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l-Hemi-Implicative Semilattices

Abstract. An l-hemi-implicative semilattice is an algebra $\mathbf{A} = (A, \wedge, \rightarrow, 1)$ such that $(A, \wedge, 1)$ is a semilattice with a greatest element 1 and satisfies: (1) for every $a, b, c \in A$, $a \leq b \rightarrow c$ implies $a \wedge b \leq c$ and (2) $a \rightarrow a = 1$. An l-hemi-implicative semilattice is commutative if if it satisfies that $a \rightarrow b = b \rightarrow a$ for every $a, b \in A$. It is shown that the class of l-hemi-implicative semilattices is a variety. These algebras provide a general framework for the study of different algebras of interest in algebraic logic. In any l-hemi-implicative semilattice it is possible to define an derived operation by $a \sim b := (a \rightarrow b) \wedge (b \rightarrow a)$. Endowing $(A, \wedge, 1)$ with the binary operation $\sim$ the algebra $(A, \wedge, \sim, 1)$ results an l-hemi-implicative semilattice, which also satisfies the identity $a \sim b = b \sim a$. In this article, we characterize the (derived) commutative l-hemi-implicative semilattices. We also provide many new examples of l-hemi-implicative semilattice on any semillatice with greatest element (possibly with bottom). Finally, we characterize congruences on the classes of l-hemi-implicative semilattices introduced earlier and we characterize the principal congruences of l-hemi-implicative semilattices.

Keywords: Bounded semilattices, Weak implications, Congruences.

1. Introduction

Recall that a structure $(A, \leq, \cdot, e)$ is said to be a partially ordered monoid if $(A, \leq)$ is a poset, $(A, \cdot, e)$ is a monoid and for all $a, b, c \in A$, if $a \leq b$ then $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$. Although commutativity does not play any special role in the discussion that follow, we shall assume in this article that all monoids are commutative.

Let us also recall that the residuum (when it exists) of the monoid operation of a partially ordered monoid $(A, \leq, \cdot, e)$ is a binary operation $\rightarrow$ on A such that for every a, b and c in A,

$$a \cdot b \leq c \text{ if and only if } a \leq b \rightarrow c.$$

Note that the previous equivalence can be seen as the conjunction of the following conditionals:

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(r) If $a \cdot b \leq c$ then $a \leq b \to c$ and

(l) If $a \leq b \to c$ then $a \cdot b \leq c$.

This suggest us to consider binary operations $\to$ satisfying either (r) or (l) above.

We say that a structure $\mathbf{A} = (A, \leq, \cdot, \to, e)$ is an *r-hemiresiduated monoid* if $(A, \leq, \cdot, e)$ is a partially ordered monoid and $\to$ is an r-hemiresiduum. Similarly, we define an *l-hemiresiduated monoid*. Clearly, every partially ordered residuated monoid is both an r-hemiresiduated monoid and an l-hemiresiduated monoid. Some examples of l-hemiresiduated monoids which in general are not residuated can be found for instance in [18].

Proposition 1.1. *Let $\mathbf{A} = (A, \leq, \cdot, \to, e)$ be a structure such that $(A, \leq, \cdot, e)$ is a partially ordered monoid. Then $\mathbf{A}$ is an l-hemiresiduated monoid if and only if $a \cdot (a \to b) \leq b$, for every $a, b \in A$.* ■

Proof. Suppose that $\mathbf{A}$ is an l-hemiresiduated monoid and let $a, b \in A$. Since $a \to b \leq a \to b$ then $a \cdot (a \to b) \leq b$. Conversely, suppose that $a \cdot (a \to b) \leq b$ for any $a, b \in A$. Let $a, b, c \in A$ be such that $a \leq b \to c$. Then $a \cdot b \leq b \cdot (b \to c) \leq c$, so $a \cdot b \leq c$. Therefore, $\mathbf{A}$ is an l-hemiresiduated monoid.

In this work we are interested in l-hemi-implicative semilattices.

Definition 1.2. An algebra $\mathbf{A} = (A, \wedge, \to, 1)$ of type $(2, 2, 0)$ is an *l-hemi-implicative semilattice* if it satisfies the following conditions:

(H1) $(A, \wedge, 1)$ is a semilattice with greatest element 1,

(H2) for every $a \in A$, $a \to a = 1$ and

(H3) for every $a, b, c \in A$, if $a \leq b \to c$ then $a \wedge b \leq c$.

An l-hemi-implicative semilatice is commutative if $a \to b = b \to a$ for every $a, b \in A$.

Notice that (H3) is the condition (l) for the case $\cdot = \wedge$. Also note that l-hemi-implicative semilattices were called weak implicative semilattices in [19]. In every l-hemi-implicative semilattice we define the binary operation $\leftrightarrow$ by $a \leftrightarrow b := (a \to b) \wedge (b \to a)$. We write hIS for the variety of l-hemi-implicative semilattices.

Remark 1.3. Let $\mathbf{A} \in$ hIS and $a, b \in A$. Then $a = b$ if and only if $a \leftrightarrow b = 1$. We also have that $1 \to a \leq a$.

Recall that an implicative semilattice [7, 15] is an algebra $(A, \wedge, \to)$ of type $(2, 2)$ such that $(A, \wedge)$ is semilattice, and for every $a, b, c \in H$ it holds that $a \wedge b \leq c$ if and only if $a \leq b \to c$. Every implicative semilattice has a greatest element. In this paper we shall include the greatest element in the language of the implicative semilattices. We write IS for the variety of implicative semilattices.

