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On the complexity of $\{k\}$-domination and k-tuple domination in graphs ☆

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ABSTRACT

We consider two types of graph domination—$\{k\}$-domination and k-tuple domination, for a fixed positive integer k—and provide new NP-complete as well as polynomial time solvable instances for their related decision problems. Regarding NP-completeness results, we solve the complexity of the $\{k\}$-domination problem on split graphs, chordal bipartite graphs and planar graphs, left open in 2008. On the other hand, by exploiting Courcelle's results on Monadic Second Order Logic, we obtain that both problems are polynomial time solvable for graphs with clique-width bounded by a constant. In addition, we give an alternative proof for the linearity of these problems on strongly chordal graphs.

1. Introduction

Due to its large range of applications, many variations and extensions of the classical domination problem in graphs have been defined and studied. In general, these problems can be stated as follows: given a graph $G = (V(G), E(G))$, $A \subseteq \mathbb{R}$ and $B = (b_v)_{v \in V(G)} \in \mathbb{R}_+^{|V(G)|}$, an *$A, B$-dominating function* of G is a function $f : V(G) \to A$ such that $f(N[v]) \geq b_v$ for each $v \in V(G)$, where $f(U) = \sum_{u \in U} f(u)$, for $U \subseteq V(G)$ and $N[v]$ is the *closed neighborhood* of v, i.e. the set of vertices at distance at most 1 from v. The *weight* of f is given by $w(f) = f(V(G))$, and let $W_{A,B}(G)$ denote the minimum possible value of $w(f)$.

This work is focused on two variations of the classical domination problem. Let $k \in \mathbb{Z}_+$ and $b_v = k$ for each $v \in V(G)$. When $A = \{0, 1\}$, f is a *k-tuple dominating function* and $W_{A,B}(G)$ is the *k-tuple domination number* of G denoted by $\gamma_{\times k}(G)$ and introduced by Harary and Haynes in [16]. Notice that a k-tuple dominating function of G can be equivalently seen as a subset D of $V(G)$ such that $|N[v] \cap D| \geq k$ for every $v \in V(G)$. When $A = \{0, 1, \ldots, k\}$, f is a *$\{k\}$-dominating function* and $W_{A,B}(G)$ is the *$\{k\}$-domination number* of G denoted by $\gamma_{\{k\}}(G)$ and introduced by Bange et al. [3]. As usual, these definitions induce the study of the following decision problems for fixed $k \in \mathbb{Z}_+$:

***k*-TUPLE DOMINATION** (k-DOM)
Instance: A graph G, $j \in \mathbb{N}$.
Question: Does G have a k-tuple dominating function of weight at most j?

$\{k\}$-DOMINATION ($\{k\}$-DOM)
Instance: A graph G, $j \in \mathbb{N}$.
Question: Does G have a $\{k\}$-dominating function of weight at most j?

When $k = 1$, both problems concern the usual notion of a dominating function. The corresponding decision problem (DOM) has been widely studied, see for instance [4,5,7,8,10–12,15].

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Table 1
Complexity table for k-DOM and $\{k\}$-DOM for fixed $k \in \mathbb{Z}_+$. "NP-c", "P" and "?" mean NP-complete, polynomial and open problem, respectively.

Class	DOM ($k = 1$)	k-DOM (fixed $k \in \mathbb{Z}_+$)	$\{k\}$-DOM (fixed $k \in \mathbb{Z}_+$)
Strongly chordal	P [7]	P [20]	P [18]
Bounded clique-width	P [11]	?	?
Split	NP-c [4]	NP-c [20]	?
Planar	NP-c [15]	NP-c ($2 \le k \le 6$) [18]	?
Chordal	NP-c [5]	NP-c [20]	?
Bipartite	NP-c [12]	NP-c [20]	NP-c [17]
Bipartite Planar	NP-c [10]	?	?
Chordal Bipartite	NP-c [8]	NP-c [18]	?

For fixed $k \in \mathbb{Z}_+$, k-DOM is NP-complete for doubly chordal graphs and for dually chordal graphs, but $\{k\}$-DOM is polynomial time solvable on both classes [18]. Regarding graph classes with a limited number of P_4-partners, the three problems are polynomial time solvable for cographs [18], DOM and k-DOM are polynomial time solvable for P_4-tidy graphs ([11] and [13], respectively), but the complexity of $\{k\}$-DOM is unknown for this class. Finally, it is known that DOM is polynomial time solvable for graphs with bounded tree-width [2] and this is also the case for both, k-DOM and $\{k\}$-DOM [1].

