers and a transmon qubit [13]. Another example of this type of architecture is the hybrid system composed of superconducting flux-qubits (SFQs) coupled to electrons in the form of NV color centers in diamond [14–25]. The theoretical model proposed by Marcos *et al* in [15] has been experimentally realized by Zhu *et al* [16]. The authors of [16], have reported a strong coherent coupling between a flux qubit and an ensemble of NV color centers in single-crystal diamond and the coherent transfer of a single quantum of energy has been demonstrated. Furthermore, several authors have explored the possibility of using a hybrid quantum architecture including coupled flux qubits and an NV center ensemble. The flux qubits can be considered as the quantum-computing processor and the NV center, due to its long coherence time, can be used as the quantum memory [26]. Novel spin-nanomechanical hybrid devices have been proposed [27, 28]. The authors of [27] have considered a hybrid system, which consists of NV centers in diamond interfaced with a suspended carbon nanotube carrying a DC current. In the same vein, the authors of [28] have shown a hybrid magneto-nano-electromechanical, which is based on a magnetic-field-induced deflection of an appropriate cantilever that oscillates between NV spins in diamond.

The analysis of the advantages of these physical systems includes the interplay between entanglement [29, 30], squeezing [31, 32], and decoherence [33–37]. Decoherence of a quantum superposition state arises from the interaction between the constituent system and the uncontrolled degrees

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of freedom of the environment. Traditionally and generally, dissipation mechanisms have been considered to have a negative effect on the performance of the quantum manipulation of mechanical modes [38]. However, in a series of recent works [39–43], it has been observed that dissipation can be used to enhance the squeezing properties of the steady state of different systems, and consequently to obtain stationary robust states against decoherence. As an example, we can mention the results that have been presented in [40]. The authors of [40] have proved that entanglement and spin squeezing, in a two-axis twisting scheme, can be amplified by including a dissipation through the linewidth in the spin states. Similar results have been reported in [25, 42, 43].

As has been demonstrated in different works, the Wigner quasi-probability distribution or Wigner function [44, 45] has proved useful to describe the quantum properties of different physical systems [46]. Among other applications, it has been extensively used in connection with the study of quantum tomography [47–52]. Also, it has become a powerful tool to understand and detect decoherence [34, 53, 54] and quantum correlations [55, 56]. In the present work, we investigate the use of the Wigner function formalism to analyze the persistence of coherence for a dissipative system of SFQ coupled to an ensemble of electrons in the form of NV color centers in diamond [25, 39, 40, 57, 58]. Also, we analyze the squeezing properties of the system by studying the entropic uncertainty relations (EURs) of its constituents [59, 60].

The work is organized as follows. In section 2, we present the Hamiltonian of the SFQ in interaction with an ensemble of NV color centers in diamond. In subsection 2.1, the formalism adopted to study the time evolution of the dissipative hybrid system is developed. In subsection 2.2, we discuss the preparation of the initial state. The EURs and associated entropic squeezing parameter are presented in 2.3. In subsection 2.4, we review the definition of the SU(2) Wigner function in connection with the decoherence properties of the system. The results of our calculations are presented and discussed in section 3. Finally, our conclusions are drawn in section 4.

2. Formalism

We shall study a hybrid system composed of an array of SFQs coupled to an ensemble of electrons in the form of NV color centers in diamond [14–17, 23, 25]. The coupling of these two subsystems is possible due to the similarity in their energy scales, which are of the order of a few gigahertz. The long coherence times of the NV-centers ensemble [61] and the experimental facilities in the construction of different SFQ arrays [62], turned this hybrid system into a potential quantum-computing device. The SFQ subsystem can be used to perform the qubit operations of a quantum computer, while the electron ensemble can be used as the memory of the device.

The coupling of a flux qubit to an ensemble of NV centers in diamond, have been achieved experimentally by the authors of [16]. They have prepared a sample of diamond containing NV centers by ion implantation in ^{12}C with (001) surface orientation. This diamond crystal was glued on top of the superconducting circuit with the ^{12}C-implanted (001) surface facing the flux qubit. The flux qubit-NV ensemble coupled system was measured by detecting the qubit state using a read-out DC SQUID.

An extension of this setup has been presented by the authors of [23]. They have proposed constructing an array of flux qubits and placing a diamond crystal with NV centers embedded on it. Adjacent flux qubits are connected via auxiliary tunable coupler qubits. Each of the flux qubits couples to the crystal. The coupler qubit does not couple to the NV, because it is far detuned from the flux qubits and consequently from the NV spins as well. The spin-flux qubit interaction is mediated by the magnetic field that stems from the persistent currents of the flux qubits. With this model in mind, the Hamiltonian of the system can be written as (see appendix)

$$\hat{H} = \hat{H}_{\rm SFQ} + \hat{H}_S + \hat{H}_{\rm int} + \hat{H}_\gamma, \tag{1}$$

with

$$\begin{aligned}\hat{H}_{\rm SFQ} = &\sum_k \frac{1}{2} E_{{\rm qb},k}\hat{\sigma}_{z,k} \\ &+ \sum_{\substack{k,k' \\ k\neq k'}}^{N_{\rm qb}} J_{k,k'}(\cos(\alpha_k)\hat{\sigma}_{z,k} + \sin(\alpha_k)\hat{\sigma}_{x,k}) \\ &\times (\cos(\alpha_{k'})\hat{\sigma}_{z,k'} + \sin(\alpha_{k'})\,\hat{\sigma}_{x,k'}),\end{aligned} \tag{2}$$

$$\hat{H}_S = D\,\hat{S}_z^2 + E\,(\hat{S}_x^2 - \hat{S}_y^2), \tag{3}$$

$$\hat{H}_{\rm int} = \frac{1}{2}\sum_k g_k(\cos(\alpha_k)\,\hat{\sigma}_{z,k} + \sin(\alpha_k)\,\hat{\sigma}_{x,k})\hat{S}_x, \tag{4}$$

$$\hat{H}_\gamma = -{\rm i}\,\gamma_s\,\hat{S}_z - {\rm i}\sum_{k=1}^{N_{\rm qb}} \gamma_{{\rm qb},k}\,\hat{\sigma}_{z,k}. \tag{5}$$

The term $\hat{H}_{\rm SFQ}$ of equation (2) is the Hamiltonian of the SFQs. The set of operators, $\{\hat{\sigma}_{x,k}, \hat{\sigma}_{y,k}, \hat{\sigma}_{z,k}\}$ obey the *su*(2) algebra. The energy of the superfluid flux qubit at site k, $E_{{\rm qb},k}$, and the parameters $\cos(\alpha_k)$ and $\sin(\alpha_k)$, can be written in terms of the energy bias ϵ_k and tunnel splitting parameter Δ_k as

$$\begin{aligned}E_{{\rm qb},k} &= \sqrt{\epsilon_k^2 + \Delta_k^2}, \\ \cos(\alpha_k) &= \epsilon_k/E_{{\rm qb},k}, \\ \sin(\alpha_k) &= -\Delta_k/E_{{\rm qb},k}.\end{aligned} \tag{6}$$

In terms of the persistent current in the qubit at site k, $I_{{\rm p},k}$, and of the external flux threading the qubit loop, $\Phi_{{\rm ex},k}$, the energy bias is given by $\epsilon_k = 2I_{{\rm p},k}(\Phi_{{\rm ex},k} - 3\Phi_0/2)$, being $\Phi_0 = 1/(2e)$ is the flux quantum. The coupling strength between qubits in the sites k and k' is given by $J_{k,k'}$.