Remark 1.4. Clearly, IS is a subvariety of hIS. Other examples of l-hemi-implicative semilattices are the $\{\wedge, \to, 1\}$-reduct of semi-Heyting algebras [16,17] (named almost Brouwerian semilattices in [4]) and the $\{\wedge, \to, 1\}$-reducts of RWH-algebras [3], in particular, the $\{\wedge, \to, 1\}$-reducts of basic algebras [1].

From a logical point of view, the variety hIS and its subvarieties fit into the general setting of semilattice-based logics, as studied in [10]. As such, the assertional logics associated to these varieties (see [10, Lemma 3]) posses interesting properties from the abstract algebraic logic theory perspective. Some particular examples of these logics, already studied in the literature are, beyond the already classic example of the $\{\wedge, \to, 1\}$-fragment of the intuitionistic logic, the Semi-Intuitionistic Logic [6] and the subintuitionistic logic corresponding to the subvariety of RWH-algebras [3].

In [12] Jenei shows that the class of BCK algebras with meet is term equivalent to the class of equivalential equality algebras, and he defines the equivalence operation $\sim$ in terms of the implication in the usual way; i.e., $a \sim b := a \leftrightarrow b$. Some of these ideas were generalized and studied for pseudo BCK-algebras [5,8,13].

In particular, the variety of implicative semilattices is term equivalent to a subvariety of that of equivalential equality algebras. Let us write ES for this subvariety. Some properties satisfied by the algebras in ES are:

(a) $a \sim b = b \sim a$,

(b) $a \sim a = 1$,

(c) $a \wedge (a \sim b) \leq b$.

On the other hand, implicative semilattices satisfy (b) and (c) above, but of course not necessarily (a). Hence there seems to be a common frame for both classes of algebras, where the algebras in IS might be seen as elements with a commutative implication and the construction $a \sim b := a \leftrightarrow b$ a sort of symmetrization of the original implication. In this paper we explore a convenient framework where the aforementioned intuitions could be made precis. Most results concerning the relation between implicative semilattices and the class ES are part of the folklore. However, for the sake of completeness and in order to motivate the introduction of these structures, we shall recall some basic results in Section 2.

In Section 3 some subvarieties of l-hemi-implicative semilattices are presented. New examples of l-hemi-implicative semilattices are provided. The relationship between the variety of l-hemi-implicative semilattices and its subvariety of commutative elements is studied.

In Section 4 we characterize congruences on the clases of l-hemi-implicative semilattices introduced in the examples of Section 3 and we describe the principal congruences of l-hemi-implicative semilattices.

2. Term Equivalence of IS and ES

We start with the following remark.

Remark 2.1. (a) In every implicative semilattice $\mathbf{A} = (A, \wedge, \rightarrow, 1)$ we have that $(A, \wedge, 1)$ is a semilattice with a greatest element and $a \leftrightarrow a = 1$ for every $a \in A$. We also have that for every $a, b, c \in A$, $c \leq a \leftrightarrow b$ if and only if $a \wedge c = b \wedge c$.

(b) Implicative semilattices satisfy $a \rightarrow b = a \leftrightarrow (a \wedge b)$ for every $a, b \in A$.

(c) Consider an algebra $\mathbf{A} = (A, \wedge, \sim, 1)$ of type $(2, 2, 0)$ such that $(A, \wedge)$ is a semilattice. For every $a, b, c \in A$ we consider the following conditions: (1) $a \wedge (a \sim b) = b \wedge (a \sim b)$ and (2) if $a \wedge c = b \wedge c$ then $c \leq a \sim b$. For every $a, b, c \in A$ conditions (1) and (2) are satisfied if and only if we have that $a \sim b = \max\,\{c \in A : a \wedge c = b \wedge c\}$ for every $a, b \in A$.

In the following proposition we consider a particular class of algebras.

Definition 2.2. The class ES consists of algebras $\mathbf{A} = (A, \wedge, \sim, 1)$ of type (2, 2, 0) that satisfy the following conditions:

(1) $(A, \wedge, 1)$ is a semilattice with a greatest element,

(2) $a \sim a = 1$,

(3) $a \wedge (a \sim b) = b \wedge (a \sim b)$,

(4) if $a \wedge c = b \wedge c$ then $c \leq a \sim b$.

Lemma 2.3. *Let $\mathbf{A} = (A, \wedge, \sim, 1)$ be an algebra wich satisfies conditions (1), (2) and (3) of Definition 2.2. Then (4) of Definition 2.2 is equivalent to the following condition:*

(4′) $c \wedge ((a \wedge c) \sim (b \wedge c)) \leq a \sim b$.

Proof. Assume the condition (4). Since $a \wedge (c \wedge ((a \wedge c) \sim (b \wedge c))) = b \wedge (c \wedge ((a \wedge c) \sim (b \wedge c)))$ then $c \wedge ((a \wedge c) \sim (b \sim c)) \leq a \sim b$, which is the condition (4′). Conversely, assume the condition (4′), and suppose that $a \wedge c = b \wedge c$. It follows from the conditions (2) and (4′) that $c = c \wedge ((a \wedge c) \sim (b \wedge c)) \leq a \sim b$, so $c \leq a \sim b$. ■

Notice that ES is a subvariety of the variety of equality algebras, which was introduced by Jenei [12]. See also [4,8,13].