The main purpose of this work is to study the unknown complexities concerning k-DOM and $\{k\}$-DOM for fixed $k \in \mathbb{Z}_+$ that are shown in Table 1. Some of these unknown results were left as open questions by Lee and Chang in [18]. Also, we focus on determining graph classes on the frontier between the hard and easy cases.

Some of the results in this work, specifically the one concerning bounded tree-width graphs, appeared without proofs in [1].

2. Background

All the graphs in this paper are finite, undirected and simple. Given a graph G, let $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. For any $v \in V(G)$, $N[v]$ is the closed neighborhood of v in G.

For any positive integers n, m, we denote by K_n and C_n the complete graph and the cycle with n vertices respectively, and by $K_{n,m}$ the complete bipartite graph on $n + m$ vertices.

Given a graph G and $S \subseteq V(G)$, the subgraph of G *induced* by S is the graph with vertex set S and all edges of G whose endpoints are both contained in S.

A *chord* of a cycle C in a graph G is an edge uv not in C such that u and v lie in C. A graph G is *chordal* if it does not contain a chordless cycle C_n, for $n \ge 4$. A bipartite graph is *chordal bipartite* if each cycle of length at least 6 has a chord.

A *clique* (*stable set*) in a graph is a set of pairwise adjacent (non-adjacent) vertices. A graph G is *split* if its vertex set admits a partition into a clique Q and a stable set S. Notice that every split graph is chordal.

For disjoint graphs G and H and $v \in V(G)$, $G[H/v]$ denotes the graph obtained by the *substitution* in G of v by H, i.e. $V(G[H/v]) = (V(G) \cup V(H)) - \{v\}$ and

$$\begin{aligned} &E(G[H/v]) \\ &\quad = E(H) \cup \{e : e \in E(G) \text{ and } e \text{ is not incident with } v\} \\ &\qquad \cup \{uw : u \in V(H),\ w \in V(G) \text{ and } w \text{ is adjacent} \\ &\qquad\quad \text{to } v \text{ in } G\}. \end{aligned}$$

For graphs G and H, the *strong product* $G \boxtimes H$ is defined on the vertex set $V(G) \times V(H)$, where two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \boxtimes H$ if and only if $u_1 = u_2$ and $v_1 v_2 \in E(H)$, or $v_1 = v_2$ and $u_1 u_2 \in E(G)$, or $v_1 v_2 \in E(H)$ and $u_1 u_2 \in E(G)$.

The vocabulary $\{E\}$ consisting of one binary relation symbol E is denoted by τ_1. For a graph G, $G(\tau_1)$ denotes the presentation of G as a τ_1-structure $< V, E >$, where V is the domain of the logical structure ($V(G)$) and E is the binary relation corresponding to the adjacency matrix of G.

Regarding graph properties, if a formula can be defined using vertices and sets of vertices of a graph, the logical operators OR, AND, NOT (denoted by $\vee$, $\wedge$, $\neg$), the logical quantifiers $\forall$ and $\exists$ over vertices and sets of vertices, the membership relation $\in$ to check whether an element belongs to a set, the equality operator $=$ for vertices and the binary adjacency relation *adj*, where $adj(u, v)$ holds if and only if vertices u and v are adjacent, then the formula is expressible in τ_1-monadic second-order logic, MSOL(τ_1) for short.

An optimization problem P is a LinEMSOL(τ_1) optimization problem over graphs, if it can be defined in the following form: Given a graph G presented as a τ_1-structure and evaluation functions $f_1, \ldots, f_m$ associating integer values to the vertices of G, find an assignment z to the free variables in θ such that

$$\sum_{\substack{1 \le i \le l \\ 1 \le j \le m}} a_{ij} |z(X_i)|_j = opt \left\{ \sum_{\substack{1 \le i \le l \\ 1 \le j \le m}} a_{ij} |z'(X_i)|_j : \theta(X_1, \ldots, X_l) \text{ is true for } G \text{ and } z' \right\},$$

where *opt* is either *Min* or *Max*, θ is an MSOL(τ_1) formula having free set variables $X_1, \ldots, X_l$, $a_{ij} : i \in \{1, \ldots, l\}$, $j \in \{1, \ldots, m\}$ are integer numbers and $|z(X_i)|_j$ is used as a short notation for $\sum_{a \in z(X_i)} f_j(a)$ ($|A|$ indicates the cardinality of the finite set A). More details can be found for example in [11].

