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Deformation by cocycles of pointed Hopf algebras over non-abelian groups

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We explore a method for explicitly constructing multiplicative 2-cocycles for bosonizations of Nichols algebras $\mathfrak{B}(V)$ over Hopf algebras H. These cocycles arise as liftings of H-invariant linear functionals on $V \otimes V$ and give a formula for deforming braided-commutator-type relations. Using this construction, we show that all known finite-dimensional pointed Hopf algebras over the dihedral groups $\mathbb{D}_m$ with $m = 4t \geq 12$, over the symmetric group $\mathbb{S}_3$, and some families over $\mathbb{S}_4$ are cocycle deformations of bosonizations of Nichols algebras.

Introduction

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. A Hopf algebra A is said to be pointed if all simple subcoalgebras are one dimensional, or equivalently, if the coradical A_0 of A coincides with the group algebra $\mathbf{k}G(A)$ over the group of group-like elements. The best method for classifying finite-dimensional pointed Hopf algebras over $\mathbf{k}$ was developed by Andruskiewitsch and Schneider, see, [7]. Shortly, the method works as follows: first find all braided vector spaces V in ${}^{A_0}_{A_0}\mathcal{YD}$ whose Nichols algebra $\mathfrak{B}(V)$ is finite dimensional, then find all pointed Hopf algebras H for which the associated coradically graded Hopf algebra $\operatorname{gr} H$ is isomorphic to the bosonization $\mathfrak{B}(V)\#A_0$, and finally prove that all Hopf algebras H obtained this way exhaust all possibilities; the latter is equivalent to proving that for every finite-dimensional pointed Hopf algebra A with the fixed coradical A_0, $\operatorname{gr} A$ is generated by its degree 0 and 1 components.

Using this method, they were able to classify in [8] all finite-dimensional pointed Hopf algebras A whose $G(A)$ is abelian and of order coprime to 210. When the group of group likes is not abelian, the problem is far from being completed. In this situation, Nichols algebras tend to be infinite dimensional, see, e.g., [2]. Nevertheless, examples on which the Nichols algebras are finite

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dimensional do exist. Over the symmetric groups $\mathbb{S}_3$ and $\mathbb{S}_4$ these algebras were determined in [6]. All of them arise from racks associated with a cocycle. In [6] and García and García Iglesias [11] the classification of all finite-dimensional pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$ is completed, respectively. Over the dihedral groups $\mathbb{D}_m$ with $m = 4t \geq 12$, the classification of finite-dimensional Nichols algebras and finite-dimensional pointed Hopf algebras was done in [10]. In this case, it turns out that all Nichols algebras are isomorphic to exterior algebras.

Among many useful tools for constructing new Hopf algebras is the use of multiplicative 2-cocycles for deforming the multiplication of a given Hopf algebra (and the dual notion of deforming its coproduct by using a twist). With this in mind, it is interesting to ask whether some non-isomorphic Hopf algebras might be cocycle deformations of each other. It has been proven that the family of finite-dimensional pointed Hopf algebras over abelian groups Γ appearing in [8] can be constructed by deforming the multiplication in $\mathfrak{B}(V)\#\mathbf{k}\Gamma$, see, [15, 17]. Also, García Iglesias and Mombelli [12] proved, using module category theory, the same result for all known finite-dimensional pointed Hopf algebras over the symmetric groups.

In these notes, we prove in Theorems 3.11 and 3.16 that all finite-dimensional pointed Hopf algebras over the dihedral groups $\mathbb{D}_m$ with $m = 4t \geq 12$ are cocycle deformations of bosonizations of finite-dimensional Nichols algebras, by giving explicitly the 2-cocycles. Moreover, using these techniques, we construct the cocycles that give the deformation of the bosoni zations of Nichols algebras over $\mathbb{S}_3$ and for some families over $\mathbb{S}_4$, see, Theorem 4.10.

To construct the 2-cocycles we apply to non-abelian groups some techniques discussed in [15] for the setting of abelian groups. Specifically, let H be a Hopf algebra with bijective antipode and $V \in {}^H_H\mathcal{YD}$. We first associate with any linear map $\eta : V \otimes V \to \mathbf{k}$ a Hochschild 2-cocycle on the Nichols algebras $\mathfrak{B}(V)$. If this map is invariant under the action of H, it gives rise to a Hochschild 2-cocycle $\tilde{\eta}$ on $A = \mathfrak{B}(V)\#H$. Applying [15, Lemma 4.1], it turns out that the exponentiation $\sigma = e^{\tilde{\eta}}$ is indeed a multiplicative 2-cocycle, provided extra conditions are satisfied, e.g., H is semisimple and the braiding of V is symmetric. As a consequence, the exponentiation of liftings of $\mathbb{D}_m$-invariant linear functionals on Yetter–Drinfeld modules with finite-dimensional Nichols algebra are multiplicative 2-cocycles and these are the ones we use to prove our first main theorems, Theorems 3.11 and 3.16.

An important consequence of using an H-invariant linear map η on $V \otimes V$ to construct a multiplicative 2-cocycle σ is that σ coincides with

η on the elements of $V \otimes V$. This gives a formula for the deformation of *braided commutator*-type relations $[x, y]_c = 0$ on $\mathfrak{B}(V)$ for $x, y \in V$ in the same connected component, which in this case also include the *power root vector*-type relations $x^2 = 0$. This simplifies the computation of the deformation, see, Lemma 2.7.

The paper is organized as follows: in Section 1, we fix the notation, make some definitions and recall some facts that are used throughout the paper. In particular, we briefly review the theory of Yetter–Drinfeld modules V over group algebras $H = \mathbf{k}\Gamma$, their Nichols algebras $\mathfrak{B}(V)$, their bosonizations $\mathfrak{B}(V)\#H$, their liftings, and their cocycle deformations. In Section 2, we give a method for constructing Hochschild 2-cocycles $\tilde{\eta}$ on $\mathfrak{B}(V)\#H$ from H-invariant linear functionals η on $V \otimes V$, and give necessary conditions for the exponential $\sigma = e^{\tilde{\eta}}$ to be a multiplicative 2-cocycle on $\mathfrak{B}(V)\#H$. In Section 3, we recall the classification of finite-dimensional pointed Hopf algebras over the dihedral groups $\mathbb{D}_m$ with $m = 4t \geq 12$ and prove that they are cocycle deformation of bosonizations by using the cocycles constructed in Section 2. Finally, in Section 4, we first recall the notion of racks and the classification of finite-dimensional pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$, and then prove our last main result, Theorem 4.10 about cocycle deformations of bosonizations of finite-dimensional Nichols algebras over the symmetric groups.

1. Preliminaries

1.1. Conventions

We work over an algebraically closed field $\mathbf{k}$ of characteristic zero. Good references for Hopf algebra theory are [19, 22].

If H is a Hopf algebra over $\mathbf{k}$ then Δ, ε and $\mathcal{S}$ denote, respectively, the comultiplication, the counit and the antipode. Comultiplication and coactions are written using the Sweedler–Heynemann notation with summation sign suppressed, e.g., $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for $h \in H$.

Let $\mathcal{C}$ be a braided monoidal category. A *braided* Hopf algebra R in $\mathcal{C}$ is an object R such that all structural maps $m_R, \Delta_R, \varepsilon_R$, and $\mathcal{S}_R$ are morphisms in the category.

The *coradical* H_0 of H is the sum of all simple sub-coalgebras of H. In particular, if $G(H)$ denotes the group of *group-like elements* of H, we have $\mathbf{k}G(H) \subseteq H_0$. A Hopf algebra is *pointed* if $H_0 = \mathbf{k}G(H)$, that is, all simple sub-coalgebras are one dimensional.

Denote by $\{H_i\}_{i\geq 0}$ the *coradical filtration* of H; if H_0 is a Hopf subalgebra of H, then by $\operatorname{gr} H = \oplus_{n\geq 0} \operatorname{gr} H(n)$ we denote the associated graded Hopf algebra, with $\operatorname{gr} H(n) = H_n/H_{n-1}$, setting $H_{-1} = 0$.

For $h, g \in G(H)$, the linear space of (h, g)-*primitives* is

$$\mathcal{P}_{h,g}(H) := \{x \in H \mid \Delta(x) = x \otimes h + g \otimes x\}.$$

If $g = 1 = h$, the linear space $\mathcal{P}(H) = \mathcal{P}_{1,1}(H)$ is called the set of primitive elements.

If M is a right H-comodule via $\delta(m) = m_{(0)} \otimes m_{(1)} \in M \otimes H$ for all $m \in M$, then the space of *right coinvariants* is $M^{\operatorname{co}\delta} = \{x \in M \mid \delta(x) = x \otimes 1\}$. In particular, if $\pi : H \to L$ is a morphism of Hopf algebras, then H is a right L-comodule via $(\operatorname{id}\otimes\pi)\Delta$ and in this case

$$H^{\operatorname{co}\pi} := H^{\operatorname{co}\ (\operatorname{id}\otimes\pi)\Delta} = \{h \in H \mid (\operatorname{id}\otimes\pi)\Delta(h) = h \otimes 1\}.$$

Left coinvariants, written ${}^{\operatorname{co}\pi}H$ are defined analogously.

Let $n, m \in \mathbb{N}$. We denote by $\mathbb{S}_n$ the symmetric group on n letters and by $\mathbb{D}_m$ the dihedral group of order $2m$.

1.2. Yetter–Drinfeld modules and Nichols algebras

In this subsection, we recall the definition of Yetter–Drinfeld modules over Hopf algebras and we describe the irreducible ones in case H is a group algebra. We also review the definition of Nichols algebras associated with them.

1.2.1. Yetter–Drinfeld modules over group algebras. Let H be a Hopf algebra. A *left Yetter–Drinfeld module* M over H is a left H-module $(M, \cdot)$ and a left H-comodule (M, δ) with $\delta(m) = m_{(-1)} \otimes m_{(0)} \in H \otimes M$ for all $m \in M$, satisfying the compatibility condition

$$\delta(h \cdot m) = h_{(1)} m_{(-1)} \mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)} \quad \forall\ m \in M, h \in H.$$

We denote by ${}^H_H\mathcal{YD}$ the category of left Yetter–Drinfeld modules over H. It is a braided monoidal category: for any $M, N \in {}^H_H\mathcal{YD}$, the braiding $c_{M,N} : M \otimes N \to N \otimes M$ is given by

$$c_{M,N}(m \otimes n) = m_{(-1)} \cdot n \otimes m_{(0)} \quad \forall\ m \in M, n \in N.$$

Assume $H = \mathbf{k}\Gamma$ with Γ a finite group. A left Yetter–Drinfeld module over $\mathbf{k}\Gamma$ is a left $\mathbf{k}\Gamma$-module and left $\mathbf{k}\Gamma$-comodule M such that

$$\delta(g.m) = ghg^{-1} \otimes g.m, \quad \forall\, m \in M_h, g, h \in \Gamma,$$

where $M_h = \{m \in M : \delta(m) = h \otimes m\}$; clearly, $M = \oplus_{h\in\Gamma} M_h$ and thus M is a Γ-graded module. We denote this category simply by ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.

Yetter–Drinfeld modules over $\mathbf{k}\Gamma$ are completely reducible. The irreducible modules are parameterized by pairs $(\mathcal{O}, \rho)$, where $\mathcal{O}$ is a conjugacy class of Γ and (ρ, V) is an irreducible representation of the centralizer $C_\Gamma(\sigma)$ of a fixed point $\sigma \in \mathcal{O}$. The corresponding Yetter–Drinfeld module is given by

$$M(\mathcal{O}, \rho) = \operatorname{Ind}_{C_\Gamma(\sigma)}^{G} V = \mathbf{k}\Gamma \otimes_{C_\Gamma(\sigma)} V.$$

Explicitly, let $\sigma_1 = \sigma, \ldots, \sigma_n$ be a numeration of $\mathcal{O}$ and let $g_i \in \Gamma$ such that $g_i \sigma g_i^{-1} = \sigma_i$ for all $1 \le i \le n$. Then $M(\mathcal{O}, \rho) = \oplus_{1\le i\le n} g_i \otimes V$. The Yetter–Drinfeld module structure is given as follows: let $g_i v := g_i \otimes v \in M(\mathcal{O}, \rho)$, $1 \le i \le n$, $v \in V$. If $v \in V$ and $1 \le i \le n$, then the action of $g \in \Gamma$ and the coaction are given by

$$g \cdot (g_i v) = g_j(\gamma \cdot v), \quad \delta(g_i v) = \sigma_i \otimes g_i v,$$

where $gg_i = g_j\gamma$, for some $1 \le j \le n$ and $\gamma \in C_\Gamma(\sigma)$. In this case, the explicit formula for the braiding is given by

$$c(g_i v \otimes g_j w) = \sigma_i \cdot (g_j w) \otimes g_i v = g_h(\gamma \cdot v) \otimes g_i v,$$

for any $1 \le i, j \le n$, $v, w \in V$, where $\sigma_i g_j = g_h \gamma$ for unique h, $1 \le h \le n$ and $\gamma \in C_\Gamma(\sigma)$. For explicit examples with $\Gamma = \mathbb{D}_m, \mathbb{S}_n$ see, Sections 3.1 and 4.1.

1.2.2. Nichols algebras. The Nichols algebra of a braided vector space (V, c) can be defined in various different ways, see, [3, 7, 16, 20]. They are connected Hopf algebras in a braided monoidal category with certain properties. In general, the computation of the Nichols algebra of an arbitrary braided vector space is a delicate issue. We are interested in Nichols algebras of braided vector spaces arising from Yetter–Drinfeld modules, hence we give the explicit definition for this case.

The notion of a Nichols algebra first appeared in the work of Nichols [20] and was later rediscovered by several authors.

Definition 1.1 ([7, Definition 2.1]). Let H be a Hopf algebra and $V \in {}^H_H\mathcal{YD}$. A braided $\mathbb{N}$-graded Hopf algebra $R = \bigoplus_{n\geq 0} R(n) \in {}^H_H\mathcal{YD}$ is called the *Nichols algebra* of V if

(i) $\mathbf{k} \simeq R(0)$, $V \simeq R(1) \in {}^H_H\mathcal{YD}$;

(ii) $R(1) = \mathcal{P}(R) = \{r \in R \mid \Delta_R(r) = r \otimes 1 + 1 \otimes r\}$, the linear space of primitive elements;

(iii) R is generated as an algebra by $R(1)$.