The electron-spin ensemble is described by the term H_S of equation (3). It takes into account that an NV center has a ground state with spin 1 and a zero-field splitting D = 2.88 GHz between the $|1, 0\rangle$ and $|1, \pm1\rangle$ states. If an external magnetic field, along the crystalline axis of the NV center, is applied an additional Zeeman splitting between $|1, \pm1\rangle$ sub-levels occurs. Then, it is possible to isolate the subsystem form by $|1, 0\rangle$ and $|1, 1\rangle$, so that the NV center can be modeled by a two-level system [15, 63–66]. The operators $\{\hat{S}_x, \hat{S}_y, \hat{S}_z\}$ of equation (3)

are Pauli spin operators, components of the total spin, $\boldsymbol{S}$, of the electrons. The electron ensemble Hamiltonian consists of a one-axe twisting (OAT) term [67], $D\hat{S}_z^2$ and a Lipkin-type interaction $E\,(\hat{S}_x^2 - \hat{S}_y^2)$ [68–77].

We shall study the mechanism of interaction among the SFQ and electron-spin ensemble proposed in [16]. It is given by the term $H_{\rm int}$ of equation (4).

To take into account dissipative effects due to the finite lifetime, equation (5), both of the the spins (electronic states) and the SFQ, we have introduced the corresponding linewidths γ_s and $\gamma_{\rm qb}$), respectively.

Finally, the proposed Hamiltonian, equation (1), can be exactly diagonalized on the basis obtained from the direct product of the states of basis of the electron ensemble, $\{|N_S,\, k_S\rangle\}$, and of the SFQ, $\{|N_{\rm qb},\, \{k_{{\rm qb},j}\rangle\}$. That is

$$\begin{aligned} |N_{\rm qb},\, \{k_{{\rm qb},j}\},\, N_S,\, k_S\rangle &= |N_{\rm qb},\, \{k_{{\rm qb},j}\}\rangle \otimes |N_S,\, k_S\rangle, \\ |N_S,\, k_S\rangle &= \mathcal{N}_S\, S_+^{k_S}|0\rangle_S, \\ |N_{\rm qb},\, \{k_{{\rm qb},j}\}\rangle &= \mathcal{N}_{\rm qb} \prod_{j=1}^{N_{\rm qb}} \sigma_{+,j}^{k_{{\rm qb},j}}|0\rangle_{\rm qb}. \end{aligned} \tag{7}$$

We have defined $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$ and $\hat{\sigma}_{\pm,j} = \hat{\sigma}_{x,j} \pm i\hat{\sigma}_{y,j}$, as usual. The states corresponding to an ensemble of N_S electrons, $\{|N_S,\, k_S>\}$, are labeled with the parameter k_S, which can run from 0 to the number of spins (electrons) of the system, N_S. The basis of the SFQ is formed by the different possible arrays, $\{k_{{\rm qb},j}\}$, of $N_{\rm qb}$ superconducting flux qubits, $|N_{\rm qb},\, \{k_{{\rm qb},j}\}\rangle$.

2.1. Time evolution

We want to study the time evolution of the hybrid system under the action of the non-Hermitian Hamiltonian H of equation (1). We shall follow the formalism of [25, 78–80].

The first step is the construction of a bi-orthonormal basis, $\{|\widetilde{\Phi}_\beta\rangle,\ |\overline{\Phi}_\alpha\rangle\}$, from the eigenvectors of H and of $H^\dagger$, respectively. Consequently, the vectors of the bi-orthonormal basis obey the condition $\langle\overline{\Phi}_\alpha|\widetilde{\Phi}_\beta\rangle = \delta_{\alpha\beta}$.

We shall consider a general initial state, $|I\rangle$. Next, we shall evaluate the mean value of an operator $\hat{O}$ at time t, which is given by

$$\langle\hat{O}\rangle = \langle\overline{I}|\mathrm{e}^{\mathrm{i}\hat{H}^\dagger t}\,\hat{O}\,\mathrm{e}^{-\mathrm{i}\hat{H}t}|\widetilde{I}\rangle. \tag{8}$$

2.2. Initial condition

Let us study the time evolution of an initial state composed by a coherent spin state for the NV-centers ensemble, $|I\rangle_S$, and of a particular state of the form $|N_{\rm qb},\, \{k_{\rm qb}\}\rangle$ (equation (7)) for the SFQ array. That is

$$|I\rangle = |I\rangle_{\rm qb} \otimes |I\rangle_S, \tag{9}$$

$$|I\rangle_{\rm qb} = |N_{\rm qb},\, \{k_{\rm qb}\}\rangle, \tag{10}$$

$$|I\rangle_S = \frac{1}{(1 + |z_S|^2)^S}\,\mathrm{e}^{z_S S_+}|S\ -S\rangle, \tag{11}$$

with $z_S = -\mathrm{e}^{-i\phi_S}\tan(\theta_S/2)$, where the angles $(\theta_S,\, \phi_S)$ define the direction $\vec{n}_S = (\sin\theta_S\cos\phi_S,\, \sin\theta_S\sin\phi_S,\, \cos\theta_S)$. The coherent state, $|I\rangle_S$, obeys the condition $\hat{\mathbf{S}}\cdot\vec{n}_S|I\rangle_S = -S|I\rangle_S$, with $\hat{\mathbf{S}} = (\hat{S}_x,\, \hat{S}_y,\, \hat{S}_z)$ and $S = N_S/2$ [81].

2.3. Spin squeezing

Squeezing is a property related to the minimization of the variance of an operator, beyond the uncertainty limit, at the expense of the enhancement of the variance of its complementary operator. The definition of a squeezing parameter is not unique, as it depends on the relevant degree of freedom of the system under consideration. Several definitions have been proposed over the years in other systems to characterize spin squeezing. Among them, we can mention the proposal of Wineland *et al* [82] in the context of spectroscopy and the extensively used squeezing parameter constructed by Sørensen *et al* [83] to detect squeezing from an entanglement criterion. Another squeezing parameter has been proposed by Kitagawa and Ueda [67], to characterize the squeezing properties of a system composed by particles with spin $s = 1/2$. More recently, the authors of [9, 84, 85], discussed the definition of the so-called nematic-spin squeezing in dealing with a system composed by particles of spin $s = 1$. The extension to a system of particles of spin $s = J$, $J > 1/2$, has been advanced in [86].

In this work, we shall look at the squeezing properties of the hybrid system by studying the information entropy of each subsystem [87–91] and its related EUR [59, 60].