It has been shown that MSOL(τ_1) is particularly useful when combined with the concept of the graph parameter clique-width as the following theorem—first stated in [11] and then reinforced in [21]—shows:

Theorem 1. *(See [11,21].) Let q be a constant and $\mathcal{C}(q)$ be a class of graphs of clique-width at most q. Then every LinEMSOL(τ_1) problem on $\mathcal{C}(q)$ can be solved in polynomial time.*

3. Polynomial instances

In [9], Brešar et al. proved that $\gamma_{\{k\}}(G) = \gamma_{\times k}(G \boxtimes K_k)$ for any graph G and $k \geq 1$. As a consequence, we can state:

Remark 2. Let $k \in \mathbb{Z}_+$ be fixed and $\mathcal{F}$ and $\mathcal{S}$ be two graph classes such that if $G \in \mathcal{F}$ then $G \boxtimes K_k \in \mathcal{S}$. If k-DOM is polynomial (linear) time solvable on $\mathcal{S}$, then $\{k\}$-DOM is polynomial (linear) time solvable on $\mathcal{F}$. Besides, if $\{k\}$-DOM is NP-complete on $\mathcal{F}$ then k-DOM is NP-complete on $\mathcal{S}$.

In the remainder of this section, we study the computational complexity of k-DOM and $\{k\}$-DOM on two graph classes that satisfy the hypothesis of Remark 2, specifically graphs with clique-width bounded by a constant and strongly chordal graphs.

The result for graphs with clique-width bounded by a constant follows from the fact that k-dominating functions can be stated in MSOL(τ_1). We can prove:

Theorem 3. *Let $k \in \mathbb{Z}_+$, q be a constant and $\mathcal{C}(q)$ be a class of graphs of clique-width at most q. Then k-DOM and $\{k\}$-DOM can be solved in polynomial time on $\mathcal{C}(q)$.*

Proof. We first prove that finding the k-tuple domination number of a given graph is a LinEMSOL(τ_1) optimization problem. In other words, we wish to prove that finding a minimum sized subset of vertices of the given graph satisfying that in every closed neighborhood there is at least k elements of this subset is a LinEMSOL(τ_1) optimization problem.

Given a graph G presented as a τ_1-structure $G(\tau_1)$ and one evaluation function (the constant function that associates 1's to the vertices of G) and denoting by $X_1(v)$ the atomic formula indicating that $v \in X_1$, find an assignment z to the free set variable X_1 in θ such that $|z(X_1)|_1 = \min\{|z'(X_1)|_1 : \theta(X_1)$ holds for G and $z'\}$, where $\theta(X_1) = \forall v \left(\bigwedge_{1\leq r\leq k} A_r(X_1, v, u_1, \ldots, u_r)\right)$, with $A_1(X_1, v, u_1) := \exists u_1 [X_1(u_1) \wedge (adj(v, u_1) \vee v = u_1)]$, and for each $r > 1$

$$A_r(X_1, v, u_1, \ldots, u_r) := \exists u_r \left[X_1(u_r) \wedge (adj(v, u_r) \vee v = u_r) \wedge \bigwedge_{1\leq i\leq r-1} \neg(u_r = u_i) \right].$$

From Theorem 1, k-DOM can be solved in polynomial time on $\mathcal{C}(q)$.

To prove the polynomiality of $\{k\}$-DOM on $\mathcal{C}(q)$, we first notice that we can obtain equivalently $G \boxtimes K_k$ by performing a substitution in G of every vertex by K_k, for each fixed $k \in \mathbb{Z}_+$. Also, we take into account that $cwd(G[H/v]) = \max\{cwd(G), cwd(H)\}$, for every pair of disjoint graphs G and H and $v \in V(G)$ [11]. Since $cwd(K_k) \leq 2$ for every k, if G is a graph in $\mathcal{C}(q)$, $q \geq 2$, then $G \boxtimes K_k$ is in $\mathcal{C}(q)$. Hence, from Remark 2, $\{k\}$-DOM can be also solved in polynomial time on $\mathcal{C}(q)$, for $q \geq 2$. As the result is trivial for $q = 1$, the result is proved. □

We turn to strongly chordal graphs, which are graphs characterized by the existence of a strong elimination ordering [14].