Proposition 2.4. *Let* $\boldsymbol{A} = (A, \wedge, \sim, 1) \in \mathrm{ES}$. *Then* $(A, \wedge, \rightarrow, 1) \in \mathrm{IS}$, *where* $\rightarrow$ *is defined by* $a \rightarrow b = a \sim (a \wedge b)$.

Proof. In order to prove that $(A, \wedge, \rightarrow, 1) \in \mathrm{IS}$, we only need to prove that for every $a, b, c \in A$, $a \leq b \rightarrow c$ if and only if $a \wedge b \leq c$. Suppose that $a \leq b \rightarrow c$, i.e., $a \leq b \sim (b \wedge c)$. Then $a \wedge b \leq b \wedge (b \sim (b \wedge c))$. It follows from (3) of Definition 2.2 that $b \wedge (b \sim (b \wedge c)) = (b \wedge c) \wedge (b \sim (b \wedge c)) \leq c$, so $a \wedge b \leq c$. Conversely, suppose that $a \wedge b \leq c$. Then $a \wedge b = a \wedge (b \wedge c)$. Taking into account (4) of Definition 2.2 we have that $a \leq b \sim (b \wedge c)$, i.e., $a \leq b \rightarrow c$. ■

The following corollary follows from Proposition 2.4 and Remark 2.1.

Corollary 2.5. *The varieties* IS *and* ES *are term equivalent. More explicitly, we have:*

(1) *If* $\mathbf{A} = (A, \wedge, \sim, 1) \in \mathrm{ES}$, *then the algebra* $\mathbf{A}^* = (A, \wedge, \rightarrow, 1) \in \mathrm{IS}$, *where* $\rightarrow$ *is defined by* $a \rightarrow b := a \sim (a \wedge b)$.
(2) *If* $\boldsymbol{A} = (A, \wedge, \rightarrow, 1) \in \mathrm{IS}$, *then the algebra* $\mathbf{A}^+ = (A, \wedge, \sim, 1) \in \mathrm{ES}$, *where* $\sim$ *is defined by* $a \sim b := a \leftrightarrow b$.
(3) *If* $\mathbf{A} \in \mathrm{ES}$, *then* $\mathbf{A}^{*+} = \mathbf{A}$,
(4) *If* $\mathbf{A} \in \mathrm{IS}$, *then* $\mathbf{A}^{+*} = \mathbf{A}$.

A straightforward argument shows that the variety ES above defined is a subvariety of the variety of equivalential equality algebras considered in [12]. Moreover, the variety of equivalential equality algebras is term equivalent to the variety of BCK-algebras with meet (see [12, Theorem 2.5]). On the other hand, as an anonymous referee make us notice, since the variety of positive implicative BCK-algebras (with meet) satisfying condition (S) is term equivalent to that of implicative semilattices (see [14, Theorem 8]), this fact together with the term equivalence for BCK-algebras with meet provides an alternative way to obtain Corollary 2.5.

3. l-Hemi-Implicative Semilattices and Commutative l-Hemi-Implicative Semilattices

In this section we study the variety of l-hemi-implicative semilattices and some of its subvarieties. We also present some general examples by defining l-hemi-implicative structures on any semilattice with a greatest element. Finally we study the variety whose algebras are commutative l-hemi-implicative semilattices. The original motivation to consider this variety follows from the properties of the algebras $(A, \wedge, \leftrightarrow, 1)$ associated to the algebras $(A, \wedge, \rightarrow, 1) \in \mathrm{hIS}$.

There are several ways of defining an l-hemi-implicative structure on any semilattice with a greatest element. Some of these ways are described in the examples below. Note that some of these procedures only apply to semilattices with a greatest element and with bottom.

Example 3.1. *Let $(A, \wedge, 1)$ be a semilattice with the largest element (with bottom 0, when necessary). We define binary operations $\rightarrow$ on A each of which makes the algebra $(A, \wedge, \rightarrow, 1)$ an l-hemi-implicative semilattice.*

$$a \rightarrow b = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} \tag{1}$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a \nleq b \end{cases} \tag{2}$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a = b \\ b & \text{if } a \neq b \end{cases} \tag{3}$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ a \wedge b & \text{if } a \nleq b \end{cases} \tag{4}$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a = b \\ a \wedge b & \text{if } a \neq b \end{cases} \tag{5}$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{if } a \nleq b \end{cases} \tag{6}$$

In Example 3.1 we define a binary operation that makes an algebra an l-hemi-implicative semilattice. In the rest of the paper we shall refer to this operation as the implication of the algebra. For $i = 1, \ldots, 6$, let K_i be the class of the algebras in hIS where the implication is given by (i). Note that every algebra of K_2 is a Hilbert algebra with infimum. The class of Hilbert algebras with infimum is a variety, as it was proved in [2,9] (see also [11]).

Remark 3.2. It can be proved that the classes K_i are not closed under products; hence, they are not quasivarieties. It would be interesting to have an answer for the following general question: which is a set of identities (quasi-identities) for the variety (quasivariety) generated by K_i?

We denote by hIS_4 and hIS_5 the subvarieties of hIS defined by the following identities respectively:

(H4) $a \to (a \wedge b) = a \to b$,

(H5) $(a \wedge b) \to b = 1$.

REMARK 3.3. Let $\mathbf{A} \in \mathrm{hIS}_4$ and $a \in A$. Then $a \to 1 = 1$, since, by (H4), we have $a \to 1 = a \to (a \wedge 1) = a \to a = 1$.