Given a general quasi-spin operator, $\hat{\mathbf{T}}$, with components $\{\hat{T}_x,\, \hat{T}_y,\, \hat{T}_z\}$, we shall define a set of orthogonal axes $\{n_{x'},\, n_{y'},\, n_{z'}\}$, such that the z'axis is along the direction of $\langle\mathbf{T}\rangle$ of the quasi-spins. In terms of the information entropy, $H(T_\sigma)$, the EUR reads [32, 92–94] as

$$\mathrm{e}^{H(T_{x'})}\mathrm{e}^{H(T_{y'})} \geqslant \frac{\mathrm{e}^{f(N)}}{\mathrm{e}^{H(T_{z'})}}, \tag{12}$$

with

$$f(N) = 2\log(2^N) - \frac{2}{2^N}\sum_k \binom{N}{k}\log\binom{N}{k}, \tag{13}$$

and being

$$H(T_\eta) = -\sum_{j=0}^{2N} P_j(\eta)\log(P_j(\eta)), \quad P_j(\eta) = \langle\eta,\, j|\rho_T(t)|\eta,\, j\rangle, \tag{14}$$

where, $P_j(\eta)$ is the expectation value of the reduced density matrix ρ_T on the j-th eigenstate of the operator T_η. As usual, we assume $\hat{\rho} = \mathrm{e}^{-\mathrm{i}\hat{H}\mathrm{t}}|I\rangle\langle I|\mathrm{e}^{\mathrm{i}\hat{H}^\dagger\mathrm{t}}$.

A state is said to be entropic squeezed in the η-direction if the information entropy $H(T_{\eta'})$ satisfies

$$\zeta_E(\eta') = \mathrm{e}^{H(T_{\eta'})}\Big/\sqrt{\mathrm{e}^{f(N)}/\mathrm{e}^{H(T_{z'})}} < 1. \tag{15}$$

We shall apply the definitions of the two previous sections to describe the squeezing properties of the electron ensemble ($\mathbf{T} = \mathbf{S}$) and qubit system ($\mathbf{T} = \boldsymbol{\sigma}$).

2.4. Decoherence and SU(2) Wigner functions

The interaction between the SFQ array and the electron ensemble should lead to the decoherence of each subsystem

[33, 34]. This effect is known as environment decoherence. It leads to a rapid cancellation of the off-diagonal terms of the local density matrix. In order to analyze the degree of decoherence of our physical system, we shall use the formalism of Wigner functions [44–46].

Since the pioneering works of Wigner [44] and Weyl [45], a lot of work has been devoted to the generalization of the Wigner function formalism in order to address different physical situations [46]. In particular, Stratanovich has introduced the quasi-distribution probability on a sphere, which has received the name SU(2) Wigner function [95]. The development of discrete Wigner function formalism has been advanced by Luis and Peřina [96]. The application of this formalism on the SU(2) group can be found in [97–100].

Let us briefly present the formalism we have followed [96, 100]. Given the reduced density matrix of each component of the hybrid system, ρ_R ($R = NV$ for the electron ensemble, $R =$ qb for the SFQ array), and its discrete SU(2) Wigner function can be computed as

$$W_R(\theta, \phi) = W_R^D(\theta, \phi) + W_R^{ND}(\theta, \phi), \tag{16}$$

$$W_R^D(\theta, \phi) = \frac{2\sqrt{\pi}}{\sqrt{[S]}} \sum_{L=0}^{2S} Y_{L0}(\theta, \phi) \frac{\sqrt{[L]}}{\sqrt{[S]}} \times \sum_{m=-S}^{S} \langle Sm\ L0|Sm\rangle \rho_R(m, m), \tag{17}$$

$$W_R^{ND}(\theta, \phi) = \frac{2\sqrt{\pi}}{\sqrt{[S]}} \sum_{L=0}^{2S} \sum_{\substack{M=-L \\ M\neq 0}}^{L} Y_{LM}^*(\theta, \phi) \frac{\sqrt{[L]}}{\sqrt{[S]}} \times \sum_{m=-S}^{S} \langle Sm\ LM|Sm + M\rangle \rho_R(m, m + M). \tag{18}$$

In the previous expression, we have separated the contribution from the diagonal and off-diagonal terms of the reduced density matrix to the SU(2) Wigner function explicitly. A cancellation of the non-diagonal component of the SU(2) Wigner function, $W_R^{ND}(\theta, \phi)$, should be a direct consequence of the effect of decoherence on the system.

3. Results and discussion

We shall present and discuss the results we have obtained from the study of the system modeled by the Hamiltonian of equation (1). The calculations have been performed by adopting the parameters given in [16, 23] for the SFQ- and the electron ensemble-sectors. We have fixed the coupling constants of the electron ensemble to the values $D = 2.88$ (GHz) and $\gamma_S = 2 \times 10^{-5}$ (GHz). The adopted value of γ_s is consistent with a coherence time of the electron ensemble $T_S \gg 10$ (μ sec) [16, 61]. For the SFQ, we have fixed the values of the coupling constants to $\Delta_k = 2.40$ (GHz) and $\epsilon_k = 1.6$ (GHz), so that $E_{\mathrm{qb},k} = \sqrt{\epsilon_k^2 + \Delta_k^2} = 2.89$ (GHz), with $k = 1, \ldots, N_{\mathrm{qb}}$ [62]. We have adopted a value of γ_{qb}, which is consistent with the time of coherence reported in [62] $T_{\mathrm{qb}} = 2$ (μ sec), $\gamma_{\mathrm{qb}} = 4 \times 10^{-3}$ (GHz). As the initial state we use a coherent state for the electron ensemble, with

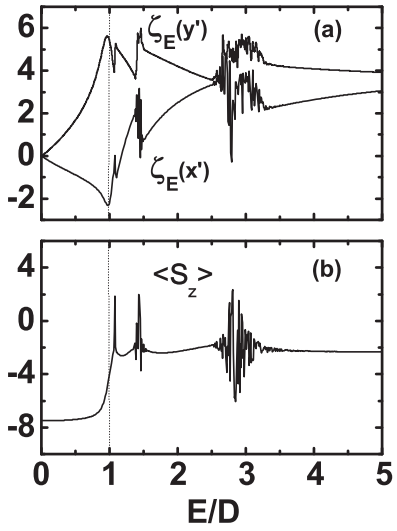

Figure 1. The stationary value of the entropic squeezing parameter, equation (15), of the electron ensemble as a function of the Lipkin coupling constant, in the absence of coupling with the flux qubits, is shown in (a). The values are given in units of decibels. The behavior of the stationary mean value of the z component of the spin for the electron ensemble, $\langle S_z\rangle$, as a function of the ratio E/D is presented in (b). We have taken $N_S = 15$ electrons, $D = 2.88$ (GHz), $\gamma_S = 2 \times 10^{-5}$ (GHz), and $t = 400$ (μ sec) ($t \gg T_S$).

$\theta_0 = \pi/4$ and $\phi_0 = 0$, and for the SFQ the state prepared as $|N_{\mathrm{qb}}, \{k_{\mathrm{qb},j}\} = \{1, 1, \ldots, 1\}\rangle$(equation (7)).

Let us first discuss the squeezing properties of each subsystem in the absence of coupling between the electron ensemble and SFQ.