It is clear that if every induced subgraph of G has a simple vertex, then every induced subgraph of $G \boxtimes K_k$ has a simple vertex, for fixed $k \in \mathbb{Z}_+$. In other words, if G is a strongly chordal graph and $k \in \mathbb{Z}_+$, then $G \boxtimes K_k$ is strongly chordal. As Table 1 shows, it is already known that $\{k\}$-DOM is linear time solvable on strongly chordal graphs, provided a strong elimination ordering of the input graph [18]. By applying Remark 2 when $\mathcal{F}$ and $\mathcal{S}$ are both the class of strongly chordal graphs and the fact that k-DOM is linear time solvable on strongly chordal graphs [20], we alternatively obtain that $\{k\}$-DOM is linear time solvable on this graph class.

In the following section, among others NP-completeness results, we prove that the linearity for $\{k\}$-DOM on strongly chordal graphs does not extend to chordal graphs, by proving that it is NP-complete on split graphs.

4. NP-completeness results

As already mentioned, $\{k\}$-DOM is NP-complete for general graphs. In this section we first analyze the computational complexity of this problem on another subclass of chordal graphs. Actually, we prove that for every $k \in \mathbb{Z}_+$, $\{k\}$-DOM remains NP-complete on split graphs. We begin by providing the following property of $\{k\}$-dominating functions.

Lemma 4. *Let G be a graph and $k \in \mathbb{Z}_+$ be fixed. Let $u, v \in V(G)$ be distinct such that $N[u] \subseteq N[v]$. There exists a minimum $\{k\}$-dominating function $\hat{f}$ of G such that $\hat{f}(u) = 0$.*

Proof. Let $u, v \in V(G)$ such that $N[u] \subseteq N[v]$. Firstly, we observe that every minimum weight $\{k\}$-dominating function f of G satisfies $f(u) + f(v) \leq k$. Else, the weight of the $\{k\}$-dominating function f' of G defined by $f'(v) = k$, $f'(u) = 0$ and $f'(w) = f(w)$ for $w \neq u$ and $w \neq v$ is strictly smaller than the weight of f.

Next, let f be a minimum weight $\{k\}$-dominating function of G. If $f(u) \neq 0$ we define $\hat{f}(v) = f(u) + f(v)$, $\hat{f}(u) = 0$, and $\hat{f}(w) = f(w)$ for $w \neq u$ and $w \neq v$. It is immediate to check that $\hat{f}$ is also a minimum weight $\{k\}$-dominating function of G, and the lemma follows. □

As a consequence of Lemma 4, we can prove:

Theorem 5. *For every fixed $k \in \mathbb{Z}_+$, $\{k\}$-DOM is NP-complete on split graphs.*

Proof. Clearly, $\{k\}$-DOM on split graphs is in NP.

We reduce $\{k\}$-DOM on a general graph to $\{k\}$-DOM on a split graph. Given a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$, we construct a split graph H with vertex partition (Q, S), where $Q = \{q_1, \ldots, q_n\}$ induces a complete graph and $S = \{s_1, \ldots, s_n\}$ is a stable set, and for all $i, j \in \{1, \ldots, n\}$, $q_i s_j \in E(H)$ if and only if $v_i v_j \in E(G)$ or $i = j$ (see Fig. 1).

We use the above construction to prove that $\gamma_{\{k\}}(G) = \gamma_{\{k\}}(H)$.

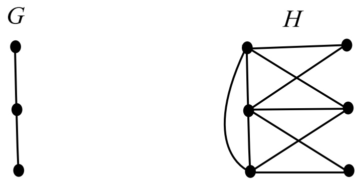

Fig. 1. Split graph H obtained from a graph G.

On the one hand, for all $i \in \{1, \ldots, n\}$, $N[s_i] \subseteq N[q_i]$. Then, from Lemma 4, there exists a minimum weight $\{k\}$-dominating function f of H such that $f(S) = 0$. We define the function $\hat{f} : V(G) \to \{0, \ldots, k\}$ by $\hat{f}(v_i) = f(q_i)$ for all $i \in \{1, \ldots, n\}$. From its definition, $\hat{f}$ is a $\{k\}$-dominating function of G. Hence, $\gamma_{\{k\}}(G) \le \gamma_{\{k\}}(H)$.