PROPOSITION 3.4. *It holds that* $\mathrm{hIS}_4 \subsetneq \mathrm{hIS}_5 \subsetneq \mathrm{hIS}$*.*

PROOF. Let $\mathbf{A} \in \mathrm{hIS}_4$ and let $a, b \in A$. Then

$$\begin{aligned}(a \wedge b) \to b &= (a \wedge b) \to ((a \wedge b) \wedge b)\\ &= (a \wedge b) \to (a \wedge b)\\ &= 1,\end{aligned}$$

so $\mathbf{A} \in \mathrm{hIS}_5$. In order to show that hIS_4 is a proper subvariety of hIS_5, consider the boolean lattice B_4 of four elements, where x and y are the atoms, and consider the implication given in (2) of Example 3.1. Then $(a \wedge b) \to b = 1$ for every a, b. However, $x \to (x \wedge y) = x \to 0 = 0$ and $x \to y = y$, so $x \to (x \wedge y) \neq x \to y$. Thus, hIS_4 is a proper subvariety of hIS_5.

Finally we shall show that hIS_5 is a proper subvariety of hIS. Consider B_4 with the implication given in (1) of Example 3.1. Then $(x \wedge y) \to y = 0$. Therefore, the equation (H5) is not satisfied. ■

Let $\mathbf{A} \in \mathrm{hIS}$ and $a, b \in A$. Notice that if $a \to b = 1$ then $a \leq b$ because $a = a \wedge 1 = a \wedge (a \to b) \leq b$. In the following corollary we characterize the l-hemi-implicative semilattices in which the converse property does hold.

COROLLARY 3.5. *Let* $\boldsymbol{A} \in \mathrm{hIS}$*. The following conditions are equivalent:*

(1) $\boldsymbol{A} \in \mathrm{hIS}_5$*.*

(2) *For every* $a, b \in A$*,* $a \leq b$ *if and only if* $a \to b = 1$*.*

PROOF. Suppose that $\mathbf{A} \in \mathrm{hIS}_5$ and let $a \leq b$. Then $1 = (a \wedge b) \to b = a \to b$. Conversely, suppose (2) holds. Since $a \wedge b \leq b$ then $(a \wedge b) \to b = 1$. Therefore, $\mathbf{A} \in \mathrm{hIS}_5$. ■

PROPOSITION 3.6. *The following conditions are satisfied:*

(1) $\mathrm{K}_i \subseteq \mathrm{hIS}$ *for* $i = 1, \ldots, 6$*.*

(2) $\mathrm{K}_i \subseteq \mathrm{hIS}_4$ *for* $i = 4, 6$*.*

(3) $\mathrm{K}_i \subseteq \mathrm{hIS}_5$ *for* $i = 2, 3$*.*

Remark 3.7. For the algebras with the implication (1) and (5) of Example 3.1, if the universe of them is not trivial, then (H5) is not satisfied because $(0 \wedge 1) \to 1 = 0$. For the algebras with the implication (2) and with the implication (3) of Example 3.1 we have that $1 \to (1 \wedge 0) = 0$ and $1 \to 1 = 1$, so (H4) is not satisfied by them.

Lemma 3.8. *Let $\boldsymbol{A} \in \mathrm{hIS}$ and $a, b \in A$. Then $a \leftrightarrow b = b \leftrightarrow a$, $a \leftrightarrow a = 1$ and $a \wedge (a \leftrightarrow b) \leq b$.*

We write ChIS for the variety of l.hemi-implicative semilattices. Let $\mathbf{A} \in \mathrm{ChIS}$ and $a, b \in A$. Define the binary operation $\Rightarrow$ by $a \Rightarrow b := a \to (a \wedge b)$.

Lemma 3.9. *Let $\boldsymbol{A} \in \mathrm{ChIS}$ and $a, b \in A$. Then*

(1) $a \Rightarrow a = 1$,

(2) $a \wedge (a \Rightarrow b) \leq b$,

(3) $a \Rightarrow (a \wedge b) = a \Rightarrow b$,

(4) *If* $b \leq a$ *then* $a \Leftrightarrow b = a \Rightarrow b = a \to b$, *where* $a \Leftrightarrow b := (a \Rightarrow b) \wedge (b \Rightarrow a)$.

Proof. Let $a, b \in A$. Then $a \Rightarrow a = a \to a = 1$. Also

$$\begin{aligned} a \wedge (a \Rightarrow b) &= a \wedge (a \to (a \wedge b)) \\ &\leq a \wedge b \\ &\leq b. \end{aligned}$$

By definition of $\Rightarrow$ we have that $a \Rightarrow (a \wedge b) = a \Rightarrow b$.

Finally we shall prove (4). In order to show this, let $b \leq a$. We have that

$$\begin{aligned} a \Leftrightarrow b &= (a \Rightarrow b) \wedge (b \Rightarrow a) \\ &= (a \to (a \wedge b)) \wedge (b \to (b \wedge a)) \\ &= (a \to b) \wedge (b \to b) \\ &= (a \to b) \wedge 1 \\ &= a \to b. \end{aligned}$$

Moreover, $a \to b = a \to (a \wedge b) = a \Rightarrow b$. Thus (4) is proved. ∎

Write $\mathrm{ChIS_E}$ for the subvariety of ChIS whose algebras $\mathbf{A} = (A, \wedge, \sim, 1)$ satisfy the following condition:

(S) $a \to b = (a \to (a \wedge b)) \wedge (b \to (a \wedge b))$.