In this case, R is denoted by $\mathfrak{B}(V) = \bigoplus_{n\geq 0} \mathfrak{B}^n(V)$.

For any $V \in {}^H_H\mathcal{YD}$ there is a Nichols algebra $\mathfrak{B}(V)$ associated with it. It can be constructed as a quotient of the tensor algebra $T(V)$ by the largest homogeneous two-sided ideal I satisfying:

- I is generated by homogeneous elements of degree ≥ 2,
- $\Delta(I) \subseteq I \otimes T(V) + T(V) \otimes I$, i.e., it is also a coideal.

In such a case, $\mathfrak{B}(V) = T(V)/I$. See [7, Section 2.1] for details.

An important observation is that the Nichols algebra $\mathfrak{B}(V)$, as algebra and as a coalgebra, is completely determined just by the braiding.

Given any braided vector space (V, c), one may construct the Nichols algebra $\mathfrak{B}(V, c) = \mathfrak{B}(V)$ in a way similar to the construction above, by taking a quotient of the tensor algebra $T(V)$ by the homogeneous two-sided ideal given by the kernel of an homogeneous symmetrizer. We now briefly describe this construction. Let $\mathbb{B}_n$ be the braid group of n letters. Since c satisfies the braid equation, it induces a representation of $\mathbb{B}_n$, $\rho_n : \mathbb{B}_n \to \mathbf{GL}(V^{\otimes n})$ for each $n \geq 2$. Consider the morphisms

$$Q_n = \sum_{\sigma\in\mathbb{S}_n} \rho_n(M(\sigma)) \in \operatorname{End}(V^{\otimes n}),$$

where $M : \mathbb{S}_n \to \mathbb{B}_n$ is the Matsumoto section corresponding to the canonical projection $\mathbb{B}_n \to \mathbb{S}_n$. Then the Nichols algebra $\mathfrak{B}(V)$ is the quotient of the tensor algebra $T(V)$ by the two-sided ideal $\mathcal{J} = \bigoplus_{n\geq 2} \operatorname{Ker} Q_n$. If $c = \tau$ is the usual flip, then $\mathfrak{B}(V) = S(V)$ is just the symmetric algebra of V; if $c = -\tau$, then $\mathfrak{B}(V) = \bigwedge V$ is the exterior algebra of V.

Let Γ be a finite group. We denote by $\mathfrak{B}(\mathcal{O}, \rho)$ the Nichols algebra associated with the irreducible Yetter–Drinfeld module $M(\mathcal{O}, \rho) \in {}^\Gamma_\Gamma\mathcal{YD}$.

Let $V \in {}^\Gamma_\Gamma\mathcal{YD}$ such that $\mathfrak{B}(V)$ is finite dimensional and let $\{x_i\}_{i\in I}$ be homogeneous primitive elements that span linearly V with $\delta(x_i) = g_i \otimes x_i$, and $g_i \in \Gamma$ for all $i \in I$. Since for all $h \in \Gamma$, $h \cdot x_i$ is again a homogeneous primitive element, from now on we assume that

$$h \cdot x_i = \chi_i(h) x_{\sigma(h)(i)}, \quad \text{for all } i \in I,\ h \in \Gamma, \tag{1}$$

where $\sigma : \Gamma \to \mathbb{S}_I$ and $\chi_i : \Gamma \to \mathbf{k}$ is a character, see, [5, Example 5.9]. This condition holds for all finite-dimensional pointed Hopf algebras over the symmetric groups and over the dihedral groups, see, Sections 3 and 4. We write $\sigma(g_i)(j) = i \triangleright j$ for all $i, j \in I$.

Remark 1.2. (a) If V is irreducible, then $V \simeq M(\mathcal{O}, \rho)$ with $\mathcal{O}$ a conjugacy class of Γ. In such a case, I can be identified with $\mathcal{O}$ and $\sigma(h)$ is just the conjugation by h.

(b) If Γ is abelian, then V is a braided space of diagonal type, i.e., $h \cdot x_i = \chi_i(h) x_i$ for all $i \in I$ and $\sigma(h) = \mathrm{id}$ for all $h \in \Gamma$.

Important examples of Nichols algebras come from the theory of quantum groups. As shown in [8], they play a crucial role in the classification of finite-dimensional pointed Hopf algebras over $\mathbf{k}$, via the lifting method. For more details on Nichols algebras see, [7, 16] and references therein; and for explicit examples with $\Gamma = \mathbb{D}_m$ and $\Gamma = \mathbb{S}_n$ see, Sections 3.1 and 4.1.

1.3. Bosonization and Hopf algebras with a projection

Let H be a Hopf algebra and R a braided Hopf algebra in ${}^H_H\mathcal{YD}$. The procedure to obtain a usual Hopf algebra from R and H is called the Majid–Radford product or *bosonization*, and it is usually denoted by $R\#H$. As vector spaces $R\#H = R \otimes H$, and the multiplication and comultiplication are given by the smash-product and smash-coproduct, respectively. That is, for all $r, s \in R$ and $g, h \in H$

$$\begin{aligned}(r\#g)(s\#h) &= r(g_{(1)} \cdot s)\#g_{(2)}h,\\ \Delta(r\#g) &= r^{(1)}\#(r^{(2)})_{(-1)}g_{(1)} \otimes (r^{(2)})_{(0)}\#g_{(2)},\end{aligned}$$

where $\Delta_R(r) = r^{(1)} \otimes r^{(2)}$ denotes the comultiplication in $R \in {}^H_H\mathcal{YD}$. If $r \in R$ and $h \in H$, then we identify $r = r\#1$ and $h = 1\#h$; in particular we have $rh = r\#h$ and $hr = h_{(1)} \cdot r\#h_{(2)}$. Clearly, the map $\iota : H \to R\#H$ given by $\iota(h) = 1\#h$ for all $h \in H$ is an injective Hopf algebra map, and the map $\pi : R\#H \to H$ given by $\pi(r\#h) = \varepsilon_R(r)h$ for all $r \in R$, $h \in H$ is a surjective Hopf algebra map such that $\pi \circ \iota = \mathrm{id}_H$. Moreover, it holds that $R = (R\#H)^{\mathrm{co}\,\pi}$.

Conversely, let A be a Hopf algebra with bijective antipode and $\pi : A \to H$ a Hopf algebra epimorphism admitting a Hopf algebra section $\iota : H \to A$ such that $\pi \circ \iota = \mathrm{id}_H$. Then $R = A^{\mathrm{co}\,\pi}$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ and $A \simeq R\#H$ as Hopf algebras.

1.4. The lifting method

The *lifting method* was introduced by Andruskiewitsch and Schneider, see, [7]. It is one of the more general results concerning the structure of Hopf algebras and became a powerful tool for classifying finite-dimensional pointed Hopf algebras, as shown in [6, 8, 10, 11], among others.

Let H be a finite-dimensional pointed Hopf algebra with coradical $H_0 = \mathbf{k}G(H)$ and $\mathrm{gr}\, H = \oplus_{n\geq 0} \mathrm{gr}\, H(n)$ the associated graded Hopf algebra. If $\pi : \mathrm{gr}\, H \to H_0$ denotes the homogeneous projection, then $R = (\mathrm{gr}\, H)^{\mathrm{co}\,\pi}$ is called the *diagram* of H. It is a braided Hopf algebra in ${}^{H_0}_{H_0}\mathcal{YD}$ and it is a graded subobject of $\mathrm{gr}\, H$. The linear space $R(1)$, with the braiding from ${}^{H_0}_{H_0}\mathcal{YD}$, is called the *infinitesimal braiding* of H and coincides with the subspace of primitive elements $\mathcal{P}(R)$. This braiding is the key to the structure of pointed Hopf algebras. It turns out that the Hopf algebra $\mathrm{gr}\, H$ is the bosonization $\mathrm{gr}\, H \simeq R\#\mathbf{k}G(H)$ and the subalgebra of R generated by V is isomorphic to the Nichols algebra $\mathfrak{B}(V)$, see, Section 1.2.

Let Γ be a finite group. The main steps of the lifting method are:

- determine all $V \in {}^\Gamma_\Gamma\mathcal{YD}$ such that the Nichols algebra $\mathfrak{B}(V)$ is finite dimensional;
- for such V, compute all Hopf algebras H such that $\mathrm{gr}\, H \simeq \mathfrak{B}(V)\#\mathbf{k}\Gamma$. We call H a *lifting* of $\mathfrak{B}(V)$ over $\Gamma \simeq G(H)$;
- prove that any finite-dimensional pointed Hopf algebras with group of group-likes Γ is generated by group likes and skew primitives.

Using this method, it was possible to classify finite-dimensional nontrivial pointed Hopf algebras over abelian groups with order relative prime to 210, over the symmetric groups $\mathbb{S}_3$ and $\mathbb{S}_4$ and over the dihedral groups $\mathbb{D}_m$ with $m = 4t$, $t \geq 3$, see, [6, 8, 10, 11].

1.5. Deforming cocycles

Let A be a Hopf algebra. Recall that a convolution invertible linear map σ in $\mathrm{Hom}_{\mathbf{k}}(A \otimes A, \mathbf{k})$ is a *normalized multiplicative 2-cocycle* if

$$\sigma(b_{(1)}, c_{(1)})\sigma(a, b_{(2)}c_{(2)}) = \sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c)$$

and $\sigma(a,1)=\varepsilon(a)=\sigma(1,a)$ for all $a,b,c\in A$, see, [5, Section 7.1]. In particular, the inverse of σ is given by $\sigma^{-1}(a,b)=\sigma(\mathcal{S}(a),b)$ for all $a,b\in A$ and the 2-cocycle condition is equivalent to

$$(\varepsilon\otimes\sigma)*\sigma(\mathrm{id}_A\otimes m)=(\sigma\otimes\varepsilon)*\sigma(m\otimes\mathrm{id}_A) \tag{2}$$

and $\sigma(\mathrm{id}_A,1)=\varepsilon=\sigma(1,\mathrm{id}_A)$.

Using a 2-cocycle σ it is possible to define a new algebra structure on A by deforming the multiplication, which we denote by A_σ. Moreover, A_σ is indeed a Hopf algebra with $A=A_\sigma$ as coalgebras, deformed multiplication $m_\sigma=\sigma*m*\sigma^{-1}:A\otimes A\to A$ given by

$$m_\sigma(a,b)=a\cdot_\sigma b=\sigma(a_{(1)},b_{(1)})a_{(2)}b_{(2)}\sigma^{-1}(a_{(3)},b_{(3)}),\qquad \text{for all } a,b\in A,$$

and antipode $\mathcal{S}_\sigma=\sigma*\mathcal{S}*\sigma^{-1}:A\to A$ given by (see, [9] for details)

$$\mathcal{S}_\sigma(a)=\sigma(a_{(1)},\mathcal{S}(a_{(2)}))\mathcal{S}(a_{(3)})\sigma^{-1}(\mathcal{S}(a_{(4)}),a_{(5)}),\qquad \text{for all } a\in A.$$

1.5.1. Deforming cocycles for graded Hopf algebras. Assume now $A=\oplus_{n\geq 0}A_n$ is a graded Hopf algebra and let $\eta\in\mathrm{Hom}_{\mathbf{k}}(A\otimes A,\mathbf{k})$ be a Hochschild 2-cocycle on A, that is

$$\varepsilon(a)\eta(b,c)+\eta(a,bc)=\eta(a,b)\varepsilon(c)+\eta(ab,c),\quad \text{for all } a,b,c\in A.$$

If we assume further that $\eta|_{A\otimes A_0+A_0\otimes A}=0$, then

$$\sigma=e^\eta=\sum_{i=0}^{\infty}\frac{\eta^{*i}}{i!}:A\otimes A\to\mathbf{k}$$

is a well-defined convolution invertible map with convolution inverse $e^{-\eta}$; moreover, *often* e^η will be a multiplicative 2-cocycle. For instance, this happens whenever $\eta(\mathrm{id}\otimes m)$ and $\eta(m\otimes\mathrm{id})$ commute (with respect to the convolution product) with $\varepsilon\otimes\eta$ and $\eta\otimes\varepsilon$, respectively. Also note that if $\eta*\eta=0$, then $e^\eta=\varepsilon+\eta$.

The following result is well known in the cocommutative setting [21] and it is a generalization of a result from [13]. Although the result holds for more general graded algebras, we state it for the case of our interest where $A=\mathfrak{B}(V)\#H$ is a graded Hopf algebra given by the bosonization of a Nichols algebra $\mathfrak{B}(V)$ with a Hopf algebra H.

Lemma 1.3 ([15, Lemma 4.1]). *Let $A=\mathfrak{B}(V)\#H$ be a bosonization of a Nichols algebra $\mathfrak{B}(V)$ with a Hopf algebra H. If $\eta\in\mathrm{Hom}_{\boldsymbol{k}}(A\otimes A,\boldsymbol{k})$ is a*

Hochschild 2*-cocycle such that* $\eta(\mathrm{id}\otimes m)$ *commutes with* $\varepsilon\otimes\eta$ *and* $\eta(m\otimes\mathrm{id})$ *commutes with* $\eta\otimes\varepsilon$ *in the convolution algebra* $\mathrm{Hom}_{\boldsymbol{k}}(A\otimes A\otimes A,\boldsymbol{k})$*, then* e^{η} *is a multiplicative* 2*-cocycle.*

2. Cocycles on bosonizations of Nichols algebras

In this section we discuss sufficient conditions for a Hochschild 2-cocycle on a bosonization $\mathfrak{B}(V)\#H$ to satisfy the conditions of Lemma 1.3; and thus inducing a multiplicative cocycle via the exponential map. We define it by extending a linear functional on $V\otimes V$ invariant under the action of H.

From now on we assume that $A=\mathfrak{B}(V)\#H$, where H is Hopf algebra with bijective antipode, $V\in {}^H_H\mathcal{YD}$ and $\mathfrak{B}(V)\in {}^H_H\mathcal{YD}$ is the Nichols algebra of V. Since $\mathfrak{B}(V)$ is graded, we have that A is also graded with the gradation given by $A_0=H$ and $A_n=\mathfrak{B}^n(V)\#H$.