Conversely, let f be a minimum weight $\{k\}$-dominating function of G. In this case, we define $f' : V(H) \to \{0, \ldots, k\}$ such that for all $i \in \{1, \ldots, n\}$ $f'(s_i) = 0$ and $f'(q_i) = f(v_i)$. It is not difficult to check that f' is a $\{k\}$-dominating function of H. Hence, $\gamma_{\{k\}}(G) \ge \gamma_{\{k\}}(H)$. $\square$

As every split graph is also a chordal graph, we obtain that $\{k\}$-DOM is NP-complete on chordal graphs.

We notice that when G is a chordal graph, $G \boxtimes K_k$ also is, for every positive integer k. On the other hand, it is clear that k-DOM on chordal graphs is in NP. Then, following Remark 2, the NP-completeness of $\{k\}$-DOM on chordal graphs implies the NP-completeness of k-DOM on chordal graphs, firstly proved in [20].

In the remainder, we address the question of determining the computational complexity of $\{k\}$-DOM for planar graphs proposed in [18], by proving that the problem is not easily solvable even for bipartite planar graphs, unless P = NP.

We begin by providing the exact value of the $\{k\}$-domination number of the complete bipartite graph $K_{2,m}$, for every $m \ge k$. Notice that $K_{2,m}$ is planar and also chordal bipartite.

Theorem 6. *For every fixed $k \in \mathbb{Z}_+$ and $m \ge k \ge 2$, $\gamma_{\{k\}}(K_{2,m}) = k + \left\lceil \frac{k}{2} \right\rceil$.*

Proof. Suppose that the bipartition of $K_{2,m}$ is $\{u^1, u^2\}$ and $\{w^1, \ldots, w^m\}$. We first remark that, for any $\{k\}$-dominating function f of $K_{2,m}$, the following facts are straightforward:

i. Since $f(N[w^j]) \ge k$ for all $j \in \{1, \ldots, m\}$,

$$f(w^j) \ge k - t, \text{ for all } j \in \{1, \ldots, m\}, \tag{1}$$

where $t = f(u^1) + f(u^2)$. Hence, $w(f) = f(u^1) + f(u^2) + \sum_{j=1}^{m} f(w^j) \ge t + (k-t)m$.

ii. Since $f(N[u^i]) = f(u^i) + \sum_{j=1}^{m} f(w^j) \ge k$ for each $i = 1, 2$,

$$\sum_{j=1}^{m} f(w^j) \ge \max\{k - f(u^1), k - f(u^2)\}. \tag{2}$$

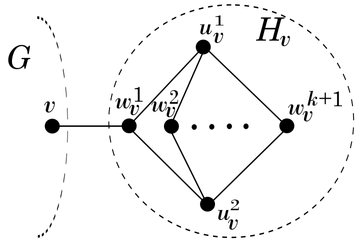

Fig. 2. $H(G)$.

On the one hand, we define the function $g : V(K_{2,m}) \to \{0, \ldots, k\}$ such that $g(u^1) = g(w^1) = \left\lceil \frac{k}{2} \right\rceil$, $g(u^2) = \left\lfloor \frac{k}{2} \right\rfloor$ and $g(w^j) = 0$ for all $j \in \{2, \ldots, m\}$. It is clear that g is a $\{k\}$-dominating function of $K_{2,m}$ with weight $w(f) = k + \left\lceil \frac{k}{2} \right\rceil$. Thus $\gamma_{\{k\}}(K_{2,m}) \le k + \left\lceil \frac{k}{2} \right\rceil$.

On the other hand, when $k \in \{2, 3\}$ and $m = k$ the result easily follows. For the remaining cases, let f be a $\{k\}$-dominating function of $K_{2,m}$. Suppose $t \le k - 1$. If $k \in \{2, 3\}$ and $m \ge k + 1$ then, from (1) $w(f) \ge t + (k-t)(k+1) = (k-t+1)k \ge 2k > k + \left\lceil \frac{k}{2} \right\rceil$. If $k \ge 4$, then, again from (1) $w(f) \ge t + (k-t)k = t + k^2 - tk = k^2 - t(k-1) \ge k^2 - (k-1)^2 > k + \left\lceil \frac{k}{2} \right\rceil$.