Corollary 3.10. *The varieties $\mathrm{hIS_4}$ and $\mathrm{ChIS_E}$ are term equivalent. More explicitly, we have:*

(1) *If* $\mathbf{A} = (A, \wedge, \rightarrow, 1) \in \mathrm{ChIS}$*, then the algebra* $\mathbf{A}^* = (A, \wedge, \Rightarrow, 1) \in \mathrm{hIS}_4$*, where* $\Rightarrow$ *is defined by* $a \Rightarrow b := a \rightarrow (a \wedge b)$*.*

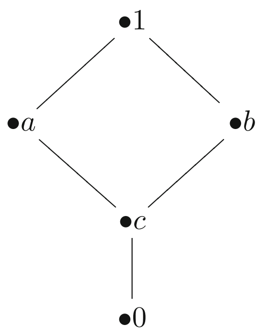
(2) *If* $\boldsymbol{A} = (A, \wedge, \rightarrow, 1) \in \mathrm{hIS}_4$*, then the algebra* $\mathbf{A}^+ = (A, \wedge, \sim, 1) \in \mathrm{ChIS}_{\mathrm{E}}$*, where* $\sim$ *is defined by* $a \sim b := a \leftrightarrow b$*.*

(3) *If* $\mathbf{A} \in \mathrm{ChIS}_{\mathrm{E}}$*, then* $\mathbf{A}^{*+} = \mathbf{A}$*,*

(4) *If* $\mathbf{A} \in \mathrm{hIS}_4$*, then* $\mathbf{A}^{+*} = \mathbf{A}$*.*

Note that $\mathrm{ChIS}_{\mathrm{E}}$ is a proper subvariety of ChIS, as the following example shows.

Example 3.11. *Let* A *be the lattice of the following figure:*



Define on A *the following binary operation:*

$\rightarrow$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	0	c	a
b	0	0	1	c	b
c	0	c	c	1	c
1	0	a	b	c	1

Straightforward computations show that the algebra $\boldsymbol{A} = (A, \wedge, \rightarrow, 1) \in \mathrm{ChIS}$*. However,* $a \rightarrow b = 0$*,* $a \rightarrow (a \wedge b) = c$ *and* $b \rightarrow (a \wedge b) = c$*. Thus, we obtain that* $a \rightarrow b \neq (a \rightarrow (a \wedge b)) \wedge (b \rightarrow (a \wedge b))$*. Therefore,* $\boldsymbol{A} \notin \mathrm{ChIS}_{\mathrm{E}}$*.*

4. Congruences

In this section we study the congruences for some subclasses of hIS. More precisely, in Section 4.1 we study the lattice of congruences for the algebras given in Example 3.1.

Let $\mathbf{A} \in \mathrm{hIS}$, $a, b \in A$ and let θ be a congruence of $\mathbf{A}$. We write a/θ to indicate the equivalence class of a associated to the congruence θ and $\theta(a, b)$

for the congruence generated by the pair (a, b). As usual, we say that F is a *filter* if it is a subset of A that satisfies the following conditions: $1 \in F$, if $a, b \in F$ then $a \wedge b \in F$, if $a \in F$ and $a \leq b$ then $b \in F$. We also consider the binary relation

$$\Theta(F) = \{(a, b) \in A \times A : a \wedge f = b \wedge f \text{ for some } f \in F\}.$$

Notice that if F is a filter in a semilattice with greatest element, then $\Theta(F)$ is the congruence generated by the filter F. For $\mathbf{A} \in \mathrm{hIS}$ and $a, b, f \in A$ we define the following element of A: $t(a, b, f) := (a \rightarrow b) \leftrightarrow ((a \wedge f) \rightarrow (b \wedge f))$.

Definition 4.1. Let $\mathbf{A} \in \mathrm{hIS}$ and F a filter of A. We say that F is a *congruent filter* if $t(a, b, f) \in F$ for every $a, b \in A$ and $f \in F$.

The following result was proved in [19].

Theorem 4.2. *Let $\boldsymbol{A} \in \mathrm{hIS}$. There exists an isomorphism between the lattice of congruences of $\boldsymbol{A}$ and the lattice of congruent filters of $\boldsymbol{A}$, which is established via the assignments $\theta \mapsto 1/\theta$ and $F \mapsto \Theta(F)$.*

Note that if $\mathbf{A} \in \mathrm{ChIS}$ and $a, b, f \in F$, then $t(a, b, f) = (a \rightarrow b) \rightarrow ((a \wedge f) \rightarrow (b \wedge f))$.

Corollary 4.3. *Let $\boldsymbol{A} \in \mathrm{ChIS}$. There exists an isomorphism between the lattice of congruences of $\boldsymbol{A}$ and the lattice of filters F of $\boldsymbol{A}$ which satisfy $(a \rightarrow b) \rightarrow ((a \wedge f) \rightarrow (b \wedge f)) \in F$ for every $a, b \in A$ and $f \in F$. The isomorphism is established via the assignments $\theta \mapsto 1/\theta$ and $F \mapsto \Theta(F)$.*

4.1. Congruences in Algebras of $\mathbf{K}_i$, $i = 1, \ldots, 6$

In this subsection we study the congruent filters associates to any of the aforementioned classes of algebras.

4.2. Congruent Filters Associated to Algebras in $\mathbf{K}_1$ or $\mathbf{K}_6$

Let F be a filter of $\mathbf{A} = (A, \wedge, \rightarrow, 1) \in \mathrm{hIS}$, where $\rightarrow$ is the implication given by (1) or (6).