2.1. Hochschild cocycles on $\mathfrak{B}(V)\#H$ from H-invariant lineal functionals on $V\otimes V$

Let $\eta: V\otimes V\to\mathbf{k}$ be a linear map. Then we can define a Hochschild 2-cocycle on $\mathfrak{B}(V)$ by

$$\eta(\mathfrak{B}^m(V)\otimes\mathfrak{B}^n(V))=0\quad\text{ if }(m,n)\neq(1,1).$$

Such a map is called an ε-*biderivation*, since it is an ε-derivation on each variable, that is, we have $\eta(1,-)=0=\eta(-,1)$ and $\eta(xy,-)=0=\eta(-,xy)$ for all $x,y\in\mathfrak{B}(V)$ such that $\varepsilon(x)=0=\varepsilon(y)$.

Since H acts on V, we have that H acts on the set of all linear maps $\eta: V\otimes V\to\mathbf{k}$ by $\eta^h(x,y)=\eta(h_{(1)}\cdot x,h_{(2)}\cdot y)$ for all $h\in H,x,y\in V$. We say that η is H-*invariant* if $\eta^h=\eta$ for all $h\in H$.

The following lemma tell us how to construct a Hochschild 2-cocycle on A from an H-invariant linear functional on $V\otimes V$. Its proof is straighforward and is left to the reader.

Lemma 2.1. *Let* $\eta: V\otimes V\to\boldsymbol{k}$ *be an* H*-invariant linear map. Then the map* $\tilde\eta\colon A\otimes A\to\boldsymbol{k}$ *defined by* $\tilde\eta(A_m\otimes A_n)=0$ *if* $(m,n)\neq(1,1)$ *and*

$$\tilde\eta(x\#h,y\#h')=\eta(x,h\cdot y)\varepsilon(h')\quad\text{for all } x,y\in V \text{ and }\ h,h'\in H$$

is an H*-invariant Hochschild* 2*-cocycle on* A *that satisfies* $\tilde\eta|_{A_0\otimes A+A\otimes A_0}=0$.

Remark 2.2. Assume V is finite dimensional and let $(x_i)_{i\in I}$ be a basis of V and $(d_i)_{i\in I}$ the dual basis of V^*. Then any linear combination of the tensor products $d_i \otimes d_j$ induces a Hochschild 2-cocycle on $\mathfrak{B}(V)$. In particular, by (1) the map $\eta = \sum_{i,j\in I} a_{ij} d_i \otimes d_j$ is Γ-invariant if

$$a_{k,\ell} = \chi_k(g)\chi_\ell(g) a_{\sigma(g)(k),\sigma(g)(\ell)} \quad \text{for all } g \in \Gamma, k, \ell \in I.$$

2.2. On the commuting conditions of *H*-invariant Hochschild 2-cocycles

In the remainder of this section, we study the question of when the exponential $e^{\tilde{\eta}}$ of a Hochschild 2-cocycle induced by an H-invariant linear functional η on $V \otimes V$ is a multiplicative 2-cocycle. We show, among other things, that this is always the case if H is semisimple and the braiding of V is symmetric.

Denote by c the braiding of $V \in {}^H_H\mathcal{YD}$. It induces an action of the braid group $\mathbb{B}_n$ on $V^{\otimes n}$. If $\pi \in \mathbb{B}_n$, we denote by $c_\pi : V_1 \otimes \cdots \otimes V_n \to V_{\pi(1)} \otimes \cdots \otimes V_{\pi(n)}$ the map induced by this action. In particular, we write

$$c_{231} = (\mathrm{id} \otimes c)(c \otimes \mathrm{id}), \quad c_{1324} = (\mathrm{id} \otimes c \otimes \mathrm{id}), \quad c_{2413} = (\mathrm{id} \otimes c \otimes \mathrm{id})(c \otimes c),$$
$$c_{1423} = (\mathrm{id} \otimes c \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes c) \quad \text{and} \quad c_{2314} = (\mathrm{id} \otimes c \otimes \mathrm{id})(c \otimes \mathrm{id} \otimes \mathrm{id}).$$

Let $\eta : V \otimes V \to \mathbf{k}$ be an H-invariant linear functional and denote again by η the map defined on $\mathfrak{B}(V)$, then we have that

$$c(\mathrm{id} \otimes \eta) = (\eta \otimes \mathrm{id})(1 \otimes c)(c \otimes 1) = (\eta \otimes \mathrm{id})c_{231}, \tag{3}$$

that is, for all $a, b, c \in V$ we have

$$\eta(b, c) \otimes a = \eta(a_{(-1)}b, c) \otimes a_{(0)} = (\eta \otimes \mathrm{id})c_{231}(a \otimes b \otimes c). \tag{4}$$

Indeed, $(\eta \otimes \mathrm{id})c_{231}(a \otimes b \otimes c) = \eta(a_{(-2)} \cdot b, a_{(-1)} \cdot c) \otimes a_{(0)} = \eta^{a_{(-1)}}(b, c) \otimes a_{(0)} = \eta(b, c) \otimes \varepsilon(a_{(-1)})a_{(0)} = c(\mathrm{id} \otimes \eta)(a \otimes b \otimes c)$ for all $a, b, c \in V$.

Using the definition of $\tilde{\eta}$ we also have for all $a, b \in A$ and $c \in V$ that

$$\tilde{\eta}(a, b) \otimes c = \tilde{\eta}(a, bc_{(-1)}) \otimes c_{(0)}. \tag{5}$$

The following lemma shows that both conditions on the commutativity of the maps in Lemma 1.3, (b) and (c) below, are equivalent. Note that, as a consequence, we need only to verify equalities on $V^{\otimes 4}$.

Lemma 2.3. *Let $\eta : V \otimes V \to \boldsymbol{k}$ be an H-invariant linear functional. The following are equivalent:*

(a) *The following conditions hold on $V^{\otimes 4}$:*

$$(\eta \otimes \eta)c_{1324} = (\eta \otimes \eta)c_{2413}, \tag{6}$$

$$(\eta \otimes \eta)c_{1423} = (\eta \otimes \eta)c_{2314}. \tag{7}$$

(b) *The following condition holds on $A^{\otimes 3}$: $(\varepsilon \otimes \tilde{\eta}) * \tilde{\eta}(\mathrm{id} \otimes m) = \tilde{\eta}(\mathrm{id} \otimes m) * (\varepsilon \otimes \tilde{\eta})$.*

(c) *The following condition holds on $A^{\otimes 3}$: $(\tilde{\eta} \otimes \varepsilon) * \tilde{\eta}(m \otimes \mathrm{id}) = \tilde{\eta}(m \otimes \mathrm{id}) * (\tilde{\eta} \otimes \varepsilon)$.*

Proof. Throughout the proof we use the fact that for $v \in V \subseteq A$ we have $\Delta(v) = v_{(1)} \otimes v_{(2)} = v_{(-1)} \otimes v_{(0)} + v \otimes 1$. We first show that (a) is equivalent to (b). Note that it is both necessary and sufficient to verify (b) by evaluating at $\mathfrak{B}(V)_m \otimes \mathfrak{B}(V)_n \otimes \mathfrak{B}(V)_p$ for all (m, n, p). Unless $(m, n, p) = (1, 2, 1)$ or $(m, n, p) = (1, 1, 2)$ both sides trivially give 0. Now evaluation at $a \otimes b \otimes cd$ for $a, b, c, d \in V$ yields that $\tilde{\eta}(b, (cd)_{(1)})\tilde{\eta}(a, (cd)_{(2)}) = \tilde{\eta}(b_{(0)}, (cd)_{(2)})\tilde{\eta}(a, b_{(-1)} (cd)_{(1)})$. After expanding $\Delta(cd)$ and using the fact that η is H-invariant we get

$$\begin{aligned}\eta(b, c)\eta(a, d) &+ \eta(b, c_{(-1)} \cdot d)\eta(a, c_{(0)}) \\ &= \eta(b_{(0)}, d)\eta(a, b_{(-1)} \cdot c) + \eta(b_{(0)}, c_{(0)})\eta(a, (b_{(-1)}c_{(-1)}) \cdot d).\end{aligned}$$

By (b) we have that $(\eta \otimes \eta)(a, d, b, c) = (\eta \otimes \eta)c_{2314}(a, b, c, d)$, this is equivalent to

$$(\eta \otimes \eta)(c_{2314} + c_{2413} - c_{1324} - c_{1423}) = 0. \tag{8}$$

We now examine evaluation of (b) at $a \otimes bc \otimes d$ for $a, b, c, d \in V$. Using (5) we get

$$\tilde{\eta}((bc)_{(1)}, d)\tilde{\eta}(a, (bc)_{(2)}) = \tilde{\eta}((bc)_{(2)}, d)\tilde{\eta}(a, (bc)_{(1)}). \tag{9}$$

Expanding $\Delta(bc)$ and using equations (4) and (5) as well as the definition of $\tilde{\eta}$ we get

$$\begin{aligned}\eta(b, c_{(-1)} \cdot d)\eta(a, c_{(0)}) &+ \eta(c, d)\eta(a, b) \\ &= \eta(c, d)\eta(a, b) + \eta(b_{(0)}, d)\eta(a, b_{(-1)} \cdot c),\end{aligned}$$

which simplifies to $\eta(b, c_{(-1)} \cdot d)\eta(a, c_{(0)}) = \eta(a, b_{(-1)} \cdot c)\eta(b_{(0)}, d)$, or equivalently, $(\eta \otimes \eta)c_{2413} = (\eta \otimes \eta)c_{1324}$, which is exactly (6). Hence we have that (b) is equivalent to (8) and (6), and consequently to (a), since (8) and (6) give (7).

The equivalence of (a) and (c) works almost exactly the same way. For the sake of completeness, we provide some intermediate steps. We verify (c) by evaluating it at $a \otimes bc \otimes d$ and $ab \otimes c \otimes d$ for $a, b, c, d \in V$. Evaluation at $a \otimes bc \otimes d$ yields (9) which is equivalent to (6). Evaluation at $ab \otimes c \otimes d$ and using that $\Delta(c) = c \otimes 1 + c_{(-1)} \otimes c_{(0)}$ yields

$$\tilde{\eta}((ab)_{(1)}, c)\tilde{\eta}((ab)_{(2)}, d) = \tilde{\eta}((ab)_{(2)}, c_{(0)})\tilde{\eta}((ab)_{(1)}, c_{(-1)} \cdot d).$$

This simplifies to

$$\begin{aligned}\eta(a, b_{(-1)} \cdot c)\eta(b_{(0)}, d) + \eta(b, c)\eta(a, d) = \eta(b_{(0)}, c_{(0)})\eta(a, b_{(-1)}c_{(-1)} \cdot d)\\ + \eta(a, c_{(0)})\eta(b, c_{(-1)} \cdot d),\end{aligned}$$

which by (4) can be written as $(\eta \otimes \eta)(c_{1324} + c_{2314} - c_{1423} - c_{2413}) = 0$. We conclude the proof by noting that this equation together with (6) are equivalent to (a). $\square$

For the following lemma, observe that using $c_{1324} = \mathrm{id} \otimes c \otimes \mathrm{id}$ and $c_{2413} = (\mathrm{id} \otimes c \otimes \mathrm{id})(c \otimes c)$, we get that (6) is equivalent to $(\eta \otimes \eta)(\mathrm{id} \otimes c \otimes \mathrm{id}) = (\eta \otimes \eta)(\mathrm{id} \otimes c \otimes \mathrm{id})(c \otimes c)$.

The next two results state that the conditions in Lemma 2.3(a) are always fulfiled when H is semisimple and the braiding is symmetric.

Lemma 2.4. *If $S_H^2 = \mathrm{id}_H$, then* (6) *is equivalent to either of the following equations on $V^{\otimes 4}$:*

$$(\eta \otimes \eta)(\mathrm{id} \otimes c \otimes \mathrm{id}) = (\eta \otimes \eta)(\mathrm{id} \otimes c^{-1} \otimes \mathrm{id}), \tag{10}$$

$$(\eta \otimes \eta)(\mathrm{id} \otimes c^2 \otimes \mathrm{id}) = \eta \otimes \eta. \tag{11}$$

In particular, (6) *is always satisfied when $c^2 = \mathrm{id}_{V \otimes V}$ and H is semisimple.*

Proof. Note that in the case S is an involution, and therefore for all $h \in H$ we have $\varepsilon(h) = S(h_{(2)})h_{(1)}$ (this is used for going from line 8 to line 9 and for going to the last line from the line above it in the computation below), we get that for all $a, b, c, d \in V$ we have:

$$\begin{aligned}(\eta\otimes\eta)c_{2413}(a,b,c,d) &= (\eta\otimes\eta)(\mathrm{id}\otimes c\otimes\mathrm{id})(a_{(-1)}\cdot b, a_{(0)}, c_{(-1)}\cdot d, c_{(0)})\\ &= \eta(a_{(-2)}\cdot b, a_{(-1)}c_{(-1)}\cdot d)\eta(a_{(0)}, c_{(0)})\\ &= \eta^{a_{(-1)}}(b, c_{(-1)}\cdot d)\eta(a_{(0)}, c_{(0)})\\ &= \eta(b, c_{(-1)}\cdot d)\eta(a, c_{(0)})\\ &= \eta^{S(c_{(-1)})}(b, c_{(-2)}\cdot d)\eta(a, c_{(0)})\\ &= \eta(S(c_{(-1)})_{(1)}\cdot b, (S(c_{(-1)})_{(2)}c_{(-2)})\cdot d)\eta(a, c_{(0)})\\ &= \eta(S(c_{(-1)_{(2)}})\cdot b, (S(c_{(-1)_{(1)}})c_{(-2)})\cdot d)\eta(a, c_{(0)})\\ &= \eta(S(c_{(-1)})\cdot b, (S(c_{(-2)})c_{(-3)})\cdot d)\eta(a, c_{(0)})\\ &= \eta(S(c_{(-1)})\cdot b, d)\eta(a, c_{(0)})\\ &= \eta(a, c_{(0)})\eta(S(c_{(-1)})\cdot b, d)\\ &= (\eta\otimes\eta)(\mathrm{id}\otimes c^{-1}\otimes\mathrm{id})(a,b,c,d).\end{aligned}$$ □

Lemma 2.5. *Let $\eta = \eta_1 \otimes \eta_2$. Then (7) is equivalent to the following equations on $V^{\otimes 4}$:*

$$(\eta\otimes\eta)(\mathrm{id}\otimes c\otimes\mathrm{id})(c\otimes\mathrm{id}\otimes\mathrm{id}) = (\eta\otimes\eta)(\mathrm{id}\otimes c\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes c), \tag{12}$$

$$\eta_1\otimes\eta\otimes\eta_2 = (\eta\otimes\eta)(\mathrm{id}\otimes c_{312}). \tag{13}$$

Moreover, if $S_H^2 = \mathrm{id}_H$ and $c^2 = \mathrm{id}_{V\otimes V}$, then (7) is always satisfied.