In figure 1 (a), we show the value of the entropic squeezing parameters of the electron ensemble, for the stationary state ($t \gg T_S$), as a function of the coupling constant E of the Lipkin sector of equation (3), for a fixed value of the OAT constant, D. The corresponding values are given in units of decibels. We have studied an electron ensemble with $N_S = 15$ electrons. The results correspond to $t = 400$ (μ sec). It is clear that there is the appearance of two well-defined regions delimited by $E/D < 1$. For values of $E/D < 1$, the stationary state of the electron ensemble evolves into a squeezed state, while for $E/D > 1$, the stationary state is no longer a squeezed state. This result can be understood by observing that, to the leading order in the number of spins, the Hamiltonian H_S of equation (3) can be mapped into a bosonic Hamiltonian of the form

$$H_B = h_0 - 2SDb^\dagger b + SE(b^{\dagger 2} + b^2).$$

H_B has the structure of the squeezed harmonic oscillator Hamiltonian. Its lowest eigenstate is a squeezed state for $E < D$ [101]. When the dissipative term H_γ is taken into account, an initial coherent state evolves into a pointer state [102], which for $E < D$ behaves as a squeezed state. For the sake of completeness, we plot the stationary behavior of the component S_z of the spin of the electron ensemble. We have checked that the mean value of the total spin, $\langle \mathbf{S}\rangle$, for the steady state points in the z direction. In the case of $E/D<$ it tends to the value $-S/2$. If $E > D$ the values of $\langle S_z\rangle$, $\Delta^2 S_x$,

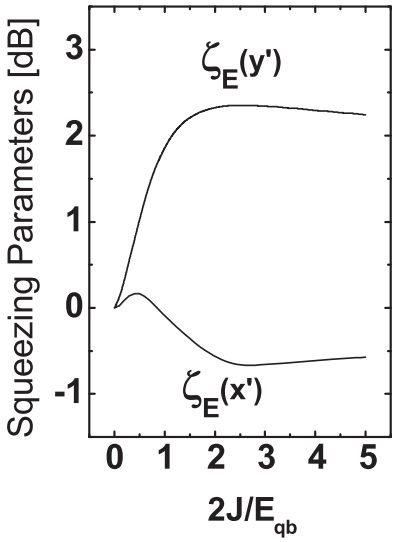

Figure 2. Stationary value of the entropic squeezing parameter of equation (15), of the SFQs as a function of the site-coupling constant, in the absence of coupling with the electron ensemble. The values are given in units of decibels. We have chosen an electron ensemble with $N_{\rm qb} = 3$ SFQs and fixed the coupling constants $\Delta_k = 2.40$ (GHz) and $\epsilon_k = 1.6$ (GHz), that is $E_{\rm qb} = 2.89$ (GHz), and with $\gamma_{\rm qb} = 4 \times 10^{-3}$ (GHz). The values have been obtained at $t = 8$ (μ sec) ($t \gg T_{\rm qb}$).

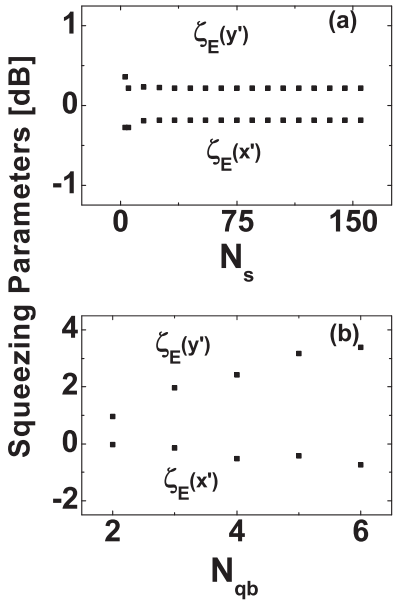

Figure 3. Stationary value of entropic squeezing parameters in the absence of coupling between the electron ensemble and SFQs ($g_k = 0$ (GHz)), as a function of the number of particles of each subsystem. In (a) the results obtained for the electron ensemble, with $N_S = 3, 5, 15, 25, \ldots, 155$ are shown. The coupling constants for the electron sector have been fixed at the values $D = 2.88$ (GHz), $E = 0.26$ (GHz) and $\gamma_S = 2 \times 10^{-5}$ (GHz). In (b), the results obtained for the SFQs are displayed. The coupling constants for the SFQ sector have been fixed at the values $\Delta_k = 2.40$ (GHz) and $\epsilon_k = 1.6$ (GHz), that is $E_{\rm qb} = 2.89$ (GHz) and $\gamma_{\rm qb} = 4 \times 10^{-3}$ (GHz), with the site-coupling constants $J_{k,k'} = J = 1.6$ (GHz).

and $\Delta^2 S_y$ vary with E/D, which is reflected in the behavior of $\zeta_E(x')$ and $\zeta_E(y')$.

Figure 2 shows the entropic squeezing parameters of the stationary state of the SFQ as a function of the site-coupling constant of equation (5), $J_{k,k'} = J$, at a fixed value of $E_{{\rm qb},k} = E_{\rm qb}$. The corresponding values are given in units of decibels. We have chosen an SFQ ensemble with $N_{\rm qb} = 3$, and the results correspond to $t = 8$ (μ sec) ($t \gg T_{\rm qb}$). The results presented in figure 2 indicate that, due to the site–site interaction of the qubits, the stationary state evolves into a squeezed state. This can be understood in terms of the SQF sector of the Hamiltonian $H_{\rm SFQ}$ of equation (2), which can be approximately written as

$$\hat{H}_{\rm SFQ} \approx \hat{H}_{\rm eff,SFQ} \approx J_{\rm eff}\ (\hat{s}_z^2 - \sum_k \hat{s}_{z,k}^2),\ \ \hat{s}_z = \sum_k \hat{s}_{z,k}.$$

The dominant term of $H_{\rm eff,SFQ}$ is an OAT interaction. This naturally leads to a stationary squeezed state.

In figure 3, we explore the behavior of the stationary state as a function of the number of particles. In (a), we show the results for the electron ensemble, with $N_S = 3, 5, 15, 25, \ldots, 155$. The coupling constant for the Lipkin-type interaction was fixed to the value $E = 0.26$ (GHz). In (b), the results obtained for the SFQ are displayed. The site–site coupling constants were fixed at $J_{k,k'} = J = 1.6$ (GHz). The squeezing properties of the stationary state of the electron ensemble, as well as those of the SFQ, persist with the increase in the number of particles.

Next, we present the squeezing properties of the hybrid system when the interaction between the electron ensemble and SFQ, equation (4) is turned on. We have fixed the coupling constant at the value $g_k = 0.1$ (GHz), for $k = 1, 2, 3$ [16].

The time dependence of the entropic squeezing parameters of each component of the hybrid system is shown in figure 4. The system consists of $N_{\rm qb} = 3$ and $N_S = 15$. Figures 4 (a) and (b) correspond to the results obtained for the electron ensemble. Both (c) and (d) display the results obtained for the SFQ. The coupling constants are fixed at the values $E = 0.26$ (GHz) and $J_{k,k'} = J = 1.6$ (GHz). The curves of (a) and (c) have been calculated in the absence of coupling between both subsystems, while those of (b) and (d) have been calculated by taking into account the interaction between the electron ensemble and the SFQ, $H_{\rm int}$ of equation (4). From these results, it is concluded that the coupling between electrons and the SFQ slightly enforces the squeezing properties of each subsystem.