Proposition 4.4. *F is a congruent filter if and only if $F = \{1\}$ or $F = A$.*

Proof. Suppose that F is a congruent filter and that $F \neq \{1\}$. We have that there exist $x, y \in A$ such that $x \neq y$ and $x/\Theta(F) = y/\Theta(F)$. Thus, there is f in F such that $x \wedge f = y \wedge f$. Suppose that $x \not\leq y$. Since F is a congruent filter we have that $(x \rightarrow y) \rightarrow 1 \in F$. But $x \not\leq y$, so $x \rightarrow y = 0$. Hence, $(x \rightarrow y) \rightarrow 1 = 0 \rightarrow 1$. However, $0 \rightarrow 1 = 0$ because $0 \not\leq 1$. Then we have that $0 \in F$. Analogously we can show that if $y \not\leq x$ then $0 \in F$ by considering $y \to x$ in place of $x \to y$. Therefore, $F = A$. It is immediate that if $F = \{1\}$ or $F = A$ then F is a congruent filter. ■

4.3. Congruent Filters of Algebras in $\mathbf{K_2}$

Let F be a filter of $\mathbf{A} = (A, \wedge, \to, 1) \in \mathrm{hIS}$, where $\to$ is the implication given by (2).

Proposition 4.5. *F is a congruent filter if and only if it satisfies the following conditions: for every $x, y \in A$ and $f \in F$,*

(1) *if $x \nleq y$ and $x \wedge f \leq y \wedge f$ then $y \in F$ and*

(2) *if $x \nleq y$, $x \wedge f \nleq y \wedge f$ and $y \nleq f$ then $y \in F$.*

Proof. Suppose that F is a congruent filter. Let $x, y \in A$ and $f \in F$. Assume $x \nleq y$ and $x \wedge f \leq y \wedge f$. Since $t(x, y, f) = y \in F$ then we have the condition (1). Suppose that $x \nleq y$, $x \wedge f \leq y \wedge f$ and $y \nleq f$. Taking into account that $t(x, y, f) = y \wedge f \in F$ we obtain the condition (2).

Conversely, suppose that the conditions (1) and (2) hold. Let $x, y \in A$ and $f \in F$. Suppose that $x \leq y$. Since $x \wedge f \leq y \wedge f$ then $t(x, y, f) = 1 \in F$. Now suppose now that $x \nleq y$. Case (a): if $x \wedge f \nleq y \wedge f$ and $y \leq f$ then $t(x, y, f) = 1 \in F$. If $x \wedge f \nleq y \wedge f$ and $y \nleq f$ then $t(x, y, f) = y \wedge f$. By (2) we have that $y \in F$, so $t(x, y, f) \in F$. Case (b): if $x \wedge f \leq y \wedge f$ then $t(x, y, f) = y$, which belongs to F by (1). Therefore, F is a congruent filter. ■

4.4. Congruent Filters of Algebras in $\mathbf{K_3}$

Let F be a filter of $\mathbf{A} = (A, \wedge, \to, 1) \in \mathrm{hIS}$, where $\to$ is the implication given by (3).

Proposition 4.6. *F is a congruent filter if and only if it satisfies the following conditions: for every $x, y \in A$ and $f \in F$,*

(1) *if $x \neq y$ and $x \wedge f = y \wedge f$ then $y \in F$ and*

(2) *if $x \neq y$, $x \wedge f \neq y \wedge f$ and $y \nleq f$ then $y \in F$.*

Proof. Assume that F is a congruent filter, and let $x, y \in A$ be such that $x \neq y$. Let $f \in F$. Then $t(x, y, f) = y \leftrightarrow ((x \wedge f) \to (y \wedge f))$. Moreover, $t(x, y, f) \in F$. If $x \wedge f = y \wedge f$ then $t(x, y, f) = y \in F$. Suppose now that $x \wedge f \neq y \wedge f$ and $y \nleq f$ (i.e., $y \neq y \wedge f$). Thus, $t(x, y, f) = y \leftrightarrow (y \wedge f) = y \wedge f \in F$, so $y \in F$.

Conversely, suppose that the conditions (1) and (2) hold. Consider $x, y \in A$ and $f \in F$. If $x = y$ then $t(x, y, f) = 1 \in F$. If $x \neq y$ then $t(x, y, f) = y \leftrightarrow ((x \wedge f) \to (y \wedge f))$. If $x \wedge f = y \wedge f$ then $t(x, y, f) = y$, which belongs to F

by condition (1). Suppose that $x \wedge f \neq y \wedge f$. Then $t(x, y, f) = y \leftrightarrow (y \wedge f)$. If $y \leq f$ then $t(x, y, f) = 1 \in F$. If $y \nleq f$ then $t(x, y, f) = y \wedge f$, which also belongs to F by condition (2) (because f, $y \in F$). Hence, F is a congruent filter. ∎

4.5. Congruent Filters of Algebras in $\mathbf{K}_4$

Let F be a filter of $\mathbf{A} = (A, \wedge, \rightarrow, 1) \in \mathrm{hIS}$, where $\rightarrow$ is the implication given by (4).

Proposition 4.7. *F is a congruent filter if and only if it satisfies the following conditions: for every $x, y \in A$ and $f \in F$,*

(1) *if $x \nleq y$ and $x \wedge f \leq y \wedge f$ then $x \wedge y \in F$ and*

(2) *if $x \nleq y$, $x \wedge f \nleq y \wedge f$ and $x \wedge y \nleq f$ then $x \wedge y \in F$.*

Proof. Suppose that F is a congruent filter. Let $x \nleq y$ and $x \wedge f \leq y \wedge f$. Then $t(x, y, f) = x \wedge y \in F$, so we have the condition (1). Suppose that $x \nleq y$, $x \wedge f \nleq y \wedge f$ and $x \wedge y \nleq f$. Then $t(x, y, f) = x \wedge y \wedge f \in F$, so $x \wedge y \in F$, which is the condition (2).