Proof. The first equation is just a translation of (7). The second equation is obtained directly from the first. The left-hand side is obtained by invoking (3); the right-hand side by using $c_{312} = c\otimes\mathrm{id}(\mathrm{id}\otimes c)$. Now if S_H and c are involutions, then by Lemma 2.4 we have that

$$[(\eta\otimes\eta)(\mathrm{id}\otimes c\otimes\mathrm{id})(c\otimes\mathrm{id}\otimes\mathrm{id})](\mathrm{id}\otimes\mathrm{id}\otimes c) = (\eta\otimes\eta)(\mathrm{id}\otimes c\otimes\mathrm{id}).$$

On the other hand,

$$\begin{aligned}&[(\eta\otimes\eta)(\mathrm{id}\otimes c\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes c)](\mathrm{id}\otimes\mathrm{id}\otimes c)\\ &\quad= (\eta\otimes\eta)(\mathrm{id}\otimes c\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{id}\otimes c^2)\\ &\quad= (\eta\otimes\eta)(\mathrm{id}\otimes c\otimes\mathrm{id}).\end{aligned}$$ □

The following corollary follows directly from the lemmas proved above.

Corollary 2.6. *Assume H is a semisimple Hopf algebra and $c^2 = \mathrm{id}_{V\otimes V}$. Let $\eta : V \otimes V \to \boldsymbol{k}$ be an H-invariant linear functional. Then $\sigma = e^{\tilde{\eta}} \in \mathrm{Hom}_{\boldsymbol{k}}(A \otimes A, \boldsymbol{k})$ is a multiplicative 2-cocycle.*

2.3. A formula for deformations of braided commutator relations

We end this section with the following lemma that will be very useful in finding liftings. In particular, it will tell us how to deform relations like $[x, y]_c = 0$, which are given by braided commutators.

Lemma 2.7. *Let $\eta : V \otimes V \to \boldsymbol{k}$ be an H-invariant linear functional such that $\sigma = e^{\tilde{\eta}}$ is a multiplicative 2-cocycle. Let $x_1, x_2 \in V$ be homogeneous elements with $\delta(x_1) = h_1 \otimes x_1$ and $\delta(x_1) = h_2 \otimes x_2$, $h_1, h_2 \in G(H)$, and denote $z_1 = x_1\#1$, $z_2 = x_2\#1 \in A$. Then $\sigma(z_1, z_2) = \eta(x_1, x_2)$. In particular, in A_σ it holds that*

$$z_1 \cdot_\sigma z_2 = \eta(x_1, x_2)(1 - h_1h_2) + z_1z_2.$$

Proof. First we show that $\tilde{\eta}^2(z_1, z_2) = 0$. Since x_1, x_2 are homogeneous we have that $\Delta(z_i) = z_i \otimes 1 + h_i \otimes z_i$, that is, z_i is $(1, h_i)$-primitive for $i = 1, 2$. Since by Lemma 2.1 we have $\tilde{\eta}|_{A_0\otimes A+A\otimes A_0} = 0$, it follows that

$$\begin{aligned}\tilde{\eta}^2(z_1, z_2) &= \tilde{\eta}([z_1]_{(1)}, [z_2]_{(1)})\tilde{\eta}([z_1]_{(2)}, [z_2]_{(2)})\\ &= \tilde{\eta}(z_1, z_2)\tilde{\eta}(1, 1) + \tilde{\eta}(z_1, h_2)\tilde{\eta}(1, z_2) + \tilde{\eta}(h_1, z_2)\tilde{\eta}(z_1, 1)\\ &\quad + \tilde{\eta}(h_1, h_2)\tilde{\eta}(z_1, z_2) = 0.\end{aligned}$$

Thus, $\sigma(z_1, z_2) = \varepsilon(z_1)\varepsilon(z_2) + \tilde{\eta}(z_1, z_2) = \eta(x_1, x_2)$; in particular, $\sigma^{-1}(z_1, z_2) = e^{-\tilde{\eta}}(z_1, z_2) = -\eta(x_1, x_2)$. Finally,

$$\begin{aligned}z_1 \cdot_\sigma z_2 &= \sigma([z_1]_{(1)}, [z_2]_{(1)})[z_1]_{(2)}[z_2]_{(2)}\sigma^{-1}([z_1]_{(3)}, [z_2]_{(3)})\\ &= \sigma(z_1, z_2)\sigma^{-1}(1, 1) + \sigma(z_1, h_2)z_2\sigma^{-1}(1, 1) + \sigma(z_1, h_2)h_2\sigma^{-1}(1, z_2)\\ &\quad + \sigma(h_1, z_2)z_1\sigma^{-1}(1, 1) + \sigma(h_1, h_2)z_1z_2\sigma^{-1}(1, 1)\\ &\quad + \sigma(h_1, h_2)z_1h_2\sigma^{-1}(1, z_2)\\ &\quad + \sigma(h_1, z_2)h_1\sigma^{-1}(z_1, 1) + \sigma(h_1, h_2)h_1z_2\sigma^{-1}(z_1, 1)\\ &\quad + \sigma(h_1, h_2)h_1h_2\sigma^{-1}(z_1, z_2)\\ &= \sigma(z_1, z_2)\sigma^{-1}(1, 1) + \sigma(h_1, h_2)z_1z_2\sigma^{-1}(1, 1)\\ &\quad + \sigma(h_1, h_2)h_1h_2\sigma^{-1}(z_1, z_2)\\ &= \eta(x_1, x_2) + z_1z_2 - h_1h_2\eta(x_1, x_2) = \eta(x_1, x_2)(1 - h_1h_2) + z_1z_2,\end{aligned}$$

which finishes the proof. □

3. On pointed Hopf algebras over dihedral groups

All pointed Hopf algebras with group of group-likes isomorphic to $\mathbb{D}_m$ with $m = 4t \geq 12$ were classified in [10]. To give the complete list we need first to introduce some terminology.

For the dihedral group $\mathbb{D}_m$ we use the following presentation by generators and relations:

$$\mathbb{D}_m := \langle g, h |\ g^2 = 1 = h^m\ ,\ gh = h^{-1}g\rangle. \tag{14}$$

Because of our purposes we assume that $m = 4t \geq 12$, $n = \frac{m}{2} = 2t$ and we fix ω an mth primitive root of unity. Recall that the non-trivial conjugacy classes of $\mathbb{D}_m$ and the corresponding centralizers are

- $\mathcal{O}_{h^n} = \{h^n\}$ and $C_{\mathbb{D}_m}(h^n) = \mathbb{D}_m$;
- $\mathcal{O}_{h^i} = \{h^i, h^{m-i}\}$ and $C_{\mathbb{D}_m}(h^i) = \langle h\rangle \simeq \mathbb{Z}/(m)$, for $1 \leq i < n$;
- $\mathcal{O}_g = \{gh^j : j \text{ even}\}$ and $C_{\mathbb{D}_m}(g) = \langle g\rangle \times \langle h^n\rangle \simeq \mathbb{Z}/(2) \times \mathbb{Z}/(2)$;
- $\mathcal{O}_{gh} = \{gh^j : j \text{ odd}\}$ and $C_{\mathbb{D}_m}(gh) = \langle gh\rangle \times \langle h^n\rangle \simeq \mathbb{Z}/(2) \times \mathbb{Z}/(2)$.

3.1. Yetter–Drinfeld modules and Nichols algebras over $\mathbb{D}_m$

The irreducible Yetter–Drinfeld modules that give rise to finite-dimensional Nichols algebras are associated with the conjugacy classes of h^n and h^i with $1 \leq i < n$, see, [10, Table 2]. Next, we describe them explicitly as well as the families of reducible Yetter–Drinfeld modules with finite-dimensional Nichols algebras associated with them. Following a suggestion of the referee we slightly changed the notation used in [10].

3.1.1. Yetter–Drinfeld modules and Nichols algebras associated $\mathcal{O}_{h^n}$. Since h^n is central, $\mathcal{O}_{h^n} = \{h^n\}$ and $C_{\mathbb{D}_m}(h^n) = \mathbb{D}_m$. The irreducible representations of $\mathbb{D}_m$ are well known and they are of degree 1 or 2. Explicitly, there are:

(i) $n - 1 = \frac{m-2}{2}$ irreducible representations of degree 2 given by $\rho_\ell : \mathbb{D}_m \to \mathbf{GL}(2, \mathbf{k})$ with

$$\rho_\ell(g^a h^b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^a \begin{pmatrix} \omega^\ell & 0 \\ 0 & \omega^{-\ell} \end{pmatrix}^b, \quad \ell \in \mathbb{N} \text{ odd with } 1 \leq \ell < n. \tag{15}$$

(ii) Four irreducible representations of degree 1. They are given by the following table:

σ	1	h^n	h^b, $1 \leq b \leq n-1$	g	gh
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^n$	$(-1)^b$	1	-1
χ_4	1	$(-1)^n$	$(-1)^b$	-1	1

The irreducible Yetter–Drinfeld modules with finite-dimensional Nichols algebra are the ones given by the two-dimensional representations ρ_ℓ with $\ell \in \mathbb{N}$ odd.

Fix $\ell \in \mathbb{N}$ odd with $1 \leq \ell < n$ and consider the two-dimensional simple representation (ρ_ℓ, V) of $\mathbb{D}_m$ described in (15) above. Let $\{v_1^\ell, v_2^\ell\}$ be a $\mathbf{k}$-basis of V. Then $M_\ell = M(\mathcal{O}_{h^n}, \rho_\ell) = \mathbf{k}\mathbb{D}_m \otimes_{\mathbf{k}\mathbb{D}_m} V$ is the Yetter–Drinfeld module spanned linearly by the elements $x_1^{(\ell)} = 1 \otimes v_1^\ell, x_2^{(\ell)} = 1 \otimes v_2^\ell$; its structure is given by

$$
\begin{aligned}
g \cdot x_1^{(\ell)} &= x_2^{(\ell)}, & h \cdot x_1^{(\ell)} &= \omega^\ell x_1^{(\ell)}, & \delta(x_1^{(\ell)}) &= h^n \otimes x_1^{(\ell)},\\
g \cdot x_2^{(\ell)} &= x_1^{(\ell)}, & h \cdot x_2^{(\ell)} &= \omega^{-\ell} x_2^{(\ell)}, & \delta(x_2^{(\ell)}) &= h^n \otimes x_2^{(\ell)}.
\end{aligned}
$$

In particular, $\dim M_\ell = 2$. By Andruskiewitsch and Fantino [1, Theorem 3.1], one has that $\mathfrak{B}(\mathcal{O}_{y^n}, \rho_\ell) \simeq \bigwedge M_\ell$ and consequently $\dim \mathfrak{B}(\mathcal{O}_{y^n}, \rho_\ell) = 4$.

Consider now the set $\mathcal{L}$ of all sequences of finite length $(\ell_1, \ldots, \ell_r)$ with $\ell_i \in \mathbb{N}$ odd and $1 \leq \ell_1, \ldots, \ell_r < n$. Then for $L = (\ell_1, \ldots, \ell_r) \in \mathcal{L}$ we define $M_L = \bigoplus_{1 \leq i \leq r} M_{\ell_i}$. Clearly, $M_L \in {}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD}$ is *reducible* and by Fantino and Garcia [10, Proposition 2.8], we have that $\mathfrak{B}(M_L) \simeq \bigwedge M_L$ and $\dim \mathfrak{B}(M_L) = 4^{|L|}$, where $|L| = r$ denotes the length of L.

Remark 3.1. Since $a \cdot x_i^{(\ell)} = \chi_i^{(\ell)}(a) x_{\sigma^{(\ell)}(a)(i)}$ for all $a \in \mathbb{D}_m$, for these Nichols algebras assumption (1) is satisfied with $\sigma^{(\ell)}(g) = (12)$, $\sigma^{(\ell)}(h) = \mathrm{id}$ and $\chi_1^{(\ell)}(g) = 1 = \chi_2^{(\ell)}(g)$, $\chi_1^{(\ell)}(h) = \omega^\ell$, $\chi_2^{(\ell)}(h) = \omega^{-\ell}$.

3.1.2. Yetter–Drinfeld modules and Nichols algebras associated $\mathcal{O}_{h^i}$. Let $1 \leq i < n$. In this case, $\mathcal{O}_{h^i} = \{h^i, h^{m-i}\}$ and $C_{\mathbb{D}_m}(h^i) = \langle h \rangle \simeq \mathbb{Z}/(m)$. For $0 \leq k < m$ denote by $\mathbf{k}_{\chi_{(k)}}$ the simple representation of $C_{\mathbb{D}_m}(h^i)$ given by the character $\chi_{(k)}(h) = \omega^k$.

Take e and g as representatives of left coclasses in $\mathbb{D}_m/\langle h \rangle$, with $h^i = eh^i e$ and $h^{m-i} = gh^i g$. Then $M_{(i,k)} = M(\mathcal{O}_{h^i}, \chi_{(k)}) = \mathbf{k}\mathbb{D}_m \otimes_{\mathbf{k}\langle h \rangle} \mathbf{k}_{\chi_{(k)}}$ is the

Yetter–Drinfeld module spanned linearly by the elements $y_1^{(i,k)} = e \otimes 1$ and $y_2^{(i,k)} = g \otimes 1$. Its structure is given by

$$\begin{aligned} g \cdot y_1^{(i,k)} &= y_2^{(i,k)}, & h \cdot y_1^{(i,k)} &= \omega^k y_1^{(i,k)}, & \delta(y_1^{(i,k)}) &= h^i \otimes y_1^{(i,k)}, \\ g \cdot y_2^{(i,k)} &= y_1^{(i,k)}, & h \cdot y_2^{(i,k)} &= \omega^{-k} y_2^{(i,k)}, & \delta(y_2^{(i,k)}) &= h^{-i} \otimes y_2^{(i,k)}. \end{aligned} \tag{16}$$

In particular, $\dim M_{(i,k)} = 2$.