In what follows, we shall discuss the decoherence properties of the proposed physical system. The theory of decoherence addresses the manner in which some quantum systems become classical due to entanglement with the environment [54]. Only one preferred state survives consecutive interactions with the environment, the so-called pointer state [102]. The rest of the states are eliminated. In this context, we shall analyze the time evolution of the discrete Wigner SU(2) function, both for the electrons and the SFQ. In particular, we are interested in the non-diagonal part of the Wigner function, $W_\rho^{ND}(\theta, \phi)$ of equation (17), which is

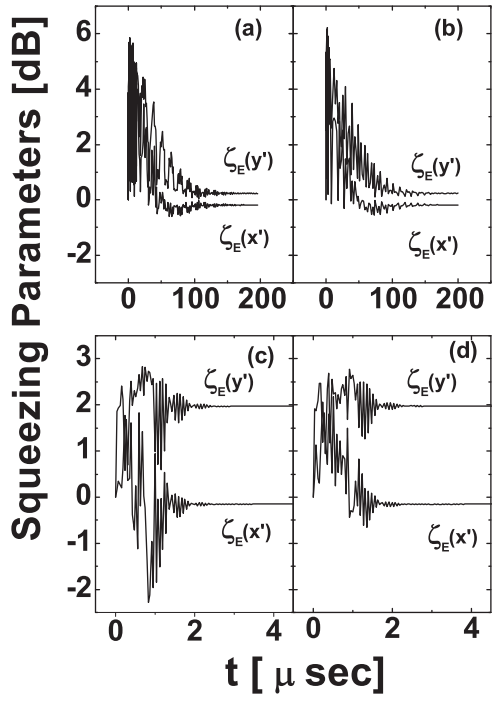

Figure 4. Behavior, as a function of time, of the entropy squeezing parameters of each component of the hybrid system. The system consists of $N_{qb} = 3$ and $N_S = 15$. The interaction coupling constants are fixed at the values $E = 0.26$ ([GHz) and $J_{k,k'} = J = 1.6$ (GHz). Both (a) and (b) correspond to the results obtained for the electron ensemble. Both (c) and (d) displayed the results obtained for the SFQ. The coupling constants are given in the captions to figure 3. The curves shown in (a) and (c) have been obtained in the absence of coupling between both subsystems, while for the results of (b) and (d) the coupling constants of H_{int} of equation (4) was fixed at the value $g_k = 0.1$ (GHz), $k = 1, 2, 3$.

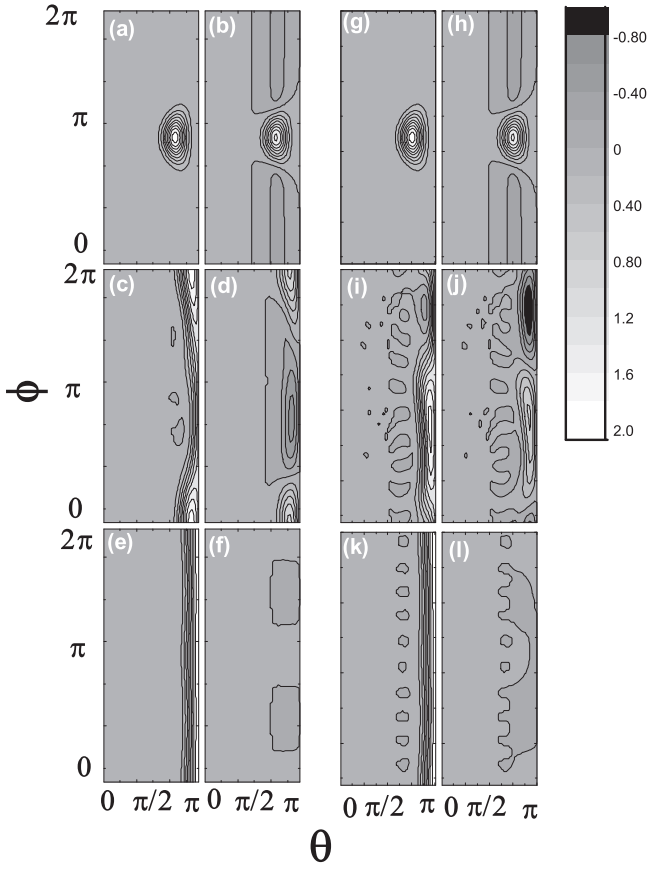

Figure 5. Behavior of the SU(2) Wigner function for the electron ensemble, at different instants of time. Diagrams (a)–(f) present the results obtained in the absence of coupling to the SFQ, while for (g)–(l) the interaction with the SFQ is taken into account. In (a), (c), (e), (g), (i), and (k) we show the results obtained for the SU(2) Wigner function. In (b), (d), (f), (h), (j), and (l) the non-diagonal contribution to the SU(2) Wigner function is displayed for the same values of time. The parameters are those of figure 4. The contour curves presented in (a), (b), (g), and (h) correspond to the results obtained at $t = 0$ (μ sec). Contour curves (c), (d), (i), and (j) correspond to $t = 62$ (μ sec). Finally, contour curves (e), (f), (k), and (l) correspond to $t = 400$ (μ sec).

computed from the off-diagonal terms of the reduced density matrix of each subsystem.

The SU(2) Wigner function for the electron ensemble, at different instants of time, is shown in figure 5. Diagrams (a)–(f) present the results obtained in the absence of coupling to the SFQ, while for (g)–(l) the interaction with the SFQ is taken into account. In (a), (c), (e), (g), (i), and (k) we show the results obtained for the SU(2) Wigner function. In (b), (d), (f), (h), (j), and (l) the non-diagonal contribution to the SU(2) Wigner function is displayed for the same values of time. The parameters are those of figure 4. The contour curves presented in (a), (b), (g), and (h) correspond to the results obtained at $t = 0$ (μ sec). Contour curves (c), (d), (i), and (j) correspond to $t = 62$ (μ sec). Finally, contour curves (e), (f), (k), and (l) correspond to $t = 400$ (μ sec).

Figure 6 shows the behavior, for different instants of time, of the SU(2) Wigner function for the SFQ. The results are presented in the same way as in figure 5, for $t = 0$ (μ sec), 1.2 (μ sec), and 4 (μ sec), respectively.

From figures 5 and 6 it can be observed that, even in the stationary state, the contribution of the off-diagonal matrix elements of the reduced density matrix does not vanish. This non-vanishing behavior of the non-diagonal terms of the Wigner function indicates that the system shows, in the stationary state, some degree of coherence. This effect is enforced by the interaction between the electron ensemble and the SFQ.

From the results presented, we observed that dissipative effects can contribute to the enhancement of entanglement and of spin-squeezing properties of the corresponding systems [40, 58]. The experimental achievement of squeezing of a canonical variable, $\zeta^2_{x'}$, implies that quantum memory and metrology applications gain by an increased signal-to-noise ratio [41, 63].