Then f cannot be a minimum $\{k\}$-dominating function. This implies that, for every minimum $\{k\}$-dominating function f of $K_{2,m}$,

$$f(u^1) + f(u^2) \ge k. \tag{3}$$

Let $\tilde{f}$ be a minimum $\{k\}$-dominating function of $K_{2,m}$. From (2), $\sum_{j=1}^{m} \tilde{f}(w^j) = \max\{k - \tilde{f}(u^1), k - \tilde{f}(u^2)\}$. Without loss of generality, we assume $\tilde{f}(u^1) \le \tilde{f}(u^2)$. Then, $\gamma_{\{k\}}(K_{2,m}) = w(\tilde{f}) = k + \tilde{f}(u^2)$, and from (3), $\gamma_{\{k\}}(K_{2,m}) \ge k + \left\lceil \frac{k}{2} \right\rceil$. $\square$

Now, we introduce another graph construction as follows: given a fixed integer $k \ge 2$ and a graph G, we define a graph $H(G)$ by adding to each vertex $v \in V(G)$ a graph H_v, isomorphic to $K_{2,k+1}$, such that the bipartition of H_v is $\{u_v^1, u_v^2\}$ and $\{w_v^1, \ldots, w_v^{k+1}\}$ and also an edge vw_v^1. An example of this construction is given in Fig. 2.

Clearly, when G is a bipartite planar graph, $H(G)$ also is. In the same way, when G is a chordal bipartite graph, $H(G)$ also is. Besides, it is clear that the construction of $H(G)$ can be done in polynomial time. In the following result we provide a formula to evaluate the $\{k\}$-domination number of $H(G)$ in terms of the $\{\left\lfloor \frac{k}{2} \right\rfloor\}$-domination number of G.

Theorem 7. *Let G be a graph and $k \ge 2$ a fixed integer. If $H(G)$ is the graph constructed from G as above (see Fig. 2), then*

$$\gamma_{\{k\}}(H(G)) = \gamma_{\{\left\lfloor \frac{k}{2} \right\rfloor\}}(G) + \left(k + \left\lceil \frac{k}{2} \right\rceil\right)|V(G)|.$$

Proof. Let $g : V(G) \to \{0, \dots, k\}$ be a minimum $\{\lfloor \frac{k}{2} \rfloor\}$-dominating function of G. We define $\hat{g} : V(H(G)) \to \{0, \dots, k\}$ as follows: for each $v \in V(G)$, $\hat{g}(v) = g(v)$, $\hat{g}(w_v^1) = \hat{g}(u_v^1) = \lceil \frac{k}{2} \rceil$, $\hat{g}(u_v^2) = \lfloor \frac{k}{2} \rfloor$ and $\hat{g}(w_v^j) = 0$ for all $j \in \{2, \dots, k+1\}$. It is not hard to see that $\hat{g}$ is a $\{k\}$-dominating function of $H(G)$. Therefore,

$$\gamma_{\{k\}}(H(G)) \le w(\hat{g}) = \gamma_{\{\lfloor \frac{k}{2} \rfloor\}}(G) + \left(k + \left\lceil \frac{k}{2} \right\rceil\right) |V(G)|.$$

To see the reverse inequality, let $\hat{h} : V(H(G)) \to \{0, \dots, k\}$ be a $\{k\}$-dominating function of $H(G)$. We construct a $\{k\}$-dominating function $\hat{f}$ of $H(G)$ such that $w(\hat{f}) \le w(\hat{h})$, according to the following procedure. For each $v \in V(G)$:

Case 1: $\hat{h}(w_v^1) \ge \lceil \frac{k}{2} \rceil$. Observe that $\hat{h}(u_v^1) + \hat{h}(u_v^2) + \hat{h}(w_v^2) \ge k$ which implies $\hat{h}(V(H_v)) \ge k + \lceil \frac{k}{2} \rceil$. Then, we define $\hat{f}(u_v^1) = \hat{f}(w_v^1) = \lceil \frac{k}{2} \rceil$, $\hat{f}(u_v^2) = \lfloor \frac{k}{2} \rfloor$, $\hat{f}(w_v^j) = 0$ for all $j \in \{2, \dots, k+1\}$, $\hat{f}(v) = \min\{\hat{h}(v) + \hat{h}(w_v^1) - \lceil \frac{k}{2} \rceil, k\}$.

Case 2: $1 \le \hat{h}(w_v^1) \le \lceil \frac{k}{2} \rceil - 1$. Denote $t = \hat{h}(u_v^1) + \hat{h}(u_v^2)$.