The irreducible Yetter–Drinfeld modules with finite-dimensional Nichols algebra are the ones given by the pairs (i,k) satisfying that $\omega^{ik} = -1$. We set $J = \{(i,k) : 1 \le i < n, 1 \le k < m \text{ such that } \omega^{ik} = -1\}$. By Andruskiewitsch and Fantino [1, Theorem 3.1], one has that $\mathfrak{B}(\mathcal{O}_{y^i}, \chi_{(k)}) \simeq \bigwedge M_{(i,k)}$, for all $(i,k) \in J$ and $\dim \mathfrak{B}(\mathcal{O}_{y^i}, \chi_{(k)}) = 4$.

Consider now the set $\mathcal{I} = ((i_1,k_1),\dots,(i_r,k_r))$ of all sequences of finite length of ordered pairs such that $(i_s,k_s) \in J$ and $\omega^{i_s k_t + i_t k_s} = 1$ for all $1 \le s,t \le r$.

For $I = ((i_1,k_1),\dots,(i_r,k_r)) \in \mathcal{I}$, we define $M_I = \bigoplus_{1 \le j \le r} M_{(i_j,k_j)}$. By Fantino and Garcia [10, Proposition 2.5], we have that $\mathfrak{B}(M_I) \simeq \bigwedge M_I$ and $\dim \mathfrak{B}(M_I) = 4^{|I|}$, where $|I| = r$ denotes the length of I.

Remark 3.2. Since $a \cdot y_j^{(i,k)} = \chi_j^{(i,k)}(a) y^{(i,k)}_{\sigma^{(i,k)}(a)(j)}$ for all $a \in \mathbb{D}_m$, for these Nichols algebras, assumption (1) is satisfied with $\sigma^{(i,k)}(g) = (12)$, $\sigma^{(i,k)}(h) = \mathrm{id}$ and $\chi_1^{(i,k)}(g) = 1 = \chi_2^{(i,k)}(g)$, $\chi_1^{(i,k)}(h) = \omega^k$, $\chi_2^{(i,k)}(h) = \omega^{-k}$.

3.1.3. Yetter–Drinfeld modules and Nichols algebras associated with mixed classes. Finally, we describe a family of reducible Yetter–Drinfeld modules given by direct sums of the modules described above.

Let $\mathcal{K}$ be the set of all pairs (I,L) with $I = ((i_1,k_1),\dots,(i_r,k_r)) \in \mathcal{I}$ and $L = (\ell_1,\dots,\ell_s) \in \mathcal{L}$ such that k_j is odd for all $1 \le j \le r$ and $\omega^{i_j \ell_t} = -1$ for all $1 \le j \le r$ and $1 \le t \le s$.

As before, for $(I,L) \in \mathcal{K}$ we define $M_{I,L} = \left(\bigoplus_{1 \le j \le s} M_{(i_j,k_j)}\right) \oplus \left(\bigoplus_{1 \le t \le s} M_{\ell_t}\right)$. By Fantino and Garcia [10, Proposition 2.12], we have that $\mathfrak{B}(M_{I,L}) \simeq \bigwedge M_{I,L}$ and $\dim \mathfrak{B}(M_{I,L}) = 4^{|I|+|L|}$.

We end this subsection recalling the following classification result.

Theorem 3.3 ([10, Theorem A]). *Let $\mathfrak{B}(M)$ be a finite-dimensional Nichols algebra in ${}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD}$. Then $\mathfrak{B}(M) \simeq \bigwedge M$, with M isomorphic either to M_I, or to M_L, or to $M_{I,L}$, with $I \in \mathcal{I}$, $L \in \mathcal{L}$ and $(I,L) \in \mathcal{K}$, respectively.*

3.2. Classification of finite-dimensional pointed Hopf algebras over $\mathbb{D}_m$

In this subsection, we present all finite-dimensional pointed Hopf algebras over $\mathbb{D}_m$ up to isomorphism. First, we introduce two families of quadratic algebras depending on families of parameters. It turns out that these quadratic algebras give all non-trivial liftings of bosonizations of finite-dimensional Nichols algebras in ${}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD}$.

Let $I \in \mathcal{I}$ and $L \in \mathcal{L}$ be as Section 3.1. By abuse of notation, if $I = ((i_1, k_1), \dots, (i_r, k_r))$ and $L = (\ell_1, \dots, \ell_s)$, we write $(i_j, k_j) \in I$ and $\ell_t \in L$ for all $1 \le j \le r$ and $1 \le t \le s$.

Consider the families of elements $\lambda = (\lambda_{p,q,i,k})_{(p,q),(i,k)\in I}$, $\gamma = (\gamma_{p,q,i,k})_{(p,q),(i,k)\in I}$, $\theta = (\theta_{p,q,\ell})_{(p,q)\in I,\ell\in L}$ and $\mu = (\mu_{p,q,\ell})_{(p,q)\in I,\ell\in L}$ in $\mathbf{k}$ satisfying

$$\lambda_{p,m-k,i,k} = \lambda_{i,k,p,m-k} \quad \text{and} \quad \gamma_{p,k,i,k} = \gamma_{i,k,p,k}. \tag{17}$$

Definition 3.4. For $I \in \mathcal{I}$, denote by $A_I(\lambda,\gamma)$ the algebra generated by the elements $g, h, a_1^{(p,q)}, a_2^{(p,q)}$ with $(p,q) \in I$ satisfying the relations:

$$\begin{aligned} g^2 = 1 = h^m, \quad ghg = h^{m-1}, \\ g a_1^{(p,q)} = a_2^{(p,q)} g, \quad h a_1^{(p,q)} = \omega^q a_1^{(p,q)} h, \quad h a_2^{(p,q)} = \omega^{-q} a_1^{(p,q)} h, \\ a_1^{(p,q)} a_1^{(i,k)} + a_1^{(i,k)} a_1^{(p,q)} = \delta_{q,m-k} \lambda_{p,q,i,k} (1 - h^{p+i}), \\ a_1^{(p,q)} a_2^{(i,k)} + a_2^{(i,k)} a_1^{(p,q)} = \delta_{q,k} \gamma_{p,q,i,k} (1 - h^{p-i}). \end{aligned}$$

It is a Hopf algebra with its structure determined by g, h being group likes and

$$\Delta(a_1^{(p,q)}) = a_1^{(p,q)} \otimes 1 + h^p \otimes a_1^{(p,q)}, \quad \Delta(a_2^{(p,q)}) = a_2^{(p,q)} \otimes 1 + h^{-p} \otimes a_2^{(p,q)}$$

for all $(p,q) \in I$. It turns out that the diagram of $A_I(\lambda,\gamma)$ is exactly $\mathfrak{B}(M_I)$, thus we call the pair (λ,γ) a *lifting datum* for $\mathfrak{B}(M_I)$. Set $\gamma = 0$ if $|I| = 1$.

Example 3.5. If $I = (i,n)$ with i odd we obtain the Hopf algebra $A_{(i,n)}(\lambda)$ generated by the elements g, h, a_1, a_2 satisfying

$$\begin{aligned} g^2 = 1 = h^m, \quad ghg = h^{m-1}, \\ g a_1 = a_2 g, \quad h a_1 = -a_1 h, \quad h a_2 = -a_2 h, \\ a_1^2 = \lambda(1 - h^{2i}), \quad a_2^2 = \lambda(1 - h^{-2i}), \quad a_1 a_2 + a_2 a_1 = 0. \end{aligned}$$

Now, we introduce the second family of quadratic algebras.

Definition 3.6. For $(I,L)\in\mathcal{K}$, denote by $B_{I,L}(\lambda,\gamma,\theta,\mu)$ the algebra generated by $g,h,a_1^{(p,q)}$, $a_2^{(p,q)},b_1^{(\ell)},b_2^{(\ell)}$ satisfying the relations:

$$\begin{aligned} g^2=1=h^m,\quad ghg=h^{m-1},\quad ga_1^{(p,q)}=a_2^{(p,q)}g,\\ ha_1^{(p,q)}=\omega^q a_1^{(p,q)}h,\quad gb_1^{(\ell)}=b_2^{(\ell)}g,\quad hb_1^{(\ell)}=\omega^\ell b_1^{(\ell)}h,\\ [a_1^{(p,q)}]^2=0=[a_2^{(p,q)}]^2,\quad b_1^{(\ell)}b_2^{(\ell')}+b_2^{(\ell')}b_1^{(\ell)}=0,\quad b_1^{(\ell)}b_1^{(\ell')}+b_1^{(\ell')}b_1^{(\ell)}=0,\\ a_1^{(p,q)}a_1^{(i,k)}+a_1^{(i,k)}a_1^{(p,q)}=\delta_{q,m-k}\lambda_{p,q,i,k}(1-h^{p+i}),\\ a_1^{(p,q)}a_2^{(i,k)}+a_2^{(i,k)}a_1^{(p,q)}=\delta_{q,k}\gamma_{p,q,i,k}(1-h^{p-i}),\\ a_1^{(p,q)}b_1^{(\ell)}+b_1^{(\ell)}a_1^{(p,q)}=\delta_{q,m-\ell}\theta_{p,q,\ell}(1-h^{n+p}),\\ a_1^{(p,q)}b_2^{(\ell)}+b_2^{(\ell)}a_1^{(p,q)}=\delta_{q,\ell}\mu_{p,q,\ell}(1-h^{n+p}).\end{aligned}$$

It is a Hopf algebra with its structure determined by g,h being group likes and

$$\begin{aligned}\Delta(a_1^{(p,q)})=a_1^{(p,q)}\otimes 1+h^p\otimes a_1^{(p,q)},\quad \Delta(a_2^{(p,q)})=a_2^{(p,q)}\otimes 1+h^{-p}\otimes a_2^{(p,q)},\\ \Delta(b_1^{(\ell)})=b_1^{(\ell)}\otimes 1+h^n\otimes b_1^{(\ell)},\qquad \Delta(b_2^{(\ell)})=b_2^{(\ell)}\otimes 1+h^n\otimes b_2^{(\ell)}\end{aligned}$$

for all $(p,q)\in I,\ell\in L$. It turns out that the diagram of $B_{I,L}(\lambda,\gamma,\theta,\mu)$ is $\mathfrak{B}(M_{I,L})$, thus we call the 4-tuple $(\lambda,\gamma,\theta,\mu)$ a lifting datum for $\mathfrak{B}(M_{I,L})$.

Example 3.7. Let $I=\{(i,k)\}$ and $L=\{m-k\}$ with $1\le k<m$ an odd number. The Hopf algebra $B_{I,L}(\theta,\mu)$ is the algebra generated by g,h,a_1,a_2, b_1,b_2 satisfying the relations

$$\begin{aligned} g^2=1=h^m,\quad ghg=h^{m-1},\\ ga_1=a_2g,\quad ha_1=\omega^k a_1h,\quad gb_1=b_2g,\quad hb_1=\omega^{-k}b_1h,\\ a_1^2=0=a_2^2,\quad b_1^2=0=b_2^2,\quad a_1a_2+a_2a_1=0,\\ b_1b_2+b_2b_1=0,\quad a_1b_1+b_1a_1=\theta(1-h^{n+i}),\quad a_1b_2+b_2a_1=\mu(1-h^{n+i}).\end{aligned}$$

The following theorem gives the classification of all finite-dimensional pointed Hopf algebras over $\mathbb{D}_m$ with $m=4t\ge 12$.

Theorem 3.8 ([10, Theorem B]). *Let H be a finite-dimensional pointed Hopf algebra with $G(H)=\mathbb{D}_m$, $m=4t\ge 12$. Then H is isomorphic to one* *of the following:*

(a) $\mathfrak{B}(M_I)\#\boldsymbol{k}\mathbb{D}_m$ *with* $I = ((i,k)) \in \mathcal{I}$, $k \neq n$, *or*

(b) $\mathfrak{B}(M_L)\#\boldsymbol{k}\mathbb{D}_m$ *with* $L \in \mathcal{L}$, *or*

(c) $A_I(\lambda,\gamma)$ *with* $I \in \mathcal{I}$, $|I| > 1$ *or* $I = ((i,n))$ *and* $\gamma \equiv 0$, *or*

(d) $B_{I,L}(\lambda,\gamma,\theta,\mu)$ *with* $(I,L) \in \mathcal{K}$, $|I| > 0$ *and* $|L| > 0$.

Conversely, any pointed Hopf algebra of the list above is a lifting of a bosonization of a finite-dimensional braided Hopf algebra in ${}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD}$.

3.3. Cocycle deformations and finite-dimensional pointed Hopf algebras over $\mathbb{D}_m$

In this subsection, we prove that all pointed Hopf algebras $A_I(\lambda,\gamma)$ and $B_{I,L}(\lambda,\gamma,\theta,\mu)$ can be obtained by deforming the multiplication of a bosonization of a Nichols algebra using a multiplicative 2-cocycle.