Similar results have been advanced in [25] and [42]. In [25], the squeezing and decoherence properties of the system modeled by the Hamiltonian of equation (1) were analyzed in terms of the squeezing parameter of Kitagawa and Ueda and of the study of the off-diagonal elements' reduced density matrix of each subsystem, respectively. The authors of [42] have studied the stationary limit of a system composed of a

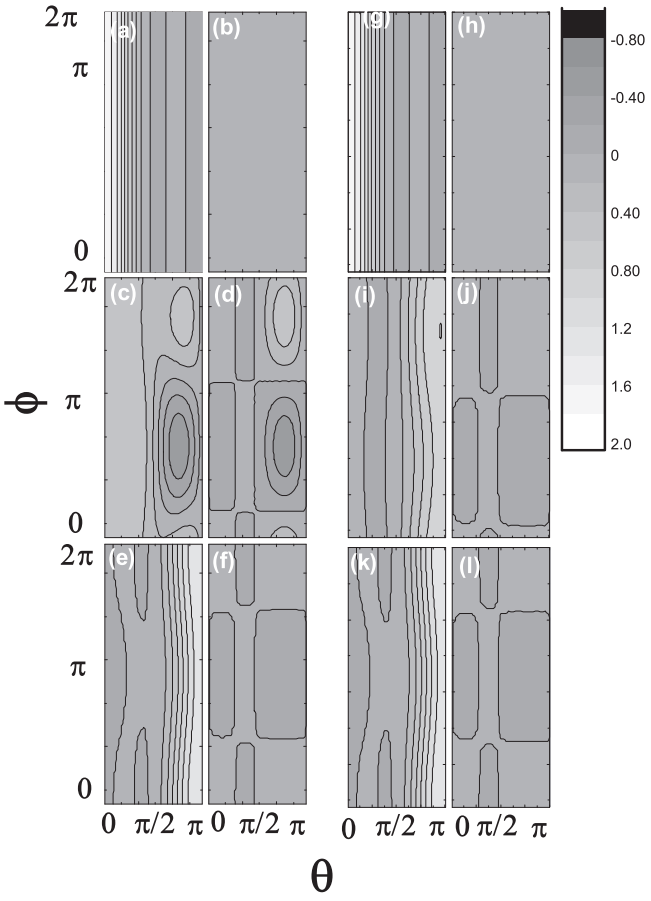

Figure 6. Behavior, for different instants of time, of the SU(2) Wigner function for the SFQ. Results are presented in the same way as in figure 5, for $t = 0$ (μ sec), 1.2 (μ sec), and 4 (μ sec), respectively.

SFQ with an ensemble of NV centers by using the master equation formalism. They concluded that dissipative effects favor both the squeezing properties and coherence of the system.

4. Conclusions

In this work, we have studied the time evolution of a hybrid dissipative system consisting of SFQs coupled to an electron ensemble in the form of NV color centers in diamond. We have taken into account the site–site interaction between the SFQ, as well as the interaction between the SFQ and the electrons. We have analyzed the squeezing properties and decoherence properties of both systems as a function of time, particularly the steady state of the system. In the absence of interaction between the SFQ and electrons, the initial SFQ state evolves into an entropic squeezed state. The steady state of the electron ensemble evolves into an entropic squeezed state depending on the ratio of the coupling constants of the Lipkin-type interaction to the OAT coupling constant, E/D. If $E/D < 1$, the stationary state is an entropic squeezed state. For realistic values of the coupling constants, the steady state of both subsystems preserves the squeezing properties as the number of particles is increased. When the interaction between the SFQ and the electron ensemble is turned on, the steady state remains squeezed. Also, the non-diagonal part of the Wigner function does not vanish. This fact indicates that the off-diagonal matrix elements of each reduced density matrix have not been completely washed out by the interaction between the SFQ and electrons. Thus, the hybrid system remains nonclassical and preserves some degree of coherence in the stationary regimen. Similar results have been advanced in [25, 42].

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Appendix

As has been proposed in different works [16, 23, 103–105], the Hamiltonian of an array of SFQs can be written, on the basis of clockwise and anticlockwise qubit-persistent-currents, as

$$\hat{H}_{\rm SFQ} = \frac{1}{2}\sum_{k=1}^{N_{\rm qb}}(\Delta_k\,\hat{s}_{x,k} + \epsilon_k\hat{s}_{z,k}) + \sum_{\substack{k,k'\\ k\neq k'}}^{N_{\rm qb}} h\,J_{k,k'}\,\hat{s}_{z,k}\hat{s}_{z,k'}, \quad (19)$$

where $\{\hat{s}_{x,k},\ \hat{s}_{y,k},\ \hat{s}_{z,k}\}$ are Pauli spin-1/2 operators. The energy bias is given by $\epsilon_k = 2I_{{\rm p},k}(\Phi_{{\rm ex},k} - 3\Phi_0/2)$, where $I_{{\rm p},k}$ is the persistent current in a qubit at site k, $\Phi_{{\rm ex},k}$ is the external flux threading the qubit loop, and $\Phi_0 = 1/(2e)$ is the flux quantum. The parameter Δ_k stands for the tunnel splitting, while $J_{k,k'}$ denotes the coupling strength between qubits in the sites k and k', respectively.

This Hamiltonian can be diagonalized by using the transformation [16, 25, 94]

$$\begin{pmatrix}\hat{s}_{z,k}\\ \hat{s}_{x,k}\\ \hat{s}_{y,k}\end{pmatrix} = \begin{pmatrix}\cos\alpha_k & \sin\alpha_k & 0\\ -\sin\alpha_k & \cos\alpha_k & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\hat{\sigma}_{z,k}\\ \hat{\sigma}_{x,k}\\ \hat{\sigma}_{y,k}\end{pmatrix}. \quad (20)$$

The set of operators, $\{\sigma_{x,k},\ \sigma_{y,k},\ \sigma_{z,k}\}$ and $\{s_{x,k},\ s_{y,k},\ s_{z,k}\}$, obey the $su(2)$ algebra, being $\cos\alpha_k = \epsilon_k/E_{{\rm qb},k}$, $\sin\alpha_k = -\Delta_k/E_{{\rm qb},k}$, and $E_{{\rm qb},k} = \sqrt{\epsilon_k^2 + \Delta_k^2}$.

Following the work of [16], we shall study the following mechanism of interaction between the SFQ and electron-spin ensemble:

$$\begin{aligned}\hat{H}_{\rm int} &= \frac{1}{2}\sum_{k=1}^{N_{\rm qb}} g_k\,\hat{s}_{z,k}S_x,\\ &= \frac{1}{2}\sum_{k=1}^{N_{\rm qb}} g_k\,(\cos(\alpha_k)\,\hat{\sigma}_{z,k} + \sin(\alpha_k)\,\hat{\sigma}_{x,k})\hat{S}_x. \end{aligned} \quad (21)$$