Assume first that $t \le k - 1$. Since $\hat{h}(u_v^1) + \hat{h}(u_v^2) + \hat{h}(w_v^j) \ge k$ for $j \in \{2, \dots, k+1\}$, then $\hat{h}(w_v^j) \ge k - t$ for $j \in \{2, \dots, k+1\}$. This implies $\hat{h}(V(H_v) - \{w_v^1\}) \ge t + (k-t)k = k^2 - t(k-1) \ge 2k - 1$. Therefore, $\hat{h}(V(H_v)) \ge 2k - 1 + \hat{h}(w_v^1) \ge 2k$.

Now, let $t \ge k$. Observe that $\sum_{j=1}^{k+1} \hat{h}(w_v^j) \ge \max\{k - f(u_v^1), k - f(u_v^2)\}$. Without loss of generality, we assume $f(u_v^1) \le f(u_v^2)$. Then, $\hat{h}(V(H_v)) \ge k + f(u_v^2) \ge k + \lceil \frac{k}{2} \rceil$.

In both cases, we define $\hat{f}(u_v^1) = \hat{f}(w_v^1) = \lceil \frac{k}{2} \rceil$, $\hat{f}(u_v^2) = \lfloor \frac{k}{2} \rfloor$, $\hat{f}(w_v^j) = 0$ for all $j \in \{2, \dots, k+1\}$ and $\hat{f}(v) = \hat{h}(v)$.

Case 3: $\hat{h}(w_v^1) = 0$. From Theorem 6 and the fact that the subgraph induced by $V(H_v) - \{w_v^1\}$ is isomorphic to $K_{2,k}$, $\hat{h}(V(H_v)) \ge k + \lceil \frac{k}{2} \rceil$.

As in the previous case, we define $\hat{f}(u_v^1) = \hat{f}(w_v^1) = \lceil \frac{k}{2} \rceil$, $\hat{f}(u_v^2) = \lfloor \frac{k}{2} \rfloor$, $\hat{f}(w_v^j) = 0$ for all $j \in \{2, \dots, k+1\}$ and $\hat{f}(v) = \hat{h}(v)$.

From its construction, $\hat{f}$ is a $\{k\}$-dominating function of $H(G)$ such that $w(\hat{f}) \le w(\hat{h})$, as desired. Besides, in all cases $\hat{f}(w_v^1) = \lceil \frac{k}{2} \rceil$ for all $v \in V(G)$ which implies that the restriction of $\hat{f}$ to G is a $\{\lfloor \frac{k}{2} \rfloor\}$-dominating function of G. As $\hat{f}(V(H_v)) = k + \lceil \frac{k}{2} \rceil$ for all $v \in V(G)$, we have $w(\hat{f}) \ge \gamma_{\{\lfloor \frac{k}{2} \rfloor\}}(G) + \left(k + \lceil \frac{k}{2} \rceil\right) |V(G)|$ which implies

$$\gamma_{\{k\}}(H(G)) \ge \gamma_{\{\lfloor \frac{k}{2} \rfloor\}}(G) + \left(k + \left\lceil \frac{k}{2} \right\rceil\right) |V(G)|. \qquad \square$$

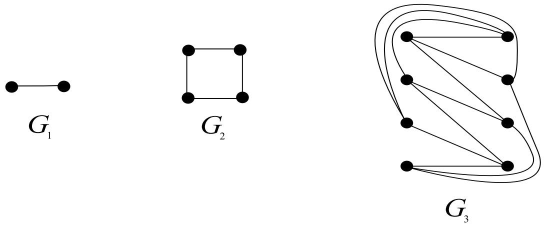

Fig. 3. Graphs G_1, G_2 and G_3.

Clearly, $\{k\}$-DOM is in NP on chordal bipartite graphs and on bipartite planar graphs. As DOM is NP-complete on both graph classes, from Theorem 7 we immediately obtain that $\{2\}$-DOM and $\{3\}$-DOM also are. Finally, applying Theorem 7 recursively, we can state:

Corollary 8. *For each $k \in \mathbb{Z}_+$ fixed, $\{k\}$-DOM is NP-complete on chordal bipartite graphs and on bipartite planar graphs.*

It is proved in [18] that k-DOM is NP-complete on planar graphs, for every fixed k, with $2 \le k \le 6$. The next theorem shows that the result holds even on bipartite planar graphs, for every fixed k, with $2 \le k \le 4$ (see the remark below).