3.3.1. Cocycle deformations and the algebras $A_I(\lambda,\gamma)$. Let $I \in \mathcal{I}$ and consider the Nichols algebra $\mathfrak{B}(M_I)$. For all $(p,q) \in I$ on M_I, consider the linear maps $d_1^{(p,q)}, d_2^{(p,q)}$ given by the rule $d_r^{(p,q)}(y_s^{(i,k)}) = \delta_{r,s}\delta_{p,i}\delta_{q,k}$ for all $r,s = 1,2$, $(p,q),(i,k) \in I$. By Section 2.1 the following map defines a Hochschild 2-cocycle on $\mathfrak{B}(M_I)$

$$\eta = \sum_{\substack{(p,q),(i,k)\in I,\\ 1\le r,s\le 2}} \alpha^{r,s}_{p,q,i,k} d_r^{(p,q)} \otimes d_s^{(i,k)}.$$

Lemma 3.9. *η is $\mathbb{D}_m$-invariant if and only if the following conditions hold:*

$$\alpha^{r,s}_{p,q,i,k} = \alpha^{s,r}_{p,q,i,k} \qquad \forall (p,q),(i,k) \in I, r,s = 1,2, \tag{18}$$

$$\alpha^{1,1}_{p,q,i,k} = \alpha^{2,2}_{p,q,i,k} \qquad \forall (p,q),(i,k) \in I, \tag{19}$$

$$\alpha^{r,r}_{p,q,i,k} = \delta_{q,m-k}\alpha^{r,r}_{p,m-k,i,k} \qquad \forall (p,q),(i,k) \in I, r = 1,2, \tag{20}$$

$$\alpha^{r,s}_{p,q,i,k} = \delta_{q,k}\alpha^{r,s}_{p,k,i,k} \qquad \forall (p,q),(i,k) \in I, 1 \le r \neq s \le 2. \tag{21}$$

Proof. To prove that η is $\mathbb{D}_m$-invariant it is enough to show that $\eta^g = \eta^h = \eta$. Since $[d_1^{(p,q)}]^g = d_2^{(p,q)}$ and $[d_2^{(p,q)}]^g = d_1^{(p,q)}$ for all $(p,q) \in I$, and η is a linear combination of tensor products of ε-derivations, we have that $\eta^g = \eta$ if and only if (18) and (19) hold. Analogously, since $[d_i^{(p,q)}]^h = \omega^{(-1)^{i-1}q} d_i^{(p,q)}$ for all $(p,q) \in I$ and $i = 1,2$ we have that $\eta^h = \eta$ if and only if

$$\eta = \sum_{\substack{(p,q),(i,k)\in I,\\ 1\le r,s\le 2}} \alpha^{r,s}_{p,q,i,k}\omega^{(-1)^{r-1}q+(-1)^{s-1}k} d^{(p,q)}_r \otimes d^{(i,k)}_s,$$

which holds if and only if $\alpha^{r,s}_{p,q,i,k} = \alpha^{r,s}_{p,q,i,k}\omega^{(-1)^{r-1}q+(-1)^{s-1}k}$ for all (p,q), $(i,k) \in I$ and $r,s = 1,2$. Thus, if $r = s$ we must have that $\alpha^{r,r}_{p,q,i,k} = 0$ or $q \equiv -k \mod m$ which gives (20) and if $r \neq s$ then $\alpha^{r,s}_{p,q,i,k} = 0$ or $q \equiv k \mod m$ which gives (21). □

Lemma 3.10. *Assume η satisfies conditions (18)–(21). Then $\sigma = e^{\tilde{\eta}}$ is a multiplicative 2-cocycle for $\mathfrak{B}(M_I)\#\boldsymbol{k}\mathbb{D}_m$.*

Proof. By assumption, we know that η is $\mathbb{D}_m$-invariant. Since by Theorem 3.3, the braiding in ${}^{\mathbb{D}_m}_{\mathbb{D}_m}\mathcal{YD}$ is symmetric, then by Lemmas 2.3–2.5, we get that $\tilde{\eta}$ fulfils the conditions in Lemma 1.3, and consequently $\sigma = e^{\tilde{\eta}}$ is a multiplicative 2-cocycle for $\mathfrak{B}(M_I)\#\mathbf{k}\mathbb{D}_m$. □

Theorem 3.11. *Let $H = \mathfrak{B}(M_I)\#\boldsymbol{k}\mathbb{D}_m$ and $\sigma = e^{\tilde{\eta}}$ be the multiplicative 2-cocycle given by Lemma 3.10. Then $H_\sigma \simeq A_I(\lambda,\gamma)$ with $\lambda_{p,q,i,k} = \alpha^{r,r}_{p,q,i,k} + \alpha^{r,r}_{i,k,p,q}$ and $\gamma_{p,q,i,k} = \alpha^{r,s}_{p,q,i,k} + \alpha^{r,s}_{i,k,p,q}$ for all $(p,q),(i,k) \in I$. In particular, $A_I(\lambda,\gamma)$ is a cocycle deformation of H for all lifting datum.*

Proof. To show that H_σ is isomorphic to $A_I(\lambda,\gamma)$ it suffices to prove that the generators of H_σ satisfy the relations given in Definition 3.4, for this would imply that there exists a Hopf algebra surjection $H_\sigma \to A_I$ and since both algebras have the same dimension they must be isomorphic.

For $(p,q) \in I$ and $1 \le r \le 2$, denote $a^{(p,q)}_r = y^{(p,q)}_r\#1 \in \mathfrak{B}(M_I)\#\mathbf{k}\mathbb{D}_m$. Then by Lemma 2.7 we have for all $(p,q),(i,k) \in I$ and $r,s = 1,2$ that

$$\begin{aligned} a^{(p,q)}_r \cdot_\sigma a^{(i,k)}_s &= \eta(y^{(p,q)}_r, y^{(i,k)}_s)(1 - h^{p(-1)^{r-1}}h^{i(-1)^{s-1}}) + a^{(p,q)}_r a^{(i,k)}_s \\ &= \alpha^{r,s}_{p,q,i,k}(1 - h^{p(-1)^{r-1}+i(-1)^{s-1}}) + a^{(p,q)}_r a^{(i,k)}_s. \end{aligned}$$

Using Lemma 3.9, we obtain that

$$\begin{aligned} a^{(p,q)}_1 \cdot_\sigma a^{(i,k)}_1 + a^{(i,k)}_1 \cdot_\sigma a^{(p,q)}_1 &= a^{(p,q)}_1 a^{(i,k)}_1 + a^{(i,k)}_1 a^{(p,q)}_1 \\ &\quad + \delta_{q,m-k}(\alpha^{1,1}_{p,q,i,k} + \alpha^{1,1}_{i,k,p,q})(1 - h^{p+i}) \\ &= \delta_{q,m-k}(\alpha^{1,1}_{p,q,i,k} + \alpha^{1,1}_{i,k,p,q})(1 - h^{p+i}), \end{aligned}$$

$$\begin{aligned} a_1^{(p,q)} \cdot_\sigma a_2^{(i,k)} + a_2^{(i,k)} \cdot_\sigma a_1^{(p,q)} &= a_1^{(p,q)} a_2^{(i,k)} + a_2^{(i,k)} a_1^{(p,q)} \\ &\quad + \delta_{q,k}(\alpha^{1,2}_{p,q,i,k} + \alpha^{2,1}_{i,k,p,q})(1-h^{p-i}) \\ &= \delta_{q,k}(\alpha^{1,2}_{p,q,i,k} + \alpha^{1,2}_{i,k,p,q})(1-h^{p-i}). \end{aligned}$$

Thus, defining $\lambda_{p,q,i,k} = \alpha^{r,r}_{p,q,i,k} + \alpha^{r,r}_{i,k,p,q}$ and $\gamma_{p,q,i,k} = \alpha^{r,s}_{p,q,i,k} + \alpha^{r,s}_{i,k,p,q}$ with $1 \leq r \neq s \leq 2$ we get that condition (17) is satisfied. Since the other relations follows from the Yetter–Drinfeld structure of M_I, the theorem is proved. □

Remark 3.12. Note that given a lifting datum (λ, γ), using Lemma 3.9 and Theorem 3.11 one is able to construct a multiplicative 2-cocycle that gives the desired deformation of $\mathfrak{B}(M_I)\#\mathbf{k}\mathbb{D}_m$.

3.3.2. Cocycle deformations and the algebras $B_{I,L}(\lambda, \gamma, \theta, \mu)$. Let $(I, L) \in \mathcal{K}$ and consider the Nichols algebra $\mathfrak{B}(M_{I,L})$. For all $(p, q) \in I, \ell \in L$, consider the linear maps $d_1^{(p,q)}, d_2^{(p,q)}$ and $d_1^{(\ell)}, d_2^{(\ell)}$ on $M_{I,L}$ given by the rules

$$\begin{aligned} d_r^{(p,q)}(y_s^{(i,k)}) &= \delta_{r,s}\delta_{p,i}\delta_{q,k}, \quad d_r^{(p,q)}(x_s^{(\ell)}) = 0, \\ d_r^{(\ell)}(x_s^{(\ell')}) &= \delta_{r,s}\delta_{\ell,\ell'}, \quad d_r^{(\ell)}(y_s^{(i,k)}) = 0 \end{aligned}$$

for all $r, s = 1, 2,$ $(p, q), (i, k) \in I,$ $\ell \in L$. By Section 2.1, the following map defines a Hochschild 2-cocycle on $\mathfrak{B}(M_{I,L})$:

$$\begin{aligned} \eta = &\sum_{\substack{(p,q),(i,k)\in I,\\ 1\leq r,s\leq 2}} \alpha^{r,s}_{p,q,i,k} d_r^{(p,q)} \otimes d_s^{(i,k)} \\ &+ \sum_{\substack{(p,q)\in I,\ell\in L\\ 1\leq r,s\leq 2}} [\beta^{r,s}_{p,q,\ell} d_r^{(p,q)} \otimes d_s^{(\ell)} + \zeta^{r,s}_{p,q,\ell} d_s^{(\ell)} \otimes d_r^{(p,q)}] \\ &+ \sum_{\substack{\ell,\ell'\in L\\ 1\leq r,s\leq 2}} \xi^{r,s}_{\ell,\ell'} d_r^{(\ell)} \otimes d_s^{(\ell')}. \end{aligned}$$

Lemma 3.13. *η is $\mathbb{D}_m$-invariant if and only if the following conditions hold:* (18)–(21) *from Lemma 3.9,*

$$\beta^{r,s}_{p,q,\ell} = \beta^{s,r}_{p,q,\ell} \qquad \forall (p,q) \in I, \ell \in L, r, s = 1, 2, \tag{22}$$

$$\beta^{1,1}_{p,q,\ell} = \beta^{2,2}_{p,q,\ell} \qquad \forall (p,q) \in I, \ell \in L, \tag{23}$$

$$\beta^{r,r}_{p,q,\ell} = \delta_{q,m-\ell}\beta^{r,r}_{p,m-\ell,\ell} \qquad \forall (p,q)\in I, \ell\in L, r=1,2, \tag{24}$$

$$\beta^{r,s}_{p,q,\ell} = \delta_{q,\ell}\beta^{r,s}_{p,\ell,\ell} \qquad \forall (p,q)\in I, \ell\in L, 1\leq r\neq s\leq 2, \tag{25}$$

$$\xi^{r,s}_{\ell,\ell'} = \xi^{s,r}_{\ell,\ell'} \qquad \forall \ell,\ell'\in L, r,s=1,2, \tag{26}$$

$$\xi^{r,r}_{\ell,\ell'} = 0 \qquad \forall \ell,\ell'\in L, r=1,2, \tag{27}$$

$$\xi^{r,s}_{\ell,\ell'} = \delta_{\ell,\ell'}\xi^{r,s}_{\ell,\ell'} \qquad \forall \ell,\ell'\in L, 1\leq r\neq s\leq 2, \tag{28}$$

and the coefficients $\zeta^{r,s}_{p,q,\ell}$ *satisfy the same conditions as the coefficients* $\beta^{r,s}_{p,q,\ell}$, *for all* $(p,q)\in I,\ \ell\in L,\ r,s=1,2$.

Remark 3.14. Note that in this case, equation (20) implies that $\alpha^{r,r}_{p,q,p,q} = 0$ for all $(p,q),(i,k)\in I$, since $m=4t$, q is odd for all $(p,q)\in I$, $(I,L)\in\mathcal{K}$ and $m-q\equiv q \mod m$ if and only if $m=2q$.

Proof. To prove that η is $\mathbb{D}_m$-invariant it is enough to show that $\eta^g=\eta^h=\eta$. Thus the first four conditions follows directly from Lemma 3.9. The proof of the remaining conditions goes along the same lines. Only note that condition (27) is different because it never holds that $\ell'\equiv m-\ell \mod m$ since $1\leq \ell,\ell'<n$ and $m=2n$. □

The proof of the following lemma is completely analogous to the proof of Lemma 3.10.

Lemma 3.15. *Assume* η *satisfies conditions* (18)–(28). *Then* $\sigma=e^{\tilde\eta}$ *is a multiplicative* 2*-cocycle for* $\mathfrak{B}(M_{I,L})\#\boldsymbol{k}\mathbb{D}_m$.

Theorem 3.16. *Let* $H=\mathfrak{B}(M_{I,L})\#\boldsymbol{k}\mathbb{D}_m$ *and* $\sigma=e^{\tilde\eta}$ *be the multiplicative* 2*-cocycle given by* Lemma 3.15. *Then* $H_\sigma\simeq B_{I,L}(\lambda,\gamma,\theta,\mu)$ *with* $\lambda_{p,q,i,k}=\alpha^{r,r}_{p,q,i,k}+\alpha^{r,r}_{i,k,p,q}$, $\gamma_{p,q,i,k}=\alpha^{r,s}_{p,q,i,k}+\alpha^{r,s}_{i,k,p,q}$, $\theta_{p,q,\ell}=\beta^{1,1}_{p,q,\ell}+\zeta^{1,1}_{p,q,\ell}$, *and* $\mu_{p,q,\ell}=\beta^{1,2}_{p,q,\ell}+\zeta^{1,2}_{p,q,\ell}$, *for all* $(p,q)\in I,\ell\in L$. *In particular,* $B_{I,L}(\lambda,\gamma,\theta,\mu)$ *is a cocycle deformation of* H *for all lifting datum.*

Proof. As in the proof of Theorem 3.11, it suffices to show that the generators of H_σ satisfy the relations given in Definition 3.6. For $(p,q)\in I,\ell\in L$ and $1\leq r\leq 2$, denote $a^{(p,q)}_r=y^{(p,q)}_r\#1$ and $b^{(\ell)}_r=x^{(\ell)}_r\#1\in\mathfrak{B}(M_{I,L})\#\mathbf{k}\mathbb{D}_m$.