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References

[1] Ladd T D, Jelezko F, Laflamme R, Nakamura Y, Monroe C and OBrien J L 2010 *Nature* **464** 45
[2] Wilson A C *et al* 2014 *Nature* **512** 57
[3] Lau H-K and Plenio M B 2016 *Phys. Rev. Lett.* **117** 100501
[4] van der Sar T *et al* 2012 *Nature* **484** 82
[5] Mishra S K, Chotorlishvili L, Rau A R P and Berakdar J 2014 *Phys. Rev.* A **90** 033817
[6] Greenland P T *et al* 2010 *Nature* **465** 1057
[7] Togan E *et al* 2010 *Nature* **466** 730
[8] Kovlakov E V, Bobrov I B, Straupe S S and Kulik S P 2017 *Phys. Rev. Lett.* **118** 030503
[9] Huang X Y, Sun F X, Zhang W, He Q Y and Sun C P 2017 *Phys. Rev.* A **95** 013605
[10] Shi-Biao Z *et al* 2015 *Phys. Rev. Lett.* **115** 260403
[11] Xiang Z-L, Ashhab S, You J Q and Nori F 2013 *Rev. Mod. Phys.* **85** 623
[12] Kubo Y *et al* 2010 *Phys. Rev. Lett.* **105** 140502
[13] Koch J *et al* 2007 *Phys. Rev.* A **76** 042319
[14] Metwally N, Wahiddin M R B and Bourennane M 2006 *Opt. Commun.* **257** 206
[15] Marcos D, Wubs M, Taylor J M, Aguado R, Lukin M D and Søorensen A S 2010 *Phys. Rev. Lett.* **105** 210501
[16] Zhu X *et al* 2011 *Nature* **478** 221
[17] Xin-You L, Ze-Liang X, Wei C, You J Q and Nori F 2013 *Phys. Rev.* A **88** 012329
[18] Qiong C, Yang W L and Mang F 2012 *Phys. Rev.* A **86** 022327
[19] Shiro S *et al* 2013 *Phys. Rev. Lett.* **111** 107008
[20] Yueyin Q, Wei X, Lin T and You J Q 2014 *Phys. Rev.* A **89** 042321
[21] Stern M *et al* 2014 *Phys. Rev. Lett.* **113** 123601
[22] Fumiki Y *et al* 2014 *Phys. Rev.* B **89** 020503(R)
[23] Hümmer T, Reuther G M, Hänggi P and Zueco D 2012 *Phys. Rev.* A **85** 052320
[24] Qiu Y, Xiong W, Tian L and You J Q 2014 *Phys. Rev.* A **89** 042321
[25] Reboiro M, Civitarese O and Ramírez R 2017 *Ann. Phys.* **378** 418
[26] Lü X-Y, Xiang Z-L, Cui W, You J Q and Nori F 2013 *Phys Rev* A **88** 012329
[27] Li P-B, Xiang Z-L, Rabl P and Nori F 2016 *PRL* **117** 015502
[28] Ma Y-H, Zhanga X-F, Song J and Wu E 2016 *Ann. Phys., NY* **369** 3644
[29] Lücke B *et al* 2014 *Phys. Rev. Lett.* **112** 155304
Giuseppe Vitagliano G, Apellaniz I, Egusquiza I L and Tóth G 2014 *Phys. Rev.* A **89** 032307
[30] Marchiolli M A, Galetti D and Debarba T B 2013 *Int. J. Quantum Information* **11** 1330001
[31] Ma J, Wang X, Sun C P and Nori F 2011 *Phys. Report* **509** 89
[32] Abdel-Khalek A, Berrada K, Obada S F and Wahidin M R B 2014 *Quantum Electronics* **44** 274
[33] Schlosshauer M 2004 *Rev. Mod. Phys.* **76** 1267 and references therein
[34] Zurek W H 1982 *Phys. Rev.* D **26** 1862
Zurek W H 1981 *Phys. Rev.* D **24** 1516
Cucchietti F M, Paz J P and Zurek W H 2005 *Phys. Rev.* A **72** 52113
Zurek W H 2007 *Progress in Mathematical Physics* **48** 1
Damski B, Quan H T and Zurek W H 2011 *Phys. Rev.* A **83** 062104
[35] Georgescu I M, Ashhab S and Nori F 2014 *Rev. Mod. Phys.* **86** 153
[36] Zhe S 2011 *Phys. Rev.* A **84** 052307
[37] Dodonov V V, Valverde C, Souza L S and Baseia B 2016 *Annals of Physics 296* **371** 296
[38] Marshall J, Campos V L and Zanardi P 2016 *Phys. Rev.* A **94** 052339
[39] Bhaktavatsala Rao D D, Sen Y and Jrg W 2017 *Phys. Rev.* A **95** 022310
[40] Tony E L, Florentin R and Nimrod M 2014 *Phys. Rev. Lett.* **113** 250401
[41] Yuichiro M *et al* 2015 *Phys. Rev. Lett.* **114** 120501
[42] Shane D, Emi Y, Yuichiro M, George C K, William J M and Kae N 2016 *New J. Phys.* **18** 053011
[43] Dong-Yang W, Cheng-Hua B, Hong-Fu W, Ai-Dong Z and Shou Z 2016 *Sci. Rep.* **6** 24421
[44] Wigner E P 1932 *Phys. Rev.* **40** 749
[45] Weyl H 1927 *Z. Phys.* **46** 1
[46] Hillery M, OConnell R F, Scully M O and Wigner E P 1984 *Phys. Rep.* **106** 121
[47] Man'ko M A and Man'ko V I 2011 *Found. Phys.* **41** 330
[48] Man'ko M A, Man'ko V I, Marmo G, Simoni A and Ventriglia F 2013 *Nuovo Cim.* C **36** 163
[49] Man'ko M A, Manko V I and Mendes V 2006 *J. Russ. Laser Res.* **27** 507
[50] Man'ko M A 2010 *Phys. Scr.* **82** 038109
[51] Man'ko M A and Man'ko V I 2012 *Phys. Scr.* T **147** 014020
[52] Adam P *et al* 2014 *J. Russ. Laser Res.* **35** 3
[53] Biswas A and Agarwal G S 2007 *Phys. Rev.* A **75** 032104
[54] Schwaha P, Querlioz D, Dollfus P, Saint-Martin J, Nedjalkov M and Selberherr S 2013 *J Comput Electron* **12** 388
[55] Tilma T, Everitt M J, Samson J H, Munro W J and Nemoto K 2016 *Phys. Rev. Lett.* **117** 180401
[56] Siyouri F, El Baz M and Hassouni Y 2016 *Int. J. of Mod. Phys.* B **30** 1650187
[57] Yong-Hong M and Xue-Feng Z 2014 *Phys. Rev.* B **89** 144113
[58] Wu L-N, Jin G-R and You L 2015 *Phys. Rev.* A **92** 033826
[59] Hirschman I I 1957 *Am. J. Math* **79** 152
[60] Coles P J, Mario B, Tomamichel M and Wehner S 2017 *Rev. Mod. Phys.* **89** 015002
[61] Mizuochi N *et al* 2009 *Phys. Rev.* B **80** 041201(R)
Stanwix P L, Pham L M, Maze J R, Le Sage D, Yeung T K, Cappellaro P, Hemmer P R, Yacoby A, Lukin M D and Walsworth R L 2010 *Phys. Rev.* B **82** 201201(R)
Farfurnik D, Jarmola A, Pham L M, Wang Z H, Dobrovitski V V, Walsworth R L, Budker D and Bar-Hill N 2015 *Phys. Rev.* B **92** 201201(R)
[62] van der Ploeg S H W *et al* 2002 *Phys. Rev. Lett.* **98** 057004
Niskanen A O *et al* 2007 *Science* **316** 723
Hime T *et al* 2006 *Science* **314** 1427
Stern M, Catelani G, Kubo Y, Grezes C, Bienfait A, Vion D, Esteve D and Bertet P 2014 *Phys. Rev. Letter.* **113** 123601
Xue Z-Y *et al* 2016 *Quantum Information Processing* **15** 721
Wen H, Xu-Bo Z and Guang G 2015 *Chin. Phys.* B **24** 064207
[63] Reiter F and Sørensen A S 2012 *Phys. Rev.* A **85** 032111
Reiter F, Kastoryano M J and Sørensen A S 2012 *New J. Phys.* **14** 053022
[64] Bennett S D, Yao N Y, Otterbach J, Zoller P, Rabl P and Lukin M D 2013 *Phys. Rev. Lett.* **110** 156402
[65] Doherty M W, Dolde F, Fedder H, Jelezko F, Wrachtrup J, Manson N B and Hollenberg L C L 2012 *Phys. Rev.* B **85** 205203
[66] Taylor J M, Capperallo P, Childress L, Jiang L, Budker D, Hemmer P R, Yacoby A, Walsworth R and Lukin M D 2008 *Nat. Phys.* **4** 810
[67] Kitagawa M and Ueda M 1993 *Phys. Rev.* A **47** 5138

[68] Lipkin H J, Meshkov N and Glick A J 1965 *Nucl. Phys.* **62** 188
[69] Ring P and Schuck P 1980 *The Nuclear Many Body Problem* (Berlin: Springer)
[70] Civitarese O, Reboiro M, Rebon L and Tielas D 2010 *Phys. Lett.* A **374** 424
[71] Civitarese O, Reboiro M, Rebón L and Tielas D 2010 *Phys. Lett.* A **374** 2117
Civitarese O, Reboiro M, Rebón L and Tielas D 2009 *Phys. Lett.* A **373** 754
[72] Dusuel S and Vidal J 2004 *Phys. Rev. Lett.* **93** 237204
[73] Dusuel S and Vidal J 2005 *Phys. Rev.* B **71** 224420
[74] Barthel T, Dusuel S and Vidal J 2006 *Phys. Rev. Lett.* **97** 220402
[75] Vidal J, Dusuel S and Barthel T 2007 *J. Stat. Mech.* **0701** P01015
[76] Orus R, Dusuel S and Vidal J 2008 *Phys. Rev. Lett.* **101** 025701
[77] Wichterich H, Vidal J and Bose S 2010 *Phys. Rev.* A **81** 032311
[78] Feshbach H 1958 *Ann. of Phys.* **5** 357
[79] Rotter I 2009 *J. Phys. A: Math. Theor.* **42** 153001
Rotter I and Bird J P 2015 *Rep. Prog. Phys.* **78** 114001
[80] Ramirez R and Reboiro M 2016 *Phys. Lett.* A **380** 1117
[81] Hecht K T 1987 *Lecture Notes in Physics* ed H Araki *et al* 290 (Berlin: Springer) (https://doi.org/10.1007/BFb0017656)
[82] Wineland D J, Bollinger J J, Itano W M, Moore F L and Heinzen D J 1992 *Phys. Rev.* A **46** R6797
Wineland D J, Bollinger J J, Itano W M and Heinzen D J 1994 *Phys. Rev.* A **50** 67
[83] Sørensen A and Mølmer K 1999 *Phys. Rev. Lett.* **83** 2274
Sørensen A and Mølmer K 2001 *Phys. Rev. Lett.* **86** 4431
Sørensen A, Duan L M, Cirac J I and Zoller P 2001 *Nature* **409** 63
[84] Yukawa E, Ueda M and Nemoto K 2013 *Phys. Rev.* A **88** 033629
Hamley C D, Gerving C S, Hoang T M, Bookjans E M and chman M S 2012 *Nat. Phys.* **8** 305
[85] Reboiro M, Civitarese O and Tielas D 2013 *Int. J. Mod. Phys.* B **27** 13501178
[86] Yukawa E and Nemoto K 2016 *J. Phys. A: Math. Theor.* **49** 255301
[87] Reboiro M, Civitarese O and Tielas D 2013 *Int. J. Mod. Phys.* B **27** 13501178
Reboiro M, Civitarese O and Tielas D 2014 *J. Russ. Laser Res.* **35** 110
Civitarese O, Reboiro M and Tielas D 2013 *J. Phys.* B **46** 065502
Reboiro M, Civitarese O and Tielas D 2015 *Phys. Scripta* **90** 074028
[88] Man'ko M A, Man'ko V I and Marmo G 2016 *Source of the Document Nuovo Cimento della Societa Italiana di Fisica* C **38** 167
Figueroa A, López J, Castaños O, López-Peña R, Man'ko M A and I Man'ko V 2015 *J. Phys. A: Math. Theor.* **48** 065301
Man'ko M A and Man'ko V I 2015 *Entropy* **17** 2876
[89] Sanchez-Ruiz J 1995 *Phys. Lett.* A **201** 125
[90] Kraus K 1987 *Phys. Rev.* D **35** 3070
[91] Maassen H and Uffink J B M 1988 *Phys. Rev. Lett.* **60** 1103
[92] Mahmoud A-A, Sebawe A M and Obada A S F 2002 *J. Phys.* B **4** S133–141 (http://stacks.iop.org/JOptB/4/S133)
[93] Liao Q, Fang G, Wangc Y, Ahmad M A and Liu S 2011 *Optik* **122** 1392
[94] Reboiro M, Civitarese O and Tielas D 2015 *Phys. Scripta* **90** 074028
[95] Stratonovich R L 1957 *Sov. Phys. JETP* **4** 891
[96] Luis A and Peřina J 1998 *J. Phys. A: Math. Gen.* **31** 1423
[97] Agarwal G S 1981 *Phys. Rev.* A **24** 2889
[98] Wolf K B 1996 *Opt. Commun.* **132** 343
Atakishiyev N M, Chumakov S M and Wolf K B 1998 *J. Math. Phys.* **39** 6247
[99] Chumakov S M, Frank A and Wolf K B 1999 *Phys. Rev.* A **60** 1817
[100] Chumakov S M, Klimov A B and Wolf K B 2000 *Phys. Rev.* A **61** 034101
Klimov A B and Chumakov S M 2002 *Rev. Mex. de Física* **48** 317
[101] Zhang Y Z 2013 *J. Phys. A: Math. Theor.* **46** 455302
Lo C F 2014 *J. Phys. A: Math. Theor.* **47** 078001
Kryuchkov S I, Suslov S K and Vega-Guzman J M 2013 *J. Phys. B: At. Mol. Opt. Phys.* **46** 104007
[102] Ferry D K, Akis R and Bird J P 2004 *Phys. Rev. Lett.* **93** 026803
[103] Orlando T P, Mooij J E, Tian L, van der Wal C H, Levitov L S, Lloyd S and Mazo J J 1999 *Phys. Rev.* B **60** 15398
[104] Plourde B L T, Zhang J, Whaley K B, Wilhelm F K, Robertson T L, Hime T, Linzen S, Reichardt P A, Wu C-E and Clarke J 2004 *Phys. Rev.* B **70** 140501
[105] Ballesteros A, Civitarese O, Herranz F J and Reboiro M 2003 *Phys. Rev.* B **68** 214519