Conversely, suppose that it holds the conditions (1) and (2). Let $x, y \in A$ and $f \in F$. If $x \leq y$ then $t(x, y, f) = 1 \in F$. Suppose that $x \nleq y$. If $x \wedge f \leq y \wedge f$ then $t(x, y, f) = x \wedge y$, which belongs to F by the condition (1). Suppose that $x \nleq y$ and $x \wedge f \nleq y \wedge f$. Then $t(x, y, f) = (x \wedge y) \leftrightarrow (x \wedge y \wedge f)$. If $x \wedge y \leq f$ then $t(x, y, f) = 1 \in F$. If $x \wedge y \nleq f$ then $t(x, y, f) = x \wedge y \wedge f$. But by condition (2) we have that $x \wedge y \in F$. Therefore, $t(x, y, f) \in F$. ∎

4.6. Congruent Filters of Algebras in $\mathbf{K}_5$

Let F be a filter of $\mathbf{A} = (A, \wedge, \rightarrow, 1) \in \mathrm{hIS}$, where $\rightarrow$ is the implication given by (5).

Proposition 4.8. *F is a congruent filter if and only if it satisfies the following conditions: for every $x, y \in A$ and $f \in F$,*

(1) *if $x \neq y$ and $x \wedge f = y \wedge f$ then $x \wedge y \in F$ and*

(2) *if $x \neq y$, $x \wedge f \neq y \wedge f$ and $x \wedge y \nleq f$ then $x \wedge y \in F$.*

Proof. Suppose that F is a congruent filter. In order to prove (1) and (2) consider $x \neq y$. Suppose that $x \wedge f = y \wedge f$. Then we have that $t(x, y, f) = (x \wedge y) \leftrightarrow 1$. If $1 \leftrightarrow (x \wedge y) = 1$ then $1 = x \wedge y$, i.e., $x = y = 1$, which is an absurd. Then $t(x, y, f) = x \wedge y \in F$. Hence we have the condition (1). Suppose now that $x \wedge f \neq y \wedge f$ and $x \wedge y \nleq f$. Hence, $t(x, y, f) = (x \wedge y) \leftrightarrow$

$(x \wedge y \wedge f)$. But $x \wedge y \nleq f$, so $t(x, y, f) = x \wedge y \wedge f \in F$. Thus $x \wedge y \in F$, which is the condition (2).

Conversely, suppose that F satisfies the conditions (1) and (2). Let $x, y \in A$ and $f \in F$. If $x = y$ then $t(x, y, f) = 1 \in F$. Suppose that $x \neq y$, so $t(x, y, f) = (x \wedge y) \leftrightarrow ((x \wedge f) \rightarrow (y \wedge f))$. If $x \wedge f = y \wedge f$ then $t(x, y, f) = x \wedge y$, which belongs to F by (1). Suppose that $x \wedge f \neq y \wedge f$, so $t(x, y, f) = (x \wedge y) \leftrightarrow (x \wedge y \wedge f)$. If $x \wedge y \leq f$ then $t(x, y, f) = 1 \in F$. If $x \wedge y \nleq f$ then by (2) we have that $x \wedge y \in F$. But $t(x, y, f) = x \wedge y \wedge f$. Thus, $t(x, y, f) \in F$. ∎

4.7. Totally Ordered Posets

Let $\mathbf{A} = (A, \wedge, \rightarrow, 1) \in$ hIS where its underlying order is total. Let $x, y, f \in A$. If $f \leq x \wedge y$ then $t(x, y, f) = (x \rightarrow y) \leftrightarrow 1$, if $y \leq f \leq x$ then $t(x, y, f) = (x \rightarrow y) \leftrightarrow (f \rightarrow y)$, if $x \leq f \leq y$ then $t(x, y, f) = (x \rightarrow y) \leftrightarrow (x \rightarrow f)$ and if $x \leq f$ and $y \leq f$ then $t(x, y, f) = 1$. Hence, we obtain the following result.

Proposition 4.9. *Let $\boldsymbol{A} \in$ hIS such that its underlying poset is a chain and let F be a filter of $\boldsymbol{A}$. Then F is a congruent filter if and only if for every $x, y \in A$ and $f \in F$ the following conditions hold:*

(a) *If $f \leq x \wedge y$ then $(x \rightarrow y) \leftrightarrow 1 \in F$.*

(b) *If $y \leq f \leq x$ then $(x \rightarrow y) \leftrightarrow (f \rightarrow y) \in F$.*

(c) *If $x \leq f \leq y$ then $(x \rightarrow y) \leftrightarrow (x \rightarrow f) \in F$.*

Corollary 4.10. *Let $\mathbf{A} \in \mathrm{K}_i$, where $i = 2, 3, 4, 5$, such that the underlying order of $\mathbf{A}$ is total. Then, every filter of $\boldsymbol{A}$ is a congruent filter.*

Proof. Let $x, y \in A$ and $f \in F$. If $f \leq x \wedge y$, then $(x \rightarrow y) \leftrightarrow 1 \in \{x \wedge y, y, 1\} \subseteq F$. If $y \leq f \leq x$ then $(x \rightarrow y) \leftrightarrow (f \rightarrow y) \in \{f, 1\} \subseteq F$. If $x \leq f \leq y$ then $(x \rightarrow y) \leftrightarrow (x \rightarrow f) \in \{f, 1\} \subseteq F$. Therefore, it follows from Proposition 4.9 that F is a congruent filter. ∎

Notice, however, that it follows from Proposition 4.4 that there are l-hemi-implicative semilattices whose order is total and in which not every filter is a congruent filter.