Since $\tilde\eta$ coincides with the multiplicative cocycle given by Lemma 3.10 when it takes values in $\{a^{(p,q)}_r:\ (p,q)\in I, r=1,2\}$, by the proof of Theorem 3.11 we have that the equations involving the generators $a^{(p,q)}_r$ are satisfied. In particular, since q is odd for all (p,q) we have that $q\not\equiv m-q$

mod m for all $(p,q)\in I$ and by Lemma 2.7

$$a_r^{(p,q)}\cdot_\sigma a_r^{(p,q)}=[a_r^{(p,q)}]^2+\delta_{q,m-q}\alpha^{r,r}_{p,q,p,q}(1-h^{2p(-1)^{r-1}})=0.$$

Moreover, again by Lemma 2.7 we get that

$$\begin{gathered} b_r^{(\ell)}\cdot_\sigma b_s^{(\ell')}=\eta(x_r^{(\ell)},x_s^{(\ell')})(1-h^nh^n)+b_r^{(\ell)}b_s^{(\ell')}=b_r^{(\ell)}b_s^{(\ell')}\\ \text{for all } \ell,\ell'\in L, r,s=1,2.\end{gathered}$$

Hence, using the relations of the Nichols algebra $\mathfrak{B}(M_{I,L})$ we have that

$$b_r^{(\ell)}\cdot_\sigma b_s^{(\ell')}+b_s^{(\ell')}\cdot_\sigma b_r^{(\ell)}=b_r^{(\ell)}b_s^{(\ell')}+b_s^{(\ell')}b_r^{(\ell)}=0\qquad\text{for all } \ell,\ell'\in L, r,s=1,2.$$

Besides, by (24) we get

$$\begin{aligned} a_1^{(p,q)}\cdot_\sigma b_1^{(\ell)}&=\eta(y_1^{(p,q)},x_1^{(\ell)})(1-h^ph^n)+a_1^{(p,q)}b_1^{(\ell)}\\ &=\delta_{q,m-\ell}\beta^{1,1}_{p,q,\ell}(1-h^{p+n})+a_1^{(p,q)}b_1^{(\ell)},\\ b_1^{(\ell)}\cdot_\sigma a_1^{(p,q)}&=\eta(x_1^{(\ell)},y_1^{(p,q)})(1-h^nh^p)+b_1^{(\ell)}a_1^{(p,q)}\\ &=\delta_{q,m-\ell}\zeta^{1,1}_{p,q,\ell}(1-h^{p+n})+b_1^{(\ell)}a_1^{(p,q)}\end{aligned}$$

for all $(p,q)\in I,\ell\in L$. Hence, using again the relations of the Nichols algebra $\mathfrak{B}(M_{I,L})$ we have

$$a_1^{(p,q)}\cdot_\sigma b_1^{(\ell)}+b_1^{(\ell)}\cdot_\sigma a_1^{(p,q)}=\delta_{q,m-\ell}(\beta^{1,1}_{p,q,\ell}+\zeta^{1,1}_{p,q,\ell})(1-h^{p+n}).$$

If we set $\theta_{p,q,\ell}=\beta^{1,1}_{p,q,\ell}+\zeta^{1,1}_{p,q,\ell}$ with $(p,q)\in I,\ell\in L$, then the condition involving the generators $a_1^{(p,q)},b_1^{(\ell)}$ is satisfied. Finally, by (25) we have that

$$\begin{aligned} a_1^{(p,q)}\cdot_\sigma b_2^{(\ell)}&=\eta(y_1^{(p,q)},x_2^{(\ell)})(1-h^ph^n)+a_1^{(p,q)}b_2^{(\ell)}\\ &=\delta_{q,\ell}\beta^{1,2}_{p,q,\ell}(1-h^{p+n})+a_1^{(p,q)}b_2^{(\ell)},\\ b_2^{(\ell)}\cdot_\sigma a_1^{(p,q)}&=\eta(x_2^{(\ell)},y_1^{(p,q)})(1-h^nh^p)+b_2^{(\ell)}a_1^{(p,q)}\\ &=\delta_{q,\ell}\zeta^{1,2}_{p,q,\ell}(1-h^{p+n})+b_2^{(\ell)}a_1^{(p,q)}\end{aligned}$$

for all $(p,q)\in I,\ell\in L$. Thus

$$a_1^{(p,q)}\cdot_\sigma b_2^{(\ell)}+b_2^{(\ell)}\cdot_\sigma a_1^{(p,q)}=\delta_{q,\ell}(\beta^{1,2}_{p,q,\ell}+\zeta^{1,2}_{p,q,\ell})(1-h^{p+n}).$$

Defining $\mu_{p,q,\ell}=\beta^{1,2}_{p,q,\ell}+\zeta^{1,2}_{p,q,\ell}$ with $(p,q)\in I,\ell\in L$, it follows that the condition involving the generators $a_1^{(p,q)},b_2^{(\ell)}$ is satisfied. Since the other relations follows from the Yetter–Drinfeld structure of $M_{I,L}$, the theorem is proved. □

Remark 3.17. Note that given a lifting datum $(\lambda, \gamma, \theta, \mu)$, using Lemma 3.13 and Theorem 3.16 one is able to construct a multiplicative 2-cocycle that gives the desired deformation.

4. On pointed Hopf algebras over symmetric groups

Finite-dimensional pointed Hopf algebras whose coradical is the group algebra of the groups $\mathbb{S}_3$ and $\mathbb{S}_4$ were classified in [6, 11], respectively. In this section, we prove that some of them are cocycle deformations by giving, as in Section 3.2, explicitly the cocycles.

4.1. Racks, Yetter–Drinfeld modules and Nichols algebras over $\mathbb{S}_n$

To present finite-dimensional Nichols algebras over $\mathbb{S}_n$ we need first to introduce the notion of racks, see, [5, Definition 1.1] for more details.

A *rack* is a pair $(X, \triangleright)$, where X is a non-empty set and $\triangleright : X \times X \to X$ is a function, such that $\phi_i = i \triangleright (\cdot) : X \to X$ is a bijection for all $i \in X$ satisfying that $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$ for all $i, j, k \in X$. A group G is a rack with $x \triangleright y = xyx^{-1}$ for all $x, y \in G$. If $G = \mathbb{S}_n$, then we denote by $\mathcal{O}_j^n$ the conjugacy class of all j-cycles in $\mathbb{S}_n$.

Let $(X, \triangleright)$ be a rack. A *rack 2-cocycle* $q : X \times X \to \mathbf{k}^\times$, $(i, j) \mapsto q_{ij}$ is a function such that $q_{i,j\triangleright k}\, q_{j,k} = q_{i\triangleright j, i\triangleright k}\, q_{i,k}$, for all $i, j, k \in X$. It determines a braiding c^q on the vector space $\mathbf{k}X$ with basis $\{x_i\}_{i\in X}$ by $c^q(x_i \otimes x_j) = q_{ij} x_{i\triangleright j} \otimes x_i$ for all $i, j \in X$. We denote by $\mathfrak{B}(X, q)$ the Nichols algebra of this braided vector space $(\mathbf{k}X, c^q)$.

Let $X = \mathcal{O}_2^n$ with $n \geq 3$ or $X = \mathcal{O}_4^4$ and consider the cocycles:

$$\begin{aligned} -1 : X \times X \to \mathbf{k}^\times, \quad & (j, i) \mapsto \operatorname{sg}(j) = -1, \\ \chi : \mathcal{O}_2^n \times \mathcal{O}_2^n \to \mathbf{k}^\times, \quad & (j, i) \mapsto \chi_i(j) = \begin{cases} 1 & \text{if } i = (a, b) \text{ and } j(a) < j(b), \\ -1 & \text{if } i = (a, b) \text{ and } j(a) > j(b) \end{cases} \end{aligned}$$

for $i, j \in X$. By Milinski and Schneider [18, Ex. 6.4], Graña [14], Andruskiewitsch and Graña [4, Theorem 6.12], García and García Iglesias

[11, Proposition 2.5], the Nichols algebras are given by

(a) $\mathfrak{B}(\mathcal{O}_2^n, -1)$; generated by the elements $\{x_{(\ell m)}\}_{1\le \ell<m\le n}$ satisfying for all $1 \le a < b < c \le n, 1 \le e < f \le n, \{a,b\} \cap \{e,f\} = \emptyset$ that

$$0 = x_{(ab)}^2 = x_{(ab)}x_{(ef)} + x_{(ef)}x_{(ab)} = x_{(ab)}x_{(bc)} + x_{(bc)}x_{(ac)} + x_{(ac)}x_{(ab)}.$$

(b) $\mathfrak{B}(\mathcal{O}_2^n, \chi)$; generated by the elements $\{x_{(\ell m)}\}_{1\le \ell<m\le n}$ satisfying for all $1 \le a < b < c \le n, 1 \le e < f \le n, \{a,b\} \cap \{e,f\} = \emptyset$ that

$$0 = x_{(ab)}^2 = x_{(ab)}x_{(ef)} - x_{(ef)}x_{(ab)} = x_{(ab)}x_{(bc)} - x_{(bc)}x_{(ac)} - x_{(ac)}x_{(ab)},$$
$$0 = x_{(bc)}x_{(ab)} - x_{(ac)}x_{(bc)} - x_{(ab)}x_{(ac)}.$$

(c) $\mathfrak{B}(\mathcal{O}_4^4, -1)$; generated by the elements $x_i, i \in \mathcal{O}_4^4$ satisfying for $ij = ki$ and $j \neq i \neq k \in \mathcal{O}_4^4$ that

$$0 = x_i^2 = x_i x_{i^{-1}} + x_{i^{-1}} x_i = x_i x_j + x_k x_i + x_j x_k.$$

Remark 4.1. These Nichols algebras can be seen as Nichols algebras over $\mathbb{S}_n$ by a principal YD-realization (see, [4, Definition 3.2], [18, Section 5]) of $(\mathcal{O}_2^n, -1)$, $(\mathcal{O}_2^n, \chi)$ over $\mathbb{S}_n$ or $(X, q) = (\mathcal{O}_4^4, -1)$ over $\mathbb{S}_4$; that is, one may describe $\mathbf{k}X$ as a Yetter–Drinfeld module over $\mathbb{S}_n$. In particular, if we denote the elements of $\mathbb{S}_n$ by h_τ and the elements of X by x_σ with $\sigma \in \mathcal{O}_k^n$, $k = 2, 4$, then the action and coaction are determined by

$$\delta(x_\tau) = h_\tau \otimes x_\tau, \quad h_\theta \cdot x_\tau = \chi_\tau(h_\theta)\, x_{\theta\triangleright\tau}, \qquad \text{for all } \tau \in X, \quad \theta \in \mathbb{S}_n, \tag{29}$$

where $(\chi_\tau)_{\tau\in X}$, with $\chi_\tau : \mathbb{S}_n \to \mathbf{k}^\times$, is a 1-cocycle, i.e., $\chi_\tau(\sigma\mu) = \chi_\tau(\mu)\chi_{\mu\triangleright\tau}$ (σ), for all $\tau \in X, \sigma, \mu \in \mathbb{S}_n$, satisfying $\chi_x(y) = q_{yx}$ for all $x, y \in X$.

4.2. Classification of finite-dimensional pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$

In this subsection, we present all finite-dimensional pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$ up to isomorphism. As before, first we introduce families of quadratic algebras. It turns out that these quadratic algebras give all non-trivial liftings of bosonizations of finite-dimensional Nichols algebras. We follow [5, Definition 3.7] and [11, Definitions 3.9, 3.10]. Let $\Lambda, \Gamma, \lambda \in \mathbf{k}$ and $t = (\Lambda, \Gamma)$. For $\theta, \tau \in \mathbb{S}_n$ denote $\theta \triangleright \tau = \theta\tau\theta^{-1}$ the conjugation in $\mathbb{S}_n$.

Definition 4.2. $\mathcal{H}(\mathcal{Q}_n^{-1}[t])$ is the algebra generated by $\{a_i, h_r : i \in \mathcal{O}_2^n, r \in \mathbb{S}_n\}$ satisfying the following relations for $r, s, j \in \mathbb{S}_n$ and $i \in \mathcal{O}_2^n$:

$$\begin{aligned} h_e = 1, \quad h_r h_s = h_{rs}, \quad h_j a_i = -a_{j\triangleright i} h_j, \quad a_{(12)}^2 = 0, \\ a_{(12)}a_{(34)} + a_{(34)}a_{(12)} = \Lambda(1 - h_{(12)}h_{(34)}), \\ a_{(12)}a_{(23)} + a_{(23)}a_{(13)} + a_{(13)}a_{(12)} = \Gamma(1 - h_{(12)}h_{(23)}). \end{aligned}$$

Definition 4.3. $\mathcal{H}(\mathcal{Q}_n^{\chi}[\lambda])$ is the algebra generated by $\{a_i, h_r : i \in \mathcal{O}_2^n, r \in \mathbb{S}_n\}$ satisfying the following relations for $r, s, j \in \mathbb{S}_n$ and $i \in \mathcal{O}_2^n$:

$$\begin{aligned} h_e = 1, \quad h_r h_s = h_{rs}, \quad h_j a_i = \chi_i(j)\, a_{j\triangleright i} h_j, \quad a_{(12)}^2 = 0, \\ a_{(12)}a_{(34)} - a_{(34)}a_{(12)} = 0, \\ a_{(12)}a_{(23)} - a_{(23)}a_{(13)} - a_{(13)}a_{(12)} = \lambda(1 - h_{(12)}h_{(23)}). \end{aligned}$$

Definition 4.4. $\mathcal{H}(\mathcal{D}[t])$ is the algebra generated by $\{a_i, h_r : i \in \mathcal{O}_4^4, r \in \mathbb{S}_4\}$ satisfying the following relations for $r, s, j \in \mathbb{S}_n$ and $i \in \mathcal{O}_4^4$:

$$\begin{aligned} h_e = 1, \quad h_r h_s = h_{rs}, \quad h_j a_i = -a_{j\triangleright i} h_j, \quad a_{(1234)}^2 = \Lambda(1 - h_{(13)}h_{(24)}), \\ a_{(1234)}a_{(1432)} + a_{(1432)}a_{(1234)} = 0, \\ a_{(1234)}a_{(1243)} + a_{(1243)}a_{(1423)} + a_{(1423)}a_{(1234)} = \Gamma(1 - h_{(12)}h_{(13)}). \end{aligned}$$

Remark 4.5. For each quadratic lifting datum $\mathcal{Q} = \mathcal{Q}_n^{-1}[t], \mathcal{Q}_n^{\chi}[\lambda], \mathcal{D}[t]$, the algebra $\mathcal{H}(\mathcal{Q})$ has a structure of a pointed Hopf algebra setting

$$\Delta(h_t) = h_t \otimes h_t \text{ and } \Delta(a_i) = a_i \otimes 1 + h_i \otimes a_i, \quad t \in \mathbb{S}_n, i \in X. \tag{30}$$

Moreover, they satisfy that $\operatorname{gr} \mathcal{H}(\mathcal{Q}) = \mathfrak{B}(X, q)\#\mathbf{k}\mathbb{S}_n$, with n as appropriate, see, [11].