First, we observe that, combining elementary properties of planar graphs and bipartite graphs, in any bipartite planar graph there is a vertex of degree less or equal 3. This implies the following fact:

Remark 9. For a bipartite planar graph and an integer $k \ge 5$, there is no k-tuple dominating function.

Theorem 10. *For every fixed $k \in \mathbb{Z}_+$ with $k \le 4$, k-DOM is NP-complete on bipartite planar graphs.*

Proof. It is known that DOM is NP-complete on bipartite planar graphs [10].

Clearly, k-DOM on bipartite planar graphs is in NP. We then reduce k-DOM on bipartite planar graphs to $(k-1)$-DOM on bipartite planar graphs in polynomial time.

Taking into account Remark 9, it only makes sense to consider k-DOM for the values $2 \le k \le 4$. Then, let $G = (V, E)$ be a bipartite planar graph and k be an integer with $2 \le k \le 4$.

Also, let G_{k-1} be the bipartite planar graphs given in Fig. 3, having vertex set $\{1, \dots, 2^{k-1}\}$.

We construct a bipartite planar graph $H(G)$ according to the following procedure:

1. To each $v \in V(G)$, we add a graph G_v with vertex set $\{1_v, \dots, 2_v^{k-1}\}$, isomorphic to G_{k-1}.
2. Without loss of generality, suppose that 1_v is a vertex in the outer face of G_v. For each $v \in V(G)$, we add the edge $v1_v$.

It is clear that the construction of $H(G)$ can be done in polynomial time.

Table 2
Complexity table for k-DOM and $\{k\}$-DOM for fixed $k \in \mathbb{Z}_+$. "NP-c", "P" mean NP-complete and polynomial problem, respectively.

Class	DOM ($k = 1$)	k-DOM (fixed $k \in \mathbb{Z}_+$)	$\{k\}$-DOM (fixed $k \in \mathbb{Z}_+$)
Bounded clique-width	P [11]	P	P
Split	NP-c [4]	NP-c [20]	NP-c
Planar	NP-c [15]	NP-c ($2 \le k \le 6$) [18]	NP-c
Chordal	NP-c [5]	NP-c [20]	NP-c
Bipartite	NP-c [12]	NP-c [20]	NP-c [17]
Bipartite Planar	NP-c [10]	NP-c ($2 \le k \le 4$)	NP-c
Chordal Bipartite	NP-c [8]	NP-c [18]	NP-c

Let f be a $(k-1)$-tuple dominating function of G and define $\tilde{f} : V(H) \to \{0,1\}$ such that $\tilde{f}(v) = f(v)$ for $v \in V(G)$ and $\tilde{f}(u) = 1$ for $u \in \bigcup_{v \in V(G)} V(G_v)$. It turns out that $\tilde{f}$ is a k-tuple dominating function of $H(G)$. Then $\gamma_{\times k}(H(G)) \le \gamma_{\times k-1}(G) + 2^{k-1}|V(G)|$.

Conversely, let $\tilde{f}$ be a k-tuple dominating function of $H(G)$. Notice that $\tilde{f}(u) = 1$ for $u \in \bigcup_{v \in V(G)} V(G_v)$. Define $f : V(G) \to \{0,1\}$ such that $f(v) = \tilde{f}(v)$ for $v \in V(G)$. It is not difficult to see that f is a $(k-1)$-tuple dominating function of G and $w(f) = w(\tilde{f}) - 2^{k-1}|V(G)|$. Thus $\gamma_{\times k-1}(G) \le \gamma_{\times k}(H(G)) - 2^{k-1}|V(G)|$.

Hence we have proved that $\gamma_{\times k}(H(G)) = \gamma_{\times k-1}(G) + 2^{k-1}|V(G)|$ and the result follows. $\square$

5. Concluding remarks

Observe that Theorem 3 not only allowed us to prove that fixed k, k-DOM and $\{k\}$-DOM can be solved in polynomial time on graphs with clique-width bounded by a constant, but also generalizes previous results, for instance the one in [19] concerning trees and also the result in [13] concerning P_4-tidy graphs.

We left as an open question whether the approach given by Remark 2 can be used to obtain the complexity of k-DOM and $\{k\}$-DOM for other proper subclasses of chordal graphs that are characterized by certain elimination ordering. Power chordal graphs [6] constitute a promising example.

To summarize, on the one hand we have enlarged the family of graph classes for which both problems are linear time solvable. On the other hand, we have solved all complexities left open in Table 1, as Table 2 shows.

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