On the other hand, it is not the case that every filter in a non totally ordered algebra of the classes considered in previous corollary is a congruent filter. Consider, for example, the boolean lattice of four elements, where x and y are the atoms, and let $F = \{x, 1\}$. We write B for the universe of this algebra. Let F be a filter of $(B, \wedge, \rightarrow, 1) \in K_i$, for $i = 1, \ldots, 6$. We write t_i for the ternary term t over the algebra $(B, \wedge, \rightarrow, 1) \in K_i$. Since $t_1(x, y, x)$,

$t_2(y,x,x)$, $t_3(0,y,x)$, $t_4(0,y,x)$, $t_5(y,x,x)$ and $t_6(x,y,x)$ are the bottom, F is not a congruent filter.

5. Principal Congruences for Algebras of hIS

In this section we characterize the principal congruences in hIS.

Let $\mathbf{A} = (A, \wedge, \rightarrow, 1) \in$ hIS and $a, b \in A$. We write $F^c(a)$ for the congruent filter generated by $\{a\}$. In [19] it was proved that if $\theta(a,b)$ is the congruence generated by (a,b), then $(x,y) \in \theta(a,b)$ if and only if $x \leftrightarrow y \in F^c(a \leftrightarrow b)$. We will give an explicit description of $F^c(a)$.

For $x, y \in A$ and $Z \subseteq A$ we define $t(x,y,Z) = \{t(x,y,z) : z \in Z\}$. Then we define $t^+(x,y,Z)$ as the the set of the elements $z \in A$ such that $z \geq t(x,y,w_1) \wedge \ldots \wedge t(x,y,w_k) \wedge w_{k+1} \wedge \ldots \wedge w_{k+t}$ for some $w_i \in Z$. In the next step we define

$$t(Z) = \bigcup_{x,y \in A} t^+(x,y,Z).$$

We also define $T_0(Z) = Z$, $T_{n+1}(Z) = t(T_n(Z))$ and $T(Z) = \bigcup_{n \in \mathbb{N}} T_n(Z)$, where $\mathbb{N}$ is the set of natural numbers. It is immediate that $a \in T(\{a\})$.

Proposition 5.1. *Let $\boldsymbol{A} \in$ hIS and $a \in A$. Then $F^c(a) = T(\{a\})$.*

Proof. Straightforward computations based on induction show that

$$T_n(\{a\}) \subseteq T_{n+1}(\{a\}) \tag{7}$$

for every n. We use this property throughout this proof.

We have that $1 \in T(\{a\})$. It follows from the construction that $T(\{a\})$ is an upset. In order to prove that this set is closed under $\wedge$, let $z, z' \in T(\{a\})$. Then there are n and m such that $z \in T_n(\{a\})$ and $z' \in T_m(\{a\})$. By (7) we have that $z, z' \in T_p(\{a\})$, where p is the greatest of n and m. Straightforward computations prove that $z \wedge z' \in T_p(\{a\})$, so $z \wedge z' \in T(\{a\})$. Hence, $T(\{a\})$ is a filter.

In order to show that $T(\{a\})$ is a congruent filter, let $z \in T(\{a\})$ and $x, y \in A$. Then, there is n such that $z \in T_n(\{a\})$. Taking into account that $t(x,y,z) \geq t(x,y,z) \wedge z$, we have that $t(x,y,z) \in t(T_n(\{a\}))$, i.e., $t(x,y,z) \in T_{n+1}(\{a\}) \subseteq T(\{a\})$. Thus, $T(\{a\})$ is a congruent filter.

Finally we show that $F^c(a) = T(\{a\})$. Let F be a congruent filter such that $a \in F$. We need to prove that $T(\{a\}) \subseteq F$, i.e., that $T_n(\{a\}) \subseteq T(\{a\})$ for every n. It is immediate that $T_0(\{a\}) \subseteq T(\{a\})$. Suppose that $T_n(\{a\}) \subseteq T(\{a\})$ for some n. We shall prove that $T_{n+1}(\{a\}) \subseteq T(\{a\})$. Let

$z \in T_{n+1}(\{a\})$. Then there are $x, y \in A$ and $w_1, \ldots, w_{k+t} \in T_n(\{a\})$ such that

$$z \geq t(x, y, w_1) \wedge \ldots \wedge t(x, y, w_k) \wedge w_{k+1} \wedge \ldots \wedge w_{k+t}.$$

But $T_n(\{a\}) \subseteq F$, and F is a congruent filter. Thus,

$$t(x, y, w_1) \wedge \ldots \wedge t(x, y, w_k) \wedge w_{k+1} \wedge \ldots \wedge w_{k+t} \in F.$$

Hence, $z \in F$. Therefore, $T_{n+1}(\{a\}) \subseteq F$, which was our aim. ■

Corollary 5.2. *Let $\boldsymbol{A} \in$ hIS and $a, b \in A$. Then $(x, y) \in \theta(a, b)$ if and only if $x \leftrightarrow y \in T(\{a \leftrightarrow b\})$.*

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