The following theorem summarizes the classification of finite-dimensional pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$, see, [6, 11].

Theorem 4.6. *Let H be a non-trivial finite-dimensional pointed Hopf algebra with $G(H) = \mathbb{S}_n$.*

(i) *If $n = 3$, then either $H \simeq \mathfrak{B}(\mathcal{O}_2^3, -1)\#\boldsymbol{k}\mathbb{S}_3$ or $H \simeq \mathcal{H}(\mathcal{Q}_3^{-1}[(0,1)])$.*

(ii) *If $n = 4$, then either $H \simeq \mathfrak{B}(X, q)\#\boldsymbol{k}\mathbb{S}_4$ with $(X, q) = (\mathcal{O}_2^4, -1)$, $(\mathcal{O}_4^4, -1)$ or $(\mathcal{O}_2^4, \chi)$, or $H \simeq \mathcal{H}(\mathcal{Q}_4^{-1}[t])$, or $H \simeq \mathcal{H}(\mathcal{Q}_4^{\chi}[1])$, or $H \simeq \mathcal{H}(\mathcal{D}[t])$ with $t \in \mathbb{P}_{\boldsymbol{k}}^1$.*

4.3. Cocycle deformations and pointed Hopf algebras over $\mathbb{S}_n$

In the following, we construct multiplicative 2-cocycles and show that some families of the pointed Hopf algebras $\mathcal{H}(\mathcal{Q}_n^{-1}[t])$ and $\mathcal{H}(\mathcal{D}[t])$ are cocycle deformations of bosonizations of Nichols algebras in ${}^{\mathbb{S}_n}_{\mathbb{S}_n}\mathcal{YD}$. As a consequence, we provide the family of cocycles needed to construct all finite-dimensional pointed Hopf algebras over $\mathbb{S}_3$ up to isomorphism.

Let $X = \mathcal{O}_2^n$ or $\mathcal{O}_4^4$ and denote the generators of $\mathfrak{B}(X,-1)$ by x_τ with $\tau \in X$. For all $\sigma, \tau \in X$, define the linear maps d_τ on $\mathbf{k}X$ by $d_\tau(x_\mu) = \delta_{\tau,\mu}$. By Section 2.1, the following map is a Hochschild 2-cocycle on $\mathfrak{B}(X,-1)$:

$$\eta = \sum_{\mu,\tau \in X} \alpha_{\tau,\mu} d_\tau \otimes d_\mu.$$

The proof of the following lemma follows by a direct computation.

Lemma 4.7. *η is $\mathbb{S}_n$-invariant if and only if $\alpha_{\tau,\mu} = \alpha_{\theta \triangleright \tau, \theta \triangleright \mu}$ for all $\tau, \mu \in X$ and $\theta \in \mathbb{S}_n$.*

Remark 4.8. Consider the set $\mathcal{T} = X \times X$. Then $\mathbb{S}_n$, and in particular X, acts by conjugation on $\mathcal{T}$ by $\theta \cdot (\tau, \mu) = (\theta \triangleright \tau, \theta \triangleright \mu)$. If we set $\alpha : \mathcal{T} \to \mathbf{k}$ with $\alpha(\tau, \mu) = \alpha_{\tau,\mu}$, then the coefficients of η are given by the function α and by Lemma 4.7, η is $\mathbb{S}_n$-invariant if and only if α is a class function, i.e., it is constant on each conjugacy class. Since (τ, μ) is conjugate to (τ', μ') if and only if $\tau\mu$ is conjugate to $\tau'\mu'$ in $\mathbb{S}_n$, if η is $\mathbb{S}_n$-invariant, we may write in the case $X = \mathcal{O}_2^n$:

$$\eta = \beta_{\mathrm{id}} \sum_{\tau \in \mathcal{O}_2^n} d_\tau \otimes d_\tau + \beta_{(123)} \sum_{\substack{\tau,\mu \in \mathcal{O}_2^n \\ \tau\mu \in \mathcal{O}_3^n}} d_\tau \otimes d_\mu + \beta_{(12)(34)} \sum_{\substack{\tau,\mu \in \mathcal{O}_2^n \\ \tau\mu \in \mathcal{O}_{2,2}^n}} d_\tau \otimes d_\mu, \tag{31}$$

with $\beta_{\mathrm{id}}, \beta_{(123)}, \beta_{(12)(34)} \in \mathbf{k}$, and in the case $X = \mathcal{O}_4^4$:

$$\eta = \gamma_{\mathrm{id}} \sum_{\tau \in \mathcal{O}_4^4} d_\tau \otimes d_{\tau^{-1}} + \gamma_{(123)} \sum_{\substack{\tau,\mu \in \mathcal{O}_4^4 \\ \tau\mu \in \mathcal{O}_3^4}} d_\tau \otimes d_\mu + \gamma_{(12)(34)} \sum_{\tau \in \mathcal{O}_4^4} d_\tau \otimes d_\tau, \tag{32}$$

with $\gamma_{\mathrm{id}}, \gamma_{(123)}, \gamma_{(12)(34)} \in \mathbf{k}$.

Assume η satisfies (31) or (32). The next lemma states that the exponentiation of the lifting of η is a multiplicative 2-cocycle if all coefficients β or γ are equal. Since the braiding on $\mathfrak{B}(X,-1)$ is not symmetric, one needs to verify equations (6) and (7) on $V = \mathbf{k}X$.

Lemma 4.9. *Assume* $\eta = \sum_{\mu,\tau\in X} \alpha_{\tau,\mu} d_\tau \otimes d_\mu$ *is* $\mathbb{S}_n$*-invariant. Then it satisfies equations* (6) *and* (7) *if and only if* $\alpha_{\tau,\mu} = \alpha_{\tau',\mu'}$ *for all* $\tau, \tau', \mu, \mu' \in X$*. In such a case,* $\sigma = e^{\tilde{\eta}}$ *is a multiplicative* 2*-cocycle for* $\mathfrak{B}(X,-1)\#\boldsymbol{k}\mathbb{S}_n$*.*

Proof. By Lemma 2.3, we need only to verify equations (6) and (7) on $V = \mathbf{k}X$. Since $\chi_\tau(h_\mu) = \mathrm{sg}(\mu) = -1$ for all $\tau, \mu \in \mathcal{O}_2^n$ and $\chi_\tau(h_\mu) = 1$ for all $\tau, \mu \in \mathcal{O}_4^4$ these equations on x_r, x_s, x_t, x_u with $r, s, t, u \in X$ equal:

$$\text{(6)} \quad \eta(x_r, x_{s\triangleright t})\eta(x_s, x_u) = \eta(x_{r\triangleright s}, x_{r\triangleright(t\triangleright u)})\eta(x_r, x_t),$$
$$\text{(7)} \quad \eta(x_r, x_{s\triangleright(t\triangleright u)})\eta(x_s, x_t) = \eta(x_{r\triangleright s}, x_{r\triangleright t})\eta(x_r, x_u).$$

It is clear that if $\eta = \lambda \sum_{\mu,\tau\in X} d_\tau \otimes d_\mu$ for some $\lambda \in \mathbf{k}$, then both equations are satisfied. Conversely, assume η satisfies (6) and (7). Since $t \triangleright -$ is a bijection for all $t \in X$, by (7) we have that $\alpha_{r,s\triangleright(t\triangleright u)}\alpha_{s,t} = \alpha_{s,t}\alpha_{r,u}$ for all $r, s, t, u \in X$. If $\alpha_{s,t} \neq 0$ for some $s, t \in X$, then $\alpha_{r,u} = \alpha_{r,s\triangleright u}$ for all $r, s, u \in X$. Since η must satisfy (31) or (32), it follows that $\eta = \lambda \sum_{\mu,\tau\in X} d_\tau \otimes d_\mu$ for some $\lambda \in \mathbf{k}$. The rest of the claim follows now by Lemma 2.3. □

Theorem 4.10. *Let* $H = \mathfrak{B}(X,-1)\#\boldsymbol{k}\mathbb{S}_n$ *and* $\sigma = e^{\tilde{\eta}}$ *be the multiplicative* 2*-cocycle given by* Lemma 4.9 *with* $\eta = \frac{\lambda}{3} \sum_{\mu,\tau\in \mathcal{O}_2^n} d_\tau \otimes d_\mu$ *and* $\lambda \in \boldsymbol{k}$*.*

(i) *If* $X = \mathcal{O}_2^n$ *then* $H_\sigma \simeq \mathcal{H}(\mathcal{Q}_3^{-1}[(0,\lambda)])$ *for* $n = 3$ *and* $H_\sigma \simeq \mathcal{H}(\mathcal{Q}_n^{-1}[(2\lambda, 3\lambda)])$ *for* $n \geq 4$*.*

(ii) *If* $X = \mathcal{O}_4^4$ *then* $H_\sigma \simeq \mathcal{H}(\mathcal{D}[(\lambda, 3\lambda)])$*.*

In particular, $\mathcal{H}(\mathcal{Q}_3^{-1}[(0,\lambda)])$ *is a cocycle deformation of* H *for all* $\lambda \in \boldsymbol{k}$*.*

Proof. As in the proof of Theorems 3.11 and 3.16, it suffices to show that the generators of H_σ satisfy the relations given in Definitions 4.2 and 4.4, respectively. For $\tau \in X$, let $a_\tau = x_\tau\#1 \in H$. Then by Lemma 2.7 we have for all $\tau, \mu \in \mathcal{O}_2^n$ that

$$a_\tau \cdot_\sigma a_\mu = \eta(x_\tau, x_\mu)(1 - h_\tau h_\mu) + a_\tau a_\mu = \lambda(1 - h_{\tau\mu}) + a_\tau a_\mu.$$

Hence, if $X = \mathcal{O}_2^n$ we get that $a_{(12)} \cdot_\sigma a_{(12)} = a_{(12)}^2 + \frac{\lambda}{3}(1 - h_{(12)(12)}) = \frac{\lambda}{3}(1 - h_e) = 0$ and

$$\begin{aligned} &a_{(12)} \cdot_\sigma a_{(23)} + a_{(23)} \cdot_\sigma a_{(13)} + a_{(13)} \cdot_\sigma a_{(12)} \\ &\quad = a_{(12)}a_{(23)} + a_{(23)}a_{(13)} + a_{(13)}a_{(12)} + \frac{\lambda}{3}(1 - h_{(12)(23)}) \end{aligned}$$

$$+ \frac{\lambda}{3}(1 - h_{(23)(13)}) + \frac{\lambda}{3}(1 - h_{(13)(12)})$$
$$= \lambda(1 - h_{(123)}) = \lambda(1 - h_{(12)(23)}).$$

Taking $\Gamma = \lambda$, this implies that $H_\sigma \simeq \mathcal{H}(\mathcal{Q}_3^{-1}[(0, \lambda)])$ if $n = 3$, since both algebras have the same dimension. For $n \geq 4$ we need to verify the extra relation involving the product of two disjoint transpositions:

$$\begin{aligned} &a_{(12)} \cdot_\sigma a_{(34)} + a_{(34)} \cdot_\sigma a_{(12)} \\ &= a_{(12)}a_{(34)} + a_{(34)}a_{(12)} + \frac{\lambda}{3}(1 - h_{(12)(34)}) + \frac{\lambda}{3}(1 - h_{(34)(12)}) \\ &= \frac{2\lambda}{3}(1 - h_{(12)(34)}). \end{aligned}$$

Thus taking $t = (\Lambda, \Gamma) = (\frac{2\lambda}{3}, \lambda)$, we have that $H_\sigma \simeq \mathcal{H}(\mathcal{Q}_n^{-1}[(2\lambda, 3\lambda)])$. Assume $X = \mathcal{O}_4^4$, then

$$\begin{aligned} &a_{(1234)} \cdot_\sigma a_{(1234)} = a^2_{(1234)} + \frac{\lambda}{3}(1 - h_{(1234)(1234)}) = \frac{\lambda}{3}(1 - h_{(13)(24)}), \\ &a_{(1234)} \cdot_\sigma a_{(1432)} + a_{(1432)} \cdot_\sigma a_{(1234)} = a_{(1234)}a_{(1432)} + a_{(1432)}a_{(1234)} \\ &\quad + \frac{\lambda}{3}(1 - h_{(1234)(1432)}) + \frac{\lambda}{3}(1 - h_{(1432)(1234)}) = \frac{2\lambda}{3}(1 - h_e) = 0, \\ &a_{(1234)} \cdot_\sigma a_{(1243)} + a_{(1243)} \cdot_\sigma a_{(1423)} + a_{(1423)} \cdot_\sigma a_{(1234)} \\ &\quad = a_{(1234)}a_{(1243)} + a_{(1243)}a_{(1423)} + a_{(1423)}a_{(1234)} + \frac{\lambda}{3}(1 - h_{(1234)(1243)}) \\ &\quad + \frac{\lambda}{3}(1 - h_{(1243)(1423)}) + \frac{\lambda}{3}(1 - h_{(1423)(1234)}) = \lambda(1 - h_{(12)(13)}). \end{aligned}$$

Therefore, taking $t = (\Lambda, \Gamma) = (\frac{\lambda}{3}, \lambda)$, we have that $H_\sigma \simeq \mathcal{H}(\mathcal{D}[(\lambda, 3\lambda)]))$. □

Remark 4.11. *Cocycle deformations and the algebras* $\mathcal{H}(\mathcal{Q}_n^\chi[\lambda])$. As shown in [12], the pointed Hopf algebras $\mathcal{H}(\mathcal{Q}_n^\chi[\lambda])$ are cocycle deformations of $\mathfrak{B}(\mathcal{O}_2^n, \chi)\#\mathbf{k}\mathbb{S}_n$. Regrettably, our construction using $\mathbb{S}_n$-invariant linear functionals on $\mathbf{k}\mathcal{O}_2^n \otimes \mathbf{k}\mathcal{O}_2^n$ only provides the trivial deformation